Associative submanifolds in Joyce's generalised Kummer constructions

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2022-01-16

Abstract

This article constructs examples of associative submanifolds in G_2–manifolds obtained by resolving G_2–orbifolds using Joyce's generalised Kummer construction. As the G_2–manifolds approach the G_2–orbifolds, the volume of the associative submanifolds tends to zero. This partially verifies a prediction due to Halverson and Morrison.

1 Introduction

The Teichmüller space

$$\mathscr{T}(Y) \coloneqq \{\phi \in \Omega^3(Y) : \phi \text{ is a } G_2\text{–structure}\}/\mathrm{Diff}_0(Y)$$

of G_2–structures on a closed 7–manifold Y is a smooth manifold of dimension $b^3(Y)$ [Joy96a, Theorem C]. The G_2 period map $\Pi\colon \mathscr{T} \to \mathrm{H}^3_{\mathrm{dR}}(Y) \oplus \mathrm{H}^4_{\mathrm{dR}}(Y)$ defined by

$$\Pi(\phi \cdot \mathrm{Diff}_0(Y)) \coloneqq ([\phi],[\psi]) \quad \text{with} \quad \psi \coloneqq *_\phi \phi$$

is a Lagrangian immersion [Joy96b, Lemma 1.1.3].[1] It is constrained by the following inequalities [Joy96b, Lemma 1.1.2; HL82, §IV.2.A and §IV.2.B]:

(1) $\int_Y \alpha \wedge \alpha \wedge \phi < 0$ for every non-zero $[\alpha] \in \mathrm{H}^2_{\mathrm{dR}}(Y)$ if $\pi_1(Y)$ is finite.

(2) $\int_Y p_1(V) \wedge \phi = -\frac{1}{4\pi^2}\mathrm{YM}(A) < 0$ for every vector bundle V which admits a non-flat G_2–instanton A; in particular, for $V = TY$ unless Y is covered by T^7.

(3) $\int_P \phi = \mathrm{vol}(P) > 0$ for every associative submanifold $P \looparrowright Y$.

(4) $\int_Q \psi = \mathrm{vol}(Q) > 0$ for every coassociative submanifold $Q \looparrowright Y$.

[1] Whether or not Π is an embedding is an open question.

These should be compared with the inequalities cutting out the Kähler cone of a Calabi–Yau 3–fold.

By analogy with Calabi–Yau 3–folds, Halverson and Morrison [HM16, §3] suggest that the above inequalities completely characterise the ideal boundary of $\mathscr{T}(Y)$. Of course, making this precise is complicated by the fact that the notions of G_2–instanton and (co)associative submanifold depend on the G_2–structure ϕ. The situation would be improved if there were invariants whose non-vanishing guaranteed the existence of G_2–instantons and (co)associative submanifolds as suggested by Donaldson and Thomas [DT98, §3]. However, their construction is fraught with enormous difficulty [DS11; Joy18; Hay17; Wal17; DW19].

A more down to earth problem is to exhibit concrete examples of degenerating families of G_2–manifolds which admit G_2–instantons whose Yang–Mills energies tend to zero [Wal13] or which admit (co)associative submanifolds whose volumes tend to zero. The purpose of this article is to present examples of the latter in G_2–manifolds arising from Joyce's generalised Kummer construction. Although these examples had been anticipated (e.g. by Halverson and Morrison [HM16, §6.2]), their rigorous construction has only recently become possibly due to the work of Platt [Pla20].

Remark. Of course, there are already numerous examples of closed associative submanifolds in the literature.

(1) Joyce [Joy96b, §4.2; Joy00, §12.6] has constructed (co)associative submanifolds in generalised Kummer constructions as fixed-point sets of involutions.

(2) Corti, Haskins, Nordström, and Pacini [CHNP15, §5.5 and §7.2.2] have constructed associative submanifolds in twisted connected sums using rigid holomorphic curves and special Lagrangians in asymptotically cylindrical Calabi–Yau 3–folds.

(3) In the physics literature, Braun, Del Zotto, Halverson, Larfors, Morrison, and Schäfer-Nameki [BDHLMS18, §4.4] have proposed a construction of infinitely many associative submanifolds in certain twisted connected sums. An important ingredient in the proof of this conjecture will be a gluing theorem for associative submanifolds in twisted connected sums analogous to [SW15]. Building on [BDHLMS18], Acharya, Braun, Svanes, and Valandro [ABSV19, §2.2 and §4.2] have constructed infinitely many associative submanifolds in certain G_2–orbifolds (without using any analytic methods).

(4) Lotay [Lot14], Kawai [Kaw15], and Ball and Madnick [BM20] have produced a wealth of examples of associative submanifolds in S^7, the squashed S^7, and the Berger space with their nearly parallel G_2–structures.

The novelty of the examples discussed in the present article is that their volumes tend to zero as the ambient G_2–manifolds degenerate. ♣

Acknowledgements. This material is based upon work supported by the Simons Collaboration "Special Holonomy in Geometry, Analysis, and Physics" (DP, TW).

2 Joyce's generalised Kummer construction

The generalised Kummer construction is a method to produce G_2–manifolds by desingularising certain closed flat G_2–orbifolds (Y_0, ϕ_0) introduced by Joyce [Joy96a; Joy96b]. Besides a rather delicate singular perturbation theory it relies on the fact that the hyperkähler 4–orbifolds $\mathbf{H}/\Gamma$, obtained as quotients of the quaternions $\mathbf{H}$ by a finite subgroup $\Gamma < \mathrm{Sp}(1)$, can be desingularised by hyperkähler 4–manifolds. The following model spaces feature prominently throughout this article.

Example 2.1 (model spaces). Let X be a hyperkähler 4–orbifold with hyperkähler form

$$\boldsymbol{\omega} \in (\operatorname{Im}\mathbf{H})^* \otimes \Omega^2(X).$$

Denote by $\mathrm{vol} \in \Omega^3(\operatorname{Im}\mathbf{H})$ and $\mathbf{1} \in \Omega^1(\operatorname{Im}\mathbf{H}) \otimes \operatorname{Im}\mathbf{H}$ the volume form and the tautological 1–form respectively.

(1) The 3–form

$$\mathrm{vol} - \langle \mathbf{1} \wedge \boldsymbol{\omega} \rangle \in \Omega^3(\operatorname{Im}\mathbf{H} \times X) \tag{2.2}$$

defines a torsion-free G_2–structure on $\operatorname{Im}\mathbf{H} \times X$. The corresponding Riemannian metric and the cross-product on $\operatorname{Im}\mathbf{H} \times X$ recover the Riemannian metric and the hypercomplex structure $\mathbf{I} \in (\operatorname{Im}\mathbf{H})^* \otimes \Gamma(\operatorname{End}(TX))$ on X.

(2) Let $G < \mathrm{SO}(\operatorname{Im}\mathbf{H}) \ltimes \operatorname{Im}\mathbf{H}$ be a Bieberbach group; that is: discrete, cocompact, and torsion-free. Let $\rho\colon G \to \operatorname{Isom}(X)$ be a homomorphism. Suppose that $\boldsymbol{\omega}$ is G–invariant; that is: for every $(R, t) \in G$

$$\big(R^* \otimes \rho(R, t)^*\big)\boldsymbol{\omega} = \boldsymbol{\omega}.$$

Set

$$Y := (\operatorname{Im}\mathbf{H} \times X)/G$$

The G_2–structure (2.2) descends to a G_2–structure

$$\phi \in \Omega^3(Y).$$

The canonical projection $p\colon Y \to B := \operatorname{Im}\mathbf{H}/G$ is a flat fibre bundle whose fibres are coassociative and diffeomorphic to X; cf. [Bar19, §3.4]. ♠

Remark 2.3 (Classification of Bieberbach groups). If $G < \mathrm{SO}(\operatorname{Im}\mathbf{H}) \ltimes \operatorname{Im}\mathbf{H}$ is a Bieberbach group, then $\Lambda := G \cap \operatorname{Im}\mathbf{H} < \operatorname{Im}\mathbf{H}$ is a lattice and $H := G/\Lambda < \mathrm{SO}(\Lambda) \times (\operatorname{Im}\mathbf{H}/\Lambda)$ is isomorphic to either $\mathbf{1}$, C_2, C_3, C_4, C_6, or C_2^2; cf. [HW35; CR03; Szc12, §3.3]. More precisely, G is among the following:

($\mathbf{1}$) Λ is arbitrary and $G = \Lambda$.

(C_2) $\Lambda = \langle \lambda_1, \lambda_2, \lambda_3 \rangle$ with

$$\langle \lambda_1, \lambda_2 \rangle = \langle \lambda_1, \lambda_3 \rangle = 0.$$

G is generated by Λ and $(R_2, \frac{1}{2}\lambda_1)$ with $R_2 \in \mathrm{SO}(\Lambda)$ as in (2.5).

(C_3) $\Lambda = \langle \lambda_1, \lambda_3, \lambda_3 \rangle$ with

$$\langle \lambda_1, \lambda_2 \rangle = \langle \lambda_1, \lambda_3 \rangle = 0 \quad \text{and} \quad |\lambda_2|^2 = |\lambda_3|^2 = -2\langle \lambda_2, \lambda_3 \rangle. \tag{2.4}$$

G is generated by Λ and $(R_3, \frac{1}{3}\lambda_1)$ with $R_3 \in \mathrm{SO}(\Lambda)$ as in (2.5).

(C_4) $\Lambda = \langle \lambda_1, \lambda_3, \lambda_3 \rangle$ with

$$\langle \lambda_1, \lambda_2 \rangle = \langle \lambda_1, \lambda_3 \rangle = 0, \quad |\lambda_2|^2 = |\lambda_3|^2, \quad \text{and} \quad \langle \lambda_2, \lambda_3 \rangle = 0.$$

G is generated by Λ and $(R_4, \frac{1}{4}\lambda_1)$ with $R_4 \in \mathrm{SO}(\Lambda)$ as in (2.5).

(C_6) $\Lambda = \langle \lambda_1, \lambda_3, \lambda_3 \rangle$ with (2.4). G is generated by Λ and $(R_6, \frac{1}{6}\lambda_1)$ with $R_6 \in \mathrm{SO}(\Lambda)$ as in (2.5).

(C_2^2) $\Lambda = \langle \lambda_1, \lambda_2, \lambda_3 \rangle$ with

$$\langle \lambda_1, \lambda_2 \rangle = \langle \lambda_2, \lambda_3 \rangle = \langle \lambda_3, \lambda_1 \rangle = 0.$$

G is generated by Λ, $(R_+, \frac{1}{2}(\lambda_1 + \lambda_2))$, and $(R_-, \frac{1}{2}(\lambda_2 + \lambda_3))$ with $R_\pm \in \mathrm{SO}(\Lambda)$ as in (2.5).

