

Castelnuovo’s bound and rigidity in almost complex geometry

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Abstract

This article is concerned with the question of whether an energy bound implies a genus bound for pseudo-holomorphic curves in almost complex manifolds. After reviewing what is known in dimensions other than 6, we establish a new result in this direction in dimension 6; in particular, for symplectic Calabi–Yau 3–folds. The proof relies on compactness and regularity theorems for pseudo-holomorphic currents.

1 Introduction

In 1889, Castelnuovo [Cas89] found a sharp upper bound for the genus of an irreducible degree d curve in \mathbf{P}^n ; see [ACGH85, Chapter III Section 2] for a proof in modern language. A corollary of this result is that for every projective variety there is an upper bound for the genus of an irreducible curve representing a given homology class. Our starting point is the question:

Are there analogues of Castelnuovo’s bound in almost complex geometry?

For plane curves Castelnuovo’s bound reduces to the degree-genus formula. The latter is a consequence of the adjunction formula, which generalizes to an inequality for almost complex 4–manifolds [MS12, Theorem 2.6.4]. The adjunction inequality directly implies the following well-known genus bound.

Proposition 1.1. *Suppose that (M, J) is an almost complex 4–manifold. If there exists a simple J –holomorphic map $u: \Sigma \rightarrow M$ representing $A \in H_2(M)$, then the genus $g(\Sigma)$ satisfies*

$$(1.2) \quad g(\Sigma) \leq \frac{1}{2} (A \cdot A - \langle c_1(M, J), A \rangle) + 1.$$

The following consequence of Gromov’s h-principle shows that in higher dimensions there cannot be a genus bound which holds for all almost complex structures.

Proposition 1.3 (cf. Li [Li05, Corollary 2.13]). *Let (M, ω) be a symplectic manifold of dimension $2n \geq 6$. For every $A \in H_2(M)$ with $\langle [\omega], A \rangle > 0$ and every $n \in \mathbf{N}$ there is an almost complex structure J compatible with ω and an embedded J -holomorphic curve C satisfying*

$$g(C) \geq n.$$

There are, however, genus bounds for *generic* almost complex structures. Here is a simple example, which follows easily from the index formula for J -holomorphic maps.

Proposition 1.4. *Let M be a manifold of dimension $2n$. Denote by \mathcal{F} the space of almost complex structures of class C^2 on M . There is a residual¹ subset $\mathcal{F}_\bullet \subset \mathcal{F}$ such that for every $J \in \mathcal{F}_\bullet$ the following holds: if there exists a simple J -holomorphic map $u: \Sigma \rightarrow M$ representing $A \in H_2(M)$, then*

$$(1.5) \quad \begin{cases} \langle c_1(M, J), A \rangle \geq 0 & \text{if } n = 3 \text{ and} \\ g(\Sigma) \leq \frac{\langle c_1(M, J), A \rangle}{n-3} + 1 & \text{if } n \geq 3. \end{cases}$$

Moreover, if M carries a symplectic form ω , then the same holds with \mathcal{F} replaced by the space $\mathcal{F}(\omega)$ of almost complex structures of class C^2 compatible with ω .

We give proofs of Proposition 1.3 and Proposition 1.4 in Appendix A.

The preceding discussion leaves open the case of almost complex manifolds of dimension six and homology classes satisfying $\langle c_1(M, J), A \rangle \geq 0$. We focus on the case

$$\langle c_1(M, J), A \rangle = 0,$$

that is: on classes for which the corresponding moduli space of J -holomorphic maps has expected dimension zero. This includes all homology classes in symplectic Calabi–Yau 3-folds. Our motivation for considering this case comes from our project to construct a symplectic analogue of the Pandharipande–Thomas invariants of projective Calabi–Yau 3-folds [DW17, Section 7]. Another motivation comes from the Gopakumar–Vafa conjecture. Bryan and Pandharipande [BP01] defined the Gopakumar–Vafa BPS invariants $n_A^g(M)$ of a symplectic Calabi–Yau 3-fold M in terms of its Gromov–Witten partition function. They conjectured that the BPS invariants $n_A^g(M)$ are integers and vanish for all but finitely many g [BP01, Conjecture 1.2]. The integrality conjecture has been proved by Ionel and Parker [IP18]. The finiteness conjecture remains open and is closely related to the question about the existence of genus bounds for symplectic Calabi–Yau 3-folds.

Motivated by Gromov–Witten theory, Bryan and Pandharipande introduced the notion of k -rigidity for almost complex structures; see Definition 2.10. Eftekhary [Eft16] proved that 4-rigidity is a generic property; see Theorem 2.13. Conjecturally, a generic almost complex structure is *super-rigid*, that is: ∞ -rigid. The main result of this article shows that k -rigidity implies a Castelnuovo bound.

¹Let X be a topological space. A subset $A \subset X$ is called **residual** if it is the intersection of countably many dense open subsets. Recall that a residual subset of a complete metric space is dense.

Theorem 1.6. *Let $k \in \mathbb{N} \cup \{\infty\}$. Let (M, J, g) be a compact almost Hermitian 6 -manifold with a k -rigid almost complex structure J . Suppose $A \in H_2(M)$ satisfies $\langle c_1(M, J), A \rangle = 0$ and has divisibility at most k . Given any $\Lambda > 0$, there are only finitely many simple J -holomorphic maps representing A and with energy at most Λ .*

Remark 1.7. Theorem 1.6 immediately implies a Castelnuovo bound for every fixed k -rigid almost complex structure J . Unlike (1.5), however, this bound may depend on J .

If J is tamed by a symplectic form ω , then imposing an upper bound for the energy is superfluous since the energy of any J -holomorphic map representing A is $\langle [\omega], A \rangle$.

Corollary 1.8. *Let (M, ω) be a compact symplectic Calabi–Yau 3 -fold. Suppose J is a super-rigid almost complex structure compatible with ω . Then for every $A \in H_2(M)$ there are only finitely many simple J -holomorphic maps representing A .*

In the situation of Theorem 1.6, Gromov’s compactness theorem [Gro85; PW93; Ye94; Hum97] shows that there are only finitely many J -holomorphic maps representing A from Riemann surfaces of fixed genus. It is thus not of much use for proving Theorem 1.6. Instead, we use the following compactness result for J -holomorphic cycles, that is: formal sums of J -holomorphic curves, with respect to geometric convergence; see Definition 4.1 and Definition 4.2.

Lemma 1.9. *Let M be a manifold and let $(J_n, g_n)_{n \in \mathbb{N}}$ be a sequence of almost Hermitian structures converging to an almost Hermitian structure (J, g) . Let $K \subset M$ be a compact subset and let $\Lambda > 0$. For each $n \in \mathbb{N}$ let C_n be a J_n -holomorphic cycle with support contained in K and of mass at most Λ . Then a subsequence of $(C_n)_{n \in \mathbb{N}}$ geometrically converges to a J -holomorphic cycle C .*

In dimension 4, this result was proved by Taubes [Tau96a]. The proof in higher dimensions relies on results in geometric measure theory; in particular, the recent work of De Lellis, Spadaro, and Spolaor [DSS17b; DSS18; DSS17a; DSS15] on the regularity of semi-calibrated currents.

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2 k -rigidity of J -holomorphic maps

We begin by briefly recalling the notion of k -rigidity as defined by Eftekhary. For a more detailed discussion we refer the reader to [Eft16, Section 2; Wen16, Section 2.1; DW18, Section 7]. Throughout, let (M, J, g) be an almost Hermitian $2n$ -manifold.

Definition 2.1. A J -holomorphic map $u: (\Sigma, j) \rightarrow (M, J)$ is a pair consisting of a closed, connected Riemann surface (Σ, j) and a smooth map $u: \Sigma \rightarrow M$ satisfying the non-linear Cauchy–Riemann equation

$$(2.2) \quad \bar{\partial}_J(u, j) := \frac{1}{2}(du + J(u) \circ du \circ j) = 0.$$

Definition 2.3. Let $u: (\Sigma, j) \rightarrow (M, J)$ be a J -holomorphic map. Let $\phi \in \text{Diff}(\Sigma)$ be a diffeomorphism. The **reparametrization** of u by ϕ is the J -holomorphic map $u \circ \phi^{-1}: (\Sigma, \phi_*j) \rightarrow (M, J)$.

Definition 2.4. Let $u: (\Sigma, j) \rightarrow (M, J)$ be a J -holomorphic map and let $\pi: (\tilde{\Sigma}, \tilde{j}) \rightarrow (\Sigma, j)$ be a holomorphic map of degree $\deg(\pi) \geq 2$. The composition $u \circ \pi: (\tilde{\Sigma}, \tilde{j}) \rightarrow (M, J)$ is said to be a **multiple cover** of u . A J -holomorphic map is **simple** if it is not constant and not a multiple cover.

Rigidity and k -rigidity are conditions on the infinitesimal deformation theory of J -holomorphic curves up to reparametrization. We will have to briefly review parts of this theory.

The index of a J -holomorphic map $u: (\Sigma, j) \rightarrow (M, J)$ is defined as

$$(2.5) \quad \text{index}(u) := (n - 3)\chi(\Sigma) + 2\langle [\Sigma], u^*c_1(M, J) \rangle.$$

This is the Fredholm index of the gauge-fixed linearization of (2.2). The restriction of this linearization to $\Gamma(u^*TM)$ is given by

$$(2.6) \quad \xi \mapsto \frac{1}{2}(\nabla\xi + J \circ (\nabla\xi) \circ j + (\nabla_\xi J) \circ du \circ j).$$

Here ∇ denotes any torsion-free connection on TM and also the induced connection on u^*TM . Since (u, j) is a J -holomorphic map, the right-hand side of (2.6) does not depend on the choice of ∇ ; see [MS12, Proposition 3.1.1].

Let $u: (\Sigma, j) \rightarrow (M, J)$ be a non-constant J -holomorphic map. There exists a unique complex subbundle

$$Tu \subset u^*TM$$

of rank one containing $du(T\Sigma)$; see [IS99, Section 1.3; Wen10, Section 3.3; DW18, Appendix A]. The generalized normal bundle of u is defined as

$$Nu := u^*TM/Tu.$$

If u is an immersion, then Nu is the usual normal bundle. If $\tilde{u} = u \circ \pi$ is a multiple cover of an immersion, then $N\tilde{u} = \pi^*Nu$. Define the normal Cauchy–Riemann operator

$$(2.7) \quad \mathfrak{d}_{u,J}^N: \Gamma(Nu) \rightarrow \Omega^{0,1}(Nu)$$

by the formula (2.6). The non-zero elements of the kernel of $\mathfrak{d}_{u,J}^N$ correspond to infinitesimal deformations of u which deform the image $u(\Sigma)$.

Definition 2.8. A non-constant J -holomorphic map u is **rigid** if $\ker \mathfrak{d}_{u,J}^N = 0$.

A multiple cover \tilde{u} of u may fail to be rigid, even if u itself is rigid.

Definition 2.9. Let $k \in \mathbb{N} \cup \{\infty\}$. A simple J -holomorphic map $u: (\Sigma, j) \rightarrow (M, J)$ is called **k -rigid** if it is rigid and all of its multiple covers of degree at most k are rigid.

It follows from [IS99, Lemma 1.5.1; Wen10, Theorem 3] that $\dim \ker \mathfrak{d}_{u,J}^N \geq \text{index}(u)$. Consequently, a k -rigid J -holomorphic map must have $\text{index}(u) \leq 0$.

Definition 2.10. Let $k \in \mathbb{N} \cup \{\infty\}$. An almost complex structure J is called **k -rigid** if the following hold:

1. Every simple J -holomorphic map of index zero is k -rigid.
2. Every simple J -holomorphic map has non-negative index.
3. Every simple J -holomorphic map of index zero is an embedding, and every two simple J -holomorphic maps of index zero either have disjoint images or are related by a reparametrization.

Remark 2.11. In dimension four, one should weaken (3) and require only that every simple J -holomorphic map of index zero is an immersion with transverse self-intersections, and that two such maps are either transverse to one another or are related by reparametrization. However, we will only be concerned with dimension (at least) six.

Definition 2.12. Let $k \in \mathbb{N} \cup \{\infty\}$. Let (M, ω) be a symplectic manifold. Denote by $\mathcal{F}(\omega)$ the space of almost complex structures on M compatible with ω . Denote by $\mathcal{R}_k(\omega)$ the subset of those almost complex structures $J \in \mathcal{F}(\omega)$ which are k -rigid.

Theorem 2.13 (Eftekhary [Eft16, Theorem 1.2]). *Let (M, ω) be a symplectic manifold. If $\dim M \geq 6$, then $\mathcal{R}_4(\omega) \subset \mathcal{F}(\omega)$ is a residual subset.*

Conjecture 2.14 (Bryan and Pandharipande [BP01, p. 290]). *Let (M, ω) be a symplectic manifold. If $\dim M \geq 6$, then $\mathcal{R}_\infty(\omega) \subset \mathcal{F}(\omega)$ is a residual subset.*

Wendl [Wen16] has made remarkable progress towards proving this conjecture. His work shows that Conjecture 2.14 holds provided generic real Cauchy Riemann operators satisfy an analytic condition known as Petri's condition; see also [DW18].

3 Real Cauchy–Riemann operators and almost complex structures

We will show that associated with every real Cauchy–Riemann operator defined on a vector bundle there is a natural almost complex structure on the total space of that bundle. This construction is inspired by [Tau96b, p. 825–826]; similar material can be found in [Wen16, Appendix B].

Definition 3.1. Let (Σ, j) be a Riemann surface. Let $\pi : E \rightarrow \Sigma$ be a Hermitian vector bundle over Σ . A first order linear differential operator $\mathfrak{d} : \Gamma(E) \rightarrow \Omega^{0,1}(\Sigma, E)$ is called a **real Cauchy–Riemann operator** if

$$(3.2) \quad \mathfrak{d}(fs) = (\bar{\partial}f)s + f\mathfrak{d}s$$

for all $f \in C^\infty(M, \mathbf{R})$. The **anti-linear part** of \mathfrak{d} is defined as

$$\mathfrak{n} = \mathfrak{n}_\mathfrak{d} := \frac{1}{2}(\mathfrak{d} + J\mathfrak{d}J) \in \Gamma(\text{Hom}(E, \overline{\text{Hom}}_{\mathbf{C}}(T\Sigma, E))).$$

Every real Cauchy–Riemann operator can be written as

$$\mathfrak{d} = \bar{\partial}_\nabla + \mathfrak{n}$$

where $\bar{\partial}_\nabla := \nabla^{0,1}$ is the Dolbeault operator associated with a Hermitian connection ∇ on E . Denote by $H_\nabla \subset TE$ the horizontal distribution of ∇ . It induces an isomorphism

$$(3.3) \quad TE = H_\nabla \oplus \pi^*E \cong \pi^*T\Sigma \oplus \pi^*E.$$

Definition 3.4. The **complex structure** J_∇ on E associated with ∇ is defined by pulling back the standard complex structure $j \oplus i$ on $\pi^*T\Sigma \oplus \pi^*E$ by the isomorphism (3.3).

It is well-known that a section $s \in \Gamma(E)$ satisfies $\bar{\partial}_\nabla s = 0$ if and only if the map $s : \Sigma \rightarrow E$ is J_∇ -holomorphic. The following proposition extends this to real Cauchy–Riemann operators.

Definition 3.5. Let $\mathfrak{d} = \bar{\partial}_\nabla + \mathfrak{n}$ be a real Cauchy–Riemann operator. Define $L_\mathfrak{n} : TE \rightarrow TE$ by

$$L_\mathfrak{n} = -2\mathfrak{n}(v)j\pi_*$$

at $v \in E$. The **almost complex structure** $J_\mathfrak{d}$ on E associated with \mathfrak{d} is defined by

$$J_\mathfrak{d} := J_\nabla + L_\mathfrak{n}.$$

Proposition 3.6. For every real Cauchy–Riemann operator $\mathfrak{d} : \Gamma(E) \rightarrow \Omega^{0,1}(E)$ the following hold:

1. $J_\mathfrak{d}$ is an almost complex structure.
2. The projection $\pi : E \rightarrow \Sigma$ is holomorphic with respect to $J_\mathfrak{d}$.
3. For every $x \in \Sigma$ the fiber $E_x = \pi^{-1}(x)$ is a $J_\mathfrak{d}$ -holomorphic submanifold of E .
4. A section $s \in \Gamma(E)$ satisfies $\mathfrak{d}s = 0$ if and only if $s : \Sigma \rightarrow E$ is a $J_\mathfrak{d}$ -holomorphic map.
5. There exists a symplectic form ω on the unit disc bundle $B_1(\Sigma) \subset N\Sigma$ which tames $J_\mathfrak{d}$.

Proof. With respect to (3.3) we have

$$(3.7) \quad J_{\mathfrak{b}} = \begin{pmatrix} j & 0 \\ -2\mathfrak{n}(v)j & i \end{pmatrix}$$

at $v \in E$. Since $\mathfrak{n}(v)$ is anti-linear,

$$\mathfrak{n}(v)j^2 + i\mathfrak{n}(v)j = 0.$$

Therefore,

$$J_{\mathfrak{b}}^2 = -\text{id};$$

that is, (1) holds.

Both (2) and (3) immediately follow from (3.7).

We prove (4). Let $s: \Sigma \rightarrow E$ be a section. The projection of ds to the first factor of (3.3) is $\pi_* \circ ds = \text{id}_{T\Sigma}$ and thus j -linear. The projection of $ds: T\Sigma \rightarrow s^*TE$ to the second factor is its covariant derivative $\nabla s: T\Sigma \rightarrow s^*E$. Therefore, the J_{∇} -antilinear part of ds is

$$\frac{1}{2}(ds + J_{\nabla} \circ ds \circ j) = (\nabla s)^{0,1} = \bar{\partial}_{\nabla} s.$$

The $J_{\mathfrak{b}}$ -antilinear part of ds is

$$\begin{aligned} \frac{1}{2}(ds + J_{\mathfrak{b}} \circ ds \circ j) &= \frac{1}{2}(ds + J_{\nabla} \circ ds \circ j + L_{\mathfrak{n}} \circ ds \circ j) \\ &= \bar{\partial}_{\nabla} s + L_{\mathfrak{n}} \circ ds \circ j \\ &= \bar{\partial}_{\nabla} s + \mathfrak{n}_{\mathfrak{b}} s = \mathfrak{d}s. \end{aligned}$$

Therefore, $ds: T\Sigma \rightarrow TE$ is $J_{\mathfrak{b}}$ -linear if and only if $\mathfrak{d}s = 0$.

The proof of (5) is standard; see, e.g., [Wen16, Lemma B.2]. Nevertheless, we include it here for completeness. Let ω_{Σ} be an area form on Σ . Let ω_E be any closed 2-form on $B_1(\Sigma)$ which is positive when restricted to the fibers of E ; that is, for all vertical tangent vectors v_E

$$(3.8) \quad \omega_E(v_E, J_{\nabla} v_E) \gtrsim |v_E|^2.$$

Such a form can be constructed by choosing local unitary trivializations of $E|_{U_i} \cong U_i \times \mathbb{C}^r$, denoting by λ_i the corresponding Liouville 1-forms on \mathbb{C}^r vanishing at zero, and setting

$$\omega_E = \text{d} \left(\sum_i \chi_i \circ \pi \cdot \lambda_i \right)$$

for a partition of unity (χ_i) . This form satisfies (3.8) on E . It remains to show that for $\tau \gg 1$ the closed 2-form $\omega = \tau\omega_{\Sigma} + \omega_E$ tames J_u on $B_1(\Sigma)$. For a tangent vector w to E at a point $(x, v) \in B_1(\Sigma)$ denote by w_H and w_E its horizontal and vertical parts in the decomposition (3.3). We have

$$\begin{aligned} \omega(w, J_{\mathfrak{b}} w) &= (\tau\omega_{\Sigma} + \omega_E)(w, (J_{\nabla} + L_{\mathfrak{n}})w) \\ &= \tau\omega_{\Sigma}(w_H, jw_H) + \omega_E(w_E, J_{\nabla} w_E) + \omega_E(w_E, L_{\mathfrak{n}} w_H). \end{aligned}$$

From $|L_n(v)| \lesssim |v| < 1$ it follows that

$$|\omega_E(w_E, L_n w_H)| \lesssim |w_E| |w_H|.$$

Since

$$\tau \omega_\Sigma(w_H, j w_H) + \omega_E(w_E, J \nabla w_E) \gtrsim \tau |w_H|^2 + |v_E|,$$

it follows that ω tames J_u provided $\tau \gg 1$. \square

The next two propositions are concerned with the following situation. Let $u: (\Sigma, j) \rightarrow (M, J)$ be a J -holomorphic embedding. Denote by $Nu \rightarrow \Sigma$ its normal bundle and by $\mathfrak{d}_{u,J}^N$ the normal Cauchy–Riemann introduced in (2.7). Write

$$(3.9) \quad J_u := J_{\mathfrak{d}_{u,J}^N}$$

for the almost complex structure on the total space of Nu associated with $\mathfrak{d}_{u,J}^N$.

Proposition 3.10. *For every $\lambda > 0$ define $\sigma_\lambda: Nu \rightarrow Nu$ by*

$$\sigma_\lambda(v) := \lambda v.$$

If $U \subset Nu$ is an open neighborhood of the zero section in Nu such that the exponential map $\exp: U \rightarrow M$ is an embedding, then

$$\sigma_\lambda^* \exp^* J \rightarrow J_u \quad \text{as } \lambda \rightarrow 0.$$

Proof. Denote by ∇ the connection on $Nu \rightarrow \Sigma$ induced by the Levi–Civita connection on M . Throughout this proof, we identify

$$TU = \pi^* T\Sigma \oplus \pi^* Nu$$

as in (3.3). The two almost complex structures J_∇ and $\exp^* J$ on $U \subset Nu$ agree along the zero section. The Taylor expansion of $\exp^* J$ is of the form

$$(3.11) \quad \exp^* J(x, v) = J_\nabla(x, 0) + \nabla_v J(x, 0) + O(|v|^2).$$

Set

$$L(x, v) := \nabla_n J(x, 0).$$

We write L as the matrix

$$L(x, v) = \begin{pmatrix} L_{11}(x, v) & L_{12}(x, v) \\ L_{21}(x, v) & L_{22}(x, v) \end{pmatrix}.$$

Here each L_{ij} is linear in v . The derivative $d\sigma_\lambda$ is given by

$$d\sigma_\lambda = \begin{pmatrix} \text{id} & \\ & \lambda \end{pmatrix}.$$

Therefore,

$$\begin{aligned} (\sigma_\lambda)^*L(x, v) &= \begin{pmatrix} \text{id} & \\ & \lambda^{-1} \end{pmatrix} \begin{pmatrix} L_{11}(x, \lambda v) & L_{12}(x, \lambda v) \\ L_{21}(x, \lambda v) & L_{22}(x, \lambda v) \end{pmatrix} \begin{pmatrix} \text{id} & \\ & \lambda \end{pmatrix} \\ &= \begin{pmatrix} \lambda L_{11}(x, v) & \lambda^2 L_{12}(x, v) \\ L_{21}(x, v) & \lambda L_{22}(x, v) \end{pmatrix}. \end{aligned}$$

As λ tend to zero, all but the bottom left entry tend to zero.

By construction, $\sigma_\lambda^*J_\nabla = J_\nabla$. As λ tends to zero, the rescalings of terms of second order and higher in (3.11) tend to zero. It remains to identify the term L_{21} . By definition,

$$L_{21}(x, v) = \pi_{Nu} \circ \nabla_v J(x, 0) \circ \pi_*.$$

Comparing (2.6), Definition 3.1, and Definition 3.5, we see that $L_{21} = L_u$. This finishes the proof. \square

Proposition 3.12. *If $\tilde{u}: (\tilde{\Sigma}, \tilde{j}) \rightarrow (Nu, J_u)$ is a simple J_u -holomorphic map whose image is not contained in the zero section, then:*

1. *the map $\varphi: (\tilde{\Sigma}, \tilde{j}) \rightarrow (\Sigma, j)$ given by $\varphi := \pi \circ \tilde{u}$ is non-constant and holomorphic, and*
2. *the J -holomorphic map $u \circ \varphi: (\tilde{\Sigma}, \tilde{j}) \rightarrow (M, J)$ is not rigid; in particular, the J -holomorphic map $u: (\Sigma, j) \rightarrow (M, J)$ is not k -rigid for $k = \deg(\varphi)$.*

Proof. By Proposition 3.6 (2), $\pi: Nu \rightarrow \Sigma$ is J_δ -holomorphic. Therefore, φ is holomorphic. The map φ is constant if and only if the image of \tilde{u} is contained in a fiber of π . This is impossible, because then \tilde{u} would be constant. This proves (1).

The normal bundle of the J -holomorphic map $u \circ \varphi$ is $N_{u \circ \varphi} = \varphi^*Nu$. The corresponding normal Cauchy–Riemann operator is

$$(3.13) \quad \mathfrak{d}_{u \circ \varphi, J}^N = \varphi^* \mathfrak{d}_{u, J}^N.$$

Since \tilde{u} takes values in Nu , for every $x \in \tilde{\Sigma}$ we have

$$\tilde{u}(x) \in Nu_{\pi(u(x))} = Nu_{\varphi(x)} = (\varphi^*Nu)_x.$$

This gives rise to the section $s \in \Gamma(\varphi^*Nu)$ defined by

$$s(x) := \tilde{u}(x) \in (\varphi^*Nu)_x.$$

This section is not the zero section, because the image of \tilde{u} is not contained in the zero section. The upcoming discussion will show that

$$\mathfrak{d}_{u \circ \varphi, J}^N s = 0.$$

In light of Proposition 3.6 (4), this will imply (2).

Denote by Z the finite set of critical values of φ . Set $\tilde{Z} := \varphi^{-1}(Z)$. The restriction $\tilde{u}: \tilde{\Sigma} \setminus \tilde{Z} \rightarrow Nu$ is a J_u -holomorphic embedding and $\varphi: \tilde{\Sigma} \setminus \tilde{Z} \rightarrow \Sigma \setminus Z$ is an unbranched holomorphic covering map. Hence, every $x \in \tilde{\Sigma} \setminus \tilde{Z}$ has an open neighborhood U such that $\tilde{u}|_U$ is an embedding and $\varphi|_U$ is biholomorphic. Therefore, $\pi|_{\tilde{u}(U)}$ maps $\tilde{u}(U)$ holomorphically to $\varphi(U)$. It follows that $\tilde{u}(U) \subset Nu$ is the graph of a J_u -holomorphic section $f: \varphi(U) \rightarrow Nu|_{\varphi(U)}$. By construction and by Proposition 3.6,

$$s|_U = \varphi^* f \quad \text{and} \quad \mathfrak{d}_{u,J}^N f = 0.$$

The relation (3.13) shows that $\mathfrak{d}_{u \circ \varphi, J}^N s = 0$ holds on U . Since $x \in \tilde{\Sigma} \setminus \tilde{Z}$ was arbitrary, it holds on all of $\tilde{\Sigma} \setminus \tilde{Z}$. In fact, since s is smooth, it holds on all of $\tilde{\Sigma}$. \square

4 J -holomorphic cycles and geometric convergence

In this section we introduce the notions of J -holomorphic cycles and geometric convergence. We then compare these with the notions of closed J -holomorphic integral currents and weak convergence. This comparison, combined with a classical compactness result in geometric measure theory, implies Lemma 1.9. Throughout, let (M, J, g) be an almost Hermitian manifold. Denote by

$$\sigma := g(J, \cdot)$$

the corresponding Hermitian form.

Definition 4.1. A J -holomorphic curve is a subset of M which is the image of a simple J -holomorphic map $u: (\Sigma, j) \rightarrow (M, J)$. A J -holomorphic cycle C is a formal linear combination

$$C = \sum_{i=1}^I m_i C_i$$

of J -holomorphic curves C_1, \dots, C_I with coefficients $m_1, \dots, m_I \in \mathbf{N}$. The **homology class** represented by C is

$$[C] := \sum_{i=1}^I m_i (u_i)_* [\Sigma_i].$$

The **support** of C is the subset

$$\text{supp}(C) := \bigcup_{i=1}^I C_i.$$

The **current** associated with C is defined by

$$\delta_C(\alpha) := \sum_{i=1}^I m_i \int_{\Sigma_i} u_i^* \alpha \quad \text{for} \quad \alpha \in \Omega_c^2(M).$$

The **mass** of C is

$$\mathbf{M}(C) := \sum_{i=1}^I m_i \text{area}(C_i) = \delta_C(\sigma).$$

We say that C is **smooth** if the J -holomorphic curves C_1, \dots, C_I are embedded and pairwise disjoint.

Definition 4.2 (Taubes [Tau98, Definition 3.1]). Let M be a manifold and let $(J_n, g_n)_{n \in \mathbf{N}}$ be a sequence of almost Hermitian structures converging to an almost Hermitian structure (J, g) . For every $n \in \mathbf{N}$ let C_n be a J_n -holomorphic cycle. We say that $(C_n)_{n \in \mathbf{N}}$ **geometrically converges** to a J -holomorphic cycle C if:

1. $(\delta_{C_n})_{n \in \mathbf{N}}$ weakly converges to δ_C ; that is:

$$\lim_{n \rightarrow \infty} \delta_{C_n}(\alpha) = \delta_C(\alpha) \quad \text{for all } \alpha \in \Omega_c^2(M)$$

and

2. $(\text{supp}(C_n))_{n \in \mathbf{N}}$ converges to $\text{supp}(C)$ in the Hausdorff distance; that is:

$$(4.3) \quad \lim_{n \rightarrow \infty} d_H(\text{supp}(C), \text{supp}(C_n)) \rightarrow 0.$$

Recall that the Hausdorff distance between two closed sets X and Y is defined by

$$d_H(X, Y) := \max \left\{ \sup_{x \in X} d(x, Y), \sup_{y \in Y} d(y, X) \right\}.$$

The following results compare J -holomorphic cycles and geometric convergence with closed integral currents on M which are calibrated by σ and weak convergence. We refer the reader to the lecture notes [Lan05] for the required background on geometric measure theory.

Proposition 4.4. *If δ_C is a closed integral current which is calibrated by σ , then there exist a J -holomorphic map $u: (\Sigma, j) \rightarrow (M, J)$, with a possibly disconnected domain Σ , and a locally constant function $k: \Sigma \rightarrow \mathbf{N}$ such that*

$$\delta_C(\alpha) = \int_{\Sigma} k \cdot u^* \alpha \quad \text{for all } \alpha \in \Omega_c^2(M).$$

In particular, there exists a J -holomorphic cycle C whose associated current is δ_C .

Proposition 4.5. *In the situation of Definition 4.2, if condition (1) holds and there exists a compact subset containing $\text{supp}(C_n)$ for every $n \in \mathbf{N}$, then condition (2) holds as well.*

We prove the second result first, since it is more elementary.

Proof of Proposition 4.5. This is contained in the proof of [Tau98, Proposition 3.3] and also a well-known fact in geometric measure theory. Let us explain the proof nevertheless. The salient point is the monotonicity formula for J -holomorphic curves; see, e.g., [PW93, Corollary 3.2; Tau98, Lemma 3.4]. It states that there are constants $c, r_0 > 0$ such that for every J -holomorphic curve C , every $x \in C$, and every $r \in [0, r_0]$

$$(4.6) \quad \mathbf{M}(\delta_C|_{B_r(x)}) = \delta_C|_{B_r(x)}(\sigma) \geq cr^2.$$

Moreover, it follows from the proof of the monotonicity formula that these constants can be chosen such that (4.6) holds for every almost Hermitian structures in the sequence $(J_n, g_n)_{n \in \mathbb{N}}$ as well as the limit (J, g) .

Condition (4.3) is equivalent to

$$(4.7) \quad \lim_{n \rightarrow \infty} \sup \{d(x, \text{supp}(C)) : x \in \text{supp}(C_n)\} = 0 \quad \text{and}$$

$$(4.8) \quad \lim_{n \rightarrow \infty} \sup \{d(x, \text{supp}(C_n)) : x \in \text{supp}(C)\} = 0.$$

If (4.7) fails, then after passing to a subsequence there exists an $\varepsilon > 0$ and a sequence of points (x_n) with

$$x_n \in \text{supp}(C_n) \quad \text{but} \quad d(x_n, \text{supp}(C)) \geq \varepsilon.$$

After passing to a further subsequence (x_n) converges to a limit $x \in M$ with $d(x, \text{supp}(C)) \geq \varepsilon$. Fix $0 < r \leq \min\{\varepsilon/2, r_0\}$. Let $\chi \in C^\infty(M, [0, 1])$ be supported in $B_{2r}(x)$ and equal to one in $B_r(x)$. By (4.6), for $n \gg 1$

$$c_0 r^2 \leq \mathbf{M}(\delta_{C_n}|_{B_r(x)}) \leq \delta_{C_n}(\chi\sigma).$$

This contradicts the weak convergence condition (1), because

$$\delta_C(\chi\sigma) = 0.$$

If (4.8) fails, then a slight variation of this argument derives another contradiction to (1). \square

Proof of Proposition 4.4. For symplectic 4-manifolds this was proved by Taubes [Tau96a, Proposition 6.1]. Taubes' argument and the work of Rivière and Tian [RT09] establish the result for general symplectic manifolds. The extension to almost Hermitian manifolds relies on the work of De Lellis, Spadaro, and Spolaor [DSS18; DSS17b; DSS17a; DSS15]. Their main result [DSS18, Theorem 0.2] implies that the singular set of δ_C is finite (since it is discrete and since δ_C is closed and of finite mass and, thus, has compact support). We need not just their main result but also the following intermediate result.

Definition 4.9. Given $k \in \mathbb{N}$, set

$$\tilde{D}^k := \{(z, w) \in \mathbb{C}^2 : z = w^k \text{ and } |z| < 1\}.$$

We consider $\tilde{D}^k \setminus \{0\}$ as oriented smooth manifold such that the map $(z, w) \mapsto z$ is an orientation-preserving local diffeomorphism. We equip it with the pull-back of the flat metric.

Definition 4.10. Let $k \in \mathbf{N}$ and $\alpha \in (0, 1)$. Let $f: \tilde{D}^k \rightarrow \mathbf{R}^{2n-2}$ be a continuous injective map which is of class $C^{3,\alpha}$ on $\tilde{D}^k \setminus \{0\}$ and satisfies $|df| \lesssim |z|^\alpha$. Define $\underline{f}: \tilde{D}^k \rightarrow \mathbf{R}^{2n}$ by

$$\underline{f}(z, w) := (z, f(w)).$$

Let $U \subset \mathbf{R}^{2n}$ be an open subset and let $\phi: U \rightarrow M$ be a chart. The **graph** of f with respect to ϕ is the integral current $G_{f,\phi}$ defined by

$$G_{f,\phi}(\alpha) := \int_{\tilde{D}^k \setminus \{0\}} \underline{f}^* \phi^* \alpha \quad \text{for } \alpha \in \Omega_c^2(M).$$

Lemma 4.11 (De Lellis, Spadaro, and Spolaor [DSS17a, Section 1]). *For every $x \in \text{supp}(\delta_C)$ there are: a neighborhood U of x , finite collections of maps f_1, \dots, f_m and charts ϕ_1, \dots, ϕ_m as in Definition 4.10, and weights $\ell_1, \dots, \ell_m \in \mathbf{N}$ such that*

$$\delta_C|_U = \sum_{i=1}^m \ell_i G_{f_i, \phi_i}.$$

Denote by $\mathring{\Sigma}$ the regular part of $\text{supp}(\delta_C)$. Since δ_C is calibrated, the tangent spaces to $\mathring{\Sigma}$ are J -invariant. Therefore, $\mathring{\Sigma}$ canonically is a Riemann surface. As mentioned earlier, the singular set $\text{sing}(\delta_C) := \text{supp}(C) \setminus \mathring{\Sigma}$ is finite. Lemma 4.11 shows that every $x \in \text{sing}(\delta_C)$ has a neighborhood U such that

$$\mathring{\Sigma} \cap U \cong \mathbf{C}^* \sqcup \dots \sqcup \mathbf{C}^*.$$

Thus, $\mathring{\Sigma}$ can be compactified to a Riemann surface Σ by adding finitely many points.

The Riemann surface Σ comes with a continuous map $u: \Sigma \rightarrow M$. Its restriction to $\mathring{\Sigma}$ is smooth and J -holomorphic. It follows from elliptic regularity that u is, in fact, smooth and J -holomorphic on all of Σ . The above discussion shows that

$$\delta_C(\alpha) = \int_{\Sigma} k \cdot u^* \alpha$$

for some locally constant function $k: \Sigma \rightarrow \mathbf{N}$. □

Proof of Lemma 1.9. The sequence of closed integral currents $(\delta_{C_n})_{n \in \mathbf{N}}$ has uniformly bounded mass. Therefore, there exists a subsequence which weakly converges to a closed integral current δ_C calibrated by σ ; see, e.g., [Fed69, Theorem 4.2.17; Sim83, Theorem 27.3; Lan05, Theorem 3.7]. By Proposition 4.4, δ_C is the current associated with a J -holomorphic cycle C . By Proposition 4.5, the sequence of pseudo-holomorphic cycles (C_n) geometrically converges to C . □

5 Proof of Theorem 1.6

Suppose that J is k -rigid and that $A \in H_2(M)$ satisfies $\langle c_1(M, J), A \rangle = 0$ and its divisibility is at most k . If the conclusion of the theorem fails, then there are infinitely many *distinct* J -holomorphic curves $C_n \subset M$ representing A and of energy at most Λ . By Lemma 1.9, after passing to a subsequence, the sequence (C_n) converges geometrically to a J -holomorphic cycle

$$C_\infty = \sum_{i=1}^I m_i C_\infty^i.$$

Proposition 5.1. *C_∞ is connected, smooth, and its multiplicity is at most the divisibility of A .*

Proof. By Definition 4.2 (1), $[C_\infty] = [A]$. Let $u_i : \Sigma_i \rightarrow M$ be a simple J -holomorphic map whose image is C_∞^i . The index formula (2.5) yields

$$\sum_{i=1}^I m_i \text{index}(u_i) = \sum_{i=1}^I 2m_i \langle c_1(M, J), [C_\infty^i] \rangle = 2 \langle c_1(M, J), [C_\infty] \rangle = 0.$$

Since J is k -rigid, by Definition 2.10 (2), there are no J -holomorphic curves of negative index. Thus, we have $\text{index}(u_i) \geq 0$ for every $i \in \{1, \dots, I\}$ and the above computation shows that

$$\text{index}(u_1) = \dots = \text{index}(u_I) = 0.$$

Therefore, by Definition 2.10 (3), the J -holomorphic curves $C_\infty^1, \dots, C_\infty^I$ are embedded and pairwise disjoint. This proves that C_∞ is smooth.

To see that C_∞ is connected, observe that if C_∞ were disconnected, then Definition 4.2 (2) would imply that C_n is disconnected for $n \gg 1$. However, C_n is a J -holomorphic curve and thus connected by definition.

Since $A = m_1[C_\infty^1]$, it follows that m_1 is at most the divisibility of A . \square

In the following, we rescale the sequence (C_n) and extract a further limit \tilde{C}_∞ . The properties of \tilde{C}_∞ will give a contradiction to J being k -rigid.

Henceforth, we denote by C_∞^1 the J -holomorphic curve underlying the J -holomorphic cycle C_∞ . Since the curves C_n are all distinct, we can assume that they are all distinct from C_∞^1 . We can also assume that every C_n is contained in a sufficiently small tubular neighborhood of C_∞^1 . By slight abuse of notation, we regard C_n as an \exp^*J -holomorphic curve in the normal bundle NC_∞^1 and C_∞^1 as the zero section in NC_∞^1 .

For every $\lambda > 0$ let σ_λ be as in Proposition 3.10. Choose (λ_n) such that such that the sets

$$\tilde{C}_n := \sigma_{\lambda_n}^{-1}(C_n)$$

satisfy

$$(5.2) \quad d_H(\tilde{C}_n, C_\infty^1) = 1/2.$$

Set

$$J_n := \sigma_{\lambda_n}^* \exp^* J.$$

By construction, the \tilde{C}_n are J_n -holomorphic. By Proposition 3.10, the sequence (J_n) converges to the almost complex structure J_u associated with the J -holomorphic map $u: C_\infty^1 \hookrightarrow M$. The sequence (\tilde{C}_n) is contained in the compact disc bundle $\bar{B}_{1/2}(C_\infty^1) \subset NC_\infty^1$. By Proposition 3.6 (5), J_u is tamed by a symplectic form ω on $B_1(C_\infty^1)$. Consequently, for $n \gg 1$ the almost complex structure J_n is tamed by ω as well. Define a Riemannian metric g on $B_1(C_\infty^1)$ by

$$g := \frac{1}{2}(\omega(J_u \cdot, \cdot) + \omega(\cdot, J_u \cdot)).$$

The analogously defined metrics g_n are Hermitian with respect to J_n and converge to g . By the energy identity [MS12, Lemma 2.2.1],

$$\lim_{n \rightarrow \infty} \mathbf{M}(\tilde{C}_n) = \lim_{n \rightarrow \infty} \delta_{\tilde{C}_n}(\omega) = \delta_{C_\infty^1}(\omega) < \infty.$$

Therefore, the mass of \tilde{C}_n with respect to g_n (and thus also g) can be bounded independent of n .

By Lemma 1.9, a subsequence of (\tilde{C}_n) geometrically converges to a J -holomorphic cycle

$$\tilde{C}_\infty = \sum_{i=1}^I \tilde{m}_i \tilde{C}_\infty^i.$$

Condition (5.2) guarantees that $\text{supp}(\tilde{C}_\infty) \neq C_\infty^1$. The argument from the proof of Proposition 5.1 shows that $I = 1$ and

$$[\tilde{C}_\infty^1] = \frac{m_1}{\tilde{m}_1} [C_\infty^1].$$

Therefore, Proposition 3.12 applies and the map φ defined there has degree at most the divisibility of A . This contradicts J being k -rigid. \square

A Proofs of Proposition 1.3 and Proposition 1.4

Proof of Proposition 1.3. By a classic theorem of Thom, there is a closed, connected, oriented surface Σ and an embedding $u_0: \Sigma \rightarrow M$ with $u_{0*}[\Sigma] = A$. After adding sufficiently many small 1-handles, we can assume that $g(\Sigma) \geq n$. Gromov's h -principle [Gro86, Section 3.4.2 Theorem (A)] implies that u_0 is C^0 -close to an embedding $u: \Sigma \rightarrow M$ with $u^* \omega > 0$.

Denote the image of u by C . The restriction of TM to C is the direct sum of the symplectic subbundle TC and its symplectic complement, which can be identified with NC . The space of complex structures on \mathbf{R}^{2n} compatible with a fixed non-degenerate 2-form form is contractible. This implies that: (1) both TC and NC admit almost complex structures compatible with ω ; hence, $TM|_C$ admits an almost complex structure compatible with ω , and (2) any such almost complex structure on $TM|_C$ can be extended to an almost complex structure on TM compatible with ω . \square

Remark A.1. If (M, ω) is a symplectic 4-manifold, then every $A \in H_2(M)$ with $\langle [\omega], A \rangle > 0$ is represented by an *immersed* symplectic surface C with transverse double points. Such a surface is J -holomorphic for an almost complex structure compatible with ω if and only if all of its self-intersections are positive.

Proof of Proposition 1.4. Denote by \mathcal{M} the component of the universal moduli space of simple pseudo-holomorphic maps from a Riemann surface of genus g to M . This is a separable Banach manifold and the projection map $\pi: \mathcal{M} \rightarrow \mathcal{F}$ is a Fredholm map of class C^1 and index

$$(n-3)(2-2g) + 2\langle c_1(M, J), A \rangle;$$

see, e.g., [Wen10, Theorem 0; IP18, Proposition 5.1]. If (1.5) is violated, then this index is negative. The result thus follows from the Sard–Smale theorem [Sma65]. \square

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