On the compactness problem for a family of generalized Seiberg–Witten equations in dimension three

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Abstract

We prove an abstract compactness theorem for a family of generalized Seiberg–Witten equations in dimension three. This result recovers Taubes' compactness theorem for stable flat $\mathbf{PSL}_2(\mathbf{C})$–connections [Tau13a] as well as the compactness theorem for Seiberg–Witten equations with multiple spinors [HW15]. Furthermore, this result implies a compactness theorem for the $\mathrm{ADHM}_{1,2}$ Seiberg–Witten equation, which partially verifies a conjecture by Doan and Walpuski [DW17, Conjecture 5.26].

1 Introduction

The study of the compactness problem for generalized Seiberg–Witten equations was pioneered by Taubes [Tau13a] with his compactness theorem for stable flat $\mathbf{PSL}_2(\mathbf{C})$–connections in dimension three. Building on the ideas developed in [Tau13a], Haydys and Walpuski [HW15] proved a compactness theorem for the Seiberg–Witten equation with multiple spinors in dimension three, and Taubes proved compactness theorems for the Kapustin–Witten equation [Tau13b], the Vafa–Witten equation [Tau17], and the Seiberg–Witten equation with multiple spinors in dimension four [Tau16]. Although the statements of these compactness theorems are very similar, many details of their proofs seem to rely heavily on the particular structure of the equation under consideration. The purpose of this article is to prove an abstract compactness theorem for generalized Seiberg–Witten equations in dimension three for which a simple analytical hypothesis holds. Our result recovers Taubes' compactness theorem for stable flat $\mathbf{PSL}_2(\mathbf{C})$–connections [Tau13a] as well as the compactness theorem for Seiberg–Witten equations with multiple spinors [HW15]. Furthermore, it also implies a compactness theorem for the $\mathrm{ADHM}_{1,2}$ Seiberg–Witten equation, which partially verifies a conjecture by Doan and Walpuski [DW17, Conjecture 5.26].

1.1 Generalized Seiberg–Witten equations

Let us review the relation between quaternionic representations and generalized Seiberg–Witten equations on an oriented Riemannian 3–manifold. For more detailed discussions we refer the reader to [Tau99; Hay14; DW19; DW17, Appendix B].

Definition 1.1. Denote by $\mathbf{H} = \mathbf{R}\langle 1, i, j, k\rangle$ the normed division algebra of the quaternions. A **quaternionic Hermitian vector space** is a left $\mathbf{H}$–module S together with an Euclidean inner product $\langle\cdot,\cdot\rangle$ such that i, j, k act by isometries. The **unitary symplectic group** $\mathrm{Sp}(S)$ is the subgroup of $\mathrm{GL}_{\mathbf{H}}(S)$ preserving $\langle\cdot,\cdot\rangle$.

Definition 1.2. A **quaternionic representation** of a Lie group H is a Lie group homomorphism $\rho\colon H \to \mathrm{Sp}(S)$ for some quaternionic Hermitian vector space S.

Let H be a compact Lie group. Denote its Lie algebra by $\mathfrak{h}$. Let $\rho\colon H \to \mathrm{Sp}(S)$ be a quaternionic representation. Abusing notation, we also denote the induced Lie algebra representation by $\rho\colon \mathfrak{h} \to \mathfrak{sp}(S)$. Define $\gamma\colon \operatorname{Im}\mathbf{H} \to \mathrm{End}(S)$, $\boldsymbol{\gamma}\colon \mathfrak{h}\otimes\operatorname{Im}\mathbf{H} \to \mathrm{End}(S)$, and $\mu\colon S \to (\mathfrak{h}\otimes\operatorname{Im}\mathbf{H})^*$ by

$$\gamma(v)\phi := v\phi, \quad \boldsymbol{\gamma}(\xi\otimes v) := \rho(\xi)\gamma(v), \quad \text{and} \quad \mu(\phi) := \frac{1}{2}\boldsymbol{\gamma}^*(\phi\phi^*), \tag{1.3}$$

respectively. The map μ is an equivariant hyperkähler moment map for the action of H on S.

Data 1.4. A set of **algebraic data** consists of:

1. a compact Lie group H with a distinguished element $-1 \in Z(H)$ satisfying $(-1)^2 = 1_H$,
2. a closed, connected, normal subgroup $G \triangleleft H$, and
3. a quaternionic representation $\rho\colon H \to \mathrm{Sp}(S)$

Having chosen a set of algebraic data, the **flavor symmetry group** is the quotient

$$K := H/(G \times \langle -1\rangle).$$

A reader who is entirely unfamiliar with generalized Seiberg–Witten equations should initially suppose that $G = H$, although this excludes many interesting cases.

Definition 1.5. Set $\mathrm{Spin}^H(3) := (\mathrm{Sp}(1)\times H)/\mathbf{Z}_2$. $\mathrm{Spin}^H(3)$ projects onto $\mathrm{Sp}(1)/\mathbf{Z}_2 = \mathrm{SO}(3)$. A **spin^H–structure** on (M, g) is a principal $\mathrm{Spin}^H(3)$–bundle $\mathfrak{s}$ together with an isomorphism

$$\mathfrak{s}\times_{\mathrm{Spin}^H(3)}\mathrm{SO}(3) \cong \mathrm{SO}(TM).$$

A spin^H–structure $\mathfrak{s}$ together with $G \triangleleft H$ and ρ induces:

1. the **flavor bundle**
$$\mathfrak{f} := \mathfrak{s}\times_{\mathrm{Spin}^H(3)} K,$$
2. the **adjoint bundle**
$$\mathrm{Ad}(\mathfrak{s}) := \mathfrak{s}\times_{\mathrm{Spin}^H(3)}\mathrm{Lie}(G),$$
3. the **spinor bundle**
$$\mathbf{S} := \mathfrak{s}\times_{\mathrm{Spin}^H(3)} S,$$

   as well as

4. maps

$$\gamma\colon TM \to \mathrm{End}(S), \quad \boldsymbol{\gamma}\colon TM \otimes \mathrm{Ad}(\mathfrak{s}) \to \mathrm{End}(S) \quad \text{and} \quad \mu\colon \mathbf{S} \to \Lambda^2 T^*M \otimes \mathrm{Ad}(\mathfrak{s}),$$

where γ and $\boldsymbol{\gamma}$ are induced directly by (1.3), and μ is induced by (1.3) and the isomorphism $\Lambda^2 T^*M \otimes \mathrm{Ad}(\mathfrak{s}) \cong T^*M \otimes \mathrm{Ad}(\mathfrak{s})^*$.

Definition 1.6. A **spin connection** on $\mathfrak{s}$ is a connection which induces the Levi–Civita connection on TM. The space of all spin connections on $\mathfrak{s}$ inducing a fixed connection B on $\mathfrak{f}$ is denoted by

$$\mathscr{A}(\mathfrak{s}, B).$$

Given a spin connection A, denote by

$$\mathrm{Ad}(A) \in \mathscr{A}(\mathrm{Ad}(\mathfrak{s}))$$

the induced connection on $\mathrm{Ad}(\mathfrak{s})$ and define the **Dirac operator** $\not{D}_A\colon \Gamma(\mathbf{S}) \to \Gamma(\mathbf{S})$ by

$$\not{D}_A \Phi \coloneqq \sum_{i=1}^{3} \gamma(e_i) \nabla_{A, e_i} \Phi$$

for e_1, e_2, e_3 a local orthonormal frame.

Data 1.7. A set of **geometric data** compatible with a given set of algebraic data (G, H, ρ) consists of:

1. an oriented Riemannian 3–manifold (M, g) together with a spin^H–structure $\mathfrak{s}$, and
2. a connection B on the flavor bundle induced by $\mathfrak{s}$.

Definition 1.8. The **generalized Seiberg–Witten equation** associated with the data (G, H, ρ) and $(M, g, \mathfrak{s}, B)$ is the following partial differential equation for $A \in \mathscr{A}(\mathfrak{s}, B)$ and $\Phi \in \Gamma(\mathbf{S})$:

$$\begin{aligned} \not{D}_A \Phi &= 0 \quad \text{and} \\ F_{\mathrm{Ad}(A)} &= \mu(\Phi). \end{aligned} \tag{1.9}$$

To illustrate the above construction, let us consider a few examples.

Example 1.10. Define the quaternionic representation $\rho\colon \mathrm{U}(1) \to \mathrm{Sp}(\mathbf{H})$ by

$$\rho(e^{i\alpha})q \coloneqq q\cos(\alpha) + qi\sin(\alpha).$$

Identifying $(i\mathbf{R} \otimes \mathrm{Im}\,\mathbf{H})^* = i\mathbf{R} \otimes \mathrm{Im}\,\mathbf{H}$, the hyperkähler moment map $\mu\colon \mathbf{H} \to (i\mathbf{R} \otimes \mathrm{Im}\,\mathbf{H})^*$ is

$$\mu(q) = -\frac{i}{2} \otimes qiq^*.$$

Splitting $\mathbf{H} = \mathbf{C} \oplus j\mathbf{C}$, we see that $\boldsymbol{\gamma}(\mu(q)) \in \mathrm{End}(\mathbf{C}^{\oplus 2})$ for $q = z + jw$ is

$$\frac{1}{2}\begin{pmatrix} |z|^2 - |w|^2 & 2z\bar{w} \\ 2\bar{z}w & |w|^2 - |z|^2 \end{pmatrix} = q\langle q, \cdot\rangle_{\mathbf{C}} - \frac{1}{2}|q|^2_{\mathbf{C}}\, \mathrm{id}_{\mathbf{C}^{\oplus 2}}. \tag{1.11}$$

Let (M, g) be an oriented Riemannian 3–manifold and let $\mathfrak{s}$ be a $\mathrm{spin}^{\mathrm{U}(1)}$–structure on M; that is: a spin^c–structure. The adjoint bundle $\mathrm{Ad}(\mathfrak{s})$ is $i\underline{\mathbf{R}}$. Denote the spinor bundles of $\mathfrak{s}$ by $\mathbf{S}$. If $A \in \mathscr{A}(\mathfrak{s})$, then it induces a connection $\det(A)$ on $\det(\mathbf{S})$ with

$$F_{\det(A)} = 2F_{\mathrm{Ad}(A)}.$$

Therefore, the generalized Seiberg–Witten equation (1.9) associated with the above data agrees with the **classical Seiberg–Witten equation**

$$\begin{aligned} \not{D}_A\Phi &= 0 \quad \text{and} \\ \frac{1}{2}\boldsymbol{\gamma}(F_{\det(A)}) &= \Phi\langle\Phi, \cdot\rangle_{\mathbf{C}} - |\Phi|^2_{\mathbf{C}}\mathrm{id}_{\mathbf{S}} \end{aligned}$$

appearing, for example, in [Wit94, Section 2; KM07, Section 1.3].

Example 1.12. Let G be a compact Lie group and set $\mathfrak{g} := \mathrm{Lie}(G)$. Choosing an G–invariant inner product on $\mathfrak{g}$ turns $S := \mathfrak{g} \otimes_{\mathbf{R}} \mathbf{H}$ into a quaternionic Hermitian vector space. The adjoint representation induces a quaternionic representation $\rho\colon G \to \mathrm{Sp}(S)$. The moment map $\mu\colon S \to \mathrm{Im}\,\mathbf{H} \otimes \mathfrak{g}$ is given by

$$\begin{aligned} \mu(\boldsymbol{\xi}) &= \frac{1}{2}[\boldsymbol{\xi}, \boldsymbol{\xi}] \\ &= ([\xi_2, \xi_3] + [\xi_0, \xi_1]) \otimes i + ([\xi_3, \xi_1] + [\xi_0, \xi_2]) \otimes j + ([\xi_1, \xi_2] + [\xi_0, \xi_3]) \otimes k \end{aligned}$$

for $\boldsymbol{\xi} = \xi_0 \otimes 1 + \xi_1 \otimes i + \xi_2 \otimes j + \xi_3 \otimes k \in \mathbf{H} \otimes_{\mathbf{R}} \mathfrak{g}$. Extend ρ to a quaternionic representation of $H := \mathrm{Sp}(1) \times G$ by declaring that $q \in \mathrm{Sp}(1)$ acts by right-multiplication with q^*. Set $-1 := (-\mathbf{1}, 1_G) \in H$.

Since

$$\mathrm{Spin}^H(3) = (\mathrm{Sp}(1) \times \mathrm{Sp}(1))/\mathbf{Z}_2 \times G = \mathrm{SO}(4) \times G,$$

a spin^H–structure is nothing but an oriented Euclidean vector bundle N of rank 4 together with an orientation-preserving isometry $\Lambda^+N \cong TM$ and a principal G–bundle P. Choosing $N = \underline{\mathbf{R}} \oplus T^*M$ and B induced by the Levi-Civita connection, the generalized Seiberg–Witten equation (1.9) associated with the above data becomes the following partial differential equation for $A \in \mathscr{A}(P)$, $a \in \Omega^1(M, \mathrm{Ad}(P))$, and $\xi \in \Gamma(\mathrm{Ad}(P))$:

$$\begin{aligned} \mathrm{d}_A^* a &= 0, \\ *\mathrm{d}_A a + \mathrm{d}_A \xi &= 0, \quad \text{and} \\ F_A &= \tfrac{1}{2}[a \wedge a] + *[\xi, a]. \end{aligned} \tag{1.13}$$

If $\xi = 0$, then (1.13) is precisely the condition for $A + ia$ to be a **stable flat $G^{\mathbf{C}}$–connection**; see [Don87; Cor88, Theorem 3.3]. In fact, if M is closed, then (1.13) implies $d_A\xi = 0$ and $[\xi, a] = 0$ and, therefore, that $A + ia$ is a stable flat $G^{\mathbf{C}}$–connection.

The compactness problem for (1.13) with $G = \mathrm{SO}(3)$ has been considered in Taubes' pioneering work [Tau13a], to which many of the techniques in this article can be traced back.

Example 1.14. For $r, k \in \mathbf{N}$, consider the quaternionic Hermitian vector space

$$S_{r,k} := \mathrm{Hom}_{\mathbf{C}}(\mathbf{C}^r, \mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^k) \oplus \mathbf{H}^* \otimes_{\mathbf{R}} \mathfrak{u}(k)$$

and

$$G = \mathrm{U}(k) \triangleleft H = \mathrm{SU}(r) \times \mathrm{Sp}(1) \times \mathrm{U}(k) \quad \text{and} \quad -1 := (\mathbf{1}, -\mathbf{1}, -\mathbf{1}).$$

If $r \geqslant 2$, then $S_{r,k}/\!\!/\!\!/G := \mu^{-1}(0)/G$ is the Uhlenbeck compactification moduli space of the moduli space of framed $\mathrm{SU}(r)$ ASD instantons of charge k on $\mathbf{R}^4$ [ADHM78]. If $r = 1$, then

$$S_{1,k}/\!\!/\!\!/G = \mathrm{Sym}^k \mathbf{H} := \mathbf{H}^k/S_k;$$

see [Nak99, Proposition 2.9; DW17, Theorem D.2].