Here $R_2, R_3, R_4, R_6, R_\pm \in \mathrm{GL}_3(\mathbf{Z})$ are defined by

$$\begin{gathered} R_2 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad R_3 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad R_4 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \\ R_6 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{and} \quad R_\pm := \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \mp 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \end{gathered} \tag{2.5}$$

$\mathrm{GL}_3(\mathbf{Z})$ is identified with $\mathrm{GL}(\Lambda)$ by the choice of generators of Λ. ♣

The generalised Kummer construction involves a choice of the following data.

Definition 2.6. Let (Y_0, ϕ_0) be a flat G_2–orbifold. Denote the connected components of the singular set of Y_0 by S_α ($\alpha \in A$). **Resolution data** $\mathfrak{R} = (\Gamma_\alpha, G_\alpha, \rho_\alpha; R_\alpha, \jmath_\alpha; \hat{X}_\alpha, \hat{\boldsymbol{\omega}}_\alpha, \hat{\rho}_\alpha, \tau_\alpha)_{\alpha \in A}$ for (Y_0, ϕ_0) consists of the following for every $\alpha \in A$:

(1) A finite subgroup $\Gamma_\alpha < \mathrm{Sp}(1)$, Bieberbach group $G_\alpha < \mathrm{SO}(\mathrm{Im}\,\mathbf{H}) \ltimes \mathrm{Im}\,\mathbf{H}$, and a homomorphism $\rho_\alpha \colon G_\alpha \to \mathrm{SO}(\mathbf{H})^{\Gamma_\alpha} \hookrightarrow \mathrm{Isom}(\mathbf{H}/\Gamma_\alpha)$ as in Example 2.1 (2) with $X := \mathbf{H}/\Gamma_\alpha$ and its canonical hyperkähler form $\boldsymbol{\omega}$.

(2) An open set

$$U_\alpha := \left(\mathrm{Im}\,\mathbf{H} \times (B_{2R_\alpha}(0)/\Gamma_\alpha)\right) \big/ G_\alpha \subset Y_\alpha$$

and an open embedding $\jmath_\alpha \colon U_\alpha \to Y_0$ satisfying $S_\alpha \subset \mathrm{im}\,\jmath_\alpha$ and

$$\jmath_\alpha^* \phi_0 = \phi_\alpha$$

with (Y_α, ϕ_α) denoting the model space associated with $\mathbf{H}/\Gamma_\alpha$, $\boldsymbol{\omega}$, G_α, and ρ_α.

(3) A hyperkähler 4–manifold $\hat{X}_\alpha$ with hyperkähler form $\hat{\boldsymbol{\omega}}_\alpha \in (\operatorname{Im}\mathbf{H})^* \otimes \Omega^2(\hat{X}_\alpha)$, a homomorphism $\hat{\rho}_\alpha \colon G_\alpha \to \operatorname{Diff}(\hat{X}_\alpha)$ with respect to which $\hat{\boldsymbol{\omega}}_\alpha$ is G_α–invariant, a compact subset $K_\alpha \subset \hat{X}_\alpha$, and a G_α–equivariant open embedding $\tau_\alpha \colon \hat{X}_\alpha \backslash K_\alpha \to \mathbf{H}/\Gamma_\alpha$ with $(\mathbf{H}\backslash B_{R_\alpha}(0))/\Gamma \subset \operatorname{im}\tau_\alpha$ and

$$|\nabla^k(\tau_* \hat{\boldsymbol{\omega}}_\alpha - \boldsymbol{\omega})| = O(r^{-4-k}) \tag{2.7}$$

for every $k \in \mathbf{N}_0$. •

Remark 2.8 (ADE classification of finite subgroups of $\mathrm{Sp}(1)$). Klein [Kle93] classified the (non-trivial) finite subgroups $\Gamma < \mathrm{Sp}(1)$. The obey an ADE classification. Γ is isomorphic to either:

(A_k) a cyclic group C_{k+1},

(D_k) a dicyclic group Dic_{k-2},

(E_6) the binary tetrahedral group $2T$,

(E_7) the binary octahedral group $2O$, or

(E_8) the binary icosahedral group $2I$. ♣

Remark 2.9. Whether or not the data in Definition 2.6 (1) and (2) exists is a property of a neighborhood of the singular set of Y_0. If it does exist, then it is essentially unique. The data in Definition 2.6 (3) involves a choice. ♣

Remark 2.10. There are many examples of closed flat G_2–orbifolds admitting resolution data in the above sense; see [Joy96b, §3; Joy00, §12; Bar06, §3; Rei17, §5.3.4 and §5.3.5]. They arise from certain crystallographic groups $G < G_2 \ltimes \mathbf{R}^7$. It would be interesting to classify these (possibly computer-aided) to grasp the full scope of Joyce's generalised Kummer construction. Partial results have been obtained by Barrett [Bar06, §3.2], and Reidegeld [Rei17, Theorem 5.3.1] observed that in Definition 2.6 (1) precisely C_2, C_3, C_4, C_6, Dic_2, Dic_3, and $2T$ can appear. ♣

Remark 2.11 (scaling resolution data). For every $(t_\alpha) \in (0,1]^A$ the data $\hat{\boldsymbol{\omega}}_\alpha$ and τ_α in Definition 2.6 (3) can be replaced with $t_\alpha^2 \hat{\boldsymbol{\omega}}_\alpha$ and $t_\alpha \tau_\alpha$. ♣

The following two remarks help to find resolution data $\mathfrak{R}$ with certain properties.

Remark 2.12 (Gibbons–Hawking construction of A_k ALE spaces). Let $k \in \mathbf{N}$. Consider the subgroup $C_k \hookrightarrow \mathrm{Sp}(1)$ generated by right multiplication with $e^{2\pi i/k}$. (Of course, i can be replaced by $\hat{\xi} \in S^2 \subset \operatorname{Im}\mathbf{H}$ throughout.) The A_k ALE hyperkähler 4–manifolds used to resolve $\mathbf{H}/C_k$ can be understood concretely using the Gibbons–Hawking construction [GH78; GRG97, §3.5].

(1) Let
$$\boldsymbol{\zeta} \in \Delta \coloneqq \mathrm{Sym}_0^k(\operatorname{Im}\mathbf{H}) \coloneqq \left\{[\zeta_1, \ldots, \zeta_k] \in (\operatorname{Im}\mathbf{H}^k)/S_k : \zeta_1 + \cdots + \zeta_k = 0\right\}.$$
Set $Z \coloneqq \{\zeta_1, \ldots, \zeta_k\}$ and $B \coloneqq \operatorname{Im}\mathbf{H}\backslash Z$. The function $V_{\boldsymbol{\zeta}} \in C^\infty(B)$ defined by

$$V_{\boldsymbol{\zeta}}(q) \coloneqq \sum_{a=1}^k \frac{1}{2|q - \zeta_a|}$$

is harmonic and

$$[*\mathrm{d}V_\zeta] \in \mathrm{im}\big(\mathrm{H}^2(B, 2\pi\mathbf{Z}) \to \mathrm{H}^2_{\mathrm{dR}}(B)\big).$$

Therefore, there is a U(1)–principal bundle $p_\zeta\colon X_\zeta^\circ \to B$ and a connection 1–form $i\theta_\zeta \in \Omega^1(X_\zeta^\circ, i\mathbf{R})$ with

$$\mathrm{d}\theta_\zeta = -p_\zeta^*(*\mathrm{d}V_\zeta). \tag{2.13}$$

Indeed, p_ζ is determined by V_ζ up to isomorphism. The Euclidean inner product on $\mathrm{Im}\,\mathbf{H}$ defines

$$\sigma \in (\mathrm{Im}\,\mathbf{H})^* \otimes \Omega^1(\mathrm{Im}\,\mathbf{H}).$$

X_ζ° is an incomplete hyperkähler manifold with hyperkähler form $\boldsymbol{\omega}_\zeta$ defined by

$$\boldsymbol{\omega}_\zeta \coloneqq \theta_\zeta \wedge p_\zeta^*\sigma + p_\zeta^*(V_\zeta \cdot *\sigma).$$

(2) The map $p_0\colon (\mathbf{H}\backslash\{0\})/\Gamma \to B$ defined by

$$p_0([x]) \coloneqq \frac{xix^*}{2k}$$

is a U(1)–principal bundle with $[x]\cdot e^{i\alpha} \coloneqq [xe^{i\alpha/k}]$. The connection 1–form $i\theta_0$ defined by

$$\theta_0([x,v]) \coloneqq \frac{\langle xi, v\rangle}{k|x|^2}$$

satisfies (2.13). Therefore, $X_0^\circ = (\mathbf{H}\backslash\{0\})/\Gamma$. A straightforward (but slightly tedious) computation reveals that $\boldsymbol{\omega}_0$ agrees with the standard hyperkähler form on $(\mathbf{H}\backslash\{0\})/\Gamma$. As a consequence, X_ζ° can be extended to a complete hyperkähler orbifold X_ζ by adding $\#Z$ points. If

$$\boldsymbol{\zeta} \in \Delta^\circ \coloneqq \big\{[\zeta_1, \ldots, \zeta_k] \in \Delta : \zeta_1, \ldots, \zeta_k \text{ are pairwise distinct}\big\},$$

then X_ζ is a manifold. Since

$$|\nabla^k(V_\zeta - V_0)\circ p_0| = O(|x|^{-4-k}) \tag{2.14}$$

for every $k \in \mathbf{N}_0$, the asymptotic decay condition (2.7) holds.

(3) Let $\boldsymbol{\zeta} \in \Delta^\circ$. If

$$\ell = \big\{\hat{\xi}t + \eta : t \in [a,b]\big\} \subset \mathrm{Im}\,\mathbf{H}$$

is a segment with $\hat{\xi} \in S^2 \subset \mathrm{Im}\,\mathbf{H}$, $\partial\ell \subset Z$ and $\ell^\circ \subset B$, then

$$\Sigma_\ell \coloneqq p_\zeta^{-1}(\ell) \subset X_\zeta$$

is $I_{\zeta,\hat{\xi}}$–holomorphic with

$$I_{\zeta,\hat{\xi}} \coloneqq \langle \mathbf{I}_\zeta, \hat{\xi}\rangle$$

and $\Sigma_\ell \cong S^2$. $\mathrm{H}_2(X_\zeta, \mathbf{Z})$ is generated by the homology classes of these curves. In fact, X_ζ retracts to a tree of these curves.

