Lectures on generalised Seiberg–Witten equations

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Lecture 1

The purpose of this lecture is to discuss some gauge-theoretic partial differential equations, mostly introduced by physicists, which are known or conjectured to be important and put them in a unifying framework.

1 Anti-self-duality in Yang–Mills theory

A resonable minimalist setup for gauge theory consists of: a connected oriented Riemannian manifold (X, g), a compact connected semi-simple Lie group G, the **structure group**, and a G–principal bundle $(\pi\colon P \to X, P \circlearrowleft G)$. The space of G–principal connections on P is denoted by $\mathscr{A}(P)$. This is an affine space modelled on $\Omega^1(X, \mathrm{Ad}(P))$ with

$$\mathrm{Ad}(P) \coloneqq P \times_{\mathrm{Ad}} \mathfrak{g} \tag{1.1}$$

denoting the **adjoint bundle**. The **curvature** of $A \in \mathscr{A}(P)$ is denoted by $F_A \in \Omega^2(X, \mathrm{Ad}(P))$. Everything is acted upon by $\mathscr{G}(P)$, the **gauge group**. See, e.g., [Bau14; Ham17, Part I] for more on this.

The **Yang–Mills functional** $\mathrm{YM}\colon \mathscr{A}(P) \to [0, \infty]$ is defined by

$$\mathrm{YM}(A) \coloneqq \frac{1}{2}\int_X |F_A|^2 \,\mathrm{vol}_g. \tag{1.2}$$

Here $|\cdot|$ is induced by minus the Killing form on $\mathfrak{g}$. The Yang–Mills functional it is $\mathscr{G}(P)$–invariant. A straight-forward computation reveals that the Euler–Lagrange equation of the Yang–Mills functional is the **Yang–Mills equation**:

$$\mathrm{d}_A^* F_A = 0. \tag{1.3}$$

It is not trivial to solve (1.3). Of course, $A \in \mathscr{A}(P)$ satisfies (1.3) if it is **flat**, that is, $F_A = 0$. The monodromy representation induces a bijection

$$\coprod_P \{A \in \mathscr{A}(P) : F_A = 0\}/\mathscr{G}(P) \cong \mathrm{Hom}(\pi_1(X, x_0), G)/G \tag{1.4}$$

with G acting by conjugation. This seems rather satisfactory since it turns a question in geometric analysis into one in algebraic topology. However, $\mathrm{Hom}(\pi_1(X, x_0), G)$ is not as easy to determine as one might expect. In fact, for 3–manifolds, it turns out to be fruitful to use (1.4) in the other direction; cf. [Tau90].

Belavin, Polyakov, Schwartz, and Tyupkin [BPST75] observed that if $\dim X = 4$ and $A \in \mathscr{A}(P)$ is an **anti-self-dual (ASD) connection**,[1] that is,

$$F_A^+ \coloneqq \frac{1}{2}(F_A + *F_A) = 0, \tag{1.5}$$

then the Bianchi identity implies the Yang–Mills equation:

$$\mathrm{d}_A^* F_A = - * \mathrm{d}_A F_A = 0. \tag{1.6}$$

In fact, something even better is true. The Yang–Mills functional satisfies the identity

$$\mathrm{YM}(A) = \int_X |F_A^+|^2 \operatorname{vol}_g + 8\pi^2 \check{h}(G) k(A) \tag{1.7}$$

with

$$k(A) \coloneqq \frac{1}{8\pi^2 \check{h}(G)} \int_X \langle F_A \wedge F_A \rangle \tag{1.8}$$

denoting the **instanton number**, and $\check{h}(G)$ denoting the dual Coxeter number of G.[2] If X is closed, then by Chern–Weil theory

$$k(A) = k(P) \coloneqq -\frac{1}{2\check{h}(G)} \langle p_1(\mathrm{Ad}(P)), [X] \rangle \tag{1.9}$$

Therefore, ASD connections are absolute minimisers of the Yang–Mills energy: they saturate a topological lower bound.

Belavin, Polyakov, Schwartz, and Tyupkin also managed to find an explicit solution of (1.5) on $\mathbf{R}^4$: the **BPST instanton**. The latter can conveniently written using the normed division algebra $\mathbf{H} = \mathbf{R}\langle 1, i, j, k \rangle$ of the **quaternions** with relations $i^2 = j^2 = k^2 = ijk = -1$. The unit sphere $\mathrm{Sp}(1) \coloneqq \{q \in \mathbf{H} : |q| = 1\}$ is a Lie group with Lie algebra $\mathfrak{sp}(1) = \operatorname{Im} \mathbf{H}$. The BPST instanton is the connection I on the trivial bundle $\mathbf{H} \times \mathrm{Sp}(1)$ defined by

$$I \coloneqq \frac{\operatorname{Im}(\bar{q}\mathrm{d}q)}{|q|^2 + 1} \in \Omega^1(\mathbf{H}, \mathfrak{sp}(1)). \tag{1.10}$$

[1] There is a similar notion of **self-dual (SD) connection**: $F_A^- \coloneqq \frac{1}{2}(F_A - *F_A) = 0$. The notion of ASD and SD are swapped by flipping the orientation of X. Indeed, the early literature in the subject is concerned with SD connections. In the light of [Don85], ASD connections appear preferable. Indeed, on a Kähler surface (X, J, ω), there is a preferred orientation and the (-1)–eigenbundle $\Lambda^+ T^* X$ of the Hodge–$*$–operator agrees with $\Lambda_0^{1,1} T^* X$. Therefore, ASD connections induce holomorphic structures.

[2] [AHS78, §8] explains meaning of the instanton number $k(P)$ and the appearance of $\check{h}(G)$ in (1.8). It denotes the dual Coxeter number of G.

with $q \in C^\infty(\mathbf{H}, \mathbf{H})$ denoting the identity map. To verify that $F_I^+ = 0$ it is convenient to use

$$
\begin{aligned}
\mathrm{d}q \wedge \mathrm{d}\bar{q} = &-2(\mathrm{d}q_0 \wedge \mathrm{d}q_1 + \mathrm{d}q_2 \wedge \mathrm{d}q_3) \otimes i \\
&- 2(\mathrm{d}q_0 \wedge \mathrm{d}q_2 + \mathrm{d}q_3 \wedge \mathrm{d}q_1) \otimes j \\
&- 2(\mathrm{d}q_0 \wedge \mathrm{d}q_3 + \mathrm{d}q_1 \wedge \mathrm{d}q_2) \otimes k
\end{aligned} \tag{1.11}
$$

and

$$
\begin{aligned}
\mathrm{d}\bar{q} \wedge \mathrm{d}q = &\,2(\mathrm{d}q_0 \wedge \mathrm{d}q_1 - \mathrm{d}q_2 \wedge \mathrm{d}q_3) \otimes i \\
&2(\mathrm{d}q_0 \wedge \mathrm{d}q_2 - \mathrm{d}q_3 \wedge \mathrm{d}q_1) \otimes j \\
&2(\mathrm{d}q_0 \wedge \mathrm{d}q_3 - \mathrm{d}q_1 \wedge \mathrm{d}q_2) \otimes k.
\end{aligned} \tag{1.12}
$$

A brief computation reveals that $k(I) = 1$.