The generalized Seiberg–Witten equation associated with the above data is called the **$\mathrm{ADHM}_{r,k}$ Seiberg–Witten equation**. It was introduced in [DW19, Example A.3; DW17, Section 5.1] and is expected to play in important role in gauge theory on G_2–manifolds [DS11; Wal17; Hay17]. For $k = 1$, this is essentially the Seiberg–Witten equation with r spinors, whose compactness problem has been considered by Haydys and Walpuski [HW15].

1.2 An abstract compactness theorem

Throughout this subsection, fix a set of algebraic data (G, H, ρ) and a compatible set of geometric data $(M, g, \mathfrak{s}, B)$ with M closed. The following result is well-known and and follows from standard elliptic theory.

Proposition 1.15. *If (A_n, Φ_n) is a sequence of solutions of (1.9) satisfying*

$$\liminf_{n\to\infty} \|\Phi_n\|_{L^2} < \infty,$$

then, after passing to a subsequence and up to gauge transformations, (A_n, Φ_n) converges to a solution (A, Φ) of (1.9) in the C^∞ topology.

Therefore, a degenerating sequence (A_n, Φ_n) of solutions of (1.9) must involve $\|\Phi_n\|_{L^2}$ becoming unbounded. In light of this, it is convenient to pass to the following equivalent equation.

Definition 1.16. The **blown-up generalized Seiberg–Witten equation** associated with the data (G, H, ρ) and $(M, g, \mathfrak{s}, B)$ is the following partial differential equation for $A \in \mathscr{A}(\mathfrak{s}, B)$, $\Phi \in \Gamma(\mathbf{S})$, and $\varepsilon \in (0, \infty)$:

$$\not{D}_A\Phi = 0, \quad \varepsilon^2 F_{\mathrm{Ad}(A)} = \mu(\Phi), \quad \text{and} \tag{1.17}$$

$$\|\Phi\|_{L^2} = 1. \tag{1.18}$$

The main result of this article is the following abstract compactness theorem.

Definition 1.19. Given $\Phi \in \Gamma(\mathbf{S})$, define $\Gamma_\Phi \colon \Lambda^2 T^*M \otimes \mathrm{Ad}(\mathfrak{s}) \to \mathbf{S}$ by

$$\Gamma_\Phi \coloneqq \boldsymbol{\gamma}(\cdot)\Phi.$$

Hypothesis 1.20. *There are constants $r_0, \delta_\mu, c > 0$ and $\Lambda \geqslant 0$ such that the following holds for every $x \in M$ and $r \in (0, r_0]$. If $A \in \mathscr{A}(\mathfrak{s}, B)$, $\Phi \in \Gamma(\mathbf{S})$, and $\varepsilon \in (0, \infty)$ satisfy (1.17),*

$$\frac{1}{2} \leqslant |\Phi| \leqslant 2, \quad \textit{and} \quad |\mu(\Phi)| \leqslant \delta_\mu,$$

on $B_r(x)$, then

$$r\int_{B_{r/2}(x)} |F_{\mathrm{Ad}(A)}|^2 \leqslant \Lambda + cr\int_{B_r(x)} |\Gamma_\Phi F_{\mathrm{Ad}(A)}|^2. \tag{1.21}$$

Theorem 1.22. *Suppose Hypothesis 1.20 holds. If $(A_n, \Phi_n, \varepsilon_n)_{n\in\mathbf{N}}$ is a sequence of solutions of (1.17) and (1.18) with ε_n tending to zero, then the following hold:*

1. *There is a closed, nowhere-dense subset $Z \subset M$, a connection $A \in \mathscr{A}(\mathfrak{s}|_{M\setminus Z}, B)$, and a spinor $\Phi \in \Gamma(M\setminus Z, \mathbf{S})$ such that the following hold:*
   (a) *A and Φ satisfy*

$$\begin{aligned} \not{D}_A\Phi &= 0, \\ \mu(\Phi) &= 0, \quad \textit{and} \\ \|\Phi\|_{L^2} &= 1. \end{aligned} \tag{1.23}$$

   (b) *The function $|\Phi|$ extends to a Hölder continuous function on all of M and*

$$Z = |\Phi|^{-1}(0).$$

2. *After passing to a subsequence and up to gauge transformations, $(A_n|_{M\setminus Z})_{n\in\mathbf{N}}$ converges to A in the weak $W^{1,2}_{\mathrm{loc}}$ topology, $(\Phi_n|_{M\setminus Z})_{n\in\mathbf{N}}$ converges to Φ in the weak $W^{2,2}_{\mathrm{loc}}$ topology, and there exists an $\alpha \in (0,1)$ such that $(|\Phi_n|)_{n\in\mathbf{N}}$ converges to $|\Phi|$ in the $C^{0,\alpha}$ topology.*

Remark 1.24. Hypothesis 1.20 holds, in particular, if the following condition is satisfied: there are constants $\delta, c > 0$ such that, for every $\Phi \in S$ with $|\Phi| = 1$ and $|\mu(\Phi)| \leqslant \delta$,

$$|\mu(\Phi)| \leqslant c|\Gamma_\Phi \mu(\Phi)|.$$

Remark 1.25. Since Γ_Φ is one-half of the adjoint of $\mathrm{d}_\Phi\mu$, the condition in Remark 1.24 is satisfied if $\mu^{-1}(0)$ is cut-out transversely away from the origin, that is: if, for every non-zero $\Phi \in \mu^{-1}(0)$, $\mathrm{d}_\Phi\mu$ is surjective. This is the case for the quaternionic representation $\mathrm{U}(1) \to \mathrm{Sp}(\mathbf{H}^n)$ which induces the Seiberg–Witten equation with multiple spinors. Therefore, Theorem 1.22 recovers [HW15, Theorem 1.5].

Remark 1.26. For the quaternionification of the adjoint representation of G, $\mu^{-1}(0)$ is never cut-out transversely away from the origin. Nevertheless, Lemma 5.1 shows that the algebraic criterion in Remark 1.24 is satisfied for SO(3) and SU(2). Therefore, Theorem 1.22 applies to stable flat $\mathbf{PSL}_2(\mathbf{C})$–connections over 3–manifolds; cf. Remark 1.33.

Remark 1.27. For many generalized Seiberg–Witten equations, including the Seiberg–Witten equation with multiple spinors and stable flat $\mathbf{PSL}_2(\mathbf{C})$–connections, a solution of (1.23) gives rise to a harmonic $\mathbf{Z}_2$ spinor whose zero locus is precisely Z [Tau14]. In this case, Zhang [Zha17, Theorem 1.4] proved that Z is $\mathscr{H}^1$–rectifiable and has finite 1–dimensional Minkowski content, cf. Section 5.4.

1.3 A compactness theorem for the ADHM$_{1,2}$ Seiberg–Witten equation

Let us discuss Example 1.14 for $r = 1$ and $k = 2$ in more detail. Decomposing $\mathfrak{u}(2) = \mathfrak{su}(2) \oplus \mathfrak{u}(1)$, $S = S_{1,2}$ can be written as

$$S = S_\circ \oplus \mathbf{H} \otimes_{\mathbf{R}} \mathfrak{u}(1) \quad \text{with} \quad S_\circ \coloneqq \mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^2 \oplus \mathbf{H} \otimes_{\mathbf{R}} \mathfrak{su}(2).$$

U(2) acts trivially on $\mathbf{H} \otimes_{\mathbf{R}} \mathfrak{u}(1)$; hence, the moment map $\mu\colon S \to \mathfrak{u}(2) \otimes \operatorname{Im} \mathbf{H}$ factors through the projection of S onto $S_\circ$.

Let (M, g) be a closed Riemannian 3–manifold. A spin$^{\mathrm{Sp}(1)\times \mathrm{U}(2)}$–structure $\mathfrak{s}$ on (M, g) is nothing but a spin$^{\mathrm{U}(2)}$–structure $\mathfrak{w}$ and a Euclidean vector bundle N of rank 4 together with an orientation-preserving isometry

$$\Lambda^+ N \cong TM.$$

Set

$$W \coloneqq \mathfrak{w} \times_{\mathrm{Spin}^{\mathrm{U}(2)}(3)} \mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^2 \quad \text{and} \quad \mathrm{Ad}(\mathfrak{w})_\circ \coloneqq \mathfrak{w} \times_{\mathrm{Spin}^{\mathrm{U}(2)}(3)} \mathfrak{su}(2).$$

The spinor bundle $\mathbf{S}$ and the flavor bundle $\mathfrak{f}$ associated with $\mathfrak{s}$ are

$$\mathbf{S} = W \oplus N \otimes \mathrm{Ad}(\mathfrak{w})_\circ \oplus N \otimes i\underline{\mathbf{R}} \quad \text{and} \quad \mathfrak{f} = \mathrm{SO}(\Lambda^- N).$$

Given a connection B on $\mathrm{SO}(\Lambda^- N)$, every connection on $\mathrm{Ad}(\mathfrak{w})$ uniquely lifts to a spin$^{\mathrm{Sp}(1)\times \mathrm{U}(2)}$–connection on $\mathfrak{s}$.

The above discussion shows that, having fixed B, the ADHM$_{1,2}$ Seiberg–Witten equations is the following partial differential equation for $A \in \mathscr{A}(\mathrm{Ad}(\mathfrak{w}))$, $\Psi \in \Gamma(W)$, and $\xi \in \Gamma(N \otimes \mathrm{Ad}(\mathfrak{w})_\circ)$,

$$\begin{aligned} \not{D}_A \Psi &= 0 \\ \not{D}_{A,B} \xi &= 0, \quad \text{and} \\ F_A &= \mu(\Psi, \xi), \end{aligned} \tag{1.28}$$

as well as the Dirac equation for $\eta \in \Gamma(N \otimes i\underline{\mathbf{R}})$:

$$\not{D}_B \eta = 0. \tag{1.29}$$

The equations (1.28) and (1.29) are completely decoupled. The compactness problem for (1.29) is trivial: after renormalization every sequence has a subsequence which converges in the C^∞ topology. Of course, Proposition 1.15 applies to the $\mathrm{ADHM}_{1,2}$ Seiberg–Witten equation (1.28). The following result concerns the case in which the hypothesis of Proposition 1.15 is not satisfied.

Theorem 1.30. *If $(A_n, \Psi_n, \xi_n)_{n\in\mathbf{N}}$ is a sequence of solutions with*

$$\liminf_{n\to\infty} \|(\Psi_n, \xi_n)\|_{L^2} = \infty,$$

then the following hold:

1. *There is a closed $\mathscr{H}^1$–rectifiable subset $Z \subset M$ with finite 1–dimensional Minkowski content, a connection $A \in \mathscr{A}(\mathfrak{w}|_{M\backslash Z})$, a spinor $\Psi \in \Gamma(M\backslash Z, W)$, a section $\xi \in \Gamma(M\backslash Z, N \otimes \mathrm{Ad}(\mathfrak{w})_\circ)$, a flat Euclidean line bundle $\mathfrak{l}$ over $M\backslash Z$, and a non-zero $\tau \in \Gamma(M\backslash Z, \mathrm{Hom}(\mathfrak{l}, \mathrm{Ad}(\mathfrak{w})_\circ))$ such that the following hold:*
   1. *(a) A and ξ satisfy*
   $$\begin{aligned} \not{D}_{A,B}\xi &= 0, \\ \mu(\xi) &= 0, \quad \text{and} \\ \|\xi\|_{L^2} &= 1. \end{aligned} \tag{1.31}$$
   2. *(b) The function $|\xi|$ extends to a Hölder continuous function on all of M and*
   $$Z = |\xi|^{-1}(0).$$
   3. *(c) The section τ is parallel with respect to A.*
   4. *(d) Set $\mathfrak{t} := \operatorname{im}\tau \oplus i\underline{\mathbf{R}} \subset \mathrm{Ad}(\mathfrak{w})|_{M\backslash Z}$ and denote by $\pi_{\mathfrak{t}}\colon \mathrm{Ad}(\mathfrak{w})|_{M\backslash Z} \to \mathfrak{t}$ the orthogonal projection onto $\mathfrak{t}$. A and Ψ satisfy*
   $$\begin{aligned} \not{D}_A\Psi &= 0 \quad \text{and} \\ F_A &= \pi_{\mathfrak{t}}\mu(\Psi). \end{aligned} \tag{1.32}$$
2. *Set*
$$\varepsilon_n := \frac{1}{\|(\Psi_n, \xi_n)\|_{L^2}}, \quad \tilde{\Psi}_n := \varepsilon_n\Psi_n, \quad \text{and} \quad \tilde{\xi}_n := \varepsilon_n\xi_n.$$
After passing to a subsequence and up to gauge transformations, $(A_n|_{M\backslash Z})_{n\in\mathbf{N}}$ converges to A in the weak $W^{1,2}_{\mathrm{loc}}$ topology, $(\Psi_n|_{M\backslash Z})_{n\in\mathbf{N}}$ converges to Ψ in the weak $W^{2,2}_{\mathrm{loc}}$ topology, $(\tilde{\xi}_n|_{M\backslash Z})_{n\in\mathbf{N}}$ converges to ξ in the weak $W^{2,2}_{\mathrm{loc}}$ topology, and there exists an $\alpha \in (0,1)$ such that $(|(\tilde{\Psi}_m, \tilde{\xi}_n)|)_{n\in\mathbf{N}}$ converges to $|\xi|$ in the $C^{0,\alpha}$ topology.

Remark 1.33. Theorem 1.30 with $\Psi_n = 0$ recovers Taubes' compactness theorem for stable flat $\mathbf{PSL}_2(\mathbf{C})$–connections over 3–manifolds [Tau13a]. In fact, it also shows that limiting connection A is flat.

Remark 1.34. Theorem 1.30 partially verifies the conjecture [DW17, Conjecture 5.26].

Conventions Throughout, fix a set of algebraic data (G, H, ρ) and a compatible set of geometric data $(M, g, \mathfrak{s}, B)$ with M closed. As is customary, $c > 0$ denotes a universal constant whose value might change from on appearance to the next and which depends only on the chosen algebraic and geometric data. Moreover, $r_0 > 0$ denotes a constant which is much smaller than the injectivity radius and at least as small as the constant appearing in Hypothesis 1.20.

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2 The Lichnerowicz–Weitzenböck formula

This section derives a number of consequences of the Lichnerowicz–Weitzenböck formula. Let us begin by reminding the reader of the latter.

Definition 2.1. Define $\mathfrak{R} \in \Omega^2(M, \mathrm{End}(\mathbf{S}))$ and $\mathfrak{K} \in \Gamma(\mathrm{End}(\mathbf{S}))$ by

$$\mathfrak{R}\Phi := \frac{1}{4}\sum_{i,j=1}^{3} \langle R(\cdot,\cdot)e_i, e_j\rangle \gamma(e_i)\gamma(e_j)\Phi + F_B \quad \text{and} \quad \mathfrak{K}\Phi := \gamma(\mathfrak{R})\Phi.$$

Proposition 2.2 (Lichnerowicz–Weitzenböck formula). *For every $A \in \mathscr{A}(\mathfrak{s}, B)$ and $\Phi \in \Gamma(\mathbf{S})$,*

$$\not{D}_A^2\Phi = \nabla_A^*\nabla_A\Phi + \boldsymbol{\gamma}(F_{\mathrm{Ad}(A)})\Phi + \mathfrak{K}\Phi. \tag{2.3}$$

If $\not{D}_A\Phi = 0$, then Proposition 2.2 implies

$$\frac{1}{2}\Delta|\Phi|^2 + |\nabla_A\Phi|^2 + 2\varepsilon^{-2}|\mu(\Phi)|^2 + \langle \mathfrak{K}\Phi, \Phi\rangle = 0. \tag{2.4}$$

The following is an immediate consequence of (2.4) and integration by parts.