(4) Define $\Lambda^+\colon \mathrm{SO}(\mathbf{H}) \to \mathrm{SO}(\operatorname{Im}\mathbf{H})$ by

$$\Lambda^+ R(\mathrm{d}q \wedge \mathrm{d}\bar{q}) \coloneqq R_*(\mathrm{d}q \wedge \mathrm{d}\bar{q}).$$

Let $\boldsymbol{\zeta} \in \Delta^\circ$. If $R \in \mathrm{SO}(\mathbf{H})^\Gamma$ satisfies $\Lambda^+ R(\boldsymbol{\zeta}) = \boldsymbol{\zeta}$, then it lifts to an isometry $\hat{R} \in \mathrm{Diff}(X_{\boldsymbol{\zeta}})$ satisfying

$$\big((\Lambda^+ R)^* \otimes \hat{R}^*\big)\boldsymbol{\omega}_{\boldsymbol{\zeta}} = \boldsymbol{\omega}_{\boldsymbol{\zeta}} \quad \text{and} \quad \hat{R}(\Sigma_\ell) = \Sigma_\ell$$

for every ℓ as in (3). ♣

Remark 2.15 (Kronheimer's construction of ALE spaces). Let $\Gamma < \mathrm{Sp}(1)$ be a finite subgroup—not necessarily cyclic. The ALE hyperkähler 4–manifolds asymptotic to $\mathbf{H}/\Gamma$ can be understood using the work of Kronheimer [Kro89b; Kro89a]. This is rather more involved that Remark 2.12 and summarised in the following. (This is only used for Example 4.6 and might be skipped at the reader's discretion.)

(1) Denote by $R \coloneqq \mathbf{C}[\Gamma] = \mathrm{Map}(\Gamma, \mathbf{C})$ the regular representation of Γ equipped with the standard Γ–invariant Hermitian inner product. Set

$$S \coloneqq (\mathbf{H} \otimes_{\mathbf{R}} \mathfrak{u}(R))^\Gamma \quad \text{and} \quad G \coloneqq \mathbf{P}U(R)^\Gamma.$$

The adjoint action of G on S has a distinguished hyperkähler moment map

$$\mu\colon S \to (\operatorname{Im}\mathbf{H})^* \otimes \mathfrak{g}^*.$$

Denote by $\mathfrak{z}^* \subset \mathfrak{g}^*$ the annihilator of $[\mathfrak{g}, \mathfrak{g}]$. For every

$$\boldsymbol{\zeta} \in \tilde{\Delta} \coloneqq (\operatorname{Im}\mathbf{H})^* \otimes \mathfrak{z}^*$$

the hyperkähler quotient

$$X_{\boldsymbol{\zeta}} \coloneqq S{/\!/\!/}_{\boldsymbol{\zeta}} G \coloneqq \mu^{-1}(\boldsymbol{\zeta})/G$$

is an ALE hyperkähler 4–orbifold asymptotic to $\mathbf{H}/\Gamma$.

(2) Set

$$\Pi \coloneqq \big\{ i\pi \in \mathfrak{u}(R)^\Gamma \backslash \{0, \mathbf{1}\} : \pi^2 = \pi \big\}.$$

(This set is in bijection with the set of non-trivial proper subrepresentations of R.) For $i\pi \in \Pi$ denote by $D_{i\pi} \coloneqq [i\pi]^0 \subset \mathfrak{z}^*$ the annihilator of $[i\pi] \in \mathfrak{g}$. If

$$\boldsymbol{\zeta} \in \tilde{\Delta}^\circ \coloneqq \tilde{\Delta} \backslash D \quad \text{with} \quad D \coloneqq \bigcup_{i\pi \in \Pi} (\operatorname{Im}\mathbf{H})^* \otimes D_{i\pi},$$

then $X_{\boldsymbol{\zeta}}$ is a manifold.

(3) Remark 2.8 associates a Dynkin diagram with Γ. According to the McKay correspondence, the non-trivial irreducible representations of Γ correspond to the vertices of this diagram. The corresponding root system Φ has a preferred set of positive roots Φ^+. The latter can be identified with Π. In particular, the hyperplanes $D_{i\pi}$ correspond to the walls of the Weyl chambers of Φ.

(4) Let $\boldsymbol{\zeta} \in \tilde{\Delta}^\circ$. Let $\alpha \in \Phi$ be a simple root. Define $\xi \in \operatorname{Im}\mathbf{H}$ by $\langle \xi, \cdot \rangle := \boldsymbol{\zeta}(\alpha)$ and set $\hat{\xi} := \xi/|\xi|$. There is a $I_{\zeta,\hat{\xi}}$–holomorphic curve
$$\Sigma_\alpha \subset X_\zeta$$
with $\Sigma_\alpha \cong S^2$. $\mathrm{H}_2(X_\zeta)$ is generated by the homology classes of these curves. In fact, X_ζ retracts to a tree of these curves. This identifies $\mathrm{H}_2(X_\zeta)$ with the root lattice $\mathbf{Z}\Phi$.

(5) Let $\boldsymbol{\zeta} \in \tilde{\Delta}^\circ$. If $R \in \mathrm{SO}(\mathbf{H})^\Gamma$ satisfies $\Lambda^+ R(\boldsymbol{\zeta}) = \boldsymbol{\zeta}$, then it lifts to an isometry $\hat{R} \in \mathrm{Diff}(X_\zeta)$ satisfying
$$(R^* \otimes \hat{R}^*)\boldsymbol{\omega}_\zeta = \boldsymbol{\omega}_\zeta \quad \text{and} \quad \hat{R}(\Sigma_\alpha) = \Sigma_\alpha.$$
Denote by W the Weyl group of Φ. Every $\sigma \in W$ induces a hyperkähler isometry $\hat{\sigma}\colon X_\zeta \cong X_{\sigma(\zeta)}$ satisfying $\hat{\sigma}(\Sigma_\alpha) = \Sigma_{\sigma(\alpha)}$. In particular, $\tilde{\Delta}$ and $\tilde{\Delta}^\circ$ can be replaced with
$$\Delta := \tilde{\Delta}/W \quad \text{and} \quad \Delta^\circ := \tilde{\Delta}^\circ/W.$$

Of course, for $\Gamma = C_k$ the above parallels Remark 2.12. ♣

The generalised Kummer construction proceeds by constructing an approximate resolution and correcting it via singular perturbation theory.

Definition 2.16 (approximate resolution). Let (Y_0, ϕ_0) be a flat G_2–orbifold together with resolution data $\mathfrak{R}$. Let $t \in (0, 1]$. Set
$$Y_0^\circ := Y_0 \backslash \bigcup_{\alpha \in A} \jmath_\alpha\Big(\big(\operatorname{Im}\mathbf{H} \times (\overline{B}_{R_\alpha}(0)/\Gamma_\alpha)\big)/G_\alpha\Big).$$
For $\alpha \in A$ denote by $(\hat{Y}_{\alpha,t}, \hat{\phi}_{\alpha,t})$ the model space associated with $\hat{X}_\alpha$, $t^2\hat{\boldsymbol{\omega}}_\alpha$, G_α, and $\hat{\rho}_\alpha$. Set
$$\hat{Y}_t^\circ := \coprod_{\alpha \in A} \hat{Y}_{\alpha,t}^\circ \quad \text{with} \quad \hat{Y}_{\alpha,t}^\circ := \Big(\operatorname{Im}\mathbf{H} \times \big(K_\alpha \cup (t\tau_\alpha)^{-1}(B_{2R_\alpha}(0)/\Gamma_\alpha)\big)\Big)\Big/ G_\alpha,$$
$$\hat{V}_t := \coprod_{\alpha \in A} \hat{V}_{\alpha,t} \quad \text{with} \quad \hat{V}_{\alpha,t} := \Big(\operatorname{Im}\mathbf{H} \times (t\tau_\alpha)^{-1}\big((B_{2R_\alpha}(0)\backslash B_{R_\alpha}(0))/\Gamma_\alpha\big)\Big)\Big/ G_\alpha, \quad \text{and}$$
$$V := \coprod_{\alpha \in A} V_\alpha \quad \text{with} \quad V_\alpha := \Big(\operatorname{Im}\mathbf{H} \times \big((B_{2R_\alpha}(0)\backslash B_{R_\alpha}(0))/\Gamma_\alpha\big)\Big)\Big/ G_\alpha.$$
Denote by $f\colon \hat{V}_t \to V$ the diffeomorphism induced by $\jmath_\alpha$ and $t\tau_\alpha$ ($\alpha \in A$). Denote by Y_t the 7–manifold obtained by gluing $\hat{Y}_t^\circ$ and Y_0° along f:
$$Y_t := \hat{Y}_t^\circ \cup_f Y_0^\circ.$$
A cut-and-paste procedure (whose details are swept under the rug [Joy96b, Proof of Theorem 2.2.1; Joy00, §11.5.3]) produces a closed 3–form
$$\tilde{\phi}_t \in \Omega^3(Y_t)$$
which agrees with $\hat{\phi}_{\alpha,t}$ on $\hat{Y}_\alpha^\circ \backslash \hat{V}_{\alpha,t}$ ($\alpha \in A$) and with ϕ_0 on $Y_0^\circ \backslash V$; moreover: if t is sufficiently small, then $\tilde{\phi}_t$ defines a G_2–structure on Y_t. •

Remark 2.17. Since $\hat{X}_\alpha$ retracts to a compact subset, there are canonical maps

$$\eta_\alpha\colon H_\bullet(\hat{Y}_{\alpha,t}, \mathbf{Z}) \cong H_\bullet(\hat{Y}^\circ_{\alpha,t}, \mathbf{Z}) \to H_\bullet(Y_t, \mathbf{Z}). \qquad ♣$$

Remark 2.18. As t tends to zero, the Riemannian metric $\tilde{g}_t$ associated with $\tilde{\phi}_t$ degenerates quite severely: $\|R_{\tilde{g}_t}\|_{L^\infty} \sim t^{-2}$ and $\operatorname{inj}(\tilde{g}_t) \sim t^{-1}$. To ameliorate this it can be convenient to pass to the Riemannian metric $t^{-2}\tilde{g}_t$ associated with $t^{-3}\tilde{\phi}_t$. ♣

The following refinement of Joyce's existence theorem for torsion-free G_2–structures [Joy96a, Theorem B; Joy00, Theorems G1 and G2] is crucial.

Theorem 2.19 (Platt [Pla20, Corollary 4.31]). *Let $\mathfrak{R}$ be resolution data for a closed flat G_2–orbifold (Y_0, ϕ_0). Let $\alpha \in (0, 1/8)$. There are $T_0 = T_0(\mathfrak{R})$, $c = c(\mathfrak{R}, \alpha) > 0$ and for every $t \in (0, T_0)$ there is a torsion-free G_2–structure $\phi_t \in \Omega^3(Y_t)$ with $[\phi_t] = [\tilde{\phi}_t] \in \mathrm{H}^3_{\mathrm{dR}}(Y_t)$ satisfying*

$$\|t^{-3}(\phi_t - \tilde{\phi}_t)\|_{C^{1,\alpha/2}} \leqslant ct^{(3-\alpha)/2}.$$

Here $\|-\|_{C^{1,\alpha}}$ is with respect to $t^{-2}\tilde{g}_t$.

3 Perturbing Morse–Bott families of associative submanifolds

Throughout, let Y be a 7–manifold with a G_2–structure $\phi \in \Omega^3(Y)$. Set

$$\psi := *\phi \in \Omega^4(Y).$$

Encode the torsion of ϕ as the section $\tau \in \Gamma(\mathfrak{gl}(TY))$ defined by

$$\nabla_v \psi =: \tau(v)^\flat \wedge \phi.$$

Here $-^\flat\colon TY \to T^*Y$ denotes the isomorphism induced by the Riemannian metric.

Definition 3.1. A closed oriented 3–dimensional immersed submanifold $P \looparrowright Y$ is **(ϕ–)associative** if

$$\phi|_P > 0 \quad \text{and} \quad (i_v\psi)|_P = 0 \quad \text{for every} \quad v \in NP$$

or, equivalently, if it is ϕ–semi-calibrated [HL82, Theorem 1.6]. •

Example 3.2. Assume the situation of Example 2.1 (2). Let $\hat{\xi} \in S^2 \subset \operatorname{Im}\mathbf{H}$, $L > 0$, and $\Sigma \subset X$. Suppose that Σ is a closed $I_{\hat{\xi}}$–holomorphic curve with $I_{\hat{\xi}} := \langle \mathbf{I}, \hat{\xi}\rangle$, $\xi := L\hat{\xi} \in \Lambda < G$ is primitive, $\mathbf{Z}\xi < G$ is normal, and, for every $g \in G$, $\rho(g)(\Sigma) = \Sigma$. In this situation, for every

$$[\eta] \in M/H \quad \text{with} \quad M := (\operatorname{Im}\mathbf{H}/\mathbf{R}\xi)/(\Lambda/\mathbf{Z}\xi) \cong T^2 \quad \text{and} \quad H := G/\Lambda$$

the submanifold

$$P_{[\eta]} := \big((\mathbf{R}\xi + \eta) \times \Sigma\big) \big/ \mathbf{Z}\xi \looparrowright Y$$

is associative and diffeomorphic to the mapping torus T_μ of $\mu = \rho(\xi) \in \operatorname{Diff}(\Sigma)$. ♠

Remark 3.3. $\mathbf{Z}\xi < G$ is normal if and only if ξ is an eigenvector of every $R \in G \cap \mathrm{SO}(\mathrm{Im}\,\mathbf{H})$. Direct inspection of Remark 2.3 reveals the following possibilities (without loss of generality):

(**1**) $H \cong \mathbf{1}$ and $\xi \in \Lambda$ is any primitive element.

(C_2^+) $H \cong C_2$ and $\xi = \lambda_1$. The orbifold M/H has 4 singularities: each with isotropy C_2.

(C_2^-) $H \cong C_2$ and $\xi = \lambda_2$. M/H is diffeomorphic to the Klein bottle $\mathbf{R}P^2\#\mathbf{R}P^2$.

(C_3) $H \cong C_3$ and $\xi = \lambda_1$. The orbifold M/H has 3 singularities: each with isotropy C_3.

(C_4) $H \cong C_4$ and $\xi = \lambda_1$. The orbifold M/H has 3 singularities: two with isotropy C_4, one with isotropy C_2.

(C_6) $H \cong C_6$ and $\xi = \lambda_1$. The orbifold M/H has 3 singularities: one with isotropy C_6, one with isotropy C_3, one with isotropy C_2.

(C_2^2) $H \cong C_2^2$ and $\xi = \lambda_1$. The orbifold M/H has 2 singularities: each with isotropy C_2. ♣

Remark 3.4. The examples discussed in Section 4 are based on Example 3.2 with $\mu = \mathrm{id}_\Sigma$. ♣

Let $\beta \in \mathrm{H}_3(Y, \mathbf{Z})$. Denote by $\mathcal{S} = \mathcal{S}(Y)$ the oribfold of closed oriented 3–dimensional immersed submanifolds $P \looparrowright Y$ with $\phi|_P > 0$ and $[P] = \beta$; cf. [KM97, §44]. Define $\delta\Upsilon = \delta\Upsilon^\psi \in \Omega^1(\mathcal{S})$ by

$$\delta\Upsilon_P(v) := \int_P i_v\psi \quad \text{for} \quad v \in T_P\mathcal{S} = \Gamma(NP).$$

By construction, $P \in \mathcal{S}$ is associative if and only if it is a critical point of $\delta\Upsilon$.

If $\mathrm{d}\psi = 0$, then $\delta\Upsilon$ is closed; indeed: there is a covering map $\pi\colon \tilde{\mathcal{S}} \to \mathcal{S}$ such that $\pi^*\delta\Upsilon$ is exact. The covering map π is the principal covering map associated with the sweep-out homomorphism

$$\mathrm{sweep}\colon \pi_1(\mathcal{S}) \to \mathrm{H}_4(Y).$$

More concretely: choose $P_0 \in \mathcal{S}$ and denote by $\tilde{S}$ the set of equivalence classes $[P, Q]$ of pairs consisting of $P \in \mathcal{S}$ and a 4–chain Q satisfying $\partial Q = P - P_0$ with respect to the equivalence relation $\sim$ defined by

$$(P_1, Q_1) \sim (P_2, Q_2) \iff \big(P_1 = P_2 \quad \text{and} \quad [Q_1 - Q_2] = 0 \in \mathrm{H}_4(Y, \mathbf{Z})\big).$$

$\tilde{\mathcal{S}}$ admits a unique smooth structure such that the canonical projection map $\pi\colon \tilde{S} \to S$ is a smooth covering map. Evidently, $\Upsilon = \Upsilon^\psi \in C^\infty(\tilde{\mathcal{S}})$ defined by

$$\Upsilon([P, Q]) := \int_Q \psi$$

satisfies

$$\mathrm{d}\Upsilon = \pi^*(\delta\Upsilon).$$

If $\mathbf{P}\colon M \to \mathcal{S}$ is a smooth map, then critical points of $\mathbf{P}^*(\delta\Upsilon)$ need not correspond to associative submanifolds. However, the following trivial observation turns out be helpful.

Lemma 3.5. *Let* $\mathbf{P}\colon M \to \mathcal{S}$ *be a smooth map. If for every* $x \in M$

$$\ker(\delta\Upsilon)_{\mathbf{P}(x)} + \operatorname{im} T_x\mathbf{P} = T_{\mathbf{P}(x)}\mathcal{S},$$

then $\mathbf{P}(x)$ *is associative if and only if* x *is a zero of* $\mathbf{P}^*(\delta\Upsilon)$. ■

Remark 3.6. Lemma 3.5 is particularly useful if there is a mechanism that forces $\mathbf{P}^*(\delta\Upsilon)$ to have zeros; e.g.:

(1) If M is closed, then $\mathbf{P}^*(\delta\Upsilon)$ has $\chi(M)$ zeros (counted with signs and multiplicities).

(2) If there is a finite group H acting on M and $\mathbf{P}^*(\delta\Upsilon)$ is H–invariant, then every isolated fixed-point is a zero.

(3) If M is closed and $\mathbf{P}^*(\delta\Upsilon)$ is exact, then it has at least two zeros. $\mathbf{P}^*(\delta\Upsilon)$ is exact if and only if the composite homomorphism

$$\pi_1(M) \xrightarrow{\pi_1(\mathbf{P})} \pi_1(\mathcal{S}) \xrightarrow{\text{sweep}} \mathrm{H}_4(Y) \xrightarrow{\langle -,[\psi]\rangle} \mathbf{R} \tag{3.7}$$

vanishes. ♣

The deformation theory of associative submanifolds is quite well-behaved. Here is a summary of the salient points.

Definition 3.8. A **tubular neighborhood** of $P \in \mathcal{S}$ is an open immersion $\jmath\colon U \looparrowright Y$ with $U \subset NP$ an open neighborhood of the zero section in NP satisfying $[0,1]\cdot U = U$. •

Let $\jmath\colon U \looparrowright V$ be a tubular neighborhood of $P \in \mathcal{S}$. Define $\mathbf{Q} = \mathbf{Q}_\jmath\colon \Gamma(U) \to \mathcal{S}$ by

$$\mathbf{Q}(v) \coloneqq \jmath(\Gamma_v) \quad \text{with} \quad \Gamma_v \coloneqq \operatorname{im} v \subset NP.$$

This map is (the inverse of) a chart of $\mathcal{S}$. Since $\Gamma(U) \subset \Gamma(NP)$ open, $\Omega^1(\Gamma(U))$ can be identified with $C^\infty(\Gamma(U), \Gamma(NP)^*)$. Therefore, it makes sense to Taylor expand $\mathbf{Q}^*(\delta\Upsilon)$. If P is associative, then the first order term is independent of $\jmath$.

Definition 3.9. Let $P \in \mathcal{S}$ be associative. Define $\gamma\colon \operatorname{Hom}(TP, NP) \to NP$ by

$$\langle \gamma(v \cdot u^\flat), w\rangle \coloneqq \phi(u, v, w).$$

Denote by $\tau^\perp \in \Gamma(\mathfrak{gl}(NP))$ the restriction of $\tau \in \Gamma(\mathfrak{gl}(TY))$. The **Fueter operator** $D = D_P\colon \Gamma(NP) \to \Gamma(NP)$ associated with P is defined by

$$D \coloneqq -\gamma\nabla + \tau^\perp.$$ •

Proposition 3.10 (McLean [McL98, §5], Akbulut and Salur [AS08, Theorem 6], Gayet [Gay14, Theorem 2.1], Joyce [Joy18, Theorem 2.12]). *Let* $P \in \mathcal{S}$ *be associative. Let* $\jmath\colon U \looparrowright V$ *be a tubular neighborhood of* P. *There are a constant* $c = c(\jmath) > 0$ *and a smooth map* $\mathcal{N} = \mathcal{N}_\jmath \in C^\infty(\Gamma(U), \Gamma(NP))$ *such that*

$$\langle \mathbf{Q}^*(\delta\Upsilon)(v), w\rangle = \langle Dv + \mathcal{N}(v), w\rangle_{L^2}.$$

and

$$\|\mathcal{N}(v) - \mathcal{N}(w)\|_{C^{0,\alpha}} \leqslant c(\|v\|_{C^{1,\alpha}} + \|w\|_{C^{1,\alpha}})\|v - w\|_{C^{1,\alpha}}.$$

Remark 3.11. If ψ is closed, then D is self-adjoint; indeed, it corresponds to the Hessian of Υ. ♣

Proof of Proposition 3.10. To ease notation, set $f \coloneqq \mathbf{P}^*(\delta\Upsilon)$. Since

$$\langle f(v), w\rangle = \int_{\Gamma_v} i_w \jmath^* \psi,$$

$T_u f\colon T_u\Gamma(U) = \Gamma(NP) \to \Gamma(NP)^*$ satisfies

$$\langle T_u f(v), w\rangle = \int_{\Gamma_u} \mathscr{L}_v i_w \jmath^* \psi.$$

Since

$$f(v) = T_0 f(v) + \underbrace{\int_0^1 (T_{tv} f - T_0 f)(v)\,\mathrm{d}t}_{=:\langle \mathscr{N}(v), -\rangle_{L^2}},$$

it remains to identify $T_0 f$ as D and estimate $\mathscr{N}(v)$.