Although (in hindsight) Belavin, Polyakov, Schwartz, and Tyupkin's observation might appear mundane, the impact that it had on mathematics can hardly be underestimated. Building on foundational work of Atiyah, Hitchin, and Singer [AHS78], Uhlenbeck [Uhl82b; Uhl82a], Taubes [Tau82], Donaldson [Don83] discovered that the moduli spaces

$$\mathscr{M}^{\mathrm{ASD}}(P) := \{A \in \mathscr{A}(P) : F_A^+ = 0\}/\mathscr{G}(P). \tag{1.13}$$

and their Uhlenbeck compactifications contain subtle information about the differential topology of X and constructed his **Donaldson polynomials** [Don90]. In the hands of Taubes, Donaldson, Floer, Kronheimer, Mrowka, etc. moduli spaces of ASD connections have become a crucial tool in the (differential) topology of 3– and 4–manifolds.

2 The Seiberg–Witten equation

Witten [Wit88] realised that Donaldson theory can be understood physically by considering the ultraviolet behaviour of a topological twist of $N = 2$ supersymmetric Yang-Mills theory in dimension four. Seiberg and Witten [SW94] studied the infrared behaviour of the same theory, and Witten [Wit94] explains that this should give a way to compute the SU(2) Donaldson invariants in terms of a U(1) gauge theory.

The setup is slightly more elaborate than that of Section 1. Instead of a G–principal bundle P, it requires a **spin^c–structure** $\mathfrak{w}$ on (X, g), that is, a reduction of the frame bundle $\mathrm{Fr} = \mathrm{Fr}_{\mathrm{SO}}(TX)$ along

$$\mathrm{Spin}^{\mathrm{U}(1)}(4) := (\mathrm{Spin}(4) \times \mathrm{U}(1))/\{\pm 1\} \to \mathrm{SO}(4). \tag{2.1}$$

The latter determines **complex spinor bundles of positive and negative chirality** $W^\pm$, both Hermitian vector bundles of rank 2, and a **Clifford multiplication**

$$\gamma\colon TX \to \mathrm{Hom}(W^+, W^-) \oplus \mathrm{Hom}(W^-, W^+). \tag{2.2}$$

The latter can be extended to $\gamma\colon \Lambda^2 T^*X \to \mathrm{End}(W^+ \oplus W^-)$. In fact, the restriction $\gamma\colon \Lambda^+ T^*X \to \mathfrak{su}(S)$ is an isomorphism. Instead of arbitrary $A \in \mathscr{A}(\mathfrak{w})$, one restricts to the subset $\mathscr{A}(\mathfrak{w}, \mathrm{LC})$

of those A which induce the Levi-Civita connection LC on Fr, a.k.a., **spin connections**. These are determined by the connection $\det(A)$ which they induce on the Hermitian line bundle $\det(W^+) = \det(W^-)$. The **Seiberg–Witten equation** for $A \in \mathscr{A}(\mathfrak{w}; \mathrm{LC})$ and $\Phi \in \Gamma(W^+)$ is:

$$\begin{aligned} D_A\Phi &= 0 \\ \frac{1}{2}F^+_{\det(A)} &= \mu(\Phi). \end{aligned} \tag{2.3}$$

Here $D_A \colon \Gamma(W^+) \to \Gamma(W^-)$ is the **Dirac operator** defined by γ, and A and $\mu \colon W^+ \to \Lambda^+T^*X \otimes i\mathbf{R}$

$$\mu(\Phi) \coloneqq \gamma^{-1}\big(\Phi\Phi^* - \tfrac{1}{2}|\Phi|^2 \cdot \mathbf{1}\big). \tag{2.4}$$

At first glance, (2.3) might appear to be more complicated (1.5); however, it is technically much easier to work with because: the structure group U(1) is abelian, bubbling is not an issue, and the spinor field Φ can be bounded a priori. [Wit94] already contains a rigorous construction of the **Seiberg–Witten invariants**. Many of the results that were previously proved using the ASD equation with hard work, were reproved with the Seiberg–Witten equation and many new results were obtained; see. [Kot95] for a report on the early developments. See also [Don96; Wit07].

Witten [Wit94, (2.14)] conjectured that the Donaldson polynomial can be expressed in terms of the Seiberg–Witten invariants. Although Pidstrigach and Tyurin [PT95] put forward a tentative proposal for establishing this conjecture, and Feehan and Leness [FL98b; FL01b; FL01a], Feehan [Fee00], and Feehan and Leness [FL98a] have worked on fleshing this out, it is still not fully resolved. Neither is the simple type conjecture.

3 The Vafa–Witten equation

$N = 4$ supersymmetric Yang–Mills theory in physics depends on a **structure group** G, an **angle** θ, a **gauge coupling constant** $e > 0$, and a geometric background X. The S**–duality** or **electric-magnetic duality** conjecture originating with Goddard, Nuyts, and Olive [GNO77] and Montonen and Olive [MO77] predicts that this theory is invariant under simultaneously inverting $4\pi/e^2$ and replacing G with its **GNO dual** or **Langlands dual** ${}^LG$.[3] [4] [5] [6] If θ and e are combined into

$$\tau \coloneqq \frac{\theta}{2\pi} + \frac{4\pi i}{e^2} \in \mathfrak{H}, \tag{3.1}$$

[3] Electric-magnetic duality is not restricted to $N = 4$ supersymmetric Yang–Mills theory, In fact, it plays some role in [SW94].

[4] The observation that the GNO and Langlands dual agree is due to Atiyah.

[5] This is only true if G is simply-laced and if the Lie algebras of $\mathfrak{g}$ and ${}^L\mathfrak{g}$ are isomorphic. In general, there is an extra factor in front of the inversion; cf. [KW07, §2.2; Wu08].