Corollary 2.5. *Let U be an open subset of M with smooth boundary and let $f \in C^\infty(\bar{U})$. If $A \in \mathscr{A}(\mathfrak{s}, B)$, $\Phi \in \Gamma(\mathbf{S})$, and $\varepsilon > 0$ satisfy (1.17) on U, then*

$$\int_U \frac{1}{2}\Delta f \cdot |\Phi|^2 + f \cdot \left(|\nabla_A\Phi|^2 + 2\varepsilon^{-2}|\mu(\Phi)|^2\right) = -\int_U f \cdot \langle \mathfrak{K}\Phi, \Phi\rangle + \frac{1}{2}\int_{\partial U} f \cdot \partial_\nu|\Phi^2| - \partial_\nu f \cdot |\Phi|^2.$$

2.1 The frequency function

The statements of the results derived in this section require the following definitions.

Definition 2.6. Given $A \in \mathscr{A}(\mathfrak{s})$, $\Phi \in \Gamma(\mathbf{S})$, $x \in M$, and $\varepsilon > 0$, define $m_x^\Phi, D_x^{A,\Phi,\varepsilon}\colon (0, r_0] \to [0, \infty)$ by

$$m_x^\Phi(r) := \frac{1}{4\pi r^2}\int_{\partial B_r(x)} |\Phi|^2 \quad \text{and}$$
$$D_x^{A,\Phi,\varepsilon}(r) := \frac{1}{4\pi r}\int_{B_r(x)} |\nabla_A\Phi|^2 + 2\varepsilon^{-2}|\mu(\Phi)|^2;$$

and, furthermore, set $r^{\Phi}_{-1,x} := \sup\{r \in (0,\infty) : m^{\Phi}_x(r) = 0\}$ and define the **frequency function** $\mathrm{N}^{A,\Phi,\varepsilon}_x : (r^{\Phi}_{-1,x}, r_0] \to [0,\infty)$ by

$$\mathrm{N}^{A,\Phi,\varepsilon}_x(r) := \frac{D^{A,\Phi,\varepsilon}_x(r)}{m^{\Phi}_x(r)}.$$

Remark 2.7. A priori, the restriction of the domain of $\mathrm{N}^{A,\Phi,\varepsilon}_x$ is necessary; however: it will shown in Proposition 3.14 that $r^{\Phi}_{-1,x} = 0$ unless $\Phi = 0$.

Remark 2.8. The frequency function was introduced by Almgren [Alm79] and is now an ubiquitous tools in the study of elliptic partial differential equations. The adaption to generalized Seiberg–Witten equations is due to [Tau13a].

For the purposes of this section we shall be content with just the above definitions. However, in Section 3, the frequency function plays a pivotal role and its properties will be studied in detail.

2.2 L^2 bounds on Φ

Proposition 2.9. *If $A \in \mathscr{A}(\mathfrak{s}, B)$, $\Phi \in \Gamma(\mathrm{S})$, and $\varepsilon > 0$ satisfy (1.17), then, for every $x \in M$ and $r \in (0, r_0]$,*

$$\frac{\pi}{2} m^{\Phi}_x\left(\frac{r}{2}\right) \leqslant r^{-3} \int_{B_r(x)} |\Phi|^2 \leqslant 4\pi m^{\Phi}_x(r).$$

Proof. Denote by $H_{x,r}$ the mean curvature of $\partial B_r(x)$. By Corollary 2.5 with $f = 1$ and $U = B_r(x)$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r} \int_{\partial B_r(x)} |\Phi|^2 &= \int_{\partial B_r(x)} H_{x,r}|\Phi|^2 + \int_{\partial B_r(x)} \partial_r |\Phi|^2 \\ &= \int_{\partial B_r(x)} H_{x,r}|\Phi|^2 + 2\int_{B_r(x)} |\nabla_A \Phi|^2 + 2\varepsilon^{-2}|\mu(\Phi)|^2 + \langle \mathfrak{R}\Phi, \Phi\rangle.\end{aligned}$$

By Hardy's inequality,

$$\int_{B_r(x)} |\Phi|^2 \leqslant cr^2 \int_{B_r(x)} |\nabla_A \Phi|^2 + cr \int_{\partial B_r(x)} |\Phi|^2.$$

Therefore and because $H_{x,r} \geqslant \frac{1}{r} - cr$, for $r \in [0, r_0]$,

$$\frac{\mathrm{d}}{\mathrm{d}r} \int_{\partial B_r(x)} |\Phi|^2 \geqslant 0.$$

This implies the assertion. □

2.3 L^∞ bounds on Φ

To state the next result, we define the following variant of the Morrey norm

$$\|f\|_{L^{p,\lambda}_\star(U)} \coloneqq \sup_{y\in U}\|r_y^{-\lambda/p} f\|_{L^p(U)}.$$

with $r_y \coloneqq d(y,\cdot)$.

Proposition 2.10. *If $A \in \mathscr{A}(\mathfrak{s}, B)$, $\Phi \in \Gamma(\mathbf{S})$, and $\varepsilon > 0$ satisfy (1.17), then*

$$\|\Phi\|_{L^\infty(M)} + \|\nabla_A\Phi\|_{L^{2,1}_\star(M)} + \varepsilon^{-1}\|\mu(\Phi)\|_{L^{2,1}_\star(M)} \leqslant c\|\Phi\|_{L^2};$$

moreover, for every $x \in M$, $r \in (0, r_0]$,

$$\|\Phi\|_{L^\infty(B_{r/2}(x))} + \|\nabla_A\Phi\|_{L^{2,1}_\star(B_{r/2}(x))} + \varepsilon^{-1}\|\mu(\Phi)\|_{L^{2,1}_\star(B_{r/2}(x))} \leqslant cm_x^\Phi(r)^{1/2}.$$

Proof. Let $\chi \in C_0^\infty(B_r(x), [0,1])$ be a cut-off function satisfying $\chi|_{B_{r/2}(x)} = 1$ and

$$r|\nabla\chi| \leqslant c \quad \text{and} \quad r^2|\nabla^2\chi| \leqslant c.$$

Denote by G the Green's kernel for $B_r(x)$ and, for $y \in B_r(x)$, set $G_y \coloneqq G(y,\cdot)$. Multiplying (2.4) with $\chi^2 G_y$ and integrating by parts yields

$$\frac{1}{2}\chi(y)^2|\Phi|^2(y) + \int_{B_r(x)} \chi^2 G_y\left(|\nabla_A\Phi|^2 + 2\varepsilon^{-2}|\mu(\Phi)|^2\right) = \int_{B_r(x)} \chi^2 G_y\langle\mathfrak{R}\Phi,\Phi\rangle + \Theta_y\cdot|\Phi|^2.$$

with

$$\Theta_y \coloneqq \langle\nabla\chi^2, \nabla G_y\rangle - \frac{1}{2}\Delta\chi^2\cdot G_y.$$

From

$$\int_{B_r(x)} G_y \leqslant cr^2 \quad \text{and} \quad \|\Theta_y\|_{L^\infty} \leqslant cr^{-3}$$

it follows that

$$\|\chi\Phi\|_{L^\infty}^2 + \sup_{y\in B_{r/2}(x)}\int_{B_{r/2}(x)} r_y^{-1}\left(|\nabla_A\Phi|^2 + \varepsilon^{-2}|\mu(\Phi)|^2\right) \leqslant cr^2\|\chi\Phi\|_{L^\infty}^2 + cr^{-3}\|\Phi\|_{L^2(B_r(x))}^2.$$

After rearranging and by Proposition 2.9, the asserted inequalities follow. □

2.4 $W^{2,2}$ bounds on Φ

Proposition 2.11. *For every $c_F, c_\Phi, c_{\mathrm{N}} > 0$ and $\delta \in (0, \frac{1}{2}]$, there is a constant $c = c(c_F, c_\Phi, c_{\mathrm{N}}) > 0$ such that the following holds for every $x \in M$, $r \in (0, r_0]$. If $A \in \mathscr{A}(\mathfrak{s}, B)$, $\Phi \in \Gamma(\mathrm{S})$, and $\varepsilon > 0$ satisfy (1.17),*

$$r \int_{B_r(x)} |F_{\mathrm{Ad}(A)}|^2 \leqslant c_F, \quad m_x^\Phi(r) \leqslant c_\Phi, \quad \text{and} \quad \mathrm{N}_x^{A,\Phi,\varepsilon}(r) \leqslant c_{\mathrm{N}},$$

then

$$r \int_{B_{r/2}(x)} |\nabla_A^2 \Phi|^2 + \left(\frac{\varepsilon}{r}\right)^2 \cdot r^3 \int_{B_{r/2}(x)} |\nabla_{\mathrm{Ad}(A)} F_{\mathrm{Ad}(A)}|^2 \leqslant c.$$

The proof relies on the following consequence of the Lichnerowicz–Weitzenböck formula (2.3).

Proposition 2.12. *If $A \in \mathscr{A}(\mathfrak{s}, B)$, $\Phi \in \Gamma(\mathrm{S})$, and $\varepsilon > 0$ satisfy (1.17), then*

$$\begin{aligned}(2.13)\quad \frac{1}{2}\Delta|\nabla_A\Phi|^2 + |\nabla_A^2\Phi|^2 + \varepsilon^{-2}|\mathrm{d}^*_{\mathrm{Ad}(A)}\mu(\Phi)|^2 + \varepsilon^{-2}|\nabla_{\mathrm{Ad}(A)}\mu(\Phi)|^2 \\ = -\varepsilon^{-2}\langle\langle\mu(\nabla_A\Phi, \nabla_A\Phi)\rangle, \mu(\Phi)\rangle + 2\varepsilon^{-2}\langle\mu(\Phi), \rho^*(\nabla_A\Phi \wedge \nabla_A\Phi^*)\rangle + \mathfrak{r}_{\nabla\Phi}\end{aligned}$$

with

$$|\mathfrak{r}_{\nabla\Phi}| \leqslant c\big(|\nabla_A\Phi|^2 + |\nabla_A\Phi||\Phi|\big).$$

The proof makes use of the following observation.

Proposition 2.14. *For every $A \in \mathscr{A}(\mathfrak{s})$ and $\Phi \in \Gamma(S)$,*

$$\begin{aligned}[\nabla_A^*\nabla_A, \nabla_A]\Phi = \rho(\mathrm{d}^*_{\mathrm{Ad}(A)}F_{\mathrm{Ad}(A)})\Phi + 2\sum_{i,j=1}^{3} e^i \otimes \rho(F_{\mathrm{Ad}(A)}(e_i, e_j))\nabla_{A,e_j}\Phi \\ + (\mathrm{d}_A^*\mathfrak{R})\Phi + 2\sum_{i,j=1}^{3} e^i \otimes \mathfrak{R}(e_i, e_j)\nabla_{A,e_j}\Phi.\end{aligned}$$

Proof. This is a consequence of the following computation

$$
\begin{aligned}
&-\sum_{j=1}^{3} \nabla_{A,e_j}\nabla_{A,e_j}\nabla_{A,e_i}\Phi \\
&= -\sum_{j=1}^{3} \nabla_{A,e_j} F_A(e_j,e_i)\Phi - \sum_{j=1}^{3} \nabla_{A,e_j}\nabla_{A,e_i}\nabla_{A,e_j}\Phi \\
&= -\sum_{j=1}^{3} \nabla_{A,e_j} F_A(e_j,e_i)\Phi - \sum_{j=1}^{3} F_A(e_j,e_i)\nabla_{A,e_j}\Phi - \sum_{j=1}^{3} \nabla_{A,e_i}\nabla_{A,e_j}\nabla_{A,e_j}\Phi \\
&= -\sum_{j=1}^{3} \Big(\nabla_{A,e_j} F_A(e_j,e_i)\Big)\Phi - 2\sum_{j=1}^{3} F_A(e_j,e_i)\nabla_{A,e_j}\Phi - \sum_{j=1}^{3} \nabla_{A,e_i}\nabla_{A,e_j}\nabla_{A,e_j}\Phi \\
&= (\mathrm{d}_A^* F_A)(e_i)\Phi - 2\sum_{j=1}^{3} F_A(e_j,e_i)\nabla_{A,e_j}\Phi - \sum_{j=1}^{3} \nabla_{A,e_i}\nabla_{A,e_j}\nabla_{A,e_j}\Phi. \qquad \square
\end{aligned}
$$

Proof of Proposition 2.12. By Proposition 2.2,

$$\nabla_A^*\nabla_A\nabla_A\Phi = [\nabla_A^*\nabla_A, \nabla_A]\Phi - \varepsilon^{-2}\boldsymbol{\gamma}(\mu(\Phi))\nabla_A\Phi - \varepsilon^{-2}\boldsymbol{\gamma}(\nabla_A\mu(\Phi))\Phi - \mathfrak{K}\nabla_A\Phi - \gamma(\nabla\mathfrak{R})\Phi.$$

By Proposition 2.14, the first term on the right-hand side can be written as

$$
\begin{aligned}
[\nabla_A^*\nabla_A, \nabla_A]\Phi &= \varepsilon^{-2}\rho(\mathrm{d}_{\mathrm{Ad}(A)}^*\mu(\Phi))\Phi + 2\varepsilon^{-2}\sum_{i,j=1}^{3} e^i \otimes \rho(\mu(\Phi)(e_i,e_j))\nabla_{A,e_j}\Phi \\
&\quad + (\mathrm{d}_A^*\mathfrak{R})\Phi + 2\sum_{i,j=1}^{3} e^i \otimes \mathfrak{R}(e_i,e_j)\nabla_{A,e_j}\Phi.
\end{aligned}
\tag{2.15}
$$

It was proved in [DW19, Proposition B.4] that if $\not{D}_A\Phi = 0$, then

$$\mathrm{d}_{\mathrm{Ad}(A)}^*\mu(\Phi) = -\rho^*(\nabla_A\Phi\Phi^*). \tag{2.16}$$

