On counting associative submanifolds and Seiberg–Witten monopoles

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Dedicated to Simon Donaldson on the occasion of his 60th birthday

Abstract

Building on ideas from [DT98; DS11; Wal17; Hay17], we outline a proposal for constructing Floer homology groups associated with a G_2–manifold generated by associative submanifolds and solutions of the ADHM Seiberg–Witten equations. The construction is motivated by the analysis of various transitions which can change the number of associative submanifolds. We discuss the relation of our proposal to Pandharipande and Thomas' stable pair invariant of Calabi–Yau 3–folds.

1 Introduction

Donaldson and Thomas [DT98, Section 3] put forward the idea of constructing enumerative invariants for G_2–manifolds by "counting" G_2–instantons. The principal difficulty in pursuing this program stems from non-compactness issues [Tia00; TT04]; in particular, G_2–instantons can degenerate by bubbling along associative submanifolds. It has been realized by Donaldson and Segal [DS11] that this phenomenon can occur along generic 1–parameter families of G_2–manifolds and, therefore, a naive count of G_2–instantons *cannot* lead to an invariant; see also [Wal17]. Donaldson and Segal proposed to compensate for this phenomenon with a counter-term consisting of a weighted count of associative submanifolds; however, they did not elaborate on how to construct a suitable coherent system of weights. Haydys and Walpuski proposed to define such weights by counting solutions to Seiberg–Witten equations associated with the ADHM construction of instantons [AHDM78], when the structure group in question is $\mathrm{SU}(r)$; see [HW15, paragraphs following Remark 1.7; Hay17; DW17a, Introduction; DW17b, Appendix B].

If one specializes to $r = 1$, that is, to trivial line bundles, then there are no non-trivial G_2–instantons and their naive count, trivially, is an invariant. However, according to the Haydys–Walpuski proposal one should still count associatives weighted by the count of solutions of the Seiberg–Witten equation on them. It is known that counting associatives by themselves does not lead to an invariant, because the following situations may arise along a generic 1–parameter family of G_2–manifolds,:

1. An embedded associative submanifold develops a self-intersection. Out of this self-intersection a new associative submanifold is created by work of Nordström [Nor13]. Topologically, this submanifold is the connected sum.

2. By analogy with special Lagrangians in Calabi–Yau 3–folds [Joy02, Section 3], it has been conjectured that, it is possible for three distinct associative submanifolds to degenerate into a singular associative submanifold with an isolated singularity modeled on T^2 [Wal13, p.154; Joy17, Conjecture 5.3]. Topologically, these three manifolds form a surgery triad.

We will argue that known vanishing results and surgery formulae for the Seiberg–Witten invariant [MT96, Proposition 4.1 and Theorem 5.3], show that the count of associatives weighted by solutions of the Seiberg–Witten equation is invariant under the transitions (1) and (2), assuming that all connected components of the associative submanifolds in question $b_1 > 1$. This restriction is needed in order to be able to avoid reducible solutions and obtain a well-defined Seiberg–Witten invariant *as an integer.*[1] We know of no natural assumption that would ensure that this restriction holds for all relevant associative submanifolds. Hence, the Haydys–Walpuski proposal cannot yield an invariant which is just an integer.

One can define a topological invariant using the Seiberg–Witten equation for any compact, oriented 3–manifold. This invariant, however, is not a number but rather a homology group, called monopole Floer homology [MW01; Man03; KM07; Frø10]. The behavior of monopole Floer homology under connected sum and in surgery triads is understood [KMOS07, Theorem 2.4; BMO; Lin15, Theorem 5]. We will explain how to construct a chain complex using the monopole chain complexes and chain maps induced by cobordisms whose homology *might be invariant* under the transitions (1) and (2).

The discussion so far only involved the classical Seiberg–Witten equation. There is a further transition that might spoil the invariance:

[1]Using spectral counter-terms, Chen [Che97; Che98] and Lim [Lim00] were able to define Seiberg–Witten invariants of 3–manifolds with $b_1 \leq 1$. These, however, are rational and cannot satisfy the necessary vanishing theorem.

3. Along generic 1–parameter families of G_2–manifolds, somewhere injective immersed associative submanifolds can degenerate by converging to a multiple cover.

We will explain why this phenomenon occurs and that it can change the number of associatives, even when weighted by counts of solutions of the Seiberg–Witten equation. This is where the Seiberg–Witten equations related to the ADHM construction, specifically that of $\mathrm{Sym}^k \mathbf{H}$, the k^{th} symmetric power of the quaternions, enter into the picture.

Counting solutions to those equations does not yield a topological invariant of 3–manifolds, and we will provide evidence for the conjecture that the change in this count exactly compensates the change in the number of associatives weighted by counts of solutions of the Seiberg–Witten equation. Based on this we will give a tentative proposal for how to obtain an enumerative invariant from associatives and solutions of ADHM Seiberg–Witten equations. Our proposal is closely related to Pandharipande and Thomas's stable pair invariant of Calabi–Yau 3–folds [PT09].

Finally, we would like to point out that an alternative approach to counting associative submanifolds has been proposed recently by Joyce [Joy17]. His proposal also does not lead to a number or a homology group, but rather a more complicated object: a super-potential up to quasi-identity morphisms.

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2 Counting associative submanifolds

We begin with a review of G_2–manifolds and associative submanifolds with a focus towards explaining what we mean by "counting associative submanifolds".

2.1 G_2–manifolds

The exceptional Lie group G_2 is the automorphism group of the octonions $\mathbf{O}$, the unique 8–dimensional normed division algebra:

$$G_2 = \mathrm{Aut}(\mathbf{O}).$$

Since any automorphism of $\mathbf{O}$ preserves the unit $1 \in \mathbf{O}$, we can think of G_2 as a subgroup of $\mathrm{SO}(7)$.

Definition 2.1. A G_2**–structure** on a 7–dimensional manifold Y is a reduction of structure group of the frame bundle of Y from $\mathrm{GL}(7)$ to G_2. An **almost** G_2**–manifold** is a 7–dimensional manifold Y equipped with a G_2–structure.

The multiplication on $\mathbf{O}$ endows $\operatorname{Im}\mathbf{O}$ with:

- an **inner product**, $g\colon S^2 \operatorname{Im}\mathbf{O} \to \mathbf{R}$, satisfying

$$g(u, v) = -\operatorname{Re}(uv),$$

- a **cross-product**, $\cdot \times \cdot\colon \Lambda^2 \operatorname{Im}\mathbf{O} \to \operatorname{Im}\mathbf{O}$, defined by

$$(u, v) \mapsto u \times v := \operatorname{Im}(uv)$$

  and a corresponding 3–form $\phi \in \Lambda^3 \operatorname{Im}\mathbf{O}^*$ defined by

$$\phi(u, v, w) := g(u \times v, w),$$

  as well as

- an **associator**, $[\cdot, \cdot, \cdot]\colon \Lambda^3 \operatorname{Im}\mathbf{O} \to \operatorname{Im}\mathbf{O}$, defined by

$$[u, v, w] := (u \times v) \times w + \langle v, w\rangle u - \langle u, w\rangle v \tag{2.2}$$

  and a corresponding 4–form $\psi \in \Lambda^4 \operatorname{Im}\mathbf{O}^*$ defined by

$$\psi(u, v, w, z) := g([u, v, w], z).$$

These are related by the identities

$$\begin{aligned} i(u)\phi \wedge i(v)\phi \wedge \phi &= 6g(u, v)\mathrm{vol}_g \quad \text{and} \\ *_g\phi &= \psi \end{aligned} \tag{2.3}$$

for a unique choice of orientation on $\operatorname{Im}\mathbf{O}$. We refer the reader to [HL82, Chapter IV; SW17] for a more detailed discussion.

A G_2–structure on Y endows TY with analogous structures:

- a Riemannian metric g,
- a cross-product $\cdot \times \cdot\colon \Lambda^2 TY \to TY$,
- a 3–form $\phi \in \Omega^3(Y)$,

- an associator $[\cdot,\cdot,\cdot]\colon \Lambda^3 TY \to TY$, and
- a 4–form $\psi \in \Omega^4(Y)$,

satisfying the same relations as above. From (2.3) it is apparent that from ϕ one can reconstruct g and thus also ψ, the cross-product, and the associator. Similarly, one can reconstruct g from ψ together with the orientation. The condition for a 3–form ϕ or a 4–form ψ to arise from a G_2–structure is that the form be definite, see [Hit01, Section 8.3; Bry06, Section 2.8]. We say that a 3–form ϕ is **definite** if the bilinear form $G_\phi \in \Gamma(S^2T^*Y \otimes \Lambda^7 T^*Y)$ defined by

$$G_\phi(u,v) := i(u)\phi \wedge i(v)\phi \wedge \phi$$

is definite, and we say that a 4–form ψ is **definite** if the bilinear form $G^*_\psi \in \Gamma(S^2TY \otimes (\Lambda^7T^*Y)^{\otimes 2})$ defined by

$$G^*_\psi(\alpha,\beta) := i(\alpha)\psi \wedge i(\beta)\psi \wedge \psi$$

is definite. Here we identify $\Lambda^4T^*Y \cong \Lambda^3TY \otimes \Lambda^7T^*Y$. Therefore, a G_2–structure can be specified either by a definite 3–form ϕ, or by a definite 4–form ψ together with an orientation.

A G_2–structure on a 7–manifold induces a spin structure through the inclusion $G_2 \subset \mathrm{Spin}(7)$. In fact, a 7–manifold admits a G_2–structure if and only if it is spin, see [Gra69, Theorems 3.1 and 3.2] and [LM89, p. 321]. This means that the existence of a G_2–structure is a soft topological condition. More rigid notions are obtained by imposing conditions on the torsion of the G_2–structure, in the sense of G–structures, see [Joy00, Section 2.6]. The most stringent and most interesting condition to impose is that the torsion vanishes.

Definition 2.4. A **G_2–manifold** is a 7–manifold equipped with a torsion-free G_2–structure.

Theorem 2.5 (Fernández and Gray [FG82, Theorem 5.2]). *A G_2–structure on a 7–manifold Y is torsion-free if and only the associated 3–form ϕ as well as the associated 4–form ψ are closed:*

$$\mathrm{d}\phi = 0 \quad \text{and} \quad \mathrm{d}\psi = 0.$$

The Riemannian metric induced by a torsion-free G_2–structure has holonomy contained in G_2—one of two exceptional holonomy groups in Berger's classification [Ber55, Theorem 3]. If Y is compact, then equality holds if and only if $\pi_1(Y)$ is finite [Joy00, Proposition 10.2.2]. We refer the reader to [Joy00, Section 10] for a thorough discussion of the properties of G_2–manifolds.

Example 2.6. If Z is a Calabi–Yau 3–fold with Kähler form ω and holomorphic volume form Ω, and t denotes the coordinate on S^1, then $S^1 \times Z$ is a G_2–manifold with

$$\phi = \mathrm{d}t \wedge \omega + \operatorname{Re}\Omega \quad \text{and} \quad \psi = \frac{1}{2}\omega \wedge \omega + \mathrm{d}t \wedge \operatorname{Im}\Omega.$$

In this case the holonomy group is contained in $\mathrm{SU}(3) \subset G_2$.