Choose a frame (e_1, e_2, e_3) on U which restricts to a positive orthonormal frame on Γ_{tu} for every $t \in [0,1]$. Denote by ∇ the Levi-Civita connection on of $\jmath^* g$ on U. To ease notation, henceforth suppress $\jmath$. Since ∇ is torsion-free,

$$\begin{aligned}(\mathscr{L}_v i_w \psi)(e_1, e_2, e_3) &= \psi(\nabla_w v, e_1, e_2, e_3) + \langle \tau v, w\rangle \phi(e_1, e_2, e_3). \\ &\quad + \psi(w, \nabla_{e_1} v, e_2, e_3) + \psi(w, e_1, \nabla_{e_2} v, e_3) + \psi(w, e_1, e_2, \nabla_{e_3} v).\end{aligned} \tag{3.12}$$

A moment's thought derives the asserted estimate on $\mathscr{N}$ from this; cf. [MS12, Remark 3.5.5].

Since P is associative, on $P = \Gamma_0$, the first term in (3.12) vanishes and the second equals $\langle \tau^\perp v, w\rangle$. To digest the second line of (3.12), define the cross-product $-\times-\colon TY \otimes TY \to TY$ and the associator $[-,-,-]\colon TY \otimes TY \otimes TY \to TY$ by

$$\langle u \times v, w\rangle \coloneqq \phi(u, v, w) \quad \text{and} \quad \psi(u, v, w, x) \coloneqq \langle [u, v, w], x\rangle.$$

These are related by

$$[u, v, w] = (u \times v) \times w + \langle v, w\rangle u - \langle u, w\rangle v.$$

Therefore,

$$\psi(w, \nabla_{e_i} v, e_j, e_k) = -\langle w, (e_j \times e_k) \times \nabla_{e_i} v\rangle;$$

cf. [SW17, §4]. Since P is associative, $e_i \times e_j = \sum_{k=1}^3 \varepsilon_{ij}{}^k e_k$. Therefore, the second line of (3.12) is

$$-\sum_{a=1}^3 \langle e_a \times \nabla_{e_a} v, w\rangle = -\langle \gamma \nabla v, w\rangle.$$ ■

In Example 3.2, the operator D, governing the infinitesimal deformation theory of $P = P_{[\eta]}$, can be understood rather concretely.

Example 3.13. Assume the situation of Example 3.2 with $\mu = \mathrm{id}_\Sigma$. Evidently,

$$TP_{[\eta]} = \mathbf{R}\xi \oplus T\Sigma \quad \text{and} \quad NP_{[\eta]} = (\mathbf{R}\xi)^\perp \oplus N\Sigma.$$

Direct inspection reveals that $\gamma(- \cdot \xi^\flat)$ defines a complex structure i on $(\mathbf{R}\xi)^\perp$ and agrees with $-I_\xi$ on $N\Sigma$; moreover, for $\zeta \cdot v^\flat \in \mathrm{Hom}(T\Sigma, (\mathbf{R}\xi)^\perp)$

$$\gamma(\zeta \cdot v^\flat) = I_\zeta v \in N\Sigma$$

A moment's thought shows that

$$\gamma(\zeta \cdot v^\flat I_\xi) = I_\xi \gamma(\zeta \cdot v^\flat) = \gamma(i\zeta \cdot v^\flat).$$

Therefore, the restriction of γ to $\mathrm{Hom}(T\Sigma, (\mathbf{R}\xi)^\perp)$ is the composition of a complex linear isomorphism

$$\kappa\colon \overline{\mathrm{Hom}}_{\mathbf{C}}(T\Sigma, (\mathbf{R}\xi)^\perp) \cong N\Sigma$$

and the projection $(-)^{0,1}\colon \mathrm{Hom}(T\Sigma, (\mathbf{R}\xi)^\perp) \to \overline{\mathrm{Hom}}_{\mathbf{C}}(T\Sigma, (\mathbf{R}\xi)^\perp)$ defined by $A^{0,1} := \frac{1}{2}(A + IAI)$. Therefore,

$$D = D_{P_{[\eta]}} = (-i \oplus I_\xi) \cdot \partial_\xi - \begin{pmatrix} 0 & \bar\partial^* \kappa^* \\ \kappa \bar\partial & 0 \end{pmatrix}$$

with the Cauchy–Riemann operator $\bar\partial\colon C^\infty(\Sigma, (\mathbf{R}\xi)^\perp) \to \Gamma\big(\overline{\mathrm{Hom}}_{\mathbf{C}}(T\Sigma, (\mathbf{R}\xi)^\perp)\big)$ defined by

$$(\bar\partial f)(v) := (\mathrm{d}f)^{0,1}(v) = \frac{1}{2}(\nabla_v f + i\nabla_{I_\xi v} f)$$

and $\bar\partial^*$ denoting its formal adjoint. In particular,

$$\ker D_{P_{[\eta]}} \cong (\mathbf{R}\xi)^\perp \oplus \mathrm{H}^{0,1}(\Sigma, (\mathbf{R}\xi)^\perp). \qquad \spadesuit$$

If $\mathrm{coker}\, D = 0$, then P is **unobstructed** and stable under perturbations of the G_2–structure ϕ. In Example 3.2, $P_{[\eta]}$ is *never* unobstructed, but does satisfy the following if $\Sigma = S^2$ because $(\mathbf{R}\xi)^\perp = T_{[\eta]}M$.[2]

Definition 3.14. A smooth map $\mathbf{P}\colon M \to \mathcal{S}$ is a **Morse–Bott** family of (ϕ–)associative submanifolds if it is an immersion and for every $x \in M$

$$(\delta\Upsilon)_{\mathbf{P}(x)} = 0 \quad \text{and} \quad \ker D_{\mathbf{P}(x)} = \mathrm{im}\, T_x\mathbf{P}. \qquad \bullet$$

Morse–Bott families of ϕ–associative submanifolds are not stable under small deformations of the G_2–structure; however, the hypothesis of Lemma 3.5 can be arranged. Most of the remainder of this section is devoted to establishing this. Henceforth, the choice of G_2–structure $\phi \in \Omega^3(Y)$ made at the beginning of this section shall be undone.

[2] If $P_{[\eta]}$ is multiply covering, then the underlying embedded associative submanifold might be unobstructed; see Remark 4.2.

Definition 3.15. Let $\mathbf{P}_0\colon M \to \mathcal{S}$ be a smooth map. Consider the fibre bundle

$$p\colon \underline{\mathbf{P}}_0 := \coprod_{x\in M} \mathbf{P}_0(x) \hookrightarrow M \times Y \to M.$$

(1) The **normal bundle** of $\mathbf{P}_0$ is the vector bundle

$$q\colon N\mathbf{P}_0 := \coprod_{x\in M} N\mathbf{P}_0(x) \to \underline{\mathbf{P}}_0.$$

There is a canonical isomorphism $N\mathbf{P}_0 \cong N\underline{\mathbf{P}}_0 := T(M\times Y)/T\underline{\mathbf{P}}_0$.

(2) A **tubular neighborhood** of $\mathbf{P}_0$ is a tubular neighborhood $\jmath\colon \mathbf{U} \hookrightarrow M \times Y$ of $\underline{\mathbf{P}}_0$ with $\mathrm{pr}_M \circ \jmath = p \circ q$. In particular, for every $x \in M$, $\jmath$ induces a tubular neighborhood $\jmath_x\colon U_x \hookrightarrow Y$ of $\mathbf{P}_0(x)$.

(3) Consider the vector bundle

$$r\colon \mathbf{E} := \coprod_{\mathbf{P}_0\in C^\infty(M,\mathcal{S})} \Gamma\big(\mathrm{Hom}(TM, \mathbf{P}_0^*T\mathcal{S})\big) \to C^\infty(M,\mathcal{S}).$$

Differentiation defines a section $T \in \Gamma(\mathbf{E})$. Let $\jmath\colon \mathbf{U} \hookrightarrow \mathbf{V}$ be a tubular neighborhood of $\mathbf{P}_0$. The map $\mathbf{Q}_\jmath\colon \Gamma(\mathbf{U}) \to C^\infty(M,\mathcal{S})$ defined by

$$\mathbf{Q}_\jmath(v)(x) := \mathbf{Q}_{\jmath_x}(v) \quad \text{with} \quad v_x := v|_{\mathbf{P}_0(x)}$$

is (the inverse of) a chart on $C^\infty(M,\mathcal{S})$. Within this chart $\mathbf{E}$ is trivialised and T is identified with a smooth map $\mathbf{T} = \mathbf{T}_\jmath \in C^\infty\big(\Gamma(\mathbf{U}), \Gamma(\mathrm{Hom}(p^*TM, N\mathbf{P}_0))\big)$; that is: the diagram

$$\begin{array}{ccc}
\Gamma(\mathbf{U}) \times \Gamma(\mathrm{Hom}(p^*TM, N\mathbf{P}_0)) & \longrightarrow & \mathbf{E} \\
{\scriptstyle (\mathrm{id},\mathbf{T})}\uparrow\downarrow & & {\scriptstyle r}\downarrow\uparrow{\scriptstyle T} \\
\Gamma(\mathbf{U}) & \xrightarrow{\quad\mathbf{Q}\quad} & C^\infty(M,\mathcal{S})
\end{array}$$

commutes.

Henceforth, suppose that ϕ_0 is a G_2–structure and that $\mathbf{P}_0(x)$ is ϕ_0–associative for every $x \in M$.

(4) Define $D = D_{\mathbf{P}_0}\colon \Gamma(N\mathbf{P}_0) \to \Gamma(N\mathbf{P}_0)$ by

$$(Dv)|_{\mathbf{P}_0(x)} := D_{\mathbf{P}_0(x)}(v|_{\mathbf{P}_0(x)}).$$

Set

$$\mathscr{V} := \{v \in \Gamma(N\mathbf{P}_0) : v|_{\mathbf{P}_0(x)} \perp \mathrm{im}\, T_x\mathbf{P}_0 \text{ for every } x \in M\}$$

with $\perp$ denoting L^2 orthogonality. Denote by $D^\perp = D^\perp_{\mathbf{P}_0}\colon \mathscr{V} \to \mathscr{V}$ the map induced by D.