These identities imply the asserted formula upon taking the inner product of (2.15) with $\nabla_A\Phi$ because

$$\langle\boldsymbol{\gamma}(\nabla_A\mu(\Phi))\Phi, \nabla_A\Phi\rangle = |\nabla_A\mu(\Phi)|^2$$

and

$$
\begin{aligned}
\sum_{i=1}^{3}\langle(\mathrm{d}_{\mathrm{Ad}(A)}^*\mu(\Phi))(e_i)\Phi, \nabla_{A,e_i}\Phi\rangle &= \sum_{i=1}^{3} -\langle\rho\rho^*\big(\nabla_{A,e_i}\Phi\Phi^*\big)\Phi, \nabla_{A,e_i}\Phi\rangle \\
&= \sum_{i=1}^{3} -\langle\rho^*\big(\nabla_{A,e_i}\Phi\Phi^*\big), \rho^*(\nabla_{A,e_i}\Phi\Phi^*)\rangle \\
&= -|\mathrm{d}_{\mathrm{Ad}(A)}^*\mu(\Phi)|^2. \qquad \square
\end{aligned}
$$

Proof of Proposition 2.11. Let $\chi \in C_0^\infty(B_r(x),[0,1])$ be a cut-off function satisfying $\chi|_{B_{r/2}(x)} = 1$ and

$$r|\nabla\chi| \leqslant c \quad \text{and} \quad r^2|\nabla^2\chi| \leqslant c.$$

Multiplying (2.13) by $r\chi^2$, integrating by parts, and using $F_{\mathrm{Ad}(A)} = \varepsilon^{-2}\mu(\Phi)$, yields

$$\begin{aligned} r\int_{B_r(x)} \chi^2\big(|\nabla_A^2\Phi|^2 + \varepsilon^2|\nabla_A F_A|^2\big) &\leqslant c(c_\Phi, c_\mathrm{N}) + cr\int_{B_r(x)} |F_A|\cdot\chi^2|\nabla_A\Phi|^2 \\ &\leqslant c(c_\Phi, c_\mathrm{N}) + \underbrace{c(c_F)\bigg(r\int_{B_r(x)} \chi^4|\nabla_A\Phi|^4\bigg)^{1/2}}_{=:(\star)}. \end{aligned}$$

By the Gagliardo–Nirenberg inequality and Cauchy–Schwarz, for every $f \in C_0^\infty(B_r(x))$ and $\sigma > 0$,

$$\begin{aligned} \|f\|_{L^4}^2 &\leqslant c\|\nabla f\|_{L^2}^{3/2}\|f\|_{L^2}^{1/2} \\ &\leqslant \sigma\|\nabla f\|_{L^2}^2 + c(\sigma)\|f\|_{L^2}^2. \end{aligned}$$

Therefore, by Kato's inequality,

$$(\star) \leqslant \frac{r}{2}\int_{B_r(x)} \chi^2|\nabla_A^2\Phi|^2 + c(c_F, c_\Phi, c_\mathrm{N}).$$

Rearranging proves the asserted inequality. □

2.5 Oscillation bounds on Φ

Proposition 2.17. *For every $c_F, c_\Phi, c_\mathrm{N} > 0$, there is a constant $c = c(c_F, c_\Phi, c_\mathrm{N}) > 0$ such that the following holds for every $x \in M$ and $r \in (0, r_0]$. If $A \in \mathscr{A}(\mathfrak{s}, B)$, $\Phi \in \Gamma(\mathrm{S})$, and $\varepsilon > 0$ satisfy (1.17),*

$$r\int_{B_r(x)} |F_A|^2 \leqslant c_F, \quad m_x^\Phi(r) \leqslant c_\Phi, \quad \textit{and} \quad \mathrm{N}_x^{A,\Phi,\varepsilon}(r) \leqslant c_\mathrm{N},$$

then, for every $y, z \in B_{r/2}(x)$,

$$\big||\Phi|(y) - |\Phi|(z)\big| \leqslant c\big[\mathrm{N}_x^{A,\Phi,\varepsilon}(r)m_x^\Phi(r)\big]^{1/8}.$$

Proof. By Proposition 2.10 and Proposition 2.11,

$$\|\Phi\|_{L^\infty(B_{r/2}(x))}^2 \leqslant c(c_\Phi) \quad \text{and} \quad r^{1/2}\|\nabla_A^2\Phi\|_{L^2(B_{r/2}(x))} \leqslant c(c_F, c_\Phi, c_\mathrm{N}).$$

Therefore, by Morrey's inequality and the Gagliardo–Nirenberg interpolation inequality,

$$\begin{aligned} r^{1/4}[|\Phi|]_{C^{0,1/4}(B_{r/2}(x))} &\leqslant cr^{1/4}\||\nabla_A\Phi|\|_{L^4(B_{r/2}(x))} \\ &\leqslant c\Big(r^{1/2}\||\nabla_A^2\Phi|\|_{L^2(B_{r/2}(x))}\Big)^{3/4}\Big(r^{-1/2}\||\nabla_A\Phi|\|_{L^2(B_{r/2}(x))}\Big)^{1/4} \\ &\quad + cr^{-1/2}\||\nabla_A\Phi|\|_{L^2(B_{r/2}(x))} \\ &\leqslant c(c_F, c_\mathrm{N}, c_\Phi)\big[\mathrm{N}_x^{A,\Phi,\varepsilon}(r)m_x^\Phi(r)\big]^{1/8}. \end{aligned}$$

This implies the assertion. ∎

2.6 L^∞ bounds on $\mu(\Phi)$

Proposition 2.18. *For every $c_F, c_\Phi, c_N > 0$, there is a constant $c = c(c_F, c_\Phi, c_N) > 0$ such that the following holds for every $x \in M$, $r \in (0, r_0]$. If $A \in \mathscr{A}(\mathfrak{s}, B)$, $\Phi \in \Gamma(\mathbf{S})$, and $\varepsilon > 0$ satisfy (1.17),*

$$r \int_{B_r(x)} |F_{\mathrm{Ad}(A)}|^2 \leqslant c_F, \quad m_x^\Phi(r) \leqslant c_\Phi, \quad \text{and} \quad \mathrm{N}_x^{A,\Phi,\varepsilon}(r) \leqslant c_N,$$

then

$$\|\mu(\Phi)\|_{L^\infty(B_{r/2}(x))} \leqslant c\left[\left(\frac{\varepsilon}{r}\right)^2 \mathrm{N}_x^{A,\Phi,\varepsilon}(r) m_x^\Phi(r)\right]^{1/32}.$$

Proof. By Morrey's inequality, the Gagliardo–Nirenberg interpolation inequality, Proposition 2.10, and Proposition 2.11,

$$\begin{aligned}
r^{1/4}[|\mu(\Phi)|^2]_{C^{0,1/4}(B_{r/2}(x))} &\leqslant c r^{1/4} \|\nabla |\mu(\Phi)|^2\|_{L^4(B_{r/2}(x))} \\
&\leqslant c\left(r^{1/2}\|\nabla^2|\mu(\Phi)|^2\|_{L^2(B_{r/2}(x))}\right)^{7/8}\left(r^{-3/2}\||\mu(\Phi)|^2\|_{L^2(B_{r/2}(x))}\right)^{1/8} \\
&\quad + c r^{-3}\||\mu(\Phi)|^2\|_{L^1(B_{r/2}(x))} \\
&\leqslant c(c_F, c_\Phi, c_N)\left(r^{-3/2}\|\mu(\Phi)\|_{L^2(B_{r/2}(x))}\right)^{1/8} \\
&\leqslant c(c_F, c_\Phi, c_N)\left[\left(\frac{\varepsilon}{r}\right)^2 \mathrm{N}_x^{A,\Phi,\varepsilon}(r) m_x^\Phi(r)\right]^{1/16}.
\end{aligned}$$

Therefore, for all $y, z \in B_{r/2}(x)$,

$$\left||\mu(\Phi)|^2(y) - |\mu(\Phi)|^2(z)\right| \leqslant c(c_F, c_\Phi, c_N)\left[\left(\frac{\varepsilon}{r}\right)^2 \mathrm{N}_x^{A,\Phi,\varepsilon}(r) m_x^\Phi(r)\right]^{1/16}.$$

This implies

$$|\mu(\Phi)|^2(y) - \fint_{B_{r/2}(x)} |\mu(\Phi)|^2 \leqslant c(c_F, c_\Phi, c_N)\left[\left(\frac{\varepsilon}{r}\right)^2 \mathrm{N}_x^{A,\Phi,\varepsilon}(r) m_x^\Phi(r)\right]^{1/16}.$$

Since

$$\fint_{B_{r/2}(x)} |\mu(\Phi)|^2 \leqslant c\left(\frac{\varepsilon}{r}\right)^2 D_x^{A,\Phi,\varepsilon}(r) = c\left(\frac{\varepsilon}{r}\right)^2 \mathrm{N}_x^{A,\Phi,\varepsilon}(r) m_x^\Phi(r),$$

the assertion follows. ∎

2.7 Curvature decay

Proposition 2.19. *Suppose that Hypothesis 1.20 holds with $\Lambda \geqslant 0$. For every $c_F > 0$, there are constants $r_{-1} = r_{-1}(c_F) > 0$ and $\delta_{\mathrm{N}} = \delta_{\mathrm{N}}(c_F) > 0$ such that the following holds for every $x \in M$ and $r \in (0, r_{-1}]$. If $A \in \mathscr{A}(\mathfrak{s}, B)$, $\Phi \in \Gamma(\mathbf{S})$, and $\varepsilon > 0$ satisfy (1.17),*

$$r \int_{B_r(x)} |F_{\mathrm{Ad}(A)}|^2 \leqslant c_F, \quad \text{and} \quad \mathrm{N}_x^{A,\Phi,\varepsilon}(r) \leqslant \delta_{\mathrm{N}},$$

then

$$\frac{r}{4} \int_{B_{r/4}(x)} |F_{\mathrm{Ad}(A)}|^2 \leqslant \Lambda + 1.$$

The proof relies on the following proposition regarding the decay of part of the curvature.

Proposition 2.20. *For every $c_F, c_{\mathrm{N}} > 0$, there is a constant $c = c(c_F, c_{\mathrm{N}}) > 0$ such that the following holds for every $x \in M$ and $r \in (0, r_0]$. If $A \in \mathscr{A}(\mathfrak{s}, B)$, $\Phi \in \Gamma(\mathbf{S})$, and $\varepsilon > 0$ satisfy (1.17),*

$$r \int_{B_r(x)} |F_{\mathrm{Ad}(A)}|^2 \leqslant c_F, \quad m_x^{\Phi}(r) = 1, \quad \text{and} \quad \mathrm{N}_x^{A,\Phi,\varepsilon}(r) \leqslant c_{\mathrm{N}},$$

then

$$r^{1/2} \|\Gamma_\Phi F_{\mathrm{Ad}(A)}\|_{L^2(B_{r/2}(x))} \leqslant c\left[\frac{\varepsilon}{r} + \mathrm{N}_x^{A,\Phi,\varepsilon}(r)^{1/8} + r^2\right].$$

Proof of Proposition 2.19. If A, Φ, ε satisfy (1.17), then so do

$$A, \quad m_x^{\Phi}(r)^{-1/2} \cdot \Phi, \quad m_x^{\Phi}(r) \cdot \varepsilon.$$

Moreover, $\mathrm{N}_x^{A,\Phi,\varepsilon}$ is invariant under this rescaling. Therefore, we can assume that

$$m_x^{\Phi}(r) = 1.$$

For $\delta_{\mathrm{N}} \ll 1$, it follows from Proposition 2.17 and Proposition 2.18 that, on $B_{r/2}(x)$,

$$\frac{1}{2} \leqslant |\Phi| \leqslant 2 \quad \text{and} \quad |\mu(\Phi)| \leqslant \delta_\mu$$

with δ_μ as in Hypothesis 1.20.

If $\varepsilon/r \ll 1$, then the desired estimate follows from Proposition 2.20 and Hypothesis 1.20; otherwise, it follows from

$$r \int_{B_r(x)} |F_{\mathrm{Ad}(A)}|^2 \leqslant \left(\frac{r}{\varepsilon}\right)^2 D_x^{A,\Phi,\varepsilon}(r) = \left(\frac{r}{\varepsilon}\right)^2 \mathrm{N}_x^{A,\Phi,\varepsilon}(r). \qquad \square$$

The proof of Proposition 2.20 relies on the following proposition, which is a consequence of the Lichnerowicz–Weitzenböck formula (2.3).

Proposition 2.21. *If $A \in \mathscr{A}(\mathfrak{s})$, $\Phi \in \Gamma(\mathrm{S})$, and $\varepsilon > 0$ satisfy (1.17), then*

$$\frac{1}{2}\Delta|\mu(\Phi)|^2 + \varepsilon^{-2}|\Gamma_\Phi\mu(\Phi)|^2 + |\nabla_A\mu(\Phi)|^2 = -2\langle\langle\mu(\nabla_A\Phi, \nabla_A\Phi)\rangle, \mu(\Phi)\rangle - \langle\Gamma_\Phi\mu(\Phi), \mathfrak{R}\Phi\rangle. \tag{2.22}$$

The proof makes use of the following identity regarding the symmetric bilinear form associated with the quadratic map μ.

Proposition 2.23. *For every $\Phi \in \Gamma(\mathrm{S})$,*

$$\mu(\boldsymbol{\gamma}(\mu(\Phi))\Phi, \Phi) = \tfrac{1}{2}\Gamma_\Phi^*\Gamma_\Phi\mu(\Phi).$$

Proof. For every $\zeta \in \Omega^2(M, \mathrm{Ad}(\mathfrak{s}))$,

$$\begin{aligned}
\langle\mu(\boldsymbol{\gamma}(\mu(\Phi))\Phi, \Phi), \zeta\rangle &= \frac{1}{2}\langle\boldsymbol{\gamma}^*(\boldsymbol{\gamma}(\mu(\Phi))\Phi\Phi^*), \zeta\rangle \\
&= \frac{1}{2}\langle\boldsymbol{\gamma}(\mu(\Phi))\Phi), \boldsymbol{\gamma}(\zeta)\Phi\rangle \\
&= \frac{1}{2}\langle\Gamma_\Phi\mu(\Phi), \Gamma_\Phi\zeta\rangle \\
&= \frac{1}{2}\langle\Gamma_\Phi^*\Gamma_\Phi\mu(\Phi), \zeta\rangle.
\end{aligned}$$

□

Proof of Proposition 2.21. By Proposition 2.2 and Proposition 2.23,

$$\begin{aligned}
\nabla_{\mathrm{Ad}(A)}^*\nabla_{\mathrm{Ad}(A)}\mu(\Phi) &= 2\mu(\nabla_A^*\nabla_A\Phi, \Phi) - 2\langle\mu(\nabla_A\Phi, \nabla_A\Phi)\rangle \\
&= -2\varepsilon^{-2}\mu(\boldsymbol{\gamma}(\mu(\Phi))\Phi, \Phi) - 2\mu(\mathfrak{R}\Phi, \Phi) - 2\langle\mu(\nabla_A\Phi, \nabla_A\Phi)\rangle \\
&= -\varepsilon^{-2}\Gamma_\Phi^*\Gamma_\Phi\mu(\Phi) - 2\mu(\mathfrak{R}\Phi, \Phi) - 2\langle\mu(\nabla_A\Phi, \nabla_A\Phi)\rangle.
\end{aligned}$$