Example 2.7. The first local, complete, and compact examples of manifolds with holonomy equal to G_2 are due to Bryant [Bry87], Bryant and Salamon [BS89], and Joyce [Joy96a; Joy96b; Joy00] respectively. Joyce's examples arise from a generalized Kummer construction based on smoothing flat G_2–orbifolds of the form T^7/Γ. This method has recently been extended to more general G_2–orbifolds by Joyce and Karigiannis [JK17]. The most fruitful construction method for G_2–manifolds to this day is the twisted connected sum construction, which was pioneered by Kovalev [Kov03] and extended and improved by Kovalev and Lee [KL11] and Corti, Haskins, Nordström, and Pacini [CHNP13; CHNP15]. It is based on gluing, in a twisted fashion, a pair of asymptotically cylindrical G_2–manifolds which are products of S^1 with asymptotically cylindrical Calabi–Yau 3–folds. Corti, Haskins, Nordström, and Pacini [CHNP15] have produced tens of millions of examples G_2–manifolds using this construction.

2.2 Associative submanifolds

Definition 2.8. Let Y be an almost G_2–manifold, let P be an oriented 3–manifold, and let $\iota\colon P \to Y$ be an immersion. We say that ι is **associative** if

$$\iota^*[\cdot,\cdot,\cdot] = 0 \in \Omega^3(P, \iota^*TY) \quad \text{and} \quad \iota^*\phi \text{ is positive.} \tag{2.9}$$

An **immersed associative submanifold** is an equivalence class $[\iota]$ of an associative immersion $\iota \in \mathrm{Imm}(P,Y)/\mathrm{Diff}_+(P)$ for some oriented 3–manifold P. Here $\mathrm{Imm}(P,Y)$ is the space of immersions $P \to Y$ and $\mathrm{Diff}_+(P)$ is the group of orientation-preserving diffeomorphisms of P.

Harvey and Lawson [HL82, Chapter IV, Theorem 1.6] proved the identity

$$\phi(u,v,w)^2 + |[u,v,w]|^2 = |u \wedge v \wedge w|^2, \tag{2.10}$$

which shows that ϕ is a semi-calibration and that associative submanifolds are calibrated by ϕ.

Proposition 2.11 (Federer [Fed69], Harvey and Lawson [HL82, Introduction]). *If $\iota\colon P \to Y$ is associative, then*

$$\iota^*\phi = \mathrm{vol}_{\iota^* g}.$$

In particular, if ϕ is closed and P is compact, then $\iota(P)$ is volume-minimizing in the homology class $\iota_*[P]$ and

$$\mathrm{vol}(P, \iota^* g) = \langle [\phi], \iota_*[P] \rangle.$$

Proposition 2.12 (see, e.g., [SW17, Lemma 4.7]). *If $\iota\colon P \to Y$ is an immersion, then the following are equivalent:*

1. $\iota^*[\cdot,\cdot,\cdot] = 0$,
2. *for all $u, v \in \iota_* T_x P$, $u \times v \in \iota_* TP$,*
3. *for all $u \in \iota_* T_x P$ and $v \in (\iota_* T_x P)^\perp$, $u \times v \in (\iota_* T_x P)^\perp$.*

Example 2.13. Let Z be a Calabi–Yau 3–fold. Equip $S^1 \times Z$ with the G_2–structure from Example 2.6. If $\Sigma \subset Z$ is a holomorphic curve, then $S^1 \times \Sigma$ is associative. If $L \subset Z$ is a special Lagrangian submanifold, then, for any $t \in S^1$, $\{t\} \times L$ is associative.

Example 2.14. Examples of associative submanifolds which arise as fixed points of involutions have been given by Joyce [Joy96b, Section 4.2]. Examples of associative submanifolds arising from holomorphic curves and special Lagrangians in asymptotically cylindrical Calabi–Yau 3–folds were constructed by Corti, Haskins, Nordström, and Pacini [CHNP15, Section 5]

2.3 The $\mathfrak{L}$ functional

Associative submanifolds can be thought of as critical points of a functional.

Definition 2.15. Define the 1–form $\delta\mathfrak{L} = \delta\mathfrak{L}_\psi \in \Omega^1(\mathrm{Imm}(P, Y))$ by[2]

$$\delta_\iota \mathfrak{L}(n) = \int_P \iota^* i(n)\psi = \int_P \langle \iota^*[\cdot,\cdot,\cdot], n \rangle.$$

for $n \in T_\iota \mathrm{Imm}(P, Y) = \Gamma(P, \iota^* TY)$.

Proposition 2.16.

1. *ι is associative if and only if $\delta_\iota \mathfrak{L} = 0$ and $\iota^*\phi$ is positive.*
2. *$\delta\mathfrak{L}$ is $\mathrm{Diff}_+(P)$–invariant.*
3. *If $\mathrm{d}\psi = 0$, then $\delta\mathfrak{L}$ is a closed 1–form. In fact, there is a $\mathrm{Diff}_+(P)$–equivariant covering space $\pi\colon \widetilde{\mathrm{Imm}}(P, Y) \to \mathrm{Imm}(P, Y)$ and a $\mathrm{Diff}_+(P)$–equivariant function $\tilde{\mathfrak{L}}\colon \widetilde{\mathrm{Imm}}(P, Y) \to \mathbf{R}$ whose differential is $\pi^*\delta\mathfrak{L}$.*[3]

[2] Although n is not a vector field on Y, by slight abuse of notation we denote by $\iota^* i(n)\psi$ the 3–form on P given by $(u, v, w) \mapsto \psi(\iota_* u, \iota_* v, \iota_* w, n)$.

[3] This justifies the notation $\delta\mathfrak{L}$ since locally it is the differential of a function.

Proof. Assertions (1) and (2) are both trivial. For $\beta \in H_3(Y, \mathbf{R})$, let $\mathrm{Imm}_\beta(P, Y)$ denote the set of immersions $\iota\colon P \to Y$ such that $\iota_*[P] = \beta$. Fix $P_0 \in \mathrm{Imm}_\beta(P, Y)$ and denote by $\widetilde{\mathrm{Imm}}_\beta(P, Y)$ the space of pairs $(\iota, [Q])$ with $\iota \in \mathrm{Imm}_\beta(P)$ and $[Q]$ an equivalence class of 4–chains in Y such that $\partial Q = P - P_0$ with $[Q] = [Q']$ if and only if $[Q - Q'] = 0 \in H_4(Y, \mathbf{Z})$. Define $\tilde{\mathfrak{L}}\colon \widetilde{\mathrm{Imm}}_\beta(P, Y) \to \mathbf{R}$ by

$$\tilde{\mathfrak{L}}(\iota, [Q]) = \int_Q \psi.$$

The function $\tilde{\mathfrak{L}}$ has the desired properties; see also [DT98, Section 8]. □

2.4 The moduli space of associatives

Definition 2.17. Let P be a compact, oriented 3–manifold and let $\beta \in H_3(Y, \mathbf{Z})$. Denote by $\mathrm{Imm}_\beta(P, Y)$ the space of immersions $\iota\colon P \to Y$ with $\iota_*[P] = \beta$. The group $\mathrm{Diff}_+(P)$ acts on $\mathrm{Imm}_\beta(P, Y)$. The **moduli space** of immersed associative submanifolds is

$$\mathfrak{A}(\psi) = \coprod_{\beta \in H_3(Y,\mathbf{Z})} \mathfrak{A}_\beta(\psi) = \coprod_{\beta \in H_3(Y,\mathbf{Z})} \coprod_P \mathfrak{A}_{P,\beta}(\psi)$$

with

$$\mathfrak{A}_{P,\beta}(\psi) \coloneqq \left\{[\iota] \in \mathrm{Imm}_\beta(P, Y)/\mathrm{Diff}_+(Y) : (2.9)\right\}.$$

Here P ranges over all diffeomorphism types of compact, oriented 3–manifolds.

Denote by $\mathscr{D}^4(Y)$ the space of definite 4–forms on Y. If $\mathscr{P}$ is a subspace of $\mathscr{D}^4(Y)$, then the **$\mathscr{P}$–universal moduli space** is

$$\mathfrak{A}(\mathscr{P}) = \bigcup_{\psi \in \mathscr{P}} \mathfrak{A}(\psi).$$

The moduli space $\mathfrak{A}(\mathscr{P})$ inherits a topology from the C^∞–topology on $\mathrm{Imm}_\beta(P, Y)$. As we will explain in the following, the infinitesimal deformation theory of associatives submanifolds is controlled by a first-order elliptic operator and $\mathfrak{A}(\mathscr{P})$ admits corresponding Kuranishi models.

Definition 2.18. Let $\iota\colon P \to Y$ be an associative immersion. Denote by

$$N\iota \coloneqq \iota^*TY/TP \cong TP^\perp \subset \iota^*TY$$

its normal bundle and by ∇ the connection on $N\iota$ induced by the Levi-Civita connection. The **Fueter operator** associated with ι is the first order differential operator $F_\iota = F_{\iota,\psi}\colon \Gamma(N\iota) \to \Gamma(N\iota)$ defined by

$$F_\iota(m) \coloneqq \sum_{i=1}^{3} \iota_* e_i \times \nabla_{e_i} m.$$

Here (e_1, e_2, e_3) is an orthonormal frame of P.

This operator arises as follows. Identify $N\iota$ with $TP^{\perp} \subset \iota^* TY$ and given a normal vector field $m \in \Gamma(N\iota)$, define $\iota_m\colon P \to Y$ by

$$\iota_m(x) \coloneqq \exp(m(x)).$$

The condition for $\iota_{\varepsilon m}$ to be associative to first order in ε is that

$$\begin{aligned} 0 &= \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} [(\iota_{\varepsilon m})_* e_1, (\iota_{\varepsilon m})_* e_2, (\iota_{\varepsilon m})_* e_3] \\ &= (\iota_* e_1 \times \iota_* e_2) \times \nabla_{e_3} m + \text{cyclic permutations} \\ &= \sum_{i=1}^{3} \iota_* e_i \times \nabla_{e_i} m, \end{aligned}$$

where we have used the definition of the associator (2.2) and the fact that $\iota\colon P \to Y$ is associative so we have $\iota_* e_1 \times \iota_* e_2 = \iota_* e_3$ (as well as all of its cyclic permutations).

Proposition 2.19 (Joyce [Joy17, paragraph after Theorem 2.12]). *If* $\mathrm{d}\psi = 0$, *then*

$$\operatorname{Hess} \tilde{\mathfrak{L}}(n, m) = \int_P \langle n, F_\iota m \rangle$$

with $\tilde{\mathfrak{L}}$ *as in Proposition 2.16(3). In particular,* F_ι *is self-adjoint.*

Theorem 2.20 (McLean [McL98], Joyce [Joy17, Theorem 2.12]). *Let* $[\iota\colon P \to Y] \in \mathfrak{A}_\beta(\psi_0)$. *Denote by* $\operatorname{Aut}(\iota)$ *the stabilizer of* ι *in* $\operatorname{Diff}_+(P)$.