(5) Let $\jmath\colon \mathbf{U} \hookrightarrow \mathbf{V}$ be a tubular neighborhood of $\mathbf{P}_0$. Define $\mathscr{N} = \mathscr{N}_\jmath\colon C^\infty(\Gamma(\mathbf{U}), \Gamma(N\mathbf{P}_0))$ by

$$(\mathscr{N}v)|_{\mathbf{P}_0(x)} := \mathscr{N}_{\jmath_x}(v)|_{\mathbf{P}_0(x)}. \qquad \bullet$$

Example 3.16. In the situation of Example 3.2 with $\mu = \mathrm{id}_\Sigma$, $M = T^2$, $\underline{\mathbf{P}} = T^2 \times (S^1 \times \Sigma)$ and $N\mathbf{P} = NT^2 \oplus N\Sigma$. $D_{\mathbf{P}(x)}$ and $\mathcal{N}_{J_x}$—for a suitable choice of J and with respect to suitable identifications—are independent of $x \in T^2$. ♠

Definition 3.17. Let $\mathbf{P}\colon M \to \mathcal{S}$ be a smooth map. Suppose that Riemannian metrics on M and Y are given. This induces a Euclidean inner product and an orthogonal covariant derivative ∇ on $N\mathbf{P}$, and an Ehresmann connection on $p\colon \underline{\mathbf{P}} \to M$. Denote by $\nabla^{1,0}$ and $\nabla^{0,1}$ the restriction of ∇ to the horizontal and vertical directions respectively. Denote by

$$\mathfrak{P} := \coprod_{x\in M} C^\infty([0,1], \pi^{-1}(x))$$

the set of vertical paths in $p\colon \underline{\mathbf{P}} \to M$. Denote by $\mathfrak{P}^+ \subset \mathfrak{P}$ the subset of non-constant paths. For $\alpha \in (0,1)$ set

$$[v]_{C^0C^{0,\alpha}} := \sup_{\gamma\in\mathfrak{P}^+} \frac{|\mathrm{tra}_\gamma(v(\gamma(0))) - v(\gamma(1))|}{\ell(\gamma)^\alpha} \quad \text{and} \quad \|v\|_{C^0C^{0,\alpha}} := \|v\|_{C^0} + [v]_{C^0C^{0,\alpha}}$$

with $\ell(\gamma)$ denoting the length of γ and tra_γ denoting parallel transport along γ. For $k, \ell \in \mathbf{N}_0$, $\alpha \in (0,1)$ define the norm $\|-\|_{C^kC^{\ell,\alpha}}$ on $\Gamma(N\mathbf{P})$ by

$$\|v\|_{C^kC^{\ell,\alpha}} := \sum_{m=0}^{k}\sum_{n=0}^{\ell} \|(\nabla^{1,0})^m(\nabla^{0,1})^n v\|_{C^0C^{0,\alpha}}. \qquad \bullet$$

Proposition 3.18. *Let $\alpha \in (0,1)$, $\beta, \gamma, c_1, c_2, c_3, c_4, c_5, R > 0$. If $2\beta > \gamma$, then there are constants $T = T(\alpha, \beta, \gamma, c_1, c_2, c_3, c_4, c_5, R) > 0$ and $c_v = c_v(\alpha, \beta, \gamma, c_1, c_2, c_3) > 0$ with the following significance. Let $\phi_0, \phi \in \Omega^3(Y)$ be two G_2–structures on Y. Let $\mathbf{P}_0\colon M \to \mathcal{S}$ be a Morse–Bott family of ϕ_0–associative submanifolds. Let $J\colon \mathbf{U} \hookrightarrow \mathbf{V}$ tubular neighborhood of $\mathbf{P}_0$. Let $t \in (0, T)$. Suppose that:*

1. *$B_R(0) \subset U_x$.*
2. *$\|J^*(\phi - \phi_0)\|_{C^{1,\alpha}(\mathbf{U})} \leqslant c_1 t^\beta$.*
3. *$D^\perp\colon \mathscr{V} \to \mathscr{V}$ is bijective and*

$$\|v\|_{C^1C^{1,\alpha}} \leqslant c_2 t^{-\gamma} \|D^\perp_x v\|_{C^1C^{0,\alpha}}.$$

4. *$\mathcal{N} \in C^\infty(\Gamma(\mathbf{U}), \Gamma(N\mathbf{P}_0))$ satisfies*

$$\|\mathcal{N}(v) - \mathcal{N}(w)\|_{C^1C^{0,\alpha}} \leqslant c_3(\|v\|_{C^1C^{1,\alpha}} + \|w\|_{C^{1,\alpha}})\|v - w\|_{C^1C^{1,\alpha}}.$$

5. *For every $\hat{x} \in TM$ and $v \in \Gamma(\mathbf{U})$*

$$|\hat{x}| \leqslant c_4 \|\mathbf{T}(0)(\hat{x})\|_{C^0} \quad \textit{and} \quad \|\mathbf{T}(v) - \mathbf{T}(0)\|_{C^0} \leqslant c_5 \|v\|_{C^1C^{1,\alpha}}.$$

In this situation, there is a $v \in \Gamma(\mathbf{U}) \subset \Gamma(N\mathbf{P}_0)$ *with* $\|v\|_{C^1C^{1,\alpha}} \leqslant c_v t^{\beta-\gamma}$ *such that the map* $\mathbf{P}\colon M \to \mathcal{S}$ *defined by*

$$\mathbf{P}(x) := \jmath_x(v_x) \quad \textit{with} \quad v_x := v|_{\mathbf{P}_0(x)}$$

satisfies the hypothesis of Lemma 3.5 with respect to ϕ*. Moreover, if* H *is a finite group acting on* M *and* Y*,* ϕ_0 *and* ϕ *are* H*−invariant, and* $\jmath$ *and* $\mathbf{P}_0$ *are* H*−equivariant, then* $\mathbf{P}$ *is* H*−equivariant.*

Proof. To ease notation, define $f_0, f \in C^\infty(\Gamma(\mathbf{U}), \Gamma(N\mathbf{P}_0))$ by

$$\langle f_0(v)|_{\mathbf{P}_0(x)}, w\rangle_{L^2} := \langle (\mathbf{Q}^*_{\jmath_x}(\delta\Upsilon^{\psi_0}))(v), w\rangle \quad \text{and} \quad \langle f(v)|_{\mathbf{P}_0(x)}, w\rangle_{L^2} := \langle (\mathbf{Q}^*_{\jmath_x}(\delta\Upsilon^{\psi}))(v), w\rangle.$$

Denote by $(-)^\perp$ the projection onto $\mathscr{V}$. For $v \in \mathscr{V}$

$$(D^\perp)^{-1} f(v)^\perp = v + \underbrace{(D^\perp)^{-1}(\mathscr{N}(v) + f(v) - f_0(v))^\perp}_{:=E(v)}.$$

By (2), (3), and (4), there is a constant $c_E = c_E(\alpha, \beta, \gamma, c_1, c_2, c_3) > 0$ such that for every $r \in (0, R)$ and $v, w \in \overline{B}_r(0) \subset C^1C^{1,\alpha}\Gamma(N\mathbf{P}_0)$

$$\|E(0)\|_{C^1C^{1,\alpha}} \leqslant c_E t^{\beta-\gamma} \quad \text{and}$$
$$\|E(v) - E(w)\|_{C^1C^{1,\alpha}} \leqslant c_E(r + t^\beta)t^{-\gamma}\|v - w\|_{C^1C^{1,\alpha}}.$$

Therefore, $-E$ defines a contraction on $\overline{B}_r(0) \subset C^1C^{1,\alpha}\Gamma(N\mathbf{P}_0)$ provided

$$c_E(r + t^\beta)t^{-\gamma} < 1 \quad \text{and} \quad c_E t^{\beta-\gamma} + c_E(r + t^\beta)t^{-\gamma} r \leqslant r.$$

These can be seen to hold for $r := 2c_E t^{\beta-\gamma}$ and $t \leqslant T \ll 1$. Denote by $v \in \overline{B}_r(0) \subset C^1C^{1,\alpha}\Gamma(N\mathbf{P}_0)$ the unique solution of

$$f(v)^\perp = 0.$$

By elliptic regularity, $v \in \Gamma(\mathbf{U})$.

It remains to prove that $\mathbf{P}$ defined by (3.18) satisfies the hypothesis of Lemma 3.5; that is: for every $x \in M$

$$\ker(\delta\Upsilon^\psi)_{\mathbf{P}(x)} + \operatorname{im} T_x\mathbf{P} = T_{\mathbf{P}(x)}\mathcal{S},$$

or, equivalently,

$$f_x(v) = 0 \quad \text{or} \quad f_x(v) \notin (\operatorname{im} \mathbf{T}_x(v))^\perp.$$

Here the subscript x indicates restriction to $\mathbf{P}_0(x)$. By construction, $f_x(v) \in \operatorname{im} \mathbf{T}_x(0)$. Therefore, the hypothesis is satisfied by (5) provided $t \leqslant T \ll 1$.

Evidently, this construction preserves H−equivariance. ■

In Example 3.2 with $\Sigma = S^2$, the following gives the required estimate on D.

Situation 3.19. Let X be a compact oriented Riemannian manifold. Let V be a Euclidean vector bundle over X. Let $A\colon \Gamma(V) \to \Gamma(V)$ be a formally self-adjoint linear elliptic differential operator of first order. Denote by $\pi\colon \Gamma(V) \to \ker A$ the L^2 orthogonal projection onto $\ker A$. Let $L > 0$. Define $\Pi\colon \Gamma((\mathbf{R}/L\mathbf{Z}) \times X, V) \to \ker A$ by

$$\Pi s := \fint_0^L \pi(i_t^* s)\,\mathrm{d}t$$

with $i_t(x) := (t, x)$. ×

Remark 3.20. In situation of Example 3.2 with $\mu = \mathrm{id}_\Sigma$, according to Example 3.13

$$D_{P_{[\eta]}} = (-i \oplus I_\xi) \cdot (\partial_\xi + A) \quad \text{with} \quad A := \begin{pmatrix} 0 & i\bar\partial^*\kappa^* \\ -\kappa\bar\partial i & 0 \end{pmatrix}. \qquad ♣$$

Proposition 3.21. *In Situation 3.19, for every $\alpha \in (0,1)$ there is a constant $c = c(A, \alpha) > 0$ such that for every $s \in \Gamma((\mathbf{R}/L\mathbf{Z}) \times X, V)$*

$$\|s\|_{C^{1,\alpha}} \leqslant c\big((L+1)\|(\partial_t + A)s\|_{C^{0,\alpha}} + \|\Pi s\|_{L^\infty}\big).$$

Proof. By interior Schauder estimates (see, e.g., [Kic06, §3])

$$\|s\|_{C^{1,\alpha}} \leqslant c_1(\|(\partial_t + A)s\|_{C^{0,\alpha}} + \|s\|_{L^\infty}).$$

Define $\hat\pi\colon \Gamma((\mathbf{R}/LZ) \times X, V) \to \Gamma((\mathbf{R}/LZ) \times X, V)$ by

$$(\hat\pi s)(t, x) := (\pi(i_t^* s))(x).$$

A contradiction argument proves that

$$\|(\mathbf{1} - \hat\pi)s\|_{L^\infty} \leqslant c_2\|(\partial_t + A)(\mathbf{1} - \hat\pi)s\|_{C^{0,\alpha}} \leqslant c_2\big(\|(\partial_t + A)s\|_{C^{0,\alpha}} + \|(\partial_t + A)\hat\pi s\|_{C^{0,\alpha}}\big);$$

cf. [Wal13, Proof of Proposition 8.5]. As a consequence of the fundamental theorem of calculus

$$\|\hat\pi s\|_{L^\infty} \leqslant L\|\partial_t \hat\pi s\|_{L^\infty} + \|\Pi s\|_{L^\infty} = L\|(\partial_t + A)\hat\pi s\|_{L^\infty} + \|\Pi s\|_{L^\infty}.$$

Therefore,

$$\|s\|_{L^\infty} \leqslant c_2\|(\partial_t + A)s\|_{C^{0,\alpha}} + (c_2 + L)\|(\partial_t + A)\hat\pi s\|_{L^\infty}.$$

Evidently,

$$\hat\pi(\partial_t + A) = (\partial_t + A)\hat\pi.$$

Therefore,

$$\|(\partial_t + A)\hat\pi s\|_{C^{0,\alpha}} \leqslant c_3\|(\partial_t + A)s\|_{C^{0,\alpha}}.$$

The above observations combine to the asserted estimate with $c = c_1(c_2 + 1)(c_3 + 1)$. ■

4 Examples

Proposition 4.1. *Let* $\mathfrak{R} = (\Gamma_\alpha, G_\alpha, \rho_\alpha; R_\alpha, \jmath_\alpha; \hat{X}_\alpha, \hat{\boldsymbol{\omega}}_\alpha, \hat{\rho}_\alpha, \tau_\alpha)_{\alpha\in A}$ *be resolution data for a closed flat* G_2*–orbifold* (Y_0, ϕ_0)*. Denote by* $(Y_t, \phi_t)_{t\in(0,T_0)}$ *the family of closed* G_2*–manifolds obtained from the generalised Kummer construction discussed in Section 2. Let* $\star \in A$*,* $\hat{\xi} \in S^2 \subset \operatorname{Im}\mathbf{H}$*,* $L > 0$*, and* $\Sigma \subset X_\star$*. Set* $\xi := L\hat{\xi}$*,* $\Lambda_\star := G_\star \cap \operatorname{Im}\mathbf{H} < \operatorname{Im}H$*,* $M_\star := (\operatorname{Im}\mathbf{H}/\mathbf{R}\xi)/(\Lambda/\mathbf{Z}\xi)$*, and* $H_\star := G_\star/\Lambda_\star$*. Denote by* n_f *the number of singularities of the orbifold* $M_\star/H_\star$ *(see Remark 3.3). Denote by* $\mathbf{I}_\star$ *the hypercomplex structure on* $X_\star$*. Set* $\beta := \eta_\star([P_{[0]}]) \in H_3(Y_t, \mathbf{Z})$ *with* η_α *as in Remark 2.17 and* $P_{[0]} \subset \hat{Y}_{\star,t}$ *as in Example 3.2. Suppose that:*

(1) Σ *is a closed* $I_{\star,\hat{\xi}}$*–holomorphic curve.* $\Sigma \cong S^2$*.*

(2) $\xi \in \Lambda_\star$ *is primitive.* $\xi \in Z(G_\star)$*.*

(3) $\rho_\star(g)(\Sigma) = \Sigma$*, for every* $g_\star \in G$*, and* $\rho(\xi)|_\Sigma = \mathrm{id}_\Sigma$*.*

In this situation, then there is a constant $T_1 \in (0, T_0]$ *and for every* $t \in (0, T_1)$ *there are at least* n_f *associative submanifolds in* (Y_t, ϕ_t) *representing the homology class* β*.*

Proof. For every $[\eta] \in M_\star/H_\star$ and $t \ll 1$, Example 3.2 constructs an associative submanifold $P_{[\eta]} \hookrightarrow \hat{Y}^\circ_{\star,t}\backslash\hat{V}_{\star,t}$. This defines an $H_\star$–invariant Morse–Bott family $\mathbf{P}_0 \colon M \to \mathscr{S}$ of $t^{-3}\tilde{\phi}_t$–associative submanifolds; see Example 3.13. With respect to the $t^{-2}\tilde{g}_t$ these submanifolds are isometric to $(\mathbf{R}/t^{-1}L\mathbf{Z}) \times \Sigma$. Theorem 2.19, Example 3.13, Remark 3.20, Proposition 3.21, and Example 3.16 verify the hypotheses of Proposition 3.18 with $\phi_0 = t^{-3}\tilde{\phi}_t$, $\phi = t^{-3}\phi_t$, $\alpha \in (0, 1/8)$, $\beta = 5/2$, $\gamma = 1$, and $c_1, c_2, c_3, c_4, c_5, R > 0$ of secondary importance. Set $T_{1/2} := \min\{T_0, T(\alpha, \beta, \gamma, c_1, c_2, c_3, c_4, c_5, R)\}$. For $t \in (0, T_{1/2})$ the resulting $H_\star$–invariant map $\mathbf{P}\colon M \to \mathscr{S}$ satisfies the hypothesis of Lemma 3.5. By Remark 3.6 (2), every isolated fixed-point of the action of $H_\star$ on $M_\star$ is a zero of $\mathbf{P}^*(\delta\Upsilon^{\phi_t})$. If $t < T_1 \ll 1$, then these map to n_f pairwise distinct elements of $\mathscr{S}$. By Lemma 3.5, each one of these is a ϕ_t–associative submanifold. ∎

Remark 4.2. If $x \in M_\star$ corresponds to an orbifold point $[x] \in M_\star/H_\star$, then $P_0 := \mathbf{P}_0(x)$ and $\mathbf{P}(x)$ are multiply covering and their deck transformation group contain the isotropy group Γ of $[x]$. The embedded associative submanifold $\check{P}_0 := P_0/\Gamma$ is unobstructed; indeed:

$$\ker D_{\check{P}_0} = (\ker D_{P_0})^\Gamma = (T_x M_\star)^\Gamma = 0.$$

This can be used to give a somewhat simpler proof of Proposition 4.1 avoiding the use of Proposition 3.18. ♣

Example 4.3. Joyce [Joy96b, Examples 4, 5, 6] constructs 7 examples of closed flat G_2–orbifolds (Y_0, ϕ_0) whose singular set has components S_α $(\alpha \in A)$. A is a disjoint union $A = A^0 \amalg A^1$ with $A^1 \neq \emptyset$. For $\alpha \in A^0$, S_α is isometric to $T^3 = \mathbf{R}^3/\mathbf{Z}^3$. For $\alpha \in A^0$, S_α is isometric to T^3/C_2. Here is a more precise description. For $\alpha \in A$ set $\Gamma_\alpha := C_2$. For $\alpha \in A^0$ set $G_\alpha := \Lambda = \langle i, j, k\rangle < \operatorname{Im}\mathbf{H}$ and denote by $\rho_\alpha \colon G_\alpha \to \operatorname{Isom}(\mathbf{H}/\Gamma_\alpha)$ the trivial homomorphism. For $\alpha \in A^1$ let $G_\alpha < \mathrm{SO}(\operatorname{Im}\mathbf{H}) \ltimes \operatorname{Im}\mathbf{H}$ be generated by Λ and $(R_2, \frac{i}{2})$, and define $\rho_\alpha \colon G_\alpha \to G_\alpha/\Lambda \to \mathrm{SO}(\mathbf{H})^{\Gamma_\alpha} \hookrightarrow \operatorname{Isom}(\mathbf{H}/\Gamma_\alpha)$ by

$$\rho_\alpha\big(R_2, \tfrac{i}{2}\big)[q] := [-iqi].$$

For every $\alpha \in A$ there is an open embedding $\jmath_\alpha\colon \big(\operatorname{Im}\mathbf{H} \times (B_{R_\alpha}(0)/\Gamma_\alpha)\big)/G_\alpha \to Y_0$ as in Definition 2.6 (2).

These can be extended to resolution data $\mathfrak{R}$ for (Y_0, ϕ_0) with the aid of the Gibbons–Hawking construction discussed in Remark 2.12. According to Remark 2.12 (2), $(X_\mathbf{0}, \boldsymbol{\omega}_\mathbf{0})$ is $\mathbf{H}/C_2$ with the standard hyperkähler from. If $\boldsymbol{\zeta} = [\zeta, -\zeta] \in \Delta^\circ$, then $(X_{\boldsymbol{\zeta}}, \boldsymbol{\omega}_{\boldsymbol{\zeta}})$ is a hyperkähler manifold and Remark 2.12 (2) provides $\tau\colon X_{\boldsymbol{\zeta}} \backslash K_{\boldsymbol{\zeta}} \to X_\mathbf{0}$. Therefore, completing the resolution data for $\alpha \in A^0$ amounts to a choice of $\zeta_\alpha \in \Delta^\circ$

For $\alpha \in A^1$ the situation is slightly complicated by the fact that ρ_α is non-trivial. The involution $R(q) \coloneqq -iqi$ lies in $\mathrm{SO}(\mathbf{H})^{\Gamma_\alpha}$ and $\Lambda^+ R = R_2$. By Remark 2.12 (4), requiring that R lifts to $X_{\boldsymbol{\zeta}}$ imposes the constraint that $\boldsymbol{\zeta}_\alpha \in (\Delta^\circ)^{R_2}$. Therefore, completing the resolution data for $\alpha \in A^1$ amounts to a choice of $\boldsymbol{\zeta}_\alpha \in (\Delta^\circ)^{R_2}$. A moment's thought shows that

$$(\Delta^\circ)^{R_2} = \big\{[\zeta, -\zeta] \in \Delta^\circ : \zeta \in \mathbf{R}i \cup (\mathbf{R}i)^\perp\big\}.$$

If $\boldsymbol{\zeta} = [\zeta, -\zeta]$ with $\zeta \in \mathbf{R}i$, then the segment joining ζ and $-\zeta$ lifts to an I_i–holomorphic curve $\Sigma \cong S^2$. Therefore, for the corresponding choices of $\mathfrak{R}$, Proposition 4.1 with $\hat{\xi} = i$ and $L = 1$ exhibits 4 associative submanifolds in (Y_t, ϕ_t) for every $t \in (0, T_1)$. ♠

Example 4.4. Joyce [Joy96b, Examples 15, 16] constructs two examples of closed flat G_2–orbifolds (Y_0, ϕ_0) whose singular set has components S_α $(\alpha \in A)$. A is a disjoint union $A = A^0 \amalg A^1$ with $A^1 = \{\star\}$. The situation is analogous to that in Example 4.3 except that $\Gamma_\star \coloneqq C_3$.

Completing the resolution data for $\star$ amounts to a choice of $\boldsymbol{\zeta}_\star \in (\Delta^\circ)^{R_2}$. A moment's thought shows that

$$(\Delta^\circ)^{R_2} = \big\{[\zeta_1, \zeta_2, \zeta_3] \in \Delta^\circ : \zeta_1 \in \mathbf{R}i \text{ and } R_2\zeta_2 = \zeta_3\big\} \cup \big\{[\zeta_1, \zeta_2, \zeta_3] \in \Delta^\circ : \zeta_1, \zeta_2, \zeta_3 \in \mathbf{R}i\big\}.$$

If $\boldsymbol{\zeta} = [\zeta_1, \zeta_2, \zeta_3]$ is in the latter component and ζ_2 is contained the the segment joining ζ_1 and ζ_2, then the segment joining ζ_1 and ζ_2 and the segment joining ζ_2 and ζ_3 lift to $I_{\star,i}$–holomorphic curves $\Sigma_1, \Sigma_2 \cong S^2$ and Proposition 4.1 (2) holds. Therefore, for the corresponding choices of $\mathfrak{R}$, Proposition 4.1 with $\hat{\xi} = i$ and $L = 1$ exhibits $8 = 2 \cdot 4$ associative submanifolds in (Y_t, ϕ_t) for every $t \in (0, T_1)$. ♠

Example 4.5. Reidegeld [Rei17, §5.3.4] constructs an example of a closed flat G_2–orbifolds (Y_0, ϕ_0) whose singular set has 16 components S_α $(\alpha \in A)$. For every $\alpha \in A$, S_α is isometric to T^3/C_2^2. Here is a more precise description. For $\alpha \in A$ set $\Gamma_\alpha \coloneqq C_2$, let $G_\alpha < \mathrm{SO}(\operatorname{Im}\mathbf{H}) \ltimes \operatorname{Im}\mathbf{H}$ be generated by $\Lambda \coloneqq \langle i, j, k\rangle$, $(R_+, \frac{i+k}{2})$, and $(R_-, \frac{j}{2})$, define $\rho_\alpha\colon G_\alpha \to G_\alpha/\Lambda \to \mathrm{SO}(\mathbf{H})^{\Gamma_\alpha} \hookrightarrow \mathrm{Isom}(\mathbf{H}/\Gamma_\alpha)$ by

$$\rho_\alpha\big(R_+, \tfrac{i+k}{2}\big)[q] \coloneqq [iqi] \quad \text{and} \quad \rho_\alpha\big(R_-, \tfrac{j}{2}\big)[q] \coloneqq [jqj].$$

These act on $\operatorname{Im}\mathbf{H}$ as R_+ and R_-. For every $\alpha \in A$ there is an open embedding $\jmath_\alpha\colon \big(\operatorname{Im}\mathbf{H} \times (B_{R_\alpha}(0)/\Gamma_\alpha)\big)/G_\alpha \to Y_0$ as in Definition 2.6 (2).

Completing the resolution data for $\alpha \in A$ amounts to a choice of $\boldsymbol{\zeta}_\alpha \in (\Delta^\circ)^{R_+,R_-}$. A moment's thought shows that

$$(\Delta^\circ)^{R_+,R_-} = \big\{[\zeta, -\zeta] \in \Delta^\circ : \zeta \in \mathbf{R}i\big\}.$$

Therefore, for *every choice* of $\mathfrak{R}$, Proposition 4.1 with $\hat{\xi} = i$ and $L = 1$ exhibits $32 = 16 \cdot 2$ associative submanifolds in (Y_t, ϕ_t) for every $t \in (0, T_1)$. ♠

Example 4.6. Here is an example that involves non-cyclic Γ and requires the use of Remark 2.15. Reidegeld [Rei17, §5.3.4] constructs an example of a closed flat G_2–orbifolds (Y_0, ϕ_0) whose singular set has 7 components S_α ($\alpha \in A$). The situation is analogous to that in Example 4.5 except that $A = A^1 \amalg A^2 \amalg A^3$ and

$$\Gamma_\alpha \coloneqq \begin{cases} C_2 & \text{if } \alpha \in A^1 \\ C_4 & \text{if } \alpha \in A^2 \\ \mathrm{Dic}_2 & \text{if } \alpha \in A^3. \end{cases}$$

Completing the resolution data for $\alpha \in A^1$ is identical to Example 4.5. For $\alpha \in A^2$ a moment's thought shows that

$$\begin{aligned}(\Delta^\circ)^{R_+,R_-} &= \left\{[\zeta, R_+\zeta, R_-\zeta, R_+R_-\zeta] \in \Delta^\circ : \zeta \notin \mathbf{R}i \cup \mathbf{R}j \cup \mathbf{R}k\right\} \\ &\cup \left\{[\zeta_1, \zeta_2, -\zeta_1, -\zeta_2] \in \Delta^\circ : \zeta_1, \zeta_2 \in \mathbf{R}i \cup \mathbf{R}j \cup \mathbf{R}k\right\}.\end{aligned}$$

If $\boldsymbol{\zeta}$ is in the latter component, then $X_{\boldsymbol{\zeta}}$ contains $I_{\boldsymbol{\zeta},\hat{\xi}_a}$–holomorphic curves $\Sigma_a \cong S^2$ with $\hat{\xi}_a = \zeta_a/|\zeta_a|$ ($a = 1, 2$). (In fact, there are more.) Therefore, for the corresponding choices of $\mathfrak{R}$, Proposition 4.1 with $\hat{\xi}_a = i$ and $L = 1$ exhibits $4 = 2 \cdot 2$ associative submanifolds in (Y_t, ϕ_t) for every $t \in (0, T_1)$.

To understand the situation for $\alpha \in A^3$, recall that the D_4 root system is

$$\Phi = \left\{\pm e_a \pm e_b \in \mathbf{R}^4 : a \neq b \in \{1, 2, 3, 4\}\right\}.$$

The standard choice of simple roots is

$$\alpha_1 \coloneqq e_1 - e_2, \quad \alpha_2 \coloneqq e_2 - e_3, \quad \alpha_3 \coloneqq e_3 - e_4, \quad \text{and} \quad \alpha_4 \coloneqq e_3 + e_4.$$

The Weyl group $W = S^4 \ltimes C_2^3$ acts by permuting and flipping the signs on an even number of the coordinates of $\mathbf{R}^4$. Therefore,

$$\Delta^\circ = \left\{[\zeta_1, \zeta_2, \zeta_3, \zeta_4] \in (\operatorname{Im}\mathbf{H} \otimes \mathbf{R}^4)/W : \zeta_a \neq \pm\zeta_b \text{ for } a \neq b \in \{1, 2, 3, 4\}\right\}.$$

A little computation reveals that

$$\begin{aligned}(\Delta^\circ)^{R_+,R_-} &= \left\{[\zeta, R_+\zeta, R_-\zeta, R_+R_-\zeta] \in \Delta^\circ : \zeta \notin (\mathbf{R}i)^\perp \cup (\mathbf{R}j)^\perp \cup (\mathbf{R}k)^\perp\right\} \\ &\cup \left\{[\zeta_1, \zeta_2, -\zeta_1, -\zeta_2] \in \Delta^\circ : \zeta_1, \zeta_2 \in (\mathbf{R}i)^\perp \cup (\mathbf{R}j)^\perp \cup (\mathbf{R}k)^\perp \backslash (\mathbf{R}i \cup \mathbf{R}j \cup \mathbf{R}k)\right\} \\ &\cup \left\{[\zeta_1, \zeta_2, \zeta_3, \zeta_4] \in \Delta^\circ : \zeta_1, \zeta_2, \zeta_3, \zeta_4 \in \mathbf{R}i \text{ or } \zeta_1, \zeta_2, \zeta_3, \zeta_4 \in \mathbf{R}j \text{ or } \zeta_1, \zeta_2, \zeta_3, \zeta_4 \in \mathbf{R}k\right\}.\end{aligned}$$

If $\boldsymbol{\zeta} = [\zeta_1, \zeta_2, \zeta_3, \zeta_4]$ is in the latter component, then, by Remark 2.15 (4), $X_{\boldsymbol{\zeta}}$ contains 4 $I_{\boldsymbol{\zeta},\hat{\xi}}$–holomorphic curves $\Sigma_a \cong S^2$ ($a \in \{1, 2, 3, 4\}$) with $\hat{\xi} \coloneqq \zeta_1/|\zeta_1|$. Therefore, for the corresponding choices of $\mathfrak{R}$, Proposition 4.1 with $\hat{\xi} = i$ and $L = 1$ exhibits $8 = 4 \cdot 2$ associative submanifolds in (Y_t, ϕ_t) for every $t \in (0, T_1)$. ♠

Remark 4.7. Reidegeld [Rei17, §5.3.4 and §5.3.5] constructed further examples of closed flat G_2–orbifolds (Y_0, ϕ_0) whose singular sets are isometric to T^3, T^3/C_2, and T^3/C_2^2 and whose transverse singularity models are $\mathbf{H}/\Gamma$ with $\Gamma \in \{C_2, C_3, C_4, C_6, \mathrm{Dic}_2, \mathrm{Dic}_3, 2T\}$. The reader might enjoy analysing these examples with the methods used above. ♣

Proposition 4.8 (TODO: Daniel+Shubham).

Example 4.9 (TODO: Daniel+Shubham). ♠

Remark 4.10. If X is a $K3$ surface with a non-symplectic involution τ, then the fixed-point locus X^τ (typically) contains a surface of genus $g \neq 1$ [Nik83, §4]. The twisted connected sum construction [Kov03; KL11; CHNP15]—in fact: a trivial version of thereof—produces closed G_2–orbifolds (Y_0, ϕ_0) from a matching pairs $(\Sigma_\pm, \tau_\pm)$ of non-symplectic $K3$ surfaces. The singular set of Y_0 is $S^1 \times M$ with $M := X_+^\tau \cup X_-^\tau$ and the transverse singularity model is $\mathbf{H}/C_2$. An extension of the generalised Kummer construction due to Joyce and Karigiannis [JK17] resolves Y_0 into a family $(Y_t, \phi_t)_{t\in(0,T_0)}$ of closed G_2–manifolds. It seems plausible that a extension of the techniques in the present article could produce $\mathbf{P}\colon M \to \mathcal{S}$ satisfying the hypothesis of Lemma 3.5. Since (typically) $\chi(M) \neq 0$, this would produce associatives in Joyce and Karigiannis's G_2–manifolds. ♣

Remark 4.11. It is also possible to construct coassociative submanifolds in G_2–manifolds obtained from the generalised Kummer construction using similar techniques. In fact, the situation is quite a bit simpler because the deformation theory of coassociative submanifolds is always unobstructed [McL98, §4] ♣

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