This implies the asserted formula upon taking the inner product with $\mu(\Phi)$ because

$$\begin{aligned}
2\langle\mu(\mathfrak{R}\Phi, \Phi), \mu(\Phi)\rangle &= \frac{1}{2}\langle\boldsymbol{\gamma}^*(\mathfrak{R}\Phi\Phi^*), \boldsymbol{\gamma}^*(\Phi\Phi^*)\rangle \\
&= \frac{1}{2}\langle\mathfrak{R}\Phi, \boldsymbol{\gamma}(\boldsymbol{\gamma}^*(\Phi\Phi^*))\Phi\rangle \\
&= \langle\mathfrak{R}\Phi, \Gamma_\Phi\mu(\Phi)\rangle.
\end{aligned}$$

□

Proof of Proposition 2.20. Let $\chi \in C_0^\infty(B_r(x), [0,1])$ be a cut-off function supported in $B_r(x)$ and satisfying $\chi|_{B_{r/2}(x)} = 1$ and

$$r|\nabla\chi| \leqslant c \quad \text{and} \quad r^2|\nabla^2\chi| \leqslant c.$$

Multiplying (2.22) by $r\varepsilon^{-2}\chi^2$, integrating by parts, and using $F_{\mathrm{Ad}(A)} = \varepsilon^{-2}\mu(\Phi)$, yields

$$\begin{aligned}
&r\int_{B_r(x)}\chi^2|\Gamma_\Phi F_{\mathrm{Ad}(A)}|^2 + \left(\frac{\varepsilon}{r}\right)^2 \cdot r^3\int_{B_r(x)}\chi^2|\nabla_{\mathrm{Ad}(A)}F_{\mathrm{Ad}(A)}|^2 \\
&\qquad\leqslant c\left(\frac{\varepsilon}{r}\right)^2 \cdot r\int_{B_r(x)}|F_{\mathrm{Ad}(A)}|^2 + rc\int_{B_r(x)}\chi^2|\nabla_A\Phi|^2|F_{\mathrm{Ad}(A)}| + rc\int_{B_r(x)}\chi^2|\Gamma_\Phi F_{\mathrm{Ad}(A)}||\Phi|.
\end{aligned}$$

By the hypotheses and using rearrangement,

$$r\int_{B_r(x)}\chi^2|\Gamma_\Phi F_{\mathrm{Ad}(A)}|^2 + r\varepsilon^2\int_{B_r(x)}\chi^2|\nabla_{\mathrm{Ad}(A)}F_{\mathrm{Ad}(A)}|^2$$
$$\leqslant c(c_F)\Big(\frac{\varepsilon}{r}\Big)^2 + c(c_F)\bigg[r\int_{B_r(x)}\chi^2|\nabla_A\Phi|^4\bigg]^{1/2} + cr^4.$$

As in the proof of Proposition 2.17, the second term on the right-hand side can be bounded by

$$c(c_F, c_{\mathrm{N}})\cdot \mathrm{N}_x^{A,\Phi,\varepsilon}(r)^{1/4}$$

Therefore,

$$r^{1/2}\|\Gamma_\Phi F_{\mathrm{Ad}(A)}\|_{L^2(B_{r/2}(x))} \leqslant c(c_F, c_{\mathrm{N}})\bigg[\frac{\varepsilon}{r} + \mathrm{N}_x^{A,\Phi,\varepsilon}(r)^{1/8} + r^2\bigg]. \qquad \square$$

3 The regularity scale

Throughout this section, suppose that Hypothesis 1.20 holds with $\Lambda \geqslant 0$.

Definition 3.1. For $\delta > 0$ as in Lemma 3.17, set

$$c_F \coloneqq \delta^{-1}(\Lambda + 1).$$

The **regularity scale** of $A \in \mathscr{A}(\mathfrak{s}, B)$ is the function $r_A\colon M \to [0, r_0]$ defined by

$$r_A(x) \coloneqq \sup\bigg\{r \in [0, r_0] : r\int_{B_r(x)}|F_A|^2 \leqslant c_F\bigg\}.$$

The following result is the key to the proof of Theorem 1.22.

Proposition 3.2. *There are constants $\delta, r_{-1}, c > 0$ such that the following holds. If $A \in \mathscr{A}(\mathfrak{s}, B)$, $\Phi \in \Gamma(\mathbf{S})$, and $\varepsilon > 0$ satisfy (1.17) and (1.18), then*

$$r_A(x) \geqslant \min\Big\{c^{-1}|\Phi|(x)^{1/\delta}, r_{-1}\Big\}.$$

The four upcoming subsections analyze the frequency function. Throughout, let $x \in M$ and let $A \in \mathscr{A}(\mathfrak{s})$, $\Phi \in \Gamma(\mathbf{S})$, and $\varepsilon > 0$ be a solution of (1.17). To simplify notation, we drop the super-scripts and simply write $r_{-1,x}$, m_x, D_x, and N_x.

3.1 Almost monotonicty of N

The following is the key result regarding the frequency function.

Proposition 3.3. *For every $r \in (r_{-1,x}, r_0]$,*

$$\mathrm{N}_x'(r) \geqslant \frac{1}{2\pi r m_x(r)} \int_{\partial B_r(x)} |\nabla_{A,\partial_r}\Phi - \frac{1}{r}\mathrm{N}_x(r)\Phi|^2 + \varepsilon^{-2}|i(\partial_r)\mu(\Phi)|^2 - cr(1 + \mathrm{N}_x(r)). \tag{3.4}$$

Before embarking on the proof of Proposition 3.3, let us record the following consequence.

Proposition 3.5. *For every $r_{-1,x} < s \leqslant r \leqslant r_0$,*

$$\mathrm{N}_x(s) \leqslant \left(1 + cr^2\right)\mathrm{N}_x(r) + cr^2.$$

Proof. By Proposition 3.3,

$$\frac{\mathrm{d}}{\mathrm{d}r} e^{\frac{1}{2}cr^2}(\mathrm{N}_x(r) + 1) \geqslant 0.$$

This implies

$$\mathrm{N}_x(s) \leqslant e^{\frac{1}{2}c(r^2-s^2)}\mathrm{N}_x(r) + e^{\frac{1}{2}c(r^2-s^2)} - 1.$$

□

The proof of of Proposition 3.3 relies on the following three propositions.

Proposition 3.6. *For every $r \in (0, r_0]$,*

$$D_x'(r) = \frac{1}{2\pi r} \int_{\partial B_r(x)} |\nabla_{A,\partial_r}\Phi|^2 + \varepsilon^{-2}|i_{\partial_r}\mu(\Phi)|^2 + \mathfrak{r}_{D'}$$

with

$$|\mathfrak{r}_{D'}| \leqslant cr(D_x(r) + m_x(r)).$$

Proof. Following Taubes [Tau13a, Proof of Lemma 5.2], define the tensor field $T \in \Gamma(S^2T^*M)$ by

$$T = T_\Phi + \varepsilon^{-2}T_\mu$$

with

$$T_\Phi(v,w) \coloneqq \langle \nabla_{A,v}\Phi, \nabla_{A,w}\Phi\rangle - \frac{1}{2}\langle v,w\rangle|\nabla_A\Phi|^2 \quad \text{and}$$
$$T_\mu(v,w) \coloneqq \langle i_v\mu(\Phi), i_w\mu(\Phi)\rangle - \langle v,w\rangle|\mu(\Phi)|^2.$$

By a straight-forward computation,

$$-2\,\mathrm{tr}\,T = |\nabla_A\Phi|^2 + 2\varepsilon^{-2}|\mu(\Phi)|^2. \tag{3.7}$$

A further computation will show that

$$|\nabla^*T| \leqslant c|\mathfrak{R}||\Phi||\nabla_A\Phi|. \tag{3.8}$$

By (3.7), the identity

$$\int_{B_r(x)} \langle \nabla^* T, \mathrm{d}r_x^2 \rangle = -2r \int_{\partial B_r(x)} T(\partial_r, \partial_r) + \int_{B_r(x)} \langle T, \mathrm{Hess}(r_x^2) \rangle$$

can be rewritten as

$$\begin{aligned}\int_{B_r(x)} 2r_x \nabla^* T(\partial_r) = -2r \int_{\partial B_r(x)} |\nabla_{A,\partial_r}\Phi|^2 + \varepsilon^{-2}|i_{\partial_r}\mu(\Phi)|^2 + r \int_{\partial B_r(x)} |\nabla_A \Phi|^2 + 2\varepsilon^{-2}|\mu(\Phi)|^2 \\ - \int_{B_r(x)} |\nabla_A \Phi|^2 + 2\varepsilon^{-2}|\mu(\Phi)|^2 + \int_{B_r(x)} \langle T, \mathfrak{r}_{\mathrm{II}} \rangle\end{aligned}$$

with

$$\mathfrak{r}_{\mathrm{II}} \coloneqq \mathrm{Hess}(r_x^2) - 2g.$$

Since

$$D_x'(r) = -\frac{1}{4\pi r^2} \int_{B_r(x)} |\nabla\Phi|^2 + 2\varepsilon^{-2}|\mu(\Phi)|^2 + \frac{1}{4\pi r} \int_{\partial B_r(x)} |\nabla\Phi|^2 + 2\varepsilon^{-2}|\mu(\Phi)|^2,$$

the inequality (3.8) implies the assertion.

It remains to prove (3.8). Let $y \in M$ be an arbitrary point of M and let e_1, e_2, e_3 be a local orthonormal frame such that $(\nabla_{e_i} e_j)(y) = 0$. All of the following computations take place at the point y. By the Lichnerowicz–Weitzenböck formula (2.3),

$$\begin{aligned}(\nabla^* T_\Phi)(e_i) &= -\sum_{j=1}^{3} \langle \nabla_{A,e_j}\nabla_{A,e_j}\Phi, \nabla_{A,e_i}\Phi \rangle + \langle \nabla_{A,e_j}\nabla_{A,e_i}\Phi, \nabla_{A,e_j}\Phi \rangle - \langle \nabla_{A,e_i}\nabla_{A,e_j}\Phi, \nabla_{A,e_j}\Phi \rangle \\ &= \langle \nabla_A^*\nabla_A\Phi, \nabla_{A,e_i}\Phi \rangle + \sum_{j=1}^{3} \langle F_A(e_i, e_j)\Phi, \nabla_{A,e_j}\Phi \rangle \\ &= -\varepsilon^{-2} \langle \boldsymbol{\gamma}(\mu(\Phi))\Phi, \nabla_{A,e_i}\Phi \rangle + \varepsilon^{-2} \sum_{j=1}^{3} \langle \rho(\mu(\Phi)(e_i, e_j))\Phi, \nabla_{A,e_j}\Phi \rangle + \mathfrak{r}_T(e_i)\end{aligned}$$

with

$$\mathfrak{r}_T(v) \coloneqq -\langle \mathfrak{K}\Phi, \nabla_{A,v}\Phi \rangle + \sum_{i=1}^{3} \langle \mathfrak{R}(v, e_i)\Phi, \nabla_{A,e_i}\Phi \rangle.$$

The first two terms on the right-hand side of the above identity can be rewritten as follows. By definition of $\mu(\Phi)$,

$$\begin{aligned}\langle \boldsymbol{\gamma}(\mu(\Phi))\Phi, \nabla_{A,e_i}\Phi \rangle &= \langle \mu(\Phi), \nabla_{\mathrm{Ad}(A),e_i}\mu(\Phi) \rangle \\ &= \frac{1}{2}\nabla_{e_i}|\mu(\Phi)|^2.\end{aligned}$$

Furthermore, the identity (2.16) implies that

$$\begin{aligned}\langle \rho(\mu(\Phi)(e_i, e_j))\Phi, \nabla_{A,e_j}\Phi\rangle &= \langle \mu(\Phi)(e_i, e_j), \rho^*((\nabla_{A,e_j}\Phi)\Phi^*)\rangle \\ &= -\langle \mu(\Phi)(e_i, e_j), (\mathrm{d}^*_{\mathrm{Ad}(A)}\mu(\Phi))(e_j)\rangle.\end{aligned}$$

Therefore,

$$(\nabla^* T_\Phi)(v) = -\tfrac{1}{2}\varepsilon^{-2}\nabla_v|\mu(\Phi)|^2 - \varepsilon^{-2}\langle \mathrm{d}^*_{\mathrm{Ad}(A)}\mu(\Phi), i(v)\mu(\Phi)\rangle + \mathfrak{r}_T(v). \tag{3.9}$$

The term $\mathfrak{r}_T$ satisfies the asserted estimate. Thus, it remains show that the first two term on the right-hand side of (3.9) are equal to $-\varepsilon^{-2}\nabla^* T_\mu$. A brief computation shows that $\mathrm{d}_A\mu(\Phi) = 0$ implies

$$\sum_{j=1}^{3}\langle \nabla_{\mathrm{Ad}(A),e_j} i(e_i)\mu(\Phi), i(e_j)\mu(\Phi)\rangle = \frac{1}{2}\nabla_{e_i}|\mu(\Phi)|^2.$$

Therefore,

$$\begin{aligned}(\nabla^* T_\mu)(e_i) &= \nabla_{e_i}|\mu(\Phi)|^2 - \sum_{j=1}^{3}\langle \nabla_{\mathrm{Ad}(A),e_j} i(e_j)\mu(\Phi), i(e_i)\mu(\Phi)\rangle + \langle \nabla_{\mathrm{Ad}(A),e_j} i(e_i)\mu(\Phi), i(e_j)\mu(\Phi)\rangle \\ &= \frac{1}{2}\nabla_{e_i}|\mu(\Phi)|^2 + \langle \mathrm{d}^*_{\mathrm{Ad}(A)}\mu(\Phi), i(e_i)\mu(\Phi)\rangle.\end{aligned}$$

This finishes the proof. □

Proposition 3.10. *For every $r \in (0, r_0]$,*

$$D_x(r) = \frac{1}{4\pi r}\int_{\partial B_r(x)} \langle \nabla_{A,\partial_r}\Phi, \Phi\rangle + \mathfrak{r}_D$$

with

$$|\mathfrak{r}_D| \leqslant cr^2 m_x(r).$$

Proof. This is a consequence of Corollary 2.5 with $f = 1$ and $U = B_r(x)$ and Proposition 2.9. □

Proposition 3.11. *For every $r \in (0, r_0]$,*

$$m_x'(r) = \frac{2D_x(r)}{r} + \mathfrak{r}_{m'}$$

with

$$|\mathfrak{r}_{m'}| \leqslant crm_x(r).$$

Proof. Denote by $H_{x,r}$ the mean curvature of $\partial B_r(x)$. By Corollary 2.5,

$$\begin{aligned} m_x(r)' &= \frac{1}{2\pi r^2}\int_{\partial B_r(x)} \big(H_{x,r} - \tfrac{1}{r}\big)|\Phi|^2 + \frac{1}{4\pi r^2}\int_{\partial B_r(x)} \partial_r |\Phi|^2 \\ &= \frac{2D_x(r)}{r} + \frac{1}{2\pi r^2}\int_{\partial B_r(x)} \big(H_{x,r} - \tfrac{1}{r}\big)|\Phi|^2 - \frac{2\mathfrak{r}_D}{r}. \end{aligned}$$

The assertion follows since $\big|H_{x,r} - \frac{1}{r}\big| \leqslant cr$. $\square$

Corollary 3.12. *For every $x \in M$ and $0 < s < r \leqslant r_0$,*

$$m_x(s) \leqslant \big(1 + cr^2\big) m_x(r).$$

Proof of Proposition 3.3. By Proposition 3.6 and Proposition 3.11,

$$\begin{aligned} \mathrm{N}_x'(r) &= \frac{D_x'(r)}{m_x(r)} - \frac{D_x(r) m_x'(r)}{m_x(r)^2} \\ &= \frac{1}{2\pi r m_x(r)} \int_{\partial B_r(x)} |\nabla_{A,\partial_r}\Phi|^2 + \varepsilon^{-2}|i_{\partial_r}\mu(\Phi)|^2 - \frac{2D_x(r)^2}{r m_x(r)^2} + \frac{\mathfrak{r}_{D'}}{m_x(r)} - \frac{\mathfrak{r}_{m'}}{m_x(r)}\mathrm{N}_x(r). \end{aligned}$$

By Proposition 3.10,

$$\begin{aligned} &\frac{1}{2\pi r m_x(r)} \int_{\partial B_r(x)} |\nabla_{A,\partial_r}\Phi - \frac{1}{r}\mathrm{N}_x(r)\Phi|^2 \\ &\quad= \frac{1}{2\pi r m_x(r)} \int_{\partial B_r(x)} |\nabla_{A,\partial_r}\Phi|^2 - \frac{\mathrm{N}_x(r)}{\pi r^2 m_x(r)} \int_{\partial B_r(x)} \langle \nabla_{A,\partial_r}\Phi, \Phi\rangle + \frac{2}{r}\mathrm{N}_x(r)^2 \\ &\quad= \frac{1}{2\pi r m_x(r)} \int_{\partial B_r(x)} |\nabla_{A,\partial_r}\Phi|^2 - \frac{2}{r}\mathrm{N}_x(r)^2 + \frac{4\mathfrak{r}_D}{r m_x(r)}\mathrm{N}_x(r). \end{aligned}$$

Therefore,

$$\begin{aligned} \mathrm{N}_x'(r) &= \frac{1}{2\pi r m_x(r)} \int_{\partial B_r(x)} |\nabla_{A,\partial_r}\Phi|^2 + \varepsilon^{-2}|i_{\partial_r}\mu(\Phi)|^2 - \frac{2}{r}\mathrm{N}_x(r)^2 + \frac{\mathfrak{r}_{D'}}{m_x(r)} - \frac{\mathfrak{r}_{m'}}{m_x(r)}\mathrm{N}_x(r) \\ &= \frac{1}{2\pi r m_x(r)} \int_{\partial B_r(x)} |\nabla_{A,\partial_r}\Phi - \frac{1}{r}\mathrm{N}_x(r)\Phi|^2 + \varepsilon^{-2}|i_{\partial_r}\mu(\Phi)|^2 \\ &\quad \underbrace{+ \frac{\mathfrak{r}_{D'}}{m_x(r)} - \left(\frac{4\mathfrak{r}_D}{r m_x(r)} + \frac{\mathfrak{r}_{m'}}{m_x(r)}\right)\mathrm{N}_x(r)}_{=:\star}. \end{aligned}$$

This completes the proof since $|\star| \leqslant cr(1 + \mathrm{N}_x(r))$. $\square$

3.2 N controls the growth of m

Proposition 3.13. *For every $x \in M$ and $0 < s < r \leqslant r_0$,*

$$\left(\frac{r}{s}\right)^{(2-cr^2)\mathrm{N}_x(r)-cr^2} m_x(s) \leqslant m_x(r) \leqslant \left(\frac{r}{s}\right)^{(2+cr^2)\mathrm{N}_x(r)+cr^2} m_x(s).$$

Proof. By Proposition 3.5 and Proposition 3.11, for $t \in [s, r]$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\log m_x(t) &\leqslant \frac{2\mathrm{N}_x(t)}{t} + ct \\ &\leqslant \frac{2(1+cr^2)}{t}\mathrm{N}_x(r) + \frac{cr^2}{t}\end{aligned}$$

as well as

$$\frac{\mathrm{d}}{\mathrm{d}t}\log m_x(t) \geqslant \frac{2(1-cr^2)}{t}\mathrm{N}_x(r) - \frac{cr^2}{t}.$$

These integrate to the asserted inequalities. □

Proposition 3.14. *If $\Phi \neq 0$, then, for every $x \in M$ and $r \in (0, r_0]$,*

$$m_x(r) > 0;$$

in particular, $r_{-1,x} = 0$.

Proof. If $m_x(r) = 0$, for some $r \in (0, r_0]$, then it follows from Proposition 3.13 that $m_x = 0$. Therefore, Φ vanishes on $B_{r_0}(x)$. This in turn implies that $m_y(r_0/2)$ vanishes for all $y \in B_{r_0/2}(x)$. Hence, Φ vanishes on $B_{\frac{3}{2}r_0}(x)$. Repeating this argument shows that Φ vanishes. □

3.3 Frequency bounds

Proposition 3.15. *For every $r_\star \in (0, r_0]$ and $\delta > 0$, if*

$$0 < s \leqslant r_\star \min\left\{1, \left(\frac{|\Phi|^2(x)}{2m_x(r_\star)}\right)^{1/\delta}\right\},$$

then

$$\mathrm{N}_x(s) \leqslant 2\delta + r_\star.$$

Proof. By Corollary 3.12 and Proposition 3.13, for every $s \in (0, r_\star]$,

$$\left(\frac{r_\star}{s}\right)^{\mathrm{N}_x(s)-r_\star} |\Phi|^2(x) \leqslant 2m_x(r_\star).$$

Therefore,

$$\mathrm{N}_x(s) \leqslant \frac{\log\left(\frac{2m_x(r_\star)}{|\Phi|^2(x)}\right)}{\log\left(\frac{r_\star}{s}\right)} + r_\star.$$

This implies the asserted inequality. □

3.4 Varying the base-point

Proposition 3.16. *There is a constant $c > 0$ such that, for every $x \in M$ and $r \in (0, r_0/4]$, if $\mathrm{N}_x(4r) \leqslant 1$, then, for every $y \in B_r(x)$ and $s \in (0, 2r]$,*

$$\mathrm{N}_y(s) \leqslant c\big(\mathrm{N}_x(4r) + r^2\big).$$

Proof. Since $\mathrm{N}_x(4r) \leqslant 1$, by Proposition 2.9 and Proposition 3.13,

$$m_x(4r) \leqslant cm_x(r/2) \leqslant cr^{-3} \int_{B_r(x)} |\Phi|^2 \leqslant cr^{-3} \int_{B_{2r}(y)} |\Phi|^2 \leqslant cm_y(2r).$$

Therefore,

$$\mathrm{N}_y(2r) \leqslant \frac{m_x(4r)}{m_y(2r)} \mathrm{N}_x(4r) \leqslant c\mathrm{N}_x(4r).$$

The assertion thus follows from Proposition 3.5. □

3.5 Decay implies interior bound

The following result is essentially contained in [Tau13a, Proof of Lemma 6.2].

Lemma 3.17. *There is a constant $\delta > 0$ such that the following holds for every $x \in M$ and $r > 0$. If $f \colon \bar{B}_r(x) \to [0, \infty)$ is an L^1 function such that, for every $y \in M$ and $s > 0$,*

$$B_s(y) \subset B_r(x) \quad \text{and} \quad s \int_{B_s(y)} f \leqslant 1 \implies \frac{s}{4} \int_{B_{s/4}(y)} f \leqslant \delta, \tag{3.18}$$

then

$$\frac{r}{2} \int_{B_{r/2}(x)} f \leqslant 1.$$

Proof. The **regularity scale** associated with f is the function $r_f \colon B_r(x) \to (0, \infty]$ defined by

$$r_f(y) \coloneqq \sup\left\{ s \geqslant 0 : s \int_{B_s(y) \cap B_r(x)} f \leqslant 1 \right\}.$$

If $r_f(x) < \frac{r}{2}$ and $\delta > 0$ is sufficiently small, then the following leads to a contradiction.

Pick a maximal sequence $x_0, x_1, \ldots, x_N$ starting with $x_0 \coloneqq x$ and such that, for every $n = 0, 1, \ldots, N-1$,

$$x_{n+1} \in B_{r_f(x_n)}(x_n) \quad \text{and} \quad r_f(x_{n+1}) < \tfrac{1}{2} r_f(x_n).$$

Such a sequence must terminate, because otherwise $(x_n)_{n \in \mathbf{N}}$ converges to a point $x_\infty \in B_r(x)$ with $r_f(x_\infty) = 0$, which is a contradiction. By maximality, $x_\star \coloneqq x_N$ is such that, for every $y \in B_{r_f(x_\star)}(x_\star)$,

$$\tfrac{1}{2} r_f(x_\star) \leqslant r_f(y). \tag{3.19}$$

There is a constant $N_c \in \mathbf{N}$ depending only on $B_r(x)$ and a finite set $\{y_1, \dots, y_{N_c}\} \subset B_{r_f(x_\star)}(x_\star)$ such that

$$B_{r_f(x_\star)}(x_\star) \subset \bigcup_{n=1}^{N_c} B_{\frac{1}{8} r_f(x_\star)}(y_n).$$

Since $r_f(x) < \frac{r}{2}$, by construction of $x_\star$,

$$d(x, x_\star) + r_f(x_\star) + \tfrac{1}{2} r_f(x_\star) < \sum_{n=0}^{N+1} \tfrac{1}{2^n} r_f(x_a) \leqslant 2 r_f(x) < r;$$

that is:

$$B_{\frac{1}{2} r_f(x_\star)}(y_n) \subset B_r(x).$$

Therefore, by (3.18) and (3.19),

$$\frac{r_f(x_\star)}{8} \int_{B_{\frac{1}{8} r_f(x_\star)}(y_n)} f \leqslant \delta.$$

Hence,

$$r_f(x_\star) \int_{B_{r_f(x_\star)}(x_\star)} f \leqslant r_f(x_\star) \sum_{n=1}^{N_c} \int_{B_{\frac{1}{8} r_f(x_\star)}(y_n)} f \leqslant 8 N_c \delta.$$

If $\delta \leqslant \frac{1}{16} N_c$, then the integral on the left-hand side is at most $\frac{1}{2}$. This, however, contradicts the definition of $r_f(x_\star)$ since $\bar{B}_{r_f(x_\star)}(x_\star) \subset B_r(x)$. □

3.6 Proof of Proposition 3.2

Without loss of generality assume $|\Phi|$ is not identically zero. Choose r_{-1} and δ_{N} as in Proposition 2.19 with c_F as in Definition 3.1. Set

$$f := \frac{|F_A|^2}{c_F}.$$

If $r_\dagger \in (0, r_{-1}]$ is such that, for every $B_s(y) \subset B_{r_\dagger}(x)$,

$$\mathrm{N}_y(s) \leqslant \delta_{\mathrm{N}},$$

then Proposition 2.19 and Lemma 3.17 imply that

$$\frac{r_\dagger}{4} \int_{B_{r_\dagger/4}(x)} |F_A|^2 \leqslant c_F;$$

therefore,

$$r_A(x) \geqslant \frac{r_\dagger}{4}.$$

Let $0 < \sigma \ll 1$. By Proposition 2.9,

$$m_x(r) \leqslant cr_0^{-3}\|\Phi\|_{L^2(M)}^2 = cr_0^{-3}.$$

By Proposition 3.15, there is a constant $c > 0$ such that

$$\mathrm{N}_x(4r_\dagger) \leqslant \sigma\delta_{\mathrm{N}} \quad \text{for} \quad r_\dagger \coloneqq c^{-1}\min\Big\{1, |\Phi|^{8/\sigma\delta_{\mathrm{N}}}(x)\Big\}.$$

By Proposition 3.16, for every $B_s(y) \subset B_{r_\dagger}(x)$,

$$\mathrm{N}_y(s) \leqslant c\Big(\sigma\delta_{\mathrm{N}} + r_\dagger^2\Big).$$

Therefore, after possibly shrinking $r_\dagger$, for every $B_s(y) \subset B_{r_\dagger}(x)$, $\mathrm{N}_y(s) \leqslant \delta_{\mathrm{N}}$. This finishes the proof. ◻

3.7 Hölder bounds

Proposition 3.20. *Suppose that Hypothesis 1.20 holds. There are constants $\alpha \in (0,1)$ and $c > 0$ such that, for every $A \in \mathscr{A}(\mathfrak{s}, B)$, $\Phi \in \Gamma(\mathrm{S})$, and $\varepsilon > 0$ satisfying (1.17) and (1.18),*

$$[\Phi]_{C^{0,\alpha}(M)} \leqslant c.$$

Proof. Let $\delta, r_{-1}, c > 0$ be as in Proposition 3.2. Set $\alpha \coloneqq \min\left\{\frac{1}{4}, \frac{1}{2}\delta\right\}$. Let $x, y \in M$ such that $|\Phi|(x) \geqslant |\Phi|(y)$.

If $d(x,y) \leqslant r_A(x)^2$, then $d(x,y) \leqslant r_A(x)/2$ because $r_A \leqslant r_0$, and by Morrey's inequality, Kato's inequality, and Proposition 2.11,

$$[|\Phi|]_{C^{0,1/2}(B_{r_A(x)/2}(x))} \leqslant c\|\nabla_A\Phi\|_{L^6(B_{r_A(x)/2}(x))} \leqslant cr_A(x)^{-1/2} \leqslant cd(x,y)^{-1/4}.$$

Hence,

$$|\Phi|(x) - |\Phi|(y) \leqslant cd(x,y)^{1/4} \leqslant cd(x,y)^{\alpha}.$$

If $d(x,y) \geqslant r_A(x)^2$, then Proposition 3.2 either $d(x,y) \geqslant r_{-1}^2$ or $d(x,y) \geqslant c^{-2}|\Phi|(x)^{2/\delta}$. In the first case, it follows from Proposition 2.10 that

$$|\Phi|(x) - |\Phi|(y) \leqslant cd(x,y)^{\alpha}.$$

In the second case,

$$|\Phi|(x) - |\Phi|(y) \leqslant 2|\Phi|(x) \leqslant c^{\delta}d(x,y)^{\delta/2}.$$ ◻

4 Proof of Theorem 1.22

Let $(A_n, \Phi_n, \varepsilon_n)$ be a sequence of solutions of (1.17) and (1.18) with ε_n tending to zero. By Proposition 2.10 and Proposition 3.20, for some $\alpha \in (0,1)$,

$$\| |\Phi_n| \|_{C^{0,\alpha}} \leqslant c.$$

Therefore, after passing to a subsequence, for every $\beta \in (0,\alpha)$, $|\Phi_n|$ converges to a limit in the $C^{0,\beta}$–topology. Denote this limit as $|\Phi|$ and set $Z := |\Phi|^{-1}(0)$. Since $\|\Phi\|_{L^2} = 1$, Z is a proper subset of M.

By Proposition 3.2, for every $x \in M\backslash Z$,

$$r_A(x) := \liminf_{n\in\mathbf{N}} r_{A_n}(x) > 0.$$

Therefore, on every compact subset of $M\backslash Z$, the L^2–norms of F_{A_n} are uniformly bounded; and, up to gauge transformations and after passing to a subsequence, (A_n) can be assumed to converge in the weak $W^{1,2}$ topology to a limit A. Moreover, by Proposition 2.11, after passing to a further subsequence, (Φ_n) converges in the weak $W^{2,2}$ topology to a limit Φ. A patching argument as in [DK90, Section 4.2.2] yields asserted convergence statement on $M\backslash Z$. By construction, the limit (A,Φ) satisfies (1.23).

It remains to prove that Z is nowhere dense. The proof of this fact relies on the following.

Proposition 4.1. *For every $x \in M$ and $r \in (0, r_0]$,*

$$\lim_{n\to\infty} \int_{B_r(x)} |\nabla_{A_n}\Phi_n|^2 + 2\varepsilon_n^{-2}|\mu(\Phi_n)|^2 = \int_{B_r(x)} |\nabla_A \Phi|^2 \quad \textit{and}$$
$$\lim_{n\to\infty} \int_{\partial B_r(x)} |\Phi_n|^2 = \int_{\partial B_r(x)} |\Phi|^2.$$

Proof. The second assertion is a consequence of the Hölder convergence. To prove the first assertion, we proceed as follows. For $\varepsilon \in (0, \frac{1}{2}]$, set $Z_\varepsilon := |\Phi|^{-1}([0,\varepsilon])$. Since weak $W^{2,2}$ convergence implies $W^{1,2}$ convergence,

$$\lim_{n\to\infty} \int_{B_r(x)\backslash Z_\varepsilon} |\nabla_{A_n}\Phi_n|^2 = \int_{B_r(x)\backslash Z_\varepsilon} |\nabla_A\Phi|^2.$$

Moreover,

$$\lim_{n\to\infty} \int_{B_r(x)\backslash Z_\varepsilon} 2\varepsilon_n^{-2}|\mu(\Phi_n)|^2 = \lim_{n\to\infty} \int_{B_r(x)\backslash Z_\varepsilon} 2\varepsilon_n^{2}|F_{\mathrm{Ad}(A)}|^2 = 0.$$

The discussion in the next paragraph shows that there exists a $\lambda > 0$ such that, for every $\varepsilon > 0$,

$$\int_{Z_\varepsilon} |\nabla_{A_n}\Phi_n|^2 + 2\varepsilon_n^{-2}|\mu(\Phi_n)|^2 \leqslant c\varepsilon^\lambda.$$

This together with the above implies the first assertion.

Fix a cut-off function $\chi \in C_0^\infty([0,2),[0,1])$ with $\chi|_{[0,1]} = 1$. Corollary 2.5 with $f = \chi(\varepsilon^{-1}|\Phi_n|)$ and $U = M$, integrating the resulting term with $\Delta|\Phi_n|^2$ by parts once and using Kato's inequality yields

$$\begin{aligned}\int_{Z_\varepsilon} |\nabla_{A_n}\Phi_n|^2 + 2\varepsilon_n^{-2}|\mu(\Phi_n)|^2 &\leqslant c\int_{Z_{2\varepsilon}} |\Phi_n|^2 + c\int_{Z_{2\varepsilon}\setminus Z_\varepsilon} \varepsilon^{-1}|\Phi_n||\nabla_A|\Phi_n||^2 \\ &\leqslant c\varepsilon^2 + c\int_{Z_{2\varepsilon}\setminus Z_\varepsilon} |\nabla_{A_n}\Phi_n|^2.\end{aligned}$$

Therefore,

$$f(\varepsilon) \coloneqq \int_{Z_\varepsilon} |\nabla_{A_n}\Phi_n|^2 + 2\varepsilon_n^{-2}|\mu(\Phi_n)|^2$$

satisfies

$$f(\varepsilon) \leqslant \sigma(\varepsilon^2 + f(2\varepsilon)) \quad \text{with} \quad \sigma \coloneqq c/(1+c).$$

Since f is bounded above and we can assume that $\sigma \geqslant 1/2$,

$$\begin{aligned}f(\varepsilon) &\leqslant \sigma\varepsilon^2\sum_{i=0}^{k-1}(4\sigma)^i + \sigma^k f(2^k\varepsilon) \\ &\leqslant \varepsilon^2\sigma\left(\frac{(4\sigma)^{k-1}-1}{4\sigma-1}\right) + c\sigma^k \\ &\leqslant c\varepsilon^2(4\sigma)^k + c\sigma^k.\end{aligned}$$

With $k \coloneqq \lfloor -\log\varepsilon/\log 2\rfloor$ this gives

$$f(\varepsilon) \leqslant c\varepsilon^{2-\log(4\sigma)/\log 2} + c\varepsilon^{-\log\sigma/\log 2} \leqslant c\varepsilon^\lambda$$

for some $\lambda > 0$ depending on σ only, since $\log(4\sigma)/\log 2 < 2$. □

If Z failed to be nowhere-dense, then we could find $x \in Z$ and $0 < r \leqslant r_0$ such that $B_r(x) \subset Z$. By the above, Proposition 3.13 applies and shows that, in fact, $B_{r_0}(x) \subset Z$. This in turn implies that $m_y^\Phi(r_0/2)$ vanishes for all $y \in B_{r_0/2}(x)$. Hence, $|\Phi|$ vanishes on $B_{\frac{3}{2}r_0}(x)$. Repeating this argument shows Z to be all of M, which contradicts $\|\Phi\|_{L^2} = 1$. □

5 Proof of Theorem 1.30

The proof of Theorem 1.30 relies on Theorem 1.22. Over the course, of the next three subsections we establish that the hypotheses of the latter hold for the $\mathrm{ADHM}_{1,2}$ Seiberg–Witten equation. The geometric and algebraic observations made in the process also enter crucially in refining the conclusion of Theorem 1.22 to obtain Theorem 1.30.

5.1 The geometry of the quaternionified adjoint representation of SU(2)

The first step towards the proof of Theorem 1.30 is to verify Hypothesis 1.20—or, more precisely: the condition in Remark 1.24—for stable flat $\mathbf{PSL}_2(\mathbf{C})$–connections.

Lemma 5.1. *There are constants $\delta_\mu, c > 0$ such that, for every $\xi \in \mathfrak{su}(2) \otimes \mathbf{H}$, if*

$$|\mu(\xi)| \leqslant \delta_\mu |\xi|^2,$$

then

$$|\xi||\mu(\xi)| \leqslant c|\Gamma_\xi \mu(\xi)|.$$

The proof of Lemma 5.1 relies on the following observation which is proved by a simple computation; see, e.g., [DW17, Proposition D.5].

Proposition 5.2. *Denote by $\mu\colon \mathfrak{g} \otimes \mathbf{H} \to \mathfrak{g} \otimes \operatorname{Im} \mathbf{H}$ the hyperkähler moment map associated with the quaternionified adjoint representation $G \to \mathrm{Sp}(\mathfrak{g} \otimes \mathbf{H})$. For every $\xi \in \mathfrak{g} \otimes \mathbf{H}$,*

$$|\mu(\xi)|^2 = \frac{1}{2} \sum_{i,j=0}^{3} \big|[\xi_i, \xi_j]\big|^2.$$

Proof of Lemma 5.1. Without loss of generality $|\xi| = 1$. The zero locus $\mu^{-1}(0)$ is a cone with smooth link. Therefore, if $|\mu(\xi)| \leqslant \delta_\mu \ll 1$, then ξ has a unique decomposition as

$$\xi = \zeta + \hat{\xi}$$

with

$$\mu(\zeta) = 0, \quad \hat{\xi} \perp T_\zeta \mu^{-1}(0), \quad \text{and} \quad |\hat{\xi}| \leqslant c|\mu(\xi)| \leqslant c\delta_\mu \ll 1.$$

By Proposition 5.2,

$$\zeta = \tau_0 \otimes v_0$$

with $|\tau_0| = 1$. Extend τ_0 to an orthonormal basis τ_0, τ_1, τ_2 of $\mathfrak{su}(2)$ such that

$$[\tau_0, \tau_1] = 2\tau_2, \quad [\tau_1, \tau_2] = 2\tau_0, \quad \text{and} \quad [\tau_2, \tau_0] = 2\tau_1. \tag{5.3}$$

Since

$$T_\zeta \mu^{-1}(0) = \langle \tau_0 \rangle \otimes \mathbf{H} + \mathfrak{su}(2) \otimes \langle v_0 \rangle,$$

it follows that

$$\mathrm{d}_\zeta \mu(\hat{\xi}) = 2\mu(\zeta, \hat{\xi}) \in \langle \tau_1, \tau_2 \rangle \otimes \operatorname{Im} \mathbf{H} \quad \text{and} \quad \mu(\hat{\xi}) \in \langle \tau_0 \rangle \otimes \operatorname{Im} \mathbf{H}. \tag{5.4}$$

A simple computation using (5.3) shows that

$$\Gamma_\zeta \mu(\hat{\xi}) = 0 \quad \text{and} \quad |\Gamma_\zeta \mu(\zeta, \hat{\xi})| = 2|\zeta||\mu(\zeta, \hat{\xi})|.$$

Therefore,

$$\begin{aligned}|\mu(\xi)|^2 &\leqslant 4|\mu(\zeta,\hat{\xi})|^2 + |\mu(\hat{\xi})|^2\\ &\leqslant c|\Gamma_\zeta\mu(\xi)|^2 + c|\mu(\xi)|^4\\ &\leqslant c|\Gamma_\xi\mu(\xi)|^2 + c\delta_\mu^2|\mu(\xi)|^2.\end{aligned}$$

A rearrangement implies the asserted estimate. □

Remark 5.5. It is crucial that the link of $\mu^{-1}(0)$ is smooth. For $\mathfrak{su}(r)$ with $r \geqslant 3$ this condition fails and, in fact, the conclusion of Lemma 5.1 does not hold in this case.

5.2 The geometry of the $\mathrm{ADHM}_{1,2}$ representation

This subsection contains a number of geometric facts regarding the quaternionic representation of $\mathrm{U}(2)$ on $\mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^2 \oplus \mathfrak{su}(2) \otimes \mathbf{H}$. These will play a crucial role in the proof of Hypothesis 1.20 for $\mathrm{ADHM}_{1,2}$ Seiberg–Witten monopoles in the next subsection.

Proposition 5.6. *Let $k \in \mathbf{N}$. Denote by $\mu\colon \mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^k \to \mathfrak{u}(k) \otimes \operatorname{Im}\mathbf{H}$ the hyperähler moment map associated with the quaternionic representation $\mathrm{U}(k) \to \mathrm{Sp}(\mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^k)$. Let $\mathfrak{t} \subset \mathfrak{u}(k)$ be a maximal torus. Denote by $\pi_{\mathfrak{t}}\colon \mathfrak{u}(k) \to \mathfrak{t}$ the orthogonal projection to $\mathfrak{t}$. For every $\Psi \in \mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^k$,*

$$|\pi_{\mathfrak{t}}\mu(\Psi)| = \frac{1}{2}|\Psi|^2.$$

Proof. For $k = 1$, $\mathfrak{t} = \mathfrak{u}(1)$. By (1.11),

$$\begin{aligned}|\mu(\Psi)|^2 &= \frac{1}{2}\langle \boldsymbol{\gamma}(\mu(\Psi))\Psi, \Psi\rangle\\ &= \frac{1}{4}|\Psi|^4.\end{aligned}$$

For $k \in \{2, 3, \dots\}$, without loss of generality $\mathfrak{t} = \mathfrak{u}(1)^{\oplus k}$. The composition $\pi_{\mathfrak{t}} \circ \mu$ is the hyperkähler moment map for the action of $\mathrm{U}(1)^k \subset \mathrm{U}(k)$ on $\mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^k$. Thus

$$\pi_{\mathfrak{t}}\mu(\Psi_1, \dots, \Psi_k) = (\mu(\Psi_1), \dots, \mu(\Psi_k))$$

and the assertion follows from the case $k = 1$. □

Proposition 5.7. *There is a constant $c > 0$ such that, for every $\Psi \in \mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^2$ and $\xi \in \mathbf{H} \otimes \mathfrak{su}(2)$,*

$$|\mu(\Psi)| + |\mu(\xi)| \leqslant c|\mu(\Psi, \xi)|$$

This is an immediate consequence of the following.

Proposition 5.8. *There is a constant $\sigma < 1$ such that, for every $\Psi \in \mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^2$ and $\xi \in \mathbf{H} \otimes \mathfrak{su}(2)$,*

$$-\langle \mu(\Psi), \mu(\xi)\rangle \leqslant \sigma|\mu(\Psi)||\mu(\xi)|.$$

The proof relies on the following fact.

Proposition 5.9 (Nakajima [Nak99, Section 2.2]; see also [DW17, Proposition D.4]). *Let $k \in \mathbf{N}$. For every $\Psi \in \mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^k$ and $\xi \in \mathbf{H} \otimes \mathfrak{su}(k)$, if $\mu(\Psi, \xi) = 0$, then $\Psi = 0$.*

Proof Proposition 5.8. Set

$$\sigma := \sup\{\langle \mu(\Psi), \mu(\xi)\rangle : |\mu(\Psi)| = |\mu(\xi)| = 1\}.$$

By Cauchy–Schwarz, $\sigma \leqslant 1$; moreover, if $\sigma = 1$, then there are

$$\mu_\Psi \in \overline{\{\mu(\Psi) : |\mu(\Psi)| = 1\}} \quad \text{and} \quad \mu_\xi \in \overline{\{\mu(\xi) : |\mu(\xi)| = 1\}}$$

with

$$\langle \mu_\Psi, \mu_\xi \rangle = -|\mu_\Psi||\mu_\xi|.$$

Therefore,

$$\mu_\Psi = -\mu_\xi.$$

The upcoming discussion proves this to be impossible.

By Proposition 5.6, the set $\{\mu(\Psi) : |\mu(\Psi)| = 1\}$ is closed. In particular,

$$\mu_\Psi = \mu(\Psi)$$

for some $\Psi \in \mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^k$. The closure of $\{\mu(\xi) : |\mu(\xi)| = 1\}$ is

$$\{\mu(\xi) : |\mu(\xi)| = 1\} \cup \left\{ \mathrm{d}_\zeta \mu(\hat{\xi}) : \mu(\zeta) = 0, |\mathrm{d}_\zeta \mu(\hat{\xi})| = 1 \right\}.$$

To see this, let $(\varepsilon_n^{-1} \xi_n)$ be a sequence with $\varepsilon_n > 0$ and $|\xi_n| = 1$ and

$$\lim_{n \to \infty} \mu(\varepsilon_n^{-1} \xi_n) = \mu_\xi.$$

After passing to a subsequence, ξ_n converges to a limit ξ and (ε_n) converges to a limit ε. If $\varepsilon \neq 0$, then, $\mu_\xi = \mu(\varepsilon^{-1}\xi)$. Otherwise, $\mu(\xi) = 0$ and, for $n \gg 1$, as in the proof of Lemma 5.1,

$$\xi_n = \zeta_n + \hat{\xi}_n \quad \text{with} \quad \mu(\zeta_n) = 0 \quad \text{and} \quad \hat{\xi}_n \perp T_{\zeta_n} \mu^{-1}(0).$$

Therefore,

$$\mu(\varepsilon_n^{-1} \xi_n) = \mathrm{d}_{\zeta_n} \mu(\varepsilon_n^{-2} \hat{\xi}_n) + \varepsilon_n^2 \mu(\varepsilon_n^{-2} \hat{\xi}_n).$$

By (5.4), $\varepsilon_n^{-2} \hat{\xi}_n$ is bounded; hence, after passing to a subsequence, it converges to a limit $\hat{\xi}$ and $\mu_\xi = \mathrm{d}_\xi \mu(\hat{\xi})$.

By Proposition 5.9 and because $\Psi \neq 0$, μ_ξ cannot be in $\{\mu(\xi) : |\mu(\xi)| = 1\}$. Therefore,

$$\mu_\xi = \mathrm{d}_\zeta \mu(\hat{\xi}).$$

By Proposition 5.2, there exists a maximal torus $\mathfrak{t} \subset \mathfrak{u}(2)$ such that

$$\zeta \in \mathfrak{t} \otimes \mathbf{H}.$$

Therefore,

$$\mu(\Psi) = -\mathrm{d}_\zeta \mu(\hat{\xi}) \in \operatorname{Im} \mathbf{H} \otimes \mathfrak{t}^\perp.$$

This, however, is impossible, because it would imply that $\Psi = 0$ by Proposition 5.6. □

Proposition 5.10. *There are constants $\delta_\mu, c > 0$ such that, for every non-zero pair $\Psi \in \mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^2$ and $\xi \in \mathbf{H} \otimes \mathfrak{su}(2)$, if*

$$|\mu(\Psi, \xi)| \leqslant \delta_\mu \big(|\Psi|^2 + |\xi|^2\big),$$

then

$$|\mu(\Psi, \xi)|^2 \leqslant c \left[\langle \mu(\Psi, \xi), \mu(\Psi) \rangle + \frac{|\Gamma_\xi \mu(\Psi, \xi)|^2}{|\Psi|^2 + |\xi|^2} \right].$$

Proof. Without loss of generality $|\Psi|^2 + |\xi|^2 = 1$. By Proposition 5.6 and Proposition 5.7,

$$|\Psi|^2 \leqslant c|\mu(\Psi, \xi)| \leqslant c\delta_\mu \quad \text{and} \quad |\mu(\xi)| \leqslant c|\mu(\Psi, \xi)| \leqslant c\delta_\mu.$$

Therefore, for $\delta_\mu \ll 1$,

$$\frac{1}{2} \leqslant |\xi| \leqslant 1.$$

If $\delta_\mu \ll 1$, then, as in the proof of Lemma 5.1, ξ uniquely decomposes as

$$\xi = \zeta + \hat{\xi}$$

with

$$\mu(\zeta) = 0, \quad \hat{\xi} \perp T_\zeta \mu^{-1}(0), \quad \text{and} \quad |\hat{\xi}| \leqslant c|\mu(\xi)| \leqslant c\delta_\mu.$$

In particular,

$$|\mu(\hat{\xi})| \leqslant c|\mu(\xi)|^2 \leqslant c\delta_\mu |\mu(\xi)|.$$

Denote by $\mathfrak{t} \subset \mathfrak{u}(2)$ the maximal torus determined by ζ and by $\pi_\mathfrak{t} \colon \mathfrak{u}(2) \to \mathfrak{t}$ the orthogonal projection onto $\mathfrak{t}$. The discussion in the proof of Lemma 5.1 shows that

$$\pi_\mathfrak{t} \mu(\xi) = \mu(\hat{\xi}) \quad \text{and} \quad (\mathbf{1} - \pi_\mathfrak{t})\mu(\xi) = 2\mu(\zeta, \hat{\xi}),$$

and, moreover,

$$|(\mathbf{1} - \pi_\mathfrak{t})\mu(\Psi, \xi)| \leqslant |\Gamma_\zeta \mu(\Psi, \xi)| \leqslant |\Gamma_\xi \mu(\Psi, \xi)| + c\delta_\mu |\mu(\Psi, \xi)|.$$

By the discussion in the preceding paragraph and Proposition 5.6,

$$\begin{aligned}\langle \mu(\Psi,\xi),\mu(\Psi)\rangle &= |\pi_{\mathfrak{t}}\mu(\Psi)|^2 + \langle \mu(\hat{\xi}),\mu(\Psi)\rangle + \langle (\mathbf{1}-\pi_{\mathfrak{t}})\mu(\Psi,\xi),\mu(\Psi)\rangle \\ &\geqslant |\pi_{\mathfrak{t}}\mu(\Psi)|^2 - c\delta_\mu|\mu(\Psi,\xi)||\Psi|^2 - c|\Gamma_\xi\mu(\Psi,\xi)||\Psi|^2 \\ &\geqslant \frac{1}{2}|\pi_{\mathfrak{t}}\mu(\Psi)|^2 - c\delta_\mu^2|\mu(\Psi,\xi)|^2 - c|\Gamma_\xi\mu(\Psi,\xi)|^2.\end{aligned}$$

Moreover,

$$\begin{aligned}|\mu(\Psi,\xi)|^2 &\leqslant |\pi_{\mathfrak{t}}\mu(\Psi)|^2 + |\mu(\hat{\xi})|^2 + |(\mathbf{1}-\pi_{\mathfrak{t}})\mu(\Psi,\xi)|^2 \\ &\leqslant c\langle\mu(\Psi,\xi),\mu(\Psi)\rangle + c\delta_\mu^2|\mu(\Psi,\xi)|^2 + c|\Gamma_\xi\mu(\Psi,\xi)|^2.\end{aligned}$$

For $\delta_\mu \ll 1$, this implies the asserted inequality by rearrangement. □

5.3 Verification of Hypothesis 1.20

Lemma 5.11. *Assume the situation of Section 1.3. There are constants $r_0, \delta_\mu, c > 0$ such that the following holds for every $x \in M$ and $r \in (0, r_0]$. If $A \in \mathscr{A}(\mathrm{Ad}(\mathfrak{w}))$, $\Psi \in \Gamma(W)$, and $\xi \in \Gamma(N \otimes \mathrm{Ad}(\mathfrak{w})_\circ)$ satisfy (1.28),*

$$\frac{1}{2} \leqslant \sqrt{|\Psi|^2 + |\xi|^2} \leqslant 2 \quad \textit{and} \quad |\mu(\Psi,\xi)| \leqslant \delta_\mu,$$

then

$$\frac{r}{2}\int_{B_{r/2}(x)} |F_A|^2 + \left(\frac{r}{\varepsilon}\right)^2 \cdot r^{-1}\int_{B_{r/2}(x)} |\nabla_A\Psi|^2 \leqslant c + cr\int_{B_r(x)} |\Gamma_\xi F_A|^2.$$

Proof. By the Lichnerowicz–Weitzenböck formula (2.3),

$$\frac{1}{2}\Delta|\Psi|^2 + |\nabla_A\Psi|^2 + \varepsilon^{-2}\langle \boldsymbol{\gamma}(\mu(\Psi,\xi)\Psi,\Psi\rangle + \langle\gamma(\mathfrak{R})\Psi,\Psi\rangle = 0.$$

Therefore, by hypothesis and Proposition 5.10,

$$|\mu(\Psi,\xi)|^2 + \varepsilon^2|\nabla_A\Psi|^2 \leqslant c_1\big(|\Gamma_\xi\mu(\Psi,\xi)|^2 + \varepsilon^2|\Psi|^2\big) - c_2\varepsilon^2\Delta|\Psi|^2.$$

Let $\chi \in C_0^\infty(B_r(x))$ be a cut-off function satisfying $\chi|_{B_{r/2}(x)} = 1$ and

$$r|\nabla\chi| \leqslant c \quad \text{and} \quad r^2|\nabla^2\chi| \leqslant c;$$

in particular,

$$|r^2\Delta\chi^4| \leqslant c\chi^2.$$

Multiplying the above by $r\varepsilon^{-4}\chi^4$ and integrating by parts, and using $F_A = \varepsilon^{-2}\mu(\Psi,\xi)$, yields

$$r\int_{B_r(x)} \chi^4|F_A|^2 + \left(\frac{r}{\varepsilon}\right)^2 r^{-1}\int_{B_r(x)} \chi^4|\nabla_A\Psi|^2 \leqslant cr\int_{B_r(x)} \chi^4|\Gamma_\xi F_A|^2 + c\varepsilon^{-2}r^{-1}\int_{B_r(x)} \chi^2|\Psi|^2.$$

By Proposition 5.6 and Proposition 5.7,

$$\begin{aligned} c\varepsilon^{-2}r^{-1}\int_{B_r(x)}\chi^2|\Psi|^2 &\leqslant \frac{r}{c_1\varepsilon^4}\int_{B_r(x)}\chi^4|\Psi|^4 + c_2 \\ &\leqslant \frac{r}{2}\int_{B_r(x)}\chi^4\varepsilon^{-4}|\mu(\Psi,\xi)|^2 + c_2 \\ &= \frac{r}{2}\int_{B_r(x)}\chi^4|F_A|^2 + c_2. \end{aligned}$$

Plugging this back in to the previous inequality and rearranging proves the asserted inequality. □

5.4 Conclusion of the proof of Theorem 1.30

Let $(A_n, \Psi_n, \xi_n)_{n\in\mathbf{N}}$ be a sequence of solutions with

$$\liminf_{n\to\infty}\|(\Psi_n,\xi_n)\|_{L^2} = \infty.$$

Set

$$\varepsilon_n := \frac{1}{\|(\Psi_n,\xi_n)\|_{L^2}}, \quad \tilde{\Psi}_n := \varepsilon_n\Psi_n, \quad \text{and} \quad \tilde{\xi}_n := \varepsilon_n\xi_n.$$

By Lemma 5.11, Theorem 1.22 applies to the sequence $(A_n, \tilde{\Psi}_n, \tilde{\xi}_n, \varepsilon_n)$. Therefore and by Proposition 5.9, the following hold:

1. There is a closed, nowhere-dense subset $Z \subset M$, a connection $A \in \mathscr{A}(\mathrm{Ad}(\mathfrak{w})|_{M\setminus Z}, B)$, and a section $\xi \in \Gamma(M\setminus Z, N \otimes \mathrm{Ad}(\mathfrak{w})_\circ)$ such that the following hold:

   (a) A and ξ satisfy

   $$\begin{aligned} \not{D}_A\xi &= 0, \\ \mu(\xi) &= 0, \quad \text{and} \\ \|\xi\|_{L^2}^2 &= 1. \end{aligned}$$

   (b) The function $|\xi|$ extends to a Hölder continuous function on all of M and

   $$Z = |\xi|^{-1}(0).$$

2. After passing to a subsequence and up to gauge transformations, $(A_n|_{M\setminus Z})_{n\in\mathbf{N}}$ converges to A in the weak $W^{1,2}_{\mathrm{loc}}$ topology, $\left(\tilde{\Psi}_n|_{M\setminus Z}\right)_{n\in\mathbf{N}}$ converges to 0 in the weak $W^{2,2}_{\mathrm{loc}}$ topology, $\left(\tilde{\xi}_n|_{M\setminus Z}\right)_{n\in\mathbf{N}}$ converges to ξ in the weak $W^{2,2}_{\mathrm{loc}}$ topology, and there exists an $\alpha \in (0,1)$ such that $\left(|(\tilde{\Psi}_n,\tilde{\xi}_n)|\right)_{n\in\mathbf{N}}$ converges to $|\xi|$ in the $C^{0,\alpha}$ topology.

The Euclidean line bundle $\mathfrak{l}$ and the parallel section τ emerge from the Haydys correspondence [DW17, Appendix C]. In the present case this abstract machinery can be made very explicit. By Proposition 5.2, away from Z, ξ can *locally* be written as

$$\xi = \tau \otimes \nu$$

where τ is a local section of $\mathrm{Ad}(\mathfrak{w})_\circ$ which is normalized such that $|\tau| = 1$, and ν is a local section of N. This decomposition is unique up to multiplying both τ and ν by -1. Decomposing ξ in this way, the equation $\not{D}_A \xi = 0$ becomes

$$0 = \tau \otimes \not{D}_B \nu + \sum_{i=1}^{3} (\nabla_{A,e_i}\tau) \otimes \gamma(e_i)\nu.$$

Since τ is normalized, the second term on the right-hand side takes values in $\langle\tau\rangle^\perp \otimes N$. Therefore, both summands on the right-hand side vanish separately. Globally, there is a flat Euclidean line bundle $\mathfrak{l}$ such that ν is a section of $N \otimes \mathfrak{l}$ and τ is a section of $\mathrm{Hom}(\mathfrak{l}, \mathrm{Ad}(\mathfrak{w})_\circ)$. The above shows that $(Z, \mathfrak{l}, \nu)$ is a harmonic $\mathbf{Z}_2$ spinor whose zero locus is precisely Z and τ is A–parallel. By [Zha17, Theorem 1.4], the former implies the asserted regularity of Z.

On every compact subset $K \subset M\backslash Z$, $\|F_{A_n}\|_{L^2(K)}$ is uniformly bounded. Therefore, by Proposition 5.6 and Proposition 5.7, $\|\Psi_n\|_{L^2(K)}$ is uniformly bounded. Since $\not{D}_{A_n}\Psi_n = 0$, it follows that, possibly after passing to a further subsequence, $(\Psi_n|_{M\backslash Z})$ converges in the weak $W^{2,2}$ to a limit Ψ satisfying $\not{D}_A\Psi = 0$. For $n \gg 1$, on K, we can decompose $\tilde{\xi}_n = \zeta_n + \hat{\xi}_n$ as in the proof of Lemma 5.1. Denote by $\mathfrak{t}_n \subset \mathrm{Ad}(\mathfrak{w})|_K$ the corresponding bundle of maximal tori. The $\mathrm{ADHM}_{1,2}$ Seiberg–Witten equation (1.28) and (5.4) imply

$$\pi_{\mathfrak{t}_n} F_{A_n} = \pi_{\mathfrak{t}_n}\mu(\Psi_n) + \varepsilon_n^{-2}\pi_{\mathfrak{t}_n}\mu(\hat{\xi}_n).$$

By (5.4) and Proposition 5.7,

$$|\hat{\xi}_n| \leqslant c\varepsilon_n^2 |F_{A_n}|.$$

Therefore,

$$|\pi_{\mathfrak{t}_n} F_{A_n} - \pi_{\mathfrak{t}_n}\mu(\Psi_n)| \leqslant c\varepsilon_n^2 |F_{A_n}|^2.$$

From this it follows that

$$F_A = \pi_{\mathfrak{t}} F_A = \pi_{\mathfrak{t}}\mu(\Psi).$$

This finishes the proof of Theorem 1.30. □

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