The group $\operatorname{Aut}(\iota)$ *is finite. The Fueter operator* F_ι *is equivariant with respect to the action* $\operatorname{Aut}(\iota)$ *on* $\Gamma(N\iota)$. *If* $\mathscr{P}$ *is a submanifold of the space of definite* 4*–forms containing* ψ_0, *then there are*

- *an* $\operatorname{Aut}(\iota)$*–invariant open subset* $U \subset \mathscr{P} \times \ker F_\iota$,
- *a smooth* $\operatorname{Aut}(\iota)$*–equivariant map* $\mathrm{ob}\colon \mathscr{P} \times U \to \operatorname{coker} F_\iota$ *with* $\mathrm{ob}(\psi_0, \cdot)$ *and its derivative vanishing at* 0,
- *an open neighborhood* V *of* $([\iota], \psi_0)$ *in* $\mathfrak{A}_\beta(\mathscr{P})$, *and*
- *a homeomorphism* $\mathfrak{x}\colon \mathrm{ob}^{-1}(0)/\operatorname{Aut}(\iota) \to V$.

Moreover, if $(\mathbf{p}, n) \in \mathrm{ob}^{-1}(0)$, *then the stabilizer of any immersion representing* $\mathfrak{x}(\mathbf{p}, n)$ *is the stabilizer of* n *in* $\operatorname{Aut}(\iota)$.

Definition 2.21. We say that an associative immersion $\iota\colon P \to Y$ is **unobstructed** (or **rigid**) if F_ι is invertible.

2.5 Transversality

Theorem 2.20 indicates that $\mathfrak{A}_\beta(\psi)$ should behave like a 0–dimensional orbifold. In fact, for generic closed ψ, a large part of $\mathfrak{A}_\beta(\psi)$ is a 0–dimensional manifold.

Definition 2.22. An immersion $\iota\colon P \to Y$ is called **somewhere injective** if each connected component of P contains a point x such that $\iota^{-1}(\iota(x)) = \{x\}$. Denote by

$$\mathfrak{A}^{\mathrm{si}}_\beta(\psi)$$

the open subset of somewhere injective immersions with respect to ψ. Given a submanifold $\mathscr{P}$ of the space of definite 4–forms, set

$$\mathfrak{A}^{\mathrm{si}}_\beta(\mathscr{P}) = \bigcup_{\psi\in\mathscr{P}} \mathfrak{A}^{\mathrm{si}}_\beta(\psi).$$

Proposition 2.23. *Denote by $\mathscr{D}^4_c(Y)$ the space of closed, definite 4–forms.*

1. *There is a residual subset $\mathscr{D}^4_{c,\mathrm{reg}} \subset \mathscr{D}^4_c(Y)$ such that for every $\psi \in \mathscr{D}^4_{c,\mathrm{reg}}$,*
   (a) *the moduli space $\mathfrak{A}^{\mathrm{si}}_\beta(\psi)$ is a 0–dimensional manifold and consists only of unobstructed associative submanifolds, and*
   (b) *$\mathfrak{A}^{\mathrm{si}}_\beta(\psi)$ consists only of embedded associative submanifolds.*
2. *If $\psi_0, \psi_1 \in \mathscr{D}^4_{c,\mathrm{reg}}(Y)$, then there is a residual subset $\boldsymbol{\mathscr{D}}^4_{c,\mathrm{reg}}(\psi_0,\psi_1)$ in the space of paths from ψ_0 to ψ_1 in $\mathscr{D}^4_c(Y)$ such that for every $(\psi_t)_{t\in[0,1]} \in \boldsymbol{\mathscr{D}}^4_{c,\mathrm{reg}}(\psi_0,\psi_1)$,*
   (a) *the universal moduli space $\mathfrak{A}^{\mathrm{si}}_\beta(\{\psi_t : t \in [0,1]\})$ is a 1–dimensional manifold, and*
   (b) *for each component $\{(\psi_t,[\iota_t]) : t \in J\}$ with $J \subset [0,1]$ an interval, there is a discrete set $J_\times \subset J$ such that:*
      i. *for $t \in J\backslash J_\times$, ι_t is an embedding.*
      ii. *for $t_\times \in J_\times$, there is a $T > 0$ and with the property that*

$$\mathbf{P} := \bigcup_{|t-t_\times|<T} \{t\}\times\iota_t(P) \subset \mathbf{R}\times Y$$

has a unique self-intersection and this intersection is transverse.

The proof of this result is similar to that of analogous results about pseudo-holomorphic curves in symplectic manifolds, cf. McDuff and Salamon [MS12, Sections 3.2 and 3.4]; in fact, our situation is simpler because we know from the outset that ι is an immersion.

Proof of Proposition 2.23. The proof relies on the following observations. The tangent space $T_\psi \mathscr{D}^4_c(Y) \subset \Omega^4(Y)$ is the space of closed 4–forms. Define

$$X_{\iota,\psi}\colon\ T_\psi \mathscr{D}^4_c(Y) \to \Gamma(N\iota)$$

by

$$\langle X_{\iota,\psi}\eta, n\rangle_{L^2} := \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \delta\mathfrak{L}_{\psi+\varepsilon\eta}(n) = \int_P \iota^* i(n)\eta \tag{2.24}$$

for every closed 4–form η on Y.

Proposition 2.25. *If $\iota\colon\ P \to Y$ is a somewhere injective associative immersion, then for every non-zero $n \in \ker F_\iota \subset \Gamma(N\iota)$, there exists $\alpha \in \Omega^3(Y)$ such that*

$$\langle X_{\iota,\psi}\mathrm{d}\alpha, n\rangle \neq 0.$$

Proof. We can assume that P is connected. Pick a point x such that $\iota^{-1}(\iota(x)) = \{x\}$. Since P is compact, there is a neighborhood U of $x \in P$ which is embedded via ι and satisfies $\iota(U) \cap \iota(P\backslash U) = \varnothing$. Choose a tubular neighborhood V of $\iota(U)$ and $\rho > 0$ such that $B_\rho(N\iota(U)) \xrightarrow{\exp} V$ is a diffeomorphism. By unique continuation, n cannot vanish identically on U. Thus we can find a function f supported in V such that $\mathrm{d}f(n) \geq 0$ and $\mathrm{d}f(n) > 0$ somewhere. Let ν be a 3–form on Y with $\nu|_U = (\mathrm{vol}_P)|_U$ and $i(n)\mathrm{d}\nu|_V = 0$. With

$$\alpha = f\nu$$

we have

$$\int_P \iota^*(i(n)\mathrm{d}\alpha) = \int_P \mathrm{d}f(n)\mathrm{vol}_P > 0. \qquad \square$$

For a somewhere injective immersed associative $[\iota\colon\ P \to Y]$, $\mathrm{Aut}(\iota)$ must be trivial. By Proposition 2.25, the linearization of the section

$$\delta\mathfrak{L} \in \Gamma(\pi^*_{\mathrm{Imm}} T^* \mathrm{Imm}_\beta(P, Y)),$$

where $\pi_{\mathrm{Imm}}\colon\ \mathrm{Imm}_\beta(P, Y) \times \mathscr{D}^4_c(Y) \to \mathrm{Imm}_\beta(P, Y)$ is the canonical projection, is surjective. Hence, it follows from the Regular Value Theorem, and the fact that there are only countably many diffeomorphism types of 3–manifolds [CK70], that the universal moduli space of immersed associatives

$$\mathfrak{A}^{\mathrm{si}}_\beta = \mathfrak{A}^{\mathrm{si}}_\beta(\mathscr{D}^4_c(Y))$$

is a smooth manifold. This directly implies (1a) and (2a) by the Sard–Smale Theorem.

Consider the moduli space of immersed associative submanifolds with n marked points

$$\mathfrak{A}^{\mathrm{si}}_{\beta,n}(\psi) := \coprod_P \left\{(\iota, x_1, \dots, x_n) \in \mathrm{Imm}_\beta(P, Y) \times P^n : [\iota] \in \mathfrak{A}^{\mathrm{si}}_\beta(\psi)\right\} \Big/ \mathrm{Diff}_+(P).$$

as well as the corresponding universal moduli space

$$\mathfrak{A}^{\mathrm{si}}_{\beta,n} := \bigcup_{\psi \in \mathscr{D}^4_c(Y)} \mathfrak{A}^{\mathrm{si}}_{\beta,n}(\psi).$$

Define the map $\mathrm{ev}\colon \mathfrak{A}^{\mathrm{si}}_{\beta,n} \to Y^n$ by

$$\mathrm{ev}([\iota, x_1, \dots, x_n], \psi) := (\iota(x_1), \dots, \iota(x_n)).$$

Proposition 2.26. *For each* $([\iota, x_1, \dots, x_n], \psi) \in \mathfrak{A}^{\mathrm{si}}_{\beta,n}$*, the derivative of* ev*,*

$$\mathrm{d}_{([\iota,x_1,\dots,x_n],\psi)}\mathrm{ev}\colon T_{([\iota,x_1,\dots,x_n],\psi)}\mathfrak{A}^{\mathrm{si}}_{\beta,n} \to \bigoplus_{i=1}^n T_{\iota(x_i)}Y,$$

is surjective.

Proof. We will show that if $(v_1, \dots, v_n) \in \bigoplus_{i=1}^n N_{x_i}\iota$, then there exist $n \in \Gamma(N\iota)$ and $\eta \in T_\psi \mathscr{D}^4_c(Y)$ such that

$$n(x_i) = v_i \quad \text{and} \quad (n, \eta) \in T_{[\iota],\psi}\mathfrak{A}^{\mathrm{si}}_\beta,$$

which immediately implies the assertion.

Denote by $\mathrm{ev}_{x_1,\dots,x_n}\colon \Gamma(N\iota) \to \bigoplus_{i=1}^n N_{x_i}\iota$ the evaluation map and define

$$\mathbf{F}^k := \left(F_\iota \oplus X_{\iota,\psi}\colon W^{k,2}\ker \mathrm{ev}_{x_1,\dots,x_n} \oplus T_\psi\mathscr{D}^4_c(Y) \to W^{k-1,2}\Gamma(N\iota)\right),$$

where F_ι is the Fueter operator and $X_{\iota,\psi}$ is defined in (2.24). We prove that the operator $\mathbf{F}^1$ is surjective, cf. McDuff and Salamon [MS12, Proof of Lemma 3.4.3]. To see this note that its image is closed and thus we need to show only that if $\nu \perp \mathrm{im}\,\mathbf{F}^1$, then $\nu = 0$. Since $\nu \perp F_\iota(W^{1,2}\ker \mathrm{ev}_{x_1,\dots,x_n})$, on $P\backslash\{x_1, \dots, x_n\}$, ν is smooth and satisfies $F_\iota \nu = 0$. We also know that $\nu \perp \mathrm{im}\,X_{\iota,\psi}$. The argument from the Proof of Proposition 2.25 shows that $\nu = 0$, because the set of points $x \in P$ satisfying $\iota^{-1}(\iota(x)) = \{x\}$ is open in P so we can choose such a point x belonging to $P\backslash\{x_1, \dots, x_n\}$). That $\mathbf{F}^k$ is surjective follows from the fact that $\mathbf{F}^1$ is surjective by elliptic regularity.