[6] The structure group G (and a maximal torus T) determine the **root datum** $(X^*, X_*, \langle\cdot,\cdot\rangle, R, R^\vee, \vee)$: with $X^* \coloneqq \mathrm{Hom}(T, \mathrm{U}(1))$ denoting the character lattice, $X_* \coloneqq \mathrm{Hom}(\mathrm{U}(1), T)$ denoting the cocharacter lattice, $\langle\cdot,\cdot\rangle$ denoting the obvious perfect pairing, $R \subset X^*$ denoting roots, $R^\vee \subset X_*$ denoting the coroots, and $\vee \colon R \to R^\vee$ denoting the bijection between roots and coroots. In fact, the latter determines G up to isomorphism. Exchanging X^* and X_*, and R and $R^\vee$ yields the root datum of ${}^LG$. Evidently, ${}^{LL}G = G$. Determining ${}^LG$ from scratch is quite a bit of work. [KW07, Table 1] is quite helpful. For example, ${}^L\mathrm{SU}(2) = \mathrm{SO}(3)$.

then the inversion of $4\pi/e^2$ corresponds to the Möbius transformation

$$S(\tau) := -\frac{1}{\tau} \tag{3.2}$$

of the upper half-plane $\mathfrak{H} := \{\tau \in \mathbf{C} : \operatorname{Im}\tau > 0\}$.

Vafa and Witten [VW94] set out to test this prediction with X an oriented closed 4–manifold. $N = 4$ supersymmetric Yang–Mills theory in dimension four admits three topological twists [Yam88; Mar95; ESW20, p. 11]. Vafa and Witten judiciously choose one of them and are led to the **Vafa–Witten equation** for $A \in \mathscr{A}(P)$, $B \in \Omega^+(X, \operatorname{Ad}(P))$, and $C \in \Omega^0(X, \operatorname{Ad}(P))$:

$$\begin{aligned} \mathrm{d}_A C + \mathrm{d}_A^* B &= 0 \\ F_A^+ &= [C, B] + \frac{1}{2}[B \times B] \end{aligned} \tag{3.3}$$

with $\cdot \times \cdot \colon \Lambda^+T^*X \otimes \Lambda^+T^*X \to \Lambda^+T^*X$ defined by

$$(\alpha \times \beta)_{ac} := \sum_{a=1}^{4} \alpha_{ac}\beta_{cb} - \beta_{ac}\alpha_{cb}. \tag{3.4}$$

After gauge fixing, (3.3) is an elliptic equation of index zero. (In fact, it has been engineered too!) Therefore, it does not seem too outlandish to count the moduli spaces

$$\mathscr{M}^{\mathrm{VW}}(P) := \{(A, B, C) \in \mathscr{A}(P) \times \Omega^+(X, \operatorname{Ad}(P)) \times \Omega^0(X, \operatorname{Ad}(P)) : (3.3)\}/\mathscr{G}(P) \tag{3.5}$$

to obtain **Vafa–Witten invariants**

$$\text{"}\#\mathscr{M}^{\mathrm{VW}}(P) \in \mathbf{Z}.\text{"} \tag{3.6}$$

To do this in earnest requires finding a suitable compactification, transversality, etc. Although there has been a lot of work on the Vafa–Witten equation, in particular, by Mares [Mar10], Tanaka [Tan15; Tan17; Tan; Tan19], and Taubes [Tau17], a rigorous mathematical definition of the Vafa–Witten invariants for arbitrary closed oriented 4–manifolds is not available (yet?). The situation in algebraic geometry is much better: Tanaka and Thomas [TT17; TT18] have defined Vafa–Witten invariants for smooth projective surfaces and $G = \mathrm{U}(n)$.

Whatever the precise and rigorous definition of $\#\mathscr{M}^{\mathrm{VW}}(P)$ might be, the **partition function** of Vafa–Witten theory should be

$$Z_X^{\mathrm{VW}}(\tau, G) := \frac{1}{\#Z(G)} \sum_P \#\mathscr{M}^{\mathrm{VW}}(P) \cdot e^{2\pi i\tau(k(P)-s(X))}. \tag{3.7}$$

with $Z(G)$ denoting the center of G, the sum being taken over all isomorphism classes of G–principal bundles P over X, $k(P)$ denoting the instanton number (1.8), and the shift in the zero point of the instanton number given by

$$s(X) := \frac{\operatorname{rk} G + 1}{24}\chi(X). \tag{3.8}$$ [7]

Finally, the prediction of S–duality is that

$$Z_X^{\mathrm{VW}}(-1/\tau, {}^L G) = \pm\left(\frac{\tau}{i}\right)^{w(X)/2} Z^{\mathrm{VW}}(\tau, G) \tag{3.9}$$

with

$$\pm := (-1)^{\frac{\mathrm{rk}\, G}{4}(\chi(X)+\sigma(X))} \quad \text{and} \quad w(X) := -\chi(X).^{8} \tag{3.10}$$

S–duality is a $\mathbf{Z}/2\mathbf{Z}$ symmetry with the latter acting on τ and G by $-1/\tau$, ${}^L G$. This can be enhanced to an (only apparently!) more awe-inspiring symmetry under the **modular group** $\mathbf{PSL}_2(\mathbf{Z}) := \mathrm{SL}_2(\mathbf{Z})/\{\pm\mathbf{1}\}$. The latter acts on $\mathfrak{H}$ by Möbius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(\tau) := \frac{a\tau + b}{c\tau + d}. \tag{3.11}$$

It is well known that

$$\mathbf{PSL}_2(\mathbf{Z}) = \langle S, T \mid S^2 = (STS)^3 = \mathbf{1}\rangle \quad \text{with} \quad S := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad T := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. \tag{3.12}$$

S acts by (3.2) and sends G to ${}^L G$. Its action Z_X^{VW} is predicted by (3.9). T acts by $T(\tau) = \tau + 1$ and trivially on G. Its effect on $Z^{\mathrm{VW}}(\tau, G)$ is easy to read off from (3.7); in fact, it is trivial if $k(P) - s(X) \in \mathbf{Z}$. In any case, there is an minimal $N(X, G) \in \mathbf{N}$ such that

$$k(P) - s(X) \in \frac{1}{N(X, G)}\mathbf{Z} \tag{3.13}$$

and $T^{N(X,G)}$ leaves Z_X^{VW} invariant.

The Langlands dual ${}^L G$ can be removed from the picture. This most straight-forward if $N(X, G) = 1$. If $N := N(X, {}^L G)$, then $ST^N S$ and T act trivially on G. Therefore, Z_X^{VW} transforms under the **Hecke congruence subgroup of level** N

$$\Gamma_0(N) := \langle ST^N S, T\rangle = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{PSL}_2(\mathbf{Z}) : c = 0 \bmod N\right\}. \tag{3.14}$$

with factor of automorphy

$$j\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tau\right) := \pm\left(\frac{c\tau + d}{i}\right)^{w(X)/2}. \tag{3.15}$$

This is almost the definition of modular form of level N and weight $w(X)/2$—except that the factor of automorphy is slightly off and no condition on holomorphicity (at ∞) is imposed.

[8] The formulae for $s(X)$, $\pm$, and $w(x)$ are from Wu [Wu08, (3.9) and (3.10)]. Vafa and Witten are rather vague about this.

After making some leaps of faith, Vafa and Witten arrive at

$$Z_{K3}^{\mathrm{VW}}(\tau,\mathrm{SU}(2)) = \frac{1}{8}G(q^2) + \frac{1}{4}G(q^{1/2}) + \frac{1}{4}G(-q^{1/2}). \tag{3.16}$$

and

$$Z_{K3}^{\mathrm{VW}}(\tau,\mathrm{SO}(3)) = \frac{1}{4}G(q^2) + 2^{21}G(q^{1/2}) + 2^{10}G(-q^{1/2}). \tag{3.17}$$

with $q = e^{2\pi i\tau}$, and $G \coloneqq \eta^{-24}$ and $\eta(q) \coloneqq q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$ denoting the **Dedekind eta function**, a modular form of level 1 and weight $\frac{1}{2}$.

It is worthwhile to give some idea for how Vafa and Witten did this. Evidently,

$$\mathscr{M}^{\mathrm{ASD}}(P) \subset \mathscr{M}^{\mathrm{VW}}(P). \tag{3.18}$$

The Vafa–Witten equation has been engineered so that if $\mathscr{M}^{\mathrm{ASD}}(P)$ agrees with $\mathscr{M}^{\mathrm{VW}}(P)$, is $\mathscr{M}^{\mathrm{ASD}}(P)$ is compact, cut-out transversely, and contains no reducible solutions, then

$$\#\mathscr{M}^{\mathrm{VW}}(P) = \chi(\mathscr{M}^{\mathrm{ASD}}(P)). \tag{3.19}$$

Of course, that is not the case for $X = K3$ and $G = \mathrm{SU}(2)$ or $\mathrm{SO}(3)$, but it is not too far off. in particular, $\mathscr{M}^{\mathrm{ASD}}(P)$ has an interpretation in complex geometry and a compactification. The Euler characterstic of the latter can be computed and this lead Vafa and Witten to the above formulae.

In general, the Vafa–Witten invariant—if it exists—cannot be an Euler characteristic of a compactified moduli space of ASD connections because the Euler characteristic is only bordism invariant mod 2.[9]

4 The Kapustin–Witten equation

The geometric Langlands conjecture predicts a fantastically complicated relation between the moduli spaces of $G^{\mathbf{C}}$ local systems over a Riemann surface Σ and the moduli space of holomorphic ${}^LG^{\mathbf{C}}$–principal bundles over Σ. The appearance of ${}^LG$ in S–duality suggests a connection between geometric Langlands conjecture and $N = 4$ super-symmetric Yang–Mills theory. Indeed, deep relations have been uncovered—starting with the work of Kapustin and Witten [KW07]. The foundation of their work is a topological twist which leads to the following family equations for $A \in \mathscr{A}(P)$ and $\phi \in \Omega^1(X, \mathrm{Ad}(P))$

$$\begin{aligned} u \cdot \left(F_A - \tfrac{1}{2}[\phi \wedge \phi]\right)^+ - v \cdot \mathrm{d}_A^+\phi &= 0 \\ v \cdot \left(F_A - \tfrac{1}{2}[\phi \wedge \phi]\right)^- + u \cdot \mathrm{d}_A^-\phi &= 0 \\ \mathrm{d}_A^*\phi &= 0 \end{aligned} \tag{4.1}$$

parametrised by $[u : v] \in \mathbf{C}P^1$. It would go far beyond the scope of these lecture notes to even start discussing Kapustin and Witten's work. It is, however, worthwhile to discuss some instances of (4.1).

[9]However, the Euler characteristic is a complex bordism invariant.

Kapustin and Witten's perspective on the geometric Langlands conjecture only requires the **Kapustin–Witten equation**, the specialisation of (4.1) to $[1:1]$:

$$\begin{aligned} F_A - \frac{1}{2}[\phi \wedge \phi] - *\mathrm{d}_A\phi &= 0 \\ \mathrm{d}_A^*\phi &= 0 \end{aligned} \tag{4.2}$$

The other equations (4.1) play a role in the quantum geometric Langlands conjecture.

For $[1:\pm i]$ the first two equations (4.1) combine to

$$F_{A\pm i\phi} = 0; \tag{4.3}$$

that is, $A + i\phi$ is a flat $G^{\mathbf{C}}$–principal connection on $P^{\mathbf{C}} := P \times_G G^{\mathbf{C}}$. Corlette [Cor88] proved that if $A + i\phi$ is **stable**, that is, its monodromy representation does not factor through a proper parabolic subgroup of $G^{\mathbf{C}}$, then its $\mathscr{G}(P^{\mathbf{C}})$–orbit contains a solution of (4.1) for $[1:\pm 1]$.

The specialisation of (4.1) at $[1:0]$ is the following **complex anti-self-duality** condition:

$$\begin{aligned} \left(F_A - \tfrac{1}{2}[\phi \wedge \phi]\right)^+ &= 0 \\ \mathrm{d}_A^-\phi &= 0 \\ \mathrm{d}_A^*\phi &= 0. \end{aligned} \tag{4.4}$$

The specialisation of (4.1) at $[0:1]$ is a complex self-duality condition.

Theorem 4.5 (Kapustin and Witten [KW07, §3.3] and Gagliardo and Uhlenbeck [GU12]). *Suppose that X is closed.*

(1) *If $\langle p_1(\mathrm{Ad}(P)), [X]\rangle \neq 0$, then (4.1) admits solutions only for $[u:v] \in \{[1:0], [0,1]\}$.*

(2) *If $\langle p_1(\mathrm{Ad}(P)), [X]\rangle = 0$ and (A, ϕ) satisfies (4.1) for any $[u, v] \in \mathbf{C}P^1$, then it also satisfies (4.1) for any $[1, \pm 1] \in \mathbf{C}P^1$.*

The Kapustin–Witten equation (4.2) is therefore most relevant if X is not closed; e.g.: if $X = I \times Y$ with I an interval and Y a closed 3–manifold. A brief computation reveals that (4.2) for $\mathbf{A} = A + \xi\mathrm{d}t$ and $\boldsymbol{\phi} = \phi + \eta\mathrm{d}t$ becomes

$$\begin{aligned} \partial_t\phi - [\xi, \phi] - *_Y F_A + \frac{1}{2} *_Y [\phi \wedge \phi] - *_Y \mathrm{d}_A\eta &= 0 \\ \partial_t A - \mathrm{d}_A\xi + [\phi, \eta] - *\mathrm{d}_A\phi &= 0 \\ \partial_t\eta + [\xi, \eta] - \mathrm{d}_A^*\phi &= 0. \end{aligned} \tag{4.6}$$

This looks more friendly if $\xi = 0$ (temporal gauge) and $\eta = 0$. It can be interpreted as the gradient flow of a complexified Chern–Simons functional.

The Kapustin–Witten equation does not just play a role in the geometric Langlands conjecture, (4.6) with $I = [0, \infty)$ is also at the heart of Witten's new approach to the Jones polynomial: the knot is encoded as a boundary condition on $\{0\} \times Y$. This will be explained in *Alfred Holmes*'s lecture on Thursday. More propaganda for the Kapustin–Witten equation can be found in [Uhl12].

Lecture 2

Schematically, the ASD equation (1.5), the Seiberg–Witten equation (2.3), the Vafa–Witten equation (3.3), and the complex ASD equation (4.4) are quite similar. Indeed, they can be understood in the unifying framework of generalised Seiberg–Witten equations. The idea of systematically generalising the Seiberg–Witten equation goes back to the work of Taubes [Tau99] and Pidstrygach [Pid04]. Several versions of have been discussed in [Hay06; Hay15; Sal13, §6; Nak16, §6(i); DW20, §2.2; WZ21, §1.1]. The following construction might, initially, seem a little intimidating, but it is really not much more complicated than what is required for the Seiberg–Witten equation (2.3).

5 Warm-up: spinors in dimension four

As a warm-up it is helpful to recall a few facts about Spin(4). The construction relies on the fact that

$$\mathrm{Spin}(4) = \mathrm{Sp}(1) \times \mathrm{Sp}(1) \tag{5.1}$$

Define $\sigma_\pm \colon \mathrm{Spin}(4) \to \mathrm{Sp}(1) < \mathrm{SO}(\mathbf{H})$, $\mathrm{Ad} \colon \mathrm{Spin}(4) \to \mathrm{SO}(4) = \mathrm{SO}(\mathbf{H})$, and $\delta_\pm \colon \mathrm{Spin}(4) \to \mathrm{SO}(3) = \mathrm{SO}(\mathrm{Im}\,\mathbf{H})$ by by

$$\sigma_\pm(p_+, p_-) \coloneqq p_\pm, \mathrm{Ad}(p_+, p_-)v \coloneqq p_- v \bar{p}_+, \quad \text{and} \quad \delta_\pm(p_+, p_-)(q) \coloneqq p_+ q p_-^{-1}. \tag{5.2}$$

Evidently, for every $(p_+, p_-) \in \mathrm{Spin}(4)$, $v, \Phi \in \mathbf{H}$

$$\mathrm{Ad}(p_+, p_-)v \cdot \sigma_+(p_+, p_-)\Phi = \sigma_-(p_+, p_-)[v \cdot \Phi]. \tag{5.3}$$

A moment's thought shows that the isomorphisms

$$\mathrm{Im}\,\mathbf{H} \to \Lambda^+\mathbf{H}^*, \quad \xi \mapsto \langle \mathrm{d}q \wedge \mathrm{d}\bar{q}, \xi \rangle \quad \text{and} \quad \mathrm{Im}\,\mathbf{H} \to \Lambda^-\mathbf{H}^*, \quad \xi \mapsto \langle \mathrm{d}\bar{q} \wedge \mathrm{d}q, \xi \rangle \tag{5.4}$$

are equivariant with respect to the actions of $\delta_\pm$ and Ad.

If $\mathfrak{s}$ is a spin structure on (X, g), then

$$\mathfrak{s} \times_{\mathrm{Ad}} \mathbf{H} \cong TX, \quad \mathfrak{s} \times_{\delta_\pm} \mathrm{Im}\,\mathbf{H} \cong \Lambda^\pm T^*X \tag{5.5}$$

and

$$S^\pm \coloneqq \mathfrak{s} \times_{\sigma^\mp} \mathbf{H} \tag{5.6}$$

are the **spinor bundles of positive and negative chirality**. Because of (5.3) multiplication of quaternions induces to a Clifford multiplication $\gamma \colon TX \to \mathrm{Hom}(S^+, S^-)$.[10]

[10] The unpleasant reversal of signs comes from the fact that the volume element acts as $\gamma(1)\gamma(i)\gamma(j)\gamma(k) = -\mathbf{1}$ on $\mathbf{H}$.

6 Generalised Seiberg–Witten equations

The setup for a generalised Seiberg–Witten equation involves choices of algebraic and geometric data.

Definition 6.1.

(1) A **quaternionic Hermitian vector space** is a left $\mathbf{H}$–module S together with an Euclidean inner product $\langle\cdot,\cdot\rangle$ such that i, j, k act by isometries. The **unitary symplectic group** $\mathrm{Sp}(S)$ is the subgroup of $\mathrm{GL}_{\mathbf{H}}(S)$ preserving $\langle\cdot,\cdot\rangle$.

(2) A **quaternionic representation** of a Lie group H is a Lie group homomorphism $\rho\colon H \to \mathrm{Sp}(S)$ for some quaternionic Hermitian vector space S.

(3) A set of **algebraic data** consists of:

   (a) A compact connected Lie group G: the **structure group**.

   (b) A compact connected Lie group K: the **auxiliary group**.

   (c) A quaternionic representation $\rho\colon G\times K \to \mathrm{Sp}(S)$ together with an $\varepsilon \in Z(G)\times Z(H)$ satisfying $\varepsilon^2 = \mathbf{1}$ and $\rho(\varepsilon) = -\mathbf{1}$. •

Choose a set of algebraic data. Define $\gamma\colon \mathbf{H} \to \mathrm{End}(S)$ and $\underline{\gamma}\colon \mathbf{H}\otimes\mathfrak{g} \to \mathrm{End}(S)$ by

$$\gamma(v)\Phi := v\cdot\Phi \quad \text{and} \quad \underline{\gamma}(v\otimes\xi)\Phi := v\cdot \mathrm{Lie}(\rho)(\xi)\Phi. \tag{6.2}$$

The quaternionic representation $\rho|_G\colon G \to \mathrm{Sp}(S)$ has a **distinguished hyperkähler moment map** $\mu\colon S \to (\mathrm{Im}\,\mathbf{H}\otimes\mathfrak{g})^*$ defined by

$$\mu(\Phi) := \frac{1}{2}\underline{\gamma}^*(\Phi\Phi^*), \tag{6.3}$$

that is,

$$\langle \mu(\Phi), v\otimes\xi\rangle = \frac{1}{2}\langle v\cdot \mathrm{Lie}(\rho)(\xi)\Phi, \Phi\rangle. \tag{6.4}$$

Set

$$\mathrm{Spin}^{G\times K}(4) := (\mathrm{Spin}(4)\times G\times K)/\{\pm 1\} \quad \text{and} \quad \mathrm{Spin}^{K}(4) := (\mathrm{Spin}(4)\times K)/\{\pm 1\}. \tag{6.5}$$

Denote by

$$\pi\colon \mathrm{Spin}^{G\times K}(4) \to \mathrm{Spin}^{K}(4) \quad \text{and} \quad \kappa\colon \mathrm{Spin}^{K}(4) \to \mathrm{SO}(4) \tag{6.6}$$

the obvious maps. Define $\tau_\pm\colon \mathrm{Spin}^{G\times K}(4) \to \mathrm{O}(S)$ by

$$\tau_\pm([p_+, p_-, g, k])\Phi := \gamma(\sigma_\pm(p_+, p_-))\rho(g,k)\Phi. \tag{6.7}$$

Evidently,

$$\gamma(\mathrm{Ad}(p_+, p_-)v)\tau_+([\mathrm{Ad}(p_+, p_-, g, k])\Phi = \tau_-([p_+, p_-, g, k])\Phi; \tag{6.8}$$

and similarly for γ. Moreover, μ is equivariant with respect to the actions of $\tau_\pm$ on S and $\delta_\pm$ and $\rho|_G$ on $(\mathrm{Im}\,\mathbf{H}\otimes\overline{\mathfrak{g}})^*$.

Definition 6.9. A set of **geometric data** consists of:

(1) A κ–reduction $\mathrm{Fr}^\star$ of Fr together with a connection $\mathrm{LC}^\star$ which induces LC on Fr.

(2) A π–reduction $\mathfrak{s}$ of $\mathrm{Fr}^\star$. •

Choose a set of geometric data. Define the **spinor bundles of positive and negative chirality** and the **relative adjoint bundle** by

$$S^\pm := \mathfrak{s} \times_{\tau_\pm} S \quad \text{and} \quad \mathrm{Ad}(\mathfrak{s}/\mathrm{Fr}^\star) := \mathfrak{s} \times_{\mathrm{Spin}^{G\times K}(4)} \mathfrak{g}. \tag{6.10}$$

For $A \in \mathscr{A}(\mathfrak{s})$ denote by

$$F_{A/\mathrm{Fr}^\star} \in \Omega^2(X, \mathrm{Ad}(\mathfrak{s}/\mathrm{Fr}^\star)) \tag{6.11}$$

the projection of $F_A \in \Omega^2(X, \mathrm{Ad}(\mathfrak{s}/\mathrm{Fr}^\star))$ to $\Omega^2(X, \mathrm{Ad}(\mathfrak{s}))$. The maps γ, $\underline{\gamma}$, and μ induce

$$\gamma\colon\ TX \to \mathrm{End}(S^+, S^-), \quad \underline{\gamma}\colon\ TX \otimes \mathrm{Ad}(\mathfrak{s}/\mathrm{Fr}^\star) \to \mathrm{End}(S^+, S^-), \quad \text{and} \tag{6.12}$$

$$\mu\colon\ S^+ \to \Lambda^+ T^*X \otimes \mathrm{Ad}(\mathfrak{s}/\mathrm{Fr}^\star). \tag{6.13}$$

Denote $\mathscr{A}(\mathfrak{s}, \mathrm{LC}^\star)$ the space of $\mathrm{Spin}^{G\times K}(4)$–principal connections on $\mathfrak{s}$ which induce $\mathrm{LC}^\star$ on $\mathrm{Fr}^\star$. This is an affine space modelled on $\Omega^1(X, \mathrm{Ad}(\mathfrak{s}/\mathrm{Fr}^\star))$. Every $A \in \mathscr{A}(\mathfrak{s}, \mathrm{LC}^\star)$ defines a Dirac operator

$$D_A\colon\ \Gamma(S^+) \to \Gamma(S^-). \tag{6.14}$$

Definition 6.15. The **generalised Seiberg–Witten equation** associated with the above is the following equation for $A \in \mathscr{A}(\mathfrak{s}/\mathrm{Fr}^\star)$ and $\Phi \in \Gamma(S^+)$:

$$\begin{aligned} D_A^+\Phi &= 0 \\ F^+_{A/\mathrm{Fr}^\star} &= \mu(\Phi). \end{aligned} \tag{6.16}$$

Of course, (6.16) is not preserved by the gauge group $\mathscr{G}(\mathfrak{s})$ of $\mathfrak{s}$, but it is preserved by $\mathscr{G}(\mathfrak{s}/\mathrm{Fr}^\star)$, the subgroup acting trivially on $\mathrm{Fr}^\star$.

Remark 6.17. It is possible to replace S by a hyperkähler manifold together with a trihamiltonian action and a permuting action. In fact, this is done in [Tau99; Pid04; Hay06; Hay15; Sal13, §6; Nak16, §6(i)]. ♣

7 Examples of generalised Seiberg–Witten equations

Example 7.1 (harmonic spinors). If $G = \{\mathbf{1}\}$, $K = \{\pm\mathbf{1}\}$, $\varepsilon = (1, -1)$, and $S = \mathbf{H}$, then $\mathrm{Fr}^\star$ is a spin structure, $\mathrm{LC}^\star$ is unique, and solutions to (6.16) are harmonic spinors of negative chirality. ♠

Example 7.2 (ASD instantons). If $K = \{\mathbf{1}\}$, $\varepsilon = (\mathbf{1}, \mathbf{1})$, and $S = 0$, then the generalised Seiberg–Witten equation (6.16) reduces to the ASD equation (1.5). ♠

Example 7.3 (The Seiberg–Witten equation). Let $G = \mathrm{U}(1)$, $K = \mathbf{1}$, $\varepsilon = (-1, \mathbf{1})$. Define the quaternionic representation $\rho\colon \mathrm{U}(1) \to \mathrm{Sp}(\mathbf{H})$ by

$$\rho(e^{i\alpha})q \coloneqq qe^{i\alpha}. \tag{7.4}$$

Identifying $(i\mathbf{R} \otimes \operatorname{Im}\mathbf{H})^* = i\mathbf{R} \otimes \operatorname{Im}\mathbf{H}$, the hyperkähler moment map $\mu\colon \mathbf{H} \to (i\mathbf{R} \otimes \operatorname{Im}\mathbf{H})^*$ is

$$\mu(q) = -\frac{i}{2} \otimes qiq^*. \tag{7.5}$$

Splitting $\mathbf{H} = \mathbf{C} \oplus j\mathbf{C}$, we see that $\underline{\gamma}(\mu(q)) \in \mathrm{End}(\mathbf{C}^{\oplus 2})$ for $q = z + jw$ is

$$\frac{1}{2}\begin{pmatrix} |z|^2 - |w|^2 & 2z\bar{w} \\ 2\bar{z}w & |w|^2 - |z|^2 \end{pmatrix} = q\langle q, \cdot\rangle_{\mathbf{C}} - \frac{1}{2}|q|_{\mathbf{C}}^2 \operatorname{id}_{\mathbf{C}^{\oplus 2}}. \tag{7.6}$$

In this situation, $\mathrm{Fr}^\star = \mathrm{Fr}$, $\mathfrak{s}$ is a spin^c structure, A is a choice of spin connection on $\mathfrak{s}$, S^+ is the complex spinor bundle of positive chirality, $F^+_{A/\mathrm{Fr}^\star} = \frac{1}{2}F_{\det(A)}$. Therefore, generalized Seiberg–Witten equation (6.16) is the (classical) Seiberg–Witten equation (2.3). ♠

Example 7.7 (The Seiberg–Witten equation with n spinors). Let $n \in \mathbf{N}$. Let $G = \mathrm{U}(1)$, $K = \mathrm{SU}(n)$, $\varepsilon = (-1, 1)$. $G \times K$ acts on $S = \mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^n = \mathbf{H}^n$. The moment map is $\mu(q_1, \ldots, q_n) = \sum_{a=1}^n \mu(q_a)$ with $\mu(q_a)$ as above. $\mathrm{Fr}^\star$ is the choice of a rank n Hermitian vector bundle E and $\mathrm{LC}^\star$ is a choice of a unitary connection on E, $\mathfrak{s}$ is the choice of a spin^c structure, and $S^+ = W^+ \otimes E$ ♠

Example 7.8 (Vafa–Witten). Let G be a compact connected semi-simple Lie group. Set $S := \mathfrak{g} \otimes_{\mathbf{R}} \mathbf{H}$. The adjoint representation induces a quaternionic representation $\rho\colon G \to \mathrm{Sp}(S)$. The distinguished moment map $\mu\colon S \to \operatorname{Im}\mathbf{H} \otimes \mathfrak{g}$ is given by

$$\begin{aligned}\mu(\xi) &= \frac{1}{2}[\xi, \xi] \\ &= ([\xi_2, \xi_3] + [\xi_0, \xi_1]) \otimes i + ([\xi_3, \xi_1] + [\xi_0, \xi_2]) \otimes j + ([\xi_1, \xi_2] + [\xi_0, \xi_3]) \otimes k\end{aligned} \tag{7.9}$$

with $\xi = \xi_0 \otimes 1 + \xi_1 \otimes i + \xi_2 \otimes j + \xi_3 \otimes k \in \mathbf{H} \otimes \mathfrak{g}$. The action of $q \in K = \mathrm{Sp}(1)$ given by right-multiplication with $\bar{q}$ commutes with ρ. Set $-\varepsilon := (-1, 1_G) \in H$. The embedding $\mathrm{SO}(4) = (\mathrm{Sp}(1) \times \mathrm{Sp}(1))/\{\pm 1\} \to \mathrm{Spin}^{\mathrm{Sp}(1)}(4)$ defined by

$$[p, q] \mapsto [p, q, p] \tag{7.10}$$

determines the choice of $\mathrm{Fr}^\star$ and $\mathrm{LC}^\star$. The choice of $\mathfrak{s}$ is equivalent to a choice of G–principal bundle P; moreover, $\mathrm{Ad}(\mathfrak{s}/\mathrm{Fr}^\star) = \mathrm{Ad}(P)$ and $\mathscr{A}(\mathfrak{s}, \mathrm{LC}^\star) = \mathscr{A}(P)$. Therefore,

$$S^+ = (\underline{\mathbf{R}} \oplus \Lambda^+ T^*X) \otimes \mathrm{Ad}(P) \quad \text{and} \quad S^- = T^*X \otimes \mathrm{Ad}(P) \tag{7.11}$$

with

$$\gamma(v)(f, \omega) = v^\flat f - i_v \omega. \tag{7.12}$$

The generalised Seiberg–Witten equation (6.16) for A and $\Phi = (C, B)$ is the Vafa–Witten equation (3.3) for (A, B, C). ♠

Example 7.13 (complex ASD). The algebraic data of Example 7.8 with the geometric data determined by the embedding $\mathrm{SO}(4) = (\mathrm{Sp}(1) \times \mathrm{Sp}(1))/\{\pm 1\} \to \mathrm{Spin}^{\mathrm{Sp}(1)}(4)$ defined by

$$[p, q] \mapsto [p, q, g] \tag{7.14}$$

leads to

$$S^+ = T^*X \otimes \mathrm{Ad}(P) \quad \text{and} \quad S^- = (\underline{\mathbf{R}} \oplus \Lambda^- T^*X) \otimes \mathrm{Ad}(P). \tag{7.15}$$

and (4.4). ♠

It can be insightful to use the above construction in infinite dimensions.

Example 7.16 (Haydys–Witten equation). Let $I \subset \mathbf{R}$ be an interval. Denote by $\mathrm{pr}_X \colon I \times X \to X$ the projection. Let H connected compact semi-simple Lie group. Set $S := C^\infty(I, \mathfrak{h} \otimes_R \mathbf{H})$ and $G := C^\infty(I, H)$. G acts on S via

$$(g\xi)(t) := g(t)\xi(t)g(t)^{-1} - \dot{g}(t)g(t)^{-1}. \tag{7.17}$$

A computation reveals that

$$\begin{aligned} \mu(\xi) = (&\dot{\xi}_1 + [\xi_2, \xi_3] + [\xi_0, \xi_1]) \otimes i \\ &+ (\dot{\xi}_2 + [\xi_3, \xi_1] + [\xi_0, \xi_2]) \otimes j \\ &+ (\dot{\xi}_3 + [\xi_1, \xi_2] + [\xi_0, \xi_3]) \otimes k. \end{aligned} \tag{7.18}$$

Choose $K = \mathrm{Sp}(1)$ and $\mathrm{Fr}^\star$ as in Example 7.8. The choice of $\mathfrak{s}$ is a choice of H–principal bundle P on X. In this situation, $\mathscr{A}(\mathfrak{s}, \mathrm{LC}^\star) = \mathscr{A}(\mathrm{pr}_X^* P)$, $\Gamma(\mathrm{Ad}(\mathfrak{s}/\mathrm{Fr}^\star)) = \Gamma(\mathrm{pr}_X^* \mathrm{Ad}(P))$, and

$$\begin{aligned} \Gamma(S^+) &= \Gamma(I \times X, \mathrm{pr}_X^*[(\underline{\mathbf{R}} \oplus \Lambda^+ T^*X) \otimes \mathrm{Ad}(P)]) \quad \text{and} \\ \Gamma(S^-) &= \Gamma(I \times X, \mathrm{pr}_x^*(T^*X \otimes \mathrm{Ad}(P))). \end{aligned} \tag{7.19}$$

The generalised Seiberg–Witten equation (6.16) for $A \in \mathscr{A}(\mathrm{pr}_X^* P)$, $B \in \Gamma(I \times X, \mathrm{pr}^*[\Lambda^+ T^*X \otimes \mathrm{Ad}(P)])$, and $C \in \Gamma(I \times X, \mathrm{pr}_X^* \mathrm{Ad}(P))$ becomes the **Haydys–Witten equation**:

$$\begin{aligned} &\mathrm{d}_A C + \mathrm{d}_A^* B - \partial_t A = 0 \\ &F_{A(t)}^+ = -\partial_t B + [C, B] + \frac{1}{2}[B \times B]. \end{aligned} \tag{7.20}$$

This is equation was introduced by [Hay14, §3] and Witten [Wit12]]. In a sense this is a gauge theory in dimension 5, but (7.20) does not make sense or arbitrary oriented 5–manifolds (at least without a further choices of structure).

Evidently, the dimensonal reduction of (7.20) along I is (3.3). The dimensonal reduction of (7.20) on $\mathbf{R} \times Y$ along Y gives (4.6). In this sense (4.6) is a generalised Seiberg–Witten equation in dimension three. This is explained in [Hay14, §4] and Witten [Wit12, §5.3.1]. ♠

The ASD equation (1.5), the Seiberg–Witten equation (2.3), the Vafa–Witten equation (3.3), and the complex ASD equation (4.4) cannot just be considered as generalised Seiberg–Witten equations. They can also be understood as arising from the Spin(7)–**instanton equation** by dimensional reduction. This is no coincidence, since (in the relevant cases) the latter itself can be seen as generalised Seiberg–Witten equation. This was observed by Haydys [Hay08; Hay12] and Donaldson and Segal [DS11, §6].

Example 7.21 (Spin(7)–instantons). For simplicity restrict to $X = \mathbf{H}$. Consider $S := \Omega^1(\mathbf{H}, \mathfrak{h})$ and $G = C^\infty(\mathbf{H}, H)$. S can be thought of as the space of connections on $P := \mathbf{H} \times G$ and G is the gauge group $\mathscr{G}(P)$. It was observed by Atiyah that

$$\mu(I) = F_I^+ \in \Omega^+(\mathbf{H}, \mathfrak{g}). \tag{7.22}$$

Since $X = \mathbf{H}$, the choice of K and $\mathrm{Fr}^\star$ is almost immaterial. The choice of $\mathfrak{s}$ is too. A section $I \in \Gamma(S^+)$ is simply a family of connections on P parametrised by a $t \in \mathbf{H}$. A connection $A \in \mathscr{A}(\mathfrak{s}, \mathrm{LC}^\star)$ can be written as

$$A = \sum_{a=0}^{3} \xi_a \mathrm{d}t_a$$

with $\xi_a \in C^\infty(\mathbf{H}, \mathfrak{h})$. A computation shows that (6.16) becomes

$$\begin{aligned} \partial_{t_0} I - \mathrm{d}_I \xi_0 - i(\partial_{t_1} I - \mathrm{d}_I \xi_1) - j(\partial_{t_2} I - \mathrm{d}_I \xi_2) - j(\partial_{t_3} I - \mathrm{d}_I \xi_3) &= 0 \\ F_A^+ &= F_I^+. \end{aligned} \tag{7.23}$$

Another computation shows that this is equivalent to $\mathbf{A} := I + \sum_{a=0}^3 \xi_a \mathrm{d}t_a$ considered as a connection on $\mathbf{H}^2 \times G$ satisfies the Spin(7)–**instanton equation**

$$*(F_A \wedge \Phi) = -F_A \wedge \Phi \tag{7.24}$$

with $\Phi \in \Omega^4(\mathbf{H}^2)$ defined by

$$\Phi := \mathrm{vol}_1 + \mathrm{vol}_2 - \frac{1}{4}\langle \mathrm{d}q_1 \wedge \mathrm{d}\bar{q}_1, \mathrm{d}q_2 \wedge \mathrm{d}\bar{q}_2 \rangle \tag{7.25}$$

with vol_a denoting the usual volume form on the $\mathbf{H}$ factors.

In general, a variation of the above can be formulated for every oriented Euclidean rank 4 bundle V over an oriented 4–manifold (X, g) together with an isometry $\Lambda^+ V \cong \Lambda^+ T^* X$. ♠

In the previous example, formally,

$$S /\!\!/\!\!/ G = \mu^{-1}(0)/G \tag{7.26}$$

is the moduli space of ASD instantons on the trivial bundle. If $H = \mathrm{SU}(r)$ one does defines S and G a little more carefully, then $S /\!\!/\!\!/ G$ turns into the framed moduli space $\mathscr{M}_{k,r}^{\mathfrak{f}}$ of $\mathrm{SU}(r)$ ASD instantons over $\mathbf{R}^4$ with instanton number k.

Theorem 7.27 (Atiyah, Drinfeld, Hitchin, and Manin [ADHM78]; see also Atiyah [Ati79]). *Let $r \in \{2, 3, \ldots\}$ and $k \in \mathbf{N}$. Consider the The quaternionic vector space*

$$S_{r,k} := \mathrm{Hom}_{\mathbf{C}}(\mathbf{C}^r, \mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^k) \oplus \mathbf{H}^* \otimes_{\mathbf{R}} \mathfrak{u}(k)$$

has an open subset $S_{r,k}^{\mathrm{reg}}$ such that

$$\mathscr{M}_{k,r} \cong S_{r,k}^{\mathrm{reg}} /\!\!/\!\!/ \mathrm{U}(k).$$

Remark 7.28. $S_{r,k} /\!\!/\!\!/ \mathrm{U}(k)$ is the Uhlenbeck compactification of $\mathscr{M}_{k,r}$ ♣

Example 7.29. For $r, k \in \mathbf{N}$, consider the quaternionic Hermitian vector space

$$S_{r,k} := \mathrm{Hom}_{\mathbf{C}}(\mathbf{C}^r, \mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^k) \oplus \mathbf{H}^* \otimes_{\mathbf{R}} \mathfrak{u}(k)$$

and

$$G = \mathrm{U}(k) \triangleleft H = \mathrm{SU}(r) \times \mathrm{Sp}(1) \times \mathrm{U}(k) \quad \text{and} \quad -1 := (\mathbf{1}, -\mathbf{1}, -\mathbf{1}).$$

If $r \geqslant 2$, then $S_{r,k} /\!\!/\!\!/ G := \mu^{-1}(0)/G$ is the Uhlenbeck compactification of the moduli space of framed $\mathrm{SU}(r)$ ASD instantons of charge k on $\mathbf{R}^4$ [ADHM78]. If $r = 1$, then

$$S_{1,k} /\!\!/\!\!/ G = \mathrm{Sym}^k \mathbf{H} := \mathbf{H}^k / S_k;$$

see [Nak99, Proposition 2.9; DW19, Theorem D.2].

The generalized Seiberg–Witten equation associated with the above data is called the **ADHM$_{r,k}$ Seiberg–Witten equation**. It was introduced in [DW20, Example A.3; DW19, Section 5.1] and is expected to play an important role in gauge theory on G_2–manifolds [DS11; Wal17; Hay17]. For $k = 1$, this is essentially the Seiberg–Witten equation with r spinors. ♠

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