Pick $n_0 \in \Gamma(N\iota)$ with

$$n_0(x_i) = v_i$$

and pick $(n_1, \eta) \in \ker \mathrm{ev}_{x_1,\dots,x_n} \oplus T_\psi \mathscr{D}^4_c(Y)$ such that

$$F_\iota n_1 + X_{\iota,\psi}(\eta) = -F_\iota n_0.$$

The pair $(n_0 + n_1, \eta) \in T_{[\iota],\psi}\mathfrak{A}^{\mathrm{si}}_\beta$ has the desired properties. □

Finally, we are in a position to prove (1b) and (2b) of Proposition 2.23. Denote by $\pi\colon \mathfrak{A}^{\mathrm{si}}_{\beta,2} \to \mathfrak{A}^{\mathrm{si}}_\beta$ the forgetful map and denote by $\Delta = \{(x,x) \in Y \times Y : x \in Y\}$ the diagonal in Y. Proposition 2.26 The universal moduli space of non-injective but somewhere injective immersed associatives is precisely

$$\pi(\mathrm{ev}^{-1}(\Delta)).$$

By Proposition 2.26, $\mathrm{ev}^{-1}(\Delta) \subset \mathfrak{A}^{\mathrm{si}}_\beta$ is a codimension 7 submanifold. Since π is a Fredholm map of index 6 and $\rho\colon \mathfrak{A}^{\mathrm{si}} \to \mathscr{D}^4_c(Y)$ is a Fredholm map of index 0, it follows that $\rho(\pi(\mathrm{ev}^{-1}(\Delta))) \subset \mathscr{D}^4_c(Y)$ is residual. This proves (1b) because an injective immersion of a compact manifold is an embedding. The proof of (2b) is similar. □

2.6 Taming

For an arbitrary definite 4–form ψ and an associative submanifold $[\iota\colon P \to Y]$ with $\iota_*[P] = \beta$, we have no control on $\mathrm{vol}(P, \iota^* g)$. In order to gain control, one can restrict to tamed definite 4–forms. This is analogous to the notion of a tamed almost complex structure in symplectic topology which guarantees area bounds for pseudo-holomorphic curves.

Definition 2.27 (Joyce [Joy17, Definition 2.6]). Let Y be an almost G_2–manifold with 3–form ϕ, 4–form ψ, and associator $[\cdot,\cdot,\cdot]$. We say that $\tau \in \Omega^3(Y)$ **tames** ψ if $\mathrm{d}\tau = 0$ and for all $x \in Y$ and $u, v, w \in T_x Y$ with $[u,v,w] = 0$ and $\phi(u,v,w) > 0$, we have $\tau(u,v,w) > 0$.

Proposition 2.28 (Donaldson and Segal [DS11, Section 3.2], Joyce [Joy17, Section 2.5]). *Let Y be a compact almost G_2–manifold. If ψ is tamed by τ, then there is a constant $c > 0$ such that for every associative immersion $\iota\colon P \to Y$ with P compact,*

$$\mathrm{vol}(P, \iota^* g) \leq c \cdot \langle [\tau], \iota_*[P] \rangle.$$

If ψ corresponds to a torsion-free G_2–structure, then any nearby 4–form is tamed by $\phi = *\psi$. One should think of tamed, closed, definite 4–forms as a softening of the notion of a definite 4–form giving rise to a torsion-free G_2–structure. The advantage of working with tamed, definite 4–forms is that the volume of any associative submanifold in $\mathfrak{A}_\beta(\psi)$ is bounded and one can, in principle, use geometric measure theory to compactify $\mathfrak{A}_\beta(\psi)$.

2.7 Enumerative invariants from associatives?

Question 2.29. Is there a residual subset of tamed, closed, definite 4–forms for which $\mathfrak{A}_\beta(\psi)$ is a compact 0–dimensional manifold (or orbifold)?

If the answer to this question is yes, then for every ψ from this residual subset we can define

$$n_\beta(\psi) := \#\mathfrak{A}_\beta(\psi). \tag{2.30}$$

Question 2.31. Is $n_\beta(\psi)$ invariant under deforming ψ?

If the answer to this question is also yes, then n_β would give rise to a deformation invariant of G_2–manifolds by defining its value on for a torsion-free G_2–structure ψ to be that on a nearby tamed, closed, definite 4–form.

It is easy to see that a naive interpretation of $\#\mathfrak{A}_\beta(\psi)$ as the cardinality of $\mathfrak{A}_\beta(\psi)$ does not lead to an invariant. Suppose that $\mathscr{P} = \{\psi_t : t \in (-1,1)\}$ is 1–parameter family of tamed, closed, definite 4–form and $[\iota_0\colon P \to Y] \in \mathfrak{A}_\beta(\psi_0)$ with $\dim\ker F_{\iota_0,\psi_0} = 1$. By Theorem 2.20, a neighborhood of $([\iota_0],\psi_0) \in \mathfrak{A}_\beta(\mathscr{P})$ is given by $\mathrm{ob}^{-1}(0)$ with ob a smooth map satisfying

$$\mathrm{ob}(t,\delta) = \lambda t + c\delta^2 + \text{higher order terms.}$$

For a generic 1–parameter family we will have $\lambda, c \neq 0$. For simplicity, let us assume that $\lambda = c = 1$. In this situation for $-1 \ll t < 0$, there are two associative submanifolds $[\iota_t^\pm\colon P \to Y]$ with respect to ψ_t near $[\iota_0]$. As t tends to 0, $[\iota_t^\pm]$ tend to $[\iota_0]$. For $t > 0$, there are no associatives near $[\iota_0]$. This means that $n_\beta(\phi)$ as defined in (2.30) changes by -2 as t passes through 0.

The origin of this problem is that $\mathfrak{A}_\beta(\psi)$ should be an oriented manifold and we should count associative immersions $[\iota] \in \mathfrak{A}_\beta(\psi)$ with signs $\varepsilon([\iota],\psi) \in \{\pm 1\}$. These signs should be such that if $\{\iota_t\colon P \to Y : t \in [0,1]\}$ is a 1–parameter family of associative immersions along a 1–parameter family of closed, definite 4–forms, then

$$\varepsilon([\iota_1],\psi_1) = (-1)^{\mathrm{SF}\left(F_{\iota_t,\psi_t}: t\in[0,1]\right)} \cdot \varepsilon([\iota_0],\psi_0). \tag{2.32}$$

In the above situation we have

$$\varepsilon([\iota_t^+],\psi_t) = -\varepsilon([\iota_t^-],\psi_t).$$

Therefore, $n_\beta(\psi)$ will be be invariant as t passes though 0 if we interpret # as as signed count, that is,

$$n_\beta(\psi) := \sum_{[\iota]\in\mathfrak{A}_\beta(\psi)} \varepsilon([\iota],\psi) \tag{2.33}$$

with some choice of ε satisfying (2.32). An almost canonical method for determining ε was recently discovered by Joyce [Joy17, Section 3]. We refer the reader to Joyce's article for a careful and detailed discussion.

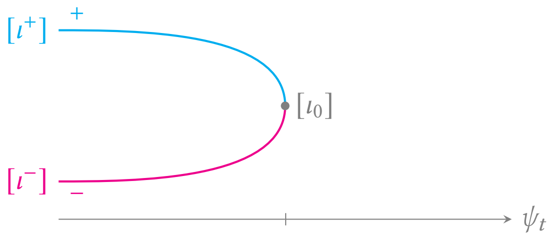

Figure 1: Two associatives submanifold with opposite signs annihilating in an obstructed associative submanifold.

3 Intersections, T^2–singularities, and the Seiberg–Witten invariant

In what follows we describe in more detail the transitions (1) and (2) from Section 1, and explain why they spoil the deformation invariance of $n_\beta(\psi)$. We then argue that the Seiberg–Witten invariant for 3–manifolds might play a role in repairing the deformation invariance.

3.1 Intersecting associative submanifolds

Let $(\psi_t)_{t\in(-T,T)}$ be a 1–parameter family of closed, tamed, definite 4–forms on Y and let $(\iota_t\colon P \to Y)_{t\in(-T,T)}$ be a 1–parameter family of somewhere injective unobstructed associative immersions. By Proposition 2.23, if (ψ_t) is generic, then we can assume that ι_t is an embedding for all $t \neq 0$ and ι_0 has a unique self-intersection as in Proposition 2.23(2b). This intersection is locally modeled on the intersection of two transverse associative subspaces of $\mathbf{R}^7$. Given any pair of transverse associative subspaces of $\mathbf{R}^7$, there is a smooth associative submanifold asymptotic to these subspaces at infinity, called the Lawlor neck. Nordström proved that out of the unique self-intersection of ι_0 a new 1–parameter family of associative submanifolds is created in Y by gluing in a Lawlor neck.

Theorem 3.1 (Nordström [Nor13]). *Let Y be a compact 7–manifold and let $(\psi_t)_{t\in(-T,T)}$ be a family of closed, definite 4–forms on Y. Let P be a compact, oriented 3–manifold. Suppose that $(\iota_t\colon P \to Y)_{t\in(-T,T)}$ is a 1–parameter family of unobstructed associative* *immersions such that*

$$\mathbf{P} := \bigcup_{t\in(-T,T)} \{t\} \times \iota_t(P) \subset \mathbf{R} \times Y$$

has a unique self-intersection which occurs for $t = 0$ and is transverse. Let $x^\pm$ denote the preimages in P of the intersection in Y and denote by $P^\sharp$ the connected sum of P at x^+ and x^-.

There is a constant $\varepsilon_0 > 0$, a continuous function $t\colon [0, \varepsilon_0] \to (-T, T)$, and a 1–parameter family of immersions $(\iota_\varepsilon^\sharp\colon P^\sharp \to Y)_{\varepsilon\in(0,\varepsilon_0]}$ such that, for each $\varepsilon \in (0, \varepsilon_0]$, $\iota_\varepsilon^\sharp$ is an unobstructed associative immersion with respect to $\psi_{t(\varepsilon)}$.

Remark 3.2. The paper [Nor13] has not yet been made available to a wider audience. A part of what goes into proving Theorem 3.1 can be found in [Joy17, Section 4.2]. There it is also argued that for a generic choice of $(\psi_t)_{t\in(-T,T)}$ the function t is expected to be of the form $t(\varepsilon) = \delta\varepsilon + O(\varepsilon^2)$ with a non-zero coefficient δ whose geometric meaning is also explained therein.

Remark 3.3. Denote by $P_1, \ldots, P_n$ the connected components of P. Let $j_\pm$ be such that $x_\pm \in P_{j_\pm}$. We have

$$P^\sharp \cong \begin{cases} \coprod_{j\neq j_\pm} P_j \sqcup (P_{j_+}\sharp P_{j_-}) & \text{for } j_+ \neq j_- \quad \text{and} \\ \coprod_{j\neq j_+} P_j \sqcup (P_{j_+}\sharp S^1 \times S^2) & \text{for } j_+ = j_-. \end{cases}$$

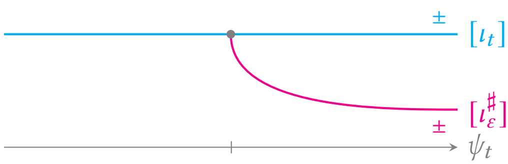

Figure 2: An associative being born out of an intersection another associative.

In the situation described in Theorem 3.1 and depicted in Figure 2, $n_\beta(\psi_t)$ as defined in (2.33) changes by ± 1 as t crosses 0. In particular, n_β is not invariant.

3.2 Associative submanifolds with T^2–singularities

Suppose that $\hat{P}$ is an associative submanifold in (Y, ψ_0) with a point singularity at $x \in \hat{P}$ modelled on the following cone over T^2:

$$\begin{aligned}\hat{L} &= \left\{(0, z_1, z_2, z_3) \in \mathbf{R} \oplus \mathbf{C}^3 : |z_1|^2 = |z_2|^2 = |z_3|^2, z_1z_2z_3 \in [0, \infty) \in \mathbf{C}\right\} \\ &= \left\{r \cdot (0, e^{i\theta_1}, e^{i\theta_2}, e^{-i\theta_1-i\theta_2}) : r \in [0, \infty), \theta_1, \theta_2 \in S^1\right\}.\end{aligned}$$

For a more formal discussion we refer the reader to Joyce [Joy17, Section 5.2], where, in particular, it is argued by analogy with the case of special Lagrangians that such singular associatives should be described by a Fredholm theory of index -1.

The singularity model $\hat{L}$ can be resolved in 3 ways:

$$\begin{aligned}
L^1_\lambda &= \{(0, z_1, z_2, z_3) \in \mathbf{R} \oplus \mathbf{C}^3 : |z_1|^2 - \lambda = |z_2|^2 = |z_3|^2, z_1z_2z_3 \in [0,\infty) \in \mathbf{C}\}, \\
L^2_\lambda &= \{(0, z_1, z_2, z_3) \in \mathbf{R} \oplus \mathbf{C}^3 : |z_1|^2 = |z_2|^2 - \lambda = |z_3|^2, z_1z_2z_3 \in [0,\infty) \in \mathbf{C}\}, \quad \text{and} \\
L^3_\lambda &= \{(0, z_1, z_2, z_3) \in \mathbf{R} \oplus \mathbf{C}^3 : |z_1|^2 = |z_2|^2 = |z_3|^2 - \lambda, z_1z_2z_3 \in [0,\infty) \in \mathbf{C}\}.
\end{aligned}$$

These are asymptotic to $\hat{L}$ at infinity and smooth, which can be seen by identifying L^i_λ with $S^1 \times \mathbf{C}$ via

$$\begin{aligned}
&S^1 \times \mathbf{C} \to L^1_\lambda, \quad (e^{i\theta}, z) \mapsto \left(0, e^{i\theta}\sqrt{|z|^2 + \lambda}, z, e^{-i\theta}\bar{z}\right), \\
&S^1 \times \mathbf{C} \to L^2_\lambda, \quad (e^{i\theta}, z) \mapsto \left(0, e^{-i\theta}\bar{z}, e^{i\theta}\sqrt{|z|^2 + \lambda}, z\right), \quad \text{and} \\
&S^1 \times \mathbf{C} \to L^3_\lambda, \quad (e^{i\theta}, z) \mapsto \left(0, z, e^{-i\theta}\bar{z}, e^{i\theta}\sqrt{|z|^2 + \lambda}\right).
\end{aligned} \tag{3.4}$$

Topologically, L^i_λ can be obtained from $\hat{L}$ via Dehn surgery.

Definition 3.5. Let P° be a 3–manifold with $\bar{\partial}P^\circ = T^2$. Let μ be a simple closed curve in T^2. The **Dehn filling** of P° along μ, denoted by P°_μ, is the 3–manifold obtained by attaching $S^1 \times D$ to P° in such a way that $\{*\} \times S^1$ is identified with μ.

Remark 3.6. Up to diffeomorphism P°_μ only depends on the homotopy class of $\mu \subset T^2$; moreover, it does not depend on the orientation of μ.

We can identify the boundary of $\hat{L}^\circ := \hat{L} \backslash B_1$ with T^2 via

$$(e^{i\theta_1}, e^{i\theta_2}) \mapsto \frac{1}{\sqrt{3}}\left(0, e^{i\theta_1}, e^{i\theta_2}, e^{-i\theta_1 - i\theta_2}\right).$$

Comparing the maps introduced in (3.4) restricted to $\{*\} \times S^1$ with the above identification, we see that L^i_λ is obtained by Dehn filling $\hat{L}^\circ$ along loops representing the homology classes

$$\mu_1 = (0,1), \quad \mu_2 = (-1,0), \quad \text{and} \quad \mu_3 = (1,-1) \tag{3.7}$$

where $(1,0)$ and $(0,1)$ are the generators of $H_1(T^2, \mathbf{Z})$ corresponding to the loops $\theta \mapsto (e^{i\theta}, 0)$ and $\theta \mapsto (0, e^{i\theta})$.

We expect that $\hat{P}$ can be resolved in three ways as well.

Conjecture 3.8 (cf. Joyce [Joy17, Conjecture 5.3]). *Let $(\psi_t)_{t\in(-T,T)}$ be 1–parameter family of closed, tamed, definite 4–forms on Y. Let $\hat{P}$ be an unobstructed singular associative submanifold in (Y,ψ_0) with a unique singularity at x which is modeled on $\hat{L}$. Associated to this data there are constants $\delta_1,\delta_2,\gamma\in\mathbf{R}$. For a generic 1–parameter family $(\psi_t)_{t\in(-T,T)}$, $\delta_1\neq 0$, $\delta_2\neq 0$, $\delta_1\neq\delta_2$ and $\gamma\neq 0$. If this holds, then there is a $\varepsilon_0>0$ and, for $i=1,2,3$, there are functions $t_i\colon [0,\varepsilon_0]\to(-T,T)$, compact, oriented 3–manifolds P^i, and 1–parameter families of immersions $(\iota^i_\varepsilon\colon P^i\to Y)_{\varepsilon\in(0,\varepsilon_0]}$ such that*

1. *ι^i_ε is an unobstructed associative immersion with respect to $\psi_{t_i(\varepsilon)}$,*
2. *$\iota^i_\varepsilon(P^i)$ is close to $\hat{P}$ away from x and close to L^i_ε near x,*
3. *P^i is diffeomorphic to the manifold obtained by Dehn filling $\hat{P}^\circ=\hat{P}\backslash B_\sigma(x)$ along μ_i where $\mu_i\in H_1(\partial\hat{P}^\circ)=H_1(T^2)$ is as in (3.7).*
4. *We have*

$$t_1(\varepsilon)=-\frac{\delta_2}{\gamma}\varepsilon+O(\varepsilon^2),\quad t_2(\varepsilon)=\frac{\delta_1}{\gamma}\varepsilon+O(\varepsilon^2),$$
$$\textit{and}\quad t_3(\varepsilon)=\frac{\delta_2-\delta_1}{\gamma}\varepsilon+O(\varepsilon^2).$$

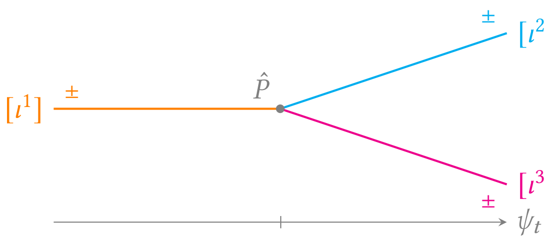

Figure 3: Three associatives emerging out of a singular associative for $\delta_2>\delta_1>0$.

In the situation described in Conjecture 3.8 and depicted in Figure 3, $n_\beta(\psi_t)$ as defined in (2.33) changes by ±1 as t crosses 0. Again, the occurrence of the phenomenon described above would preclude n_β from being an invariant.

3.3 The Seiberg–Witten invariant of 3–manifolds

If there were a topological invariant $w(P)\in\mathbf{Z}$ defined for every compact, oriented 3–manifold and satisfying

$$w(P_1\sharp P_2)=0\quad\text{and}$$
$$\varepsilon_1 w(P^\circ_{\mu_1})+\varepsilon_2 w(P^\circ_{\mu_2})+\varepsilon_3 w(P^\circ_{\mu_3})=0 \tag{3.9}$$

with μ_1, μ_2, μ_3 as in (3.7) and some choice of $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{\pm 1\}$, then

$$n_\beta(\psi) := \sum_{[\iota] \in \mathfrak{A}_\beta(\psi)} \varepsilon([\iota], \psi) w(P) \tag{3.10}$$

would be invariant along the transition discussed in Section 3.1 and also along the transition discussed in Section 3.2 provided the signs work out correctly.

It is easy to see that the only such invariant defined for all 3–manifolds is trivial since $w(P) = w(P \sharp S^3) = 0$ for all oriented 3–manifolds P. However, for those 3–manifolds P with $b_1(P_j) > 1$ for all connected components P_j non-trivial invariants satisfying (3.9) do exist. One example of such an invariant is the **Seiberg–Witten invariant** $\mathrm{SW}(P)$. We refer the reader to [MT96, Section 2] for a detailed discussion of the construction of $\mathrm{SW}(P)$. For the moment, it shall suffice to think of $\mathrm{SW}(P)$ as the signed count of all gauge-equivalence classes of solutions to the Seiberg–Witten equation; that is, pairs of $(\Psi, A) \in \Gamma(W) \times \mathscr{A}(\det(W))$ satisfying

$$\begin{aligned} \not{D}_A \Psi &= 0 \quad \text{and} \\ \frac{1}{2} F_A &= \mu(\Psi). \end{aligned} \tag{3.11}$$

Here W is the spinor bundle of a spin^c–structure $\mathfrak{w}$ on P, $\not{D}_A$ is the twisted Dirac operator, and $\mu(\Psi) = \Psi\Psi^* - \frac{1}{2}|\Psi|^2 \,\mathrm{id}_W$ is identified with an imaginary-valued 2–form using the Clifford multiplication.

Remark 3.12. The actual definition of $\mathrm{SW}(P)$ involves perturbing (3.11) by a closed 2–form η in order to ensure that the moduli space of solutions is cut-out transversely and contains no reducible solutions. The necessity to choose η and the fact that $H^2(P, \mathbf{Z})$ has codimension $b_1(P)$ in $H^2(P, \mathbf{R})$, where the cohomology class of η lies, is responsible for the restriction $b_1(P) > 1$.

Remark 3.13. $\mathrm{SW}(P)$ has a refinement $\underline{\mathrm{SW}}(P)$ defined for oriented 3–manifolds P with $b_1(P) > 0$; roughly speaking, it is an integer-valued function on the set of the isomorphism classes of spin^c–structures $\mathfrak{w}$ on P. When $b_1 > 1$, it is zero for all but finitely many $\mathfrak{w}$ and we can take $\mathrm{SW}(P)$ to be the sum of the invariants over all spin^c–structures. We will come back to this point in Section 9.2.

Theorem 3.14 (Meng and Taubes [MT96, Proposition 4.1]). *If P_1, P_2 are two compact, connected, oriented 3–manifolds with $b_1(P_i) \geq 1$, then*

$$\mathrm{SW}(P_1 \sharp P_2) = 0.$$

Theorem 3.15 (Meng and Taubes [MT96, Theorem 5.3]). *Let P° be a compact, connected, oriented 3–manifold with $\partial P^\circ = T^2$. If $\mu_1, \mu_2, \mu_3 \in H_1(\partial P^\circ)$ are such that*

$$\mu_1 \cdot \mu_2 = \mu_2 \cdot \mu_3 = \mu_3 \cdot \mu_1 = -1$$

(with $T^2 = \partial P^\circ$ oriented as the boundary of P°), then

$$\varepsilon_1 \cdot \mathrm{SW}(P^\circ_{\mu_1}) + \varepsilon_2 \cdot \mathrm{SW}(P^\circ_{\mu_2}) + \varepsilon_3 \cdot \mathrm{SW}(P^\circ_{\mu_3}) = 0$$

for suitable choices of $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{\pm 1\}$, provided $b_1(P^\circ_{\mu_i}) > 1$ for all $i = 1, 2, 3$.

Remark 3.16. The formulation of [MT96, Theorem 5.3] is in terms of p/q–surgery on a link L which is rationally trivial in homology. The discussion in [KM07, Section 42.1] explains how this is related to Dehn filling, and from this it is clear that the surgery formula given by Meng and Taubes implies the above theorem.

Remark 3.17. The Seiberg–Witten invariant is often defined only for compact, connected, oriented 3–manifolds P. If P has connected components $P_1, \dots, P_m$, then $\mathrm{SW}(P) = \prod_{j=1}^m \mathrm{SW}(P_j)$.

Let us temporarily assume that all associative immersions $\iota\colon P \to Y$ with $\iota_*[P] = \beta$ happen to be such that all connected components P_j satisfy $b_1(P_j) > 1$. If we defined n_β by (3.10) with the weight $w = \mathrm{SW}$, then n_β would be invariant in the situations described in Section 3.1 and Section 3.2, at least if the signs work out correctly, or modulo 2. Defining n_β in this way really amounts to counting a much larger moduli space than $\mathfrak{A}_\beta(\psi)$, namely:

$$\mathfrak{A}^{\mathrm{SW}}_\beta(\psi) = \coprod_P \coprod_{\mathfrak{w}} \mathfrak{A}^{\mathrm{SW}}_{P,\beta,\mathfrak{w}}(\psi)$$

with

$$\mathfrak{A}^{\mathrm{SW}}_{P,\beta,\mathfrak{w}}(\psi) := \frac{\left\{ (\iota, \Psi, A) \in \mathrm{Imm}_\beta(P, Y) \times \Gamma(W) \times \mathscr{A}(\det W) : \begin{array}{l} \iota \text{ satisfies (2.9) and} \\ (\Psi, A) \text{ satisfies (3.11)} \\ \text{with respect to } \iota^* g_\psi \end{array} \right\}}{\mathrm{Diff}_+(P) \ltimes C^\infty(P, \mathrm{U}(1))}.$$

Here $\mathfrak{w}$ ranges over all isomorphism classes of spin^c–structures on P and W denotes the spinor bundle. The non-invariance of n_β as defined in (2.33) can be traced back to the completion of $\mathfrak{A}_\beta(\{\psi_t\})$ not being a 1–manifold. The moduli space $\mathfrak{A}^{\mathrm{SW}}_\beta(\{\psi_t\})$ smooths out the singularities in the completion of $\mathfrak{A}_\beta(\{\psi_t\})$ encountered in the situations described in Section 3.1 and Section 3.2; see Figure 4.

The issue with defining a topological invariant $w(P) \in \mathbf{Z}$ with the properties described in (3.9) means that there is indeed no invariant $n_\beta(\psi) \in \mathbf{Z}$ defined by a formula of the form (3.10). That is unless all associatives with $\iota_*[P] = \beta$ happen to be such that all connected components P_j satisfy $b_1(P_j) > 1$, but we do *not* believe this holds for any reasonable class of closed, tamed, definite 4–forms ψ or choice of β. (The situation is somewhat better for associatives arising from holomorphic curves in Calabi–Yau 3–folds. We will discuss this case in Section 9.)

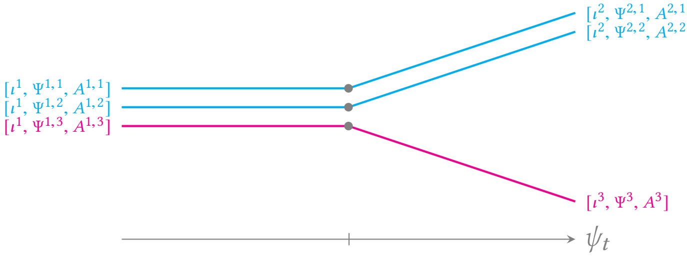

Figure 4: An example of how counting with Seiberg–Witten solutions can smooth out the situation depicted in Figure 3.

3.4 Putative Floer theory

Although there is no topological invariant $w(P) \in \mathbf{Z}$, defined for all closed, oriented 3–manifolds, satisfying the properties described in (3.9), there are Seiberg–Witten–Floer homology theories satisfying analogues of (3.9), see Marcolli and Wang [MW01], Manolescu [Man03], Kronheimer and Mrowka [KM07], Frøyshov [Frø10]. We focus on one of the variants defined by Kronheimer and Mrowka. To each closed, oriented 3–manifold P they assign a homology group

$$\widehat{\mathrm{HM}}(P) = H\big(\widehat{\mathrm{CM}}(P, \clubsuit), \hat{\partial}\big).$$

Very roughly, the chain complexes $\widehat{\mathrm{CM}}(P, \clubsuit)$ are the $C^\infty(P, \mathrm{U}(1))$–equivariant Morse complexes of the **Chern–Simons–Dirac functional** $\mathrm{CSD}\colon\ \Gamma(W) \times \mathscr{A}(\det W) \to R$ defined by

$$\mathrm{CSD}(\Psi, A) = \frac{1}{2}\int_P (A - A_0) \wedge F_A + \int_P \langle \not{D}_A \Psi, \Psi \rangle \, \mathrm{vol}. \tag{3.18}$$

on the configuration space

$$\mathscr{C}(P) = \coprod_{\mathfrak{w}} \mathscr{C}(P, \mathfrak{w}) \quad \text{with} \quad \mathscr{C}(P, \mathfrak{w}) = \Gamma(W) \otimes \mathscr{A}(\det W)$$

(The fact that $C^\infty(P, \mathrm{U}(1))$ does not act freely is a significant problem, which Kronheimer and Mrowka resolve by blowing up $\mathscr{C}(P)$ to a manifold with boundary and defining corresponding Morse complexes adapted to this situation.) The chain complexes $\widehat{\mathrm{CM}}(P, \clubsuit)$ depend on choices of additional data $\clubsuit$, in particular, a Riemannian metric on P as well as the choice of a suitable perturbation of the equation). Different choices of $\clubsuit$, however, lead to quasi-isomorphic chain complexes. We denote by $\widehat{\mathrm{CM}}(P)$ quasi-isomorphism class of $\widehat{\mathrm{CM}}(P, \clubsuit)$, or rather its isomorphism class in the derived category of chain complexes. If Q is a 4–dimensional cobordism with $\partial Q = P_1 - P_2$, then Kronheimer and Mrowka define an induced chain map

$$\widehat{\mathrm{CM}}(Q)\colon\ \widehat{\mathrm{CM}}(P_1) \to \widehat{\mathrm{CM}}(P_2).$$

If $Q = [0,1]\times P$, then $\widehat{\mathrm{CM}}(Q)$ is simply the differential $\hat\partial$ on $\widehat{\mathrm{CM}}(P)$. The construction of $\widehat{\mathrm{HM}}$ involves the choice of coefficients. For the upcoming results to hold on needs to work with $\mathbf{Z}_2$ coefficients (or suitable local systems). The monopole homology groups are then $\mathbf{Z}_2[\![U]\!]$–modules. Here one should think U as the same U as in $H^\bullet(BU(1)) = \mathbf{Z}[U]$.

The following results are the analogues of the vanishing theorem Theorem 3.14 and the surgery formula Theorem 3.15.

Theorem 3.19 (Bloom, Mrowka, and Ozsváth [BMO]; Lin [Lin15, Theorem 5]). *Let P^+ and P^- be a two compact, connected, oriented 3–manifolds. Denote by $P^+\sharp P^-$ their connected sum and by Q the surgery cobordism from $P^+ \sqcup P^-$ to $P^+\sharp P^-$. There is an exact triangle*[4]

$$\widehat{\mathrm{CM}}(P^+ \sqcup P^-) \xrightarrow{\widehat{\mathrm{CM}}(Q)} \widehat{\mathrm{CM}}(P^+\sharp P^-) \to \widehat{\mathrm{CM}}(P^+ \sqcup P^-) \to \widehat{\mathrm{CM}}(P^+ \sqcup P^-)[-1];$$

in particular,

$$\widehat{\mathrm{HM}}(P^+ \sqcup P^-)) \cong H\big(\mathrm{cone}\big(\widehat{\mathrm{CM}}(P^+ \sqcup P^-) \xrightarrow{\widehat{\mathrm{CM}}(Q)} \widehat{\mathrm{CM}}(P^+\sharp P^-)\big)\big). \tag{3.20}$$

Remark 3.21. In [Lin15, Theorem 5], Theorem 3.19 is stated and proved as

$$\widehat{\mathrm{HM}}(P^+\sharp P^-) \cong H\big(\mathrm{cone}\big(\widehat{\mathrm{CM}}(P^+) \otimes \widehat{\mathrm{CM}}(P^-)[1] \xrightarrow{\mathrm{id}\otimes U + U\otimes \mathrm{id}} \widehat{\mathrm{CM}}(P^+) \otimes \widehat{\mathrm{CM}}(P^-)\big)\big)$$

with the isomorphism induced by Q. This formulation is much more useful for actual computations of $\widehat{\mathrm{HM}}(P^+\sharp P^-)$, but we need (3.20) for our purposes. The equivalence of these statements follows by observing that once we identify

$$\widehat{\mathrm{CM}}(P^+ \sqcup P^-) = \widehat{\mathrm{CM}}(P^+) \otimes \widehat{\mathrm{CM}}(P^-)$$

the map $\widehat{\mathrm{CM}}(P^+ \sqcup P^-) \to \widehat{\mathrm{CM}}(P^+ \sqcup P^-)[-1]$ is given by $\mathrm{id} \otimes U + U \otimes \mathrm{id}$ and rotating the above exact triangle.

Remark 3.22. More generally, if $P^\sharp$ is obtained by performing a connect sum at two points $x^\pm$ in P and Q denotes the surgery cobordism from P to $P^\sharp$, then we expect there to be an exact triangle

$$\widehat{\mathrm{CM}}(P) \xrightarrow{\widehat{\mathrm{CM}}(Q)} \widehat{\mathrm{CM}}(P^\sharp) \to \widehat{\mathrm{CM}}(P) \to \widehat{\mathrm{CM}}(P)[-1].$$

Theorem 3.19 asserts that this is holds if the points $x^\pm$ lie in different connected components of P.

[4]We use square brackets to denote the translation $C[p]_n = C_{p+n}$, see [Wei94, Translation 1.2.8].

Theorem 3.23 (Kronheimer, Mrowka, Ozsváth, and Szabó [KMOS07, Theorem 2.4]; see also [KM07, Theorem 42.2.1]). *Let P° be a compact, connected, oriented 3–manifold with $\partial P^\circ = T^2$. Let $\mu_1, \mu_2, \mu_3 \in H_1(\partial P^\circ)$ be such that*

$$\mu_1 \cdot \mu_2 = \mu_2 \cdot \mu_3 = \mu_3 \cdot \mu_1 = -1$$

(with $T^2 = \partial P^\circ$ oriented as the boundary of P°.) Denote by Q_{ij} the surgery cobordism from $P^\circ_{\mu_i}$ to $P^\circ_{\mu_j}$. There is an exact triangle

$$\widehat{\mathrm{CM}}(P^\circ_{\mu_2}) \xrightarrow{\widehat{\mathrm{CM}}(Q_{23})} \widehat{\mathrm{CM}}(P^\circ_{\mu_3}) \to \widehat{\mathrm{CM}}(P^\circ_{\mu_1}) \to \widehat{\mathrm{CM}}(P^\circ_{\mu_2})[-1];$$

in particular,

$$\widehat{\mathrm{HM}}(P^\circ_{\mu_1}) \cong H\big(\mathrm{cone}\big(\widehat{\mathrm{CM}}(P^\circ_{\mu_2}) \xrightarrow{\widehat{\mathrm{CM}}(Q_{23})} \widehat{\mathrm{CM}}(P^\circ_{\mu_3})\big)\big). \tag{3.24}$$

Remark 3.25. While Theorem 3.23 holds for all three version of monopole homology defined by Kronheimer and Mrowka, Theorem 3.19 only holds form $\widehat{\mathrm{HM}}$; see [Lin15, paragraph after Equation (13)]. This is why we restricted to this version from the outset.

Associative submanifolds are critical points of the functional $\mathfrak{L}$ defined in Proposition 2.16. Gradient flow lines of the functional $\mathfrak{L}$ can naturally be identified with immersions $\boldsymbol{\iota}\colon \mathbf{R} \times P \to \mathbf{R} \times Y$ such that

$$\boldsymbol{\iota}^*(\psi + \mathrm{d}t \wedge \phi) = \mathrm{vol}_{\boldsymbol{\iota}^* g}$$

and $\pi_{\mathbf{R}} \circ \boldsymbol{\iota}(t, x) = t$; see, e.g., [SW17, Lemma 12.6].

Definition 3.26. Let $\iota^\pm\colon P^\pm \to Y$ be associative immersions with respect to ψ. A **Cayley cobordism** in $\mathbf{R} \times Y$ from ι_- to ι_+ is an oriented 4–manifold Q together with an immersion $\boldsymbol{\iota}\colon Q \to \mathbf{R} \times Y$ such that

$$\boldsymbol{\iota}^*(\psi + \mathrm{d}t \wedge \phi) = \mathrm{vol}_{\boldsymbol{\iota}^* g}$$

and there are two open subsets $U_\pm \subset Q$ such that $Q \backslash (U_+ \cup U_-)$ is compact, constants $T_\pm, c > 0$, and diffeomorphisms $\phi_+\colon (T_+, \infty) \times P^+ \to U_+$ and $\phi_-\colon (-\infty, T_-) \times P^- \to U_-$ such that

$$\mathrm{dist}(\boldsymbol{\iota} \circ \phi_\pm(t, x), (t, \iota^\pm(x))) = O(e^{-c|t|}) \quad \text{as } t \to \pm\infty.$$

The **truncation** of a Cayley cobordism is (the diffeomorphism type of)

$$\bar{Q} := Q \backslash (\phi_-(-\infty, T_- - 1) \cup \phi_+(T_+ + 1, \infty))$$

Question 3.27. In the situation of Theorem 3.1, does there exist a Cayley cobordism $\boldsymbol{\iota}\colon Q \to \mathbf{R} \times Y$ from $\iota_{t(\varepsilon)}$ to $\iota_\varepsilon^\sharp$, for all $\varepsilon \in (0, \varepsilon_0)$, whose truncation $\bar{Q}$ is the surgery cobordism from P to $P^\sharp$?

Question 3.28. In the situation of Conjecture 3.8, if $\delta_2 > \delta_1 > 0$, does there exist a Cayley cobordism $\boldsymbol{\iota}\colon Q \to \mathbf{R} \times Y$ from ι_t^2 to ι_t^3 with $\bar{Q}$ being the surgery cobordism from $P^\circ_{\mu_2}$ to $P^\circ_{\mu_3}$ for each $t \in (0, T)$? (Similarly for the cases $\delta_1 > \delta_2 > 0$, $\delta_2 < \delta_1 < 0$, and $\delta_1 < \delta_2 < 0$.)

We *hope* that the answer to these questions is yes. For the sake of argument, let us assume that this is indeed the case. Define

$$\mathrm{CMA}_\beta(\psi) := \bigoplus_P \bigoplus_{[\iota] \in \mathfrak{A}_{P,\beta}(\psi)} \mathrm{CMA}_{\beta,[\iota]}(\psi) \quad \text{with} \quad \mathrm{CMA}_{\beta,[\iota]}(\psi) := \widehat{\mathrm{CM}}(P) \tag{3.29}$$

and define a differential on $\mathrm{CMA}_\beta(\psi)$ by declaring

$$\big(\partial\colon \mathrm{CMA}_{\beta,[\iota_-]}(\psi) \to \mathrm{CMA}_{\beta,[\iota_+]}(\psi)\big) := \sum_{[\boldsymbol{\iota}]} \widehat{\mathrm{CM}}(\bar{Q})$$

where $[\boldsymbol{\iota}\colon Q \to \mathbf{R} \times Y]$ ranges over all equivalence classes of Cayley cobordisms from $[\iota_-]$ to $[\iota_+]$.

Since $\widehat{\mathrm{CM}}([0,1] \times P)$ is just the differential $\hat{\partial}$ on $\widehat{\mathrm{CM}}(P)$, in the situation of Theorem 3.1 with $\delta > 0$ as in Remark 3.2 (and assuming that there no other Cayley cobordism involving $[\iota_t]$ or $[\iota_t^\sharp]$), for $t < 0$, the chain complex $\mathrm{CMA}_\beta(\psi_t)$ contains the contribution

$$\mathrm{CMA}_\beta^\times(\psi_t) = \widehat{\mathrm{CM}}(P) \quad \text{with} \quad \partial = \hat{\partial};$$

for $t > 0$ this changes to

$$\mathrm{CMA}_\beta^\times(\psi_t) = \widehat{\mathrm{CM}}(P) \oplus \widehat{\mathrm{CM}}(P^\sharp) \quad \text{with} \quad \partial = \begin{pmatrix} \hat{\partial} & 0 \\ \widehat{\mathrm{CM}}(Q) & \hat{\partial} \end{pmatrix}$$

with Q the surgery cobordism from P to $P^\sharp$. The latter is simply the mapping cone

$$\operatorname{cone}\big(\widehat{\mathrm{CM}}(P) \xrightarrow{\widehat{\mathrm{CM}}(Q)} \widehat{\mathrm{CM}}(P^\sharp)\big).$$

Therefore, it follows from Theorem 3.19, that the homology group

$$H(\mathrm{CMA}_\beta^\times(\psi_t), \partial)$$

does not change as t passes through zero. Similarly, in the situation of Conjecture 3.8, Theorem 3.23 the relevant contribution to $H(\mathrm{CMA}_\beta(\psi_t), \partial)$ does not change as t passes through zero.

3.5 A coupled Chern–Simons–Dirac functional

An immersion $\iota \in \mathrm{Imm}_\beta(P, Y)$ is associative if and only if it is a critical point of $\mathfrak{L}$. Likewise, a pair $(\Psi, A) \in \Gamma(W) \times \mathscr{A}(\det W)$ is a solution of the Seiberg–Witten equation (3.11) if and only if it is a critical point of the Chern–Simons–Dirac functional CSD. The coupled condition that ι is associative and (Ψ, A) satisfies the Seiberg–Witten equation with respect to $\iota^* g$ is not equivalent to (ι, Ψ, A) being a critical point of any naturally defined functional. However, it does appear as the decoupling limit of a family of critical point equations.

Definition 3.30. Given a **coupling constant** $\kappa > 0$, we define the **$\mathfrak{L}$–Chern–Simons–Dirac functional** $\mathfrak{L}\mathrm{CSD}_\kappa \colon \widetilde{\mathrm{Imm}}_\beta(P, Y) \times \Gamma(W) \times \mathscr{A}(\det W) \to \mathbf{R}$ by

$$\mathfrak{L}\mathrm{CSD}_\kappa(\iota, [Q], \Psi, A) := \int_Q \psi + \kappa \left(\frac{1}{2} \int_P (A - A_0) \wedge F_A + \int_P \langle \not{D}_{A, \iota^* g} \Psi, \Psi \rangle \mathrm{vol}_{\iota^* g} \right).$$

Proposition 3.31. *Given $\kappa > 0$, $(\iota, [Q], \Psi, A)$ is a critical point of $\mathfrak{L}CSD_\kappa$ if and only if the pair (Ψ, A) satisfies the Seiberg–Witten equation on P with respect to the Riemannian metric $\iota^* g$:*

$$\begin{aligned} \not{D}_{A, \iota^* g} \Psi &= 0 \quad \textit{and} \\ \frac{1}{2} F_A &= \mu_{\iota^* g}(\Psi), \end{aligned} \tag{3.32}$$

and for every positive orthonormal basis (e_1, e_2, e_3) and $n \in N_x \iota$,

$$\langle [\iota_* e_1, \iota_* e_2, \iota_* e_3], n \rangle = \frac{\kappa}{2} \sum_{i=1}^{3} \langle \gamma(e_i) \nabla_{A, \mathrm{II}_\iota(n) e_i} \Psi, \Psi \rangle. \tag{3.33}$$

Here γ denotes the Clifford multiplication and $\mathrm{II}_\iota \in \Gamma(S^2 T^* P \otimes N\iota) \subset \Gamma(N\iota^* \otimes \mathrm{End}(TP))$ the second fundamental form of the immersion ι.

Remark 3.34. If $\kappa = 0$, then (3.33) is simply the condition that ι is associative.

Proof of Proposition 3.31. Let $(\iota, [Q], \Psi, A) \in \widetilde{\mathrm{Imm}}_\beta(P, Y) \times \Gamma(W) \times \mathscr{A}(\det W)$. Denote by $H_\iota = \mathrm{tr}\, \mathrm{II}_\iota \in \Gamma(N\iota)$ the mean curvature of ι. We define the **stress-energy tensor** $T_{\Psi, \iota^* g} \in \Gamma(S^2 T^* P)$ of Ψ with respect to $\iota^* g$ by

$$T_{\Psi, \iota^* g}(v, w) := \langle \gamma(v) \nabla_{A, w} \Psi + \gamma(w) \nabla_{A, v} \Psi, \Psi \rangle.$$

Using the variation formula for the Dirac operator, cf. Bourguignon and Gauduchon

[BG92, Thèorème 21] and Branding [Bra16, Lemma 2.3], we obtain

$$
\begin{aligned}
&\mathrm{d}_{(\iota,[Q],\Psi,A)}\mathfrak{L}\mathrm{CSD}_\kappa(n,\hat{\psi},a) \\
&\quad = \int_P \iota^* i(n)\hat{\psi} - \frac{\kappa}{4}\int_P \langle T_{\Psi,\iota^*g}, \mathrm{II}_\iota(n)\rangle \,\mathrm{vol}_{\iota^*g} + \frac{\kappa}{2}\int_P \langle H_\iota, n\rangle\langle \not{D}_{A,\iota^*g}\Psi,\Psi\rangle\,\mathrm{vol}_{\iota^*g} \\
&\qquad + \kappa\left(\int_P a\wedge F_A + \langle\bar{\gamma}(a)\Psi,\Psi\rangle + 2\langle\not{D}_{A,\iota^*g}\Psi,\psi\rangle\right);
\end{aligned}
$$

If $(\iota,[Q],\Psi,A)$ is a critical point of $\mathfrak{L}\mathrm{CSD}_\kappa$, then setting $n=0$ and varying $\hat{\psi}$ and a shows that (Ψ,A) satisfies the Seiberg–Witten equation with respect to ι^*g. In particular, $\not{D}_{A,\iota^*}\Psi = 0$ and the third term vanishes. The resulting condition on ι is

$$
\langle[\iota_*e_1,\iota_*e_2,\iota_*e_3],n\rangle = \frac{\kappa}{2}\sum_{i=1}^{3}\langle\gamma(e_i)\nabla_{A,\mathrm{II}_\iota(n)e_i}\Psi,\Psi\rangle. \qquad \square
$$

Equation (3.33) has some similarities with Dirac-harmonic maps introduced by Chen, Jost, Li, and Wang [CJLW04]. It is an interesting question whether the more strongly coupled equations (3.33) and (3.32) for $\kappa > 0$ instead $\kappa = 0$ might have a regularizing effect. In particular, it would be interesting to understand what happens to the singularities encountered in Section 3.1 and Section 3.2 when κ is deformed from $\kappa = 0$ to $\kappa > 0$.

4 Multiple covers of associative submanifolds

4.1 Collapsing of immersions of multiple covers

A further problem with counting associatives arises from multiple covers. To see this, consider the following situation. Let $\iota_0\colon P \to Y$ be an associative immersion with $(\iota_0)_*[P] = \beta \in H_3(Y)$ with respect to $\psi_0 \in \mathscr{D}^4_c(Y)$. Let $\pi\colon \tilde{P} \to P$ be an orientation preserving k–fold unbranched normal cover with deck transformation group $\mathrm{Aut}(\pi)$. The composition

$$
\kappa_0 \coloneqq \iota_0\circ\pi\colon \tilde{P} \to Y
$$

is an associative immersion with

$$
(\kappa_0)_*[\tilde{P}] = k\cdot\beta \quad\text{and}\quad \mathrm{Aut}(\pi)\subset\mathrm{Aut}(\kappa_0).
$$

Suppose that $[\iota_0]$ is unobstructed but

$$
\ker F_{\kappa_0} = \mathbf{R}\langle n\rangle \subset \Gamma(N\kappa_0).
$$

We expect that this situation can arise along generic paths $(\psi_t)_{t\in(-T,T)}$ in $\mathscr{D}^4_c(Y)$. A neighborhood of $([\kappa_0],\psi_0)$ in the 1–parameter family of moduli spaces $\bigcup_t \mathfrak{M}_{k\cdot\beta}(\psi_t)$ can be analyzed using Theorem 2.20.

The stabilizer of κ_0 plays an important role in this analysis. Since $\mathrm{Aut}(\kappa_0)$ acts on $N\kappa_0$ and F_{κ_0} is $\mathrm{Aut}(\kappa_0)$–equivariant, $\mathrm{Aut}(\kappa_0)$ acts on $\ker F_{\kappa_0}$. This yields a homomorphism $\mathrm{sign}\colon \mathrm{Aut}(\kappa_0) \to \{\pm 1\}$ such that

$$f \cdot n = \mathrm{sign}(f)n \tag{4.1}$$

for all $f \in \mathrm{Aut}(\kappa_0)$. The homomorphism sign must be non-trivial, for otherwise n would be $\mathrm{Aut}(\pi)$–invariant and descend to a non-trivial element of $\ker F_{\iota_0}$.

To summarize, $\kappa_0\colon P \to Y$ is an associative immersion with respect to $\psi_0 \in \mathscr{D}^4_c(Y)$ such that:

1. $\mathrm{Aut}(\kappa_0)$ is non-trivial,
2. $[\kappa_0]$ is obstructed; more precisely: $\ker F_{\kappa_0} = \mathbf{R}\langle n\rangle$, and
3. the homomorphism $\mathrm{sign}\colon \mathrm{Aut}(\kappa_0) \to \{\pm 1\}$ defined by (4.1) is non-trivial.

In this situation, if $(\psi_t)_{t\in(-T,T)}$ is generic, then the obstruction map ob from Theorem 2.20, whose zero set models a neighborhood of $([\kappa_0],\psi_0)$ in $\bigcup_t \mathfrak{M}_{k\cdot\beta}(\psi_t)$, will be of the form

$$\mathrm{ob}(t,\delta) = \lambda t\delta + c\delta^3 + \text{higher order terms}.$$

We can assume that $\lambda = c = 1$. Ignoring the higher order terms, $\mathrm{ob}^{-1}(0)$ consists of the line $\{\delta = 0\}$ and the parabola $\{t + \delta^2 = 0\}$. Since $[\iota_0]$ is unobstructed, for each $|t| \ll 1$, there is an associative immersion $\iota_t\colon P \to Y$ with respect to ψ_t near ι_0. The line $\{\delta = 0\}$ corresponds to the unobstructed associative immersions $[\kappa_t] \coloneqq [\iota_t \circ \pi]$ for $|t| \ll 1$. By Theorem 2.20, for each $-1 \ll t < 0$ there are also associative immersions $[\kappa_t^\pm : \tilde{P} \to Y]$ with respect to ψ_t near $[\kappa_0]$; these correspond to the two branches of the parabola $\{t + \delta^2 = 0\}$. As t tends to 0, $[\kappa_t^\pm]$ tends to $[\kappa_0]$; and $\mathrm{Aut}(\kappa_t^\pm)$ is the stabilizer of n in $\mathrm{Aut}(\kappa_0)$. Since $\mathrm{sign}\colon \mathrm{Aut}(\kappa_0) \to \{\pm 1\}$ is non-trivial, there is an $f \in \mathrm{Aut}(\kappa_0)$ such that

$$f_* n = -n$$

and, therefore, κ_τ^+ and κ_τ^- differ by a diffeomorphism of $\tilde{P}$ and give rise to the same element in the moduli space of associatives:

$$[\tilde{\iota}_t] \coloneqq [\kappa_t^+] = [\kappa_t^-].$$

Thus, the neighborhood $\mathrm{ob}^{-1}(0)/\mathrm{Aut}(\kappa_0)$ of $([\kappa_0],\psi_0)$ in $\bigcup_t \mathfrak{M}_{k\cdot\beta}(\psi_t)$ is homeomorphic to the figure depicted in Figure 5. Consequently, $n_{k\cdot\beta}(\psi_t)$ as in (3.10) with the weight $w = \mathrm{SW}$ changes by $\pm\mathrm{SW}(\tilde{P})$ as t crosses zero. Similarly, if one were to adopt the approach described in Section 3.4, part of the chain complex $\mathrm{CMA}_{k\cdot\beta}(\psi_t)$ would disappear as t crosses zero.

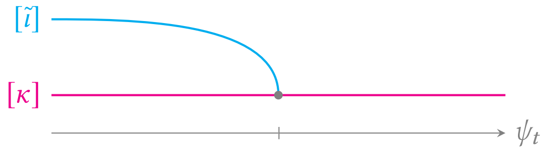

Figure 5: An family of associative immersions collapsing to a multiple cover.

4.2 Counting orbifolds points

The standard way to deal with the issue of multiple covers is to count the immersions $[\kappa]$ and $[\tilde{\iota}]$ described before as orbifold points in the moduli space; that is, to define

$$n_\beta(\psi) := \sum_{[\iota]\in\mathfrak{M}_\beta(\psi)} \frac{\varepsilon([\iota],\psi)w(P)}{|\mathrm{Aut}(\iota)|}. \tag{4.2}$$

Since $[\kappa_0]$ is obstructed, more precisely, since the Fueter operator associated with κ_0 has a 1–dimensional kernel, equation (2.32) implies that the sign $\varepsilon([\kappa_t],\psi_t) \in \{\pm 1\}$ flips as t passes through 0. Moreover,

$$\mathrm{Aut}(\tilde{\iota}) = \ker \mathrm{sign} \subset \mathrm{Aut}(\kappa),$$

where $\mathrm{sign}\colon\ \mathrm{Aut}(\kappa_0) \longrightarrow \{\pm 1\}$ is the homomorphism introduced above, and thus

$$|\mathrm{Aut}(\kappa)| = 2 \cdot |\mathrm{Aut}(\tilde{\iota})|.$$

Consequently, for $0 < t \ll 1$, we have

$$\frac{\varepsilon([\kappa_{-t}],\psi_{-t})w(\tilde{P})}{|\mathrm{Aut}(\kappa_{-t})|} + \frac{\varepsilon([\tilde{\iota}_{-t}],\psi_{-t})w(\tilde{P})}{|\mathrm{Aut}(\tilde{\iota})|} = \frac{\varepsilon([\kappa_{+t}],\psi_{+t})w(\tilde{P})}{|\mathrm{Aut}(\kappa_{+t})|} \in \mathbf{Q}.$$

This attempt works well for unbranched covers, but we believe that similar situations can occur with branched covers $\pi\colon\ \tilde{P} \longrightarrow P$. If π is a branched cover (with non-empty branching locus), then $\kappa := \iota \circ \pi$ is not an immersion and thus the theory from Section 2 does not apply. What exactly replaces this theory is unclear to us; the work of Smith [Smi11] might be a starting point. Nevertheless, one would need to count $[\kappa]$ to be able to compensate the jump. The crucial point is that, for any given 3–manifold P and $k \in \mathbf{N}$, infinitely many diffeomorphism types of 3–manifolds might be realized as k–fold branched covers of P. This is illustrated by the following result.

Theorem 4.3 (Hilden [Hil74; Hil76] and Montesinos [Mon74]). *Every compact, connected, orientable 3–manifold is a 3–fold branched cover of S^3.*

Therefore, if $\iota\colon S^3 \to Y$ is an associative immersion in (Y,ψ), then, for every compact, connected, oriented 3–manifold $\tilde{P}$, there is a 3–fold branched cover $\pi\colon \tilde{P} \to P$, and $[\iota \circ \pi]$ would have to contribute to (4.2). This leads to an infinite contribution from branched covers.

4.3 Counting embeddings with multiplicty

We believe that the origin of the problem is that all the associative submanifolds $[\iota \circ \pi]$ represent the same geometric object, namely, "k times $\mathrm{im}(\iota)$". Instead of trying to count immersions and their compositions with branched covers with weights, we should count embeddings with multiplicity, and embeddings with with multiplicity one should be weighted by the Seiberg–Wittten invariant, as in Section 3.3 or Section 3.4. In what follows we briefly outline an approach for defining the weights with which to count embeddings with multiplicity k larger than one. A more detailed discussion will be given over the next four sections.

Remark 4.4. Our approach should be compared with holomorphic curve counting via Donaldson–Thomas/Pandharipande–Thomas theory in algebraic geometry where one counts embedded subschemes, including contributions from thickened subschemes, rather than images of maps. We will elaborate on the relationship of this approach with Pandharipande–Thomas theory in Section 9.

To set the stage, let us go back to the situation described at the beginning of this section; that is, we have an unobstructed associative embedding $\iota\colon P \to Y$ and an orientation preserving k–fold unbranched cover $\pi\colon \tilde{P} \to P$ such that

$$\kappa \coloneqq \iota \circ \pi\colon \tilde{P} \to Y$$

is an obstructed associative immersion with $\dim\ker F_\kappa = 1$. Denote by $\tilde{\iota}\colon \tilde{P} \to Y$ the associative immersion which is the deformation of κ that does not come from deforming ι. (For simplicity's sake, we dropped the subscripts t from the notation.) Consider the bundle of stratified spaces

$$\mathrm{Sym}^k N\iota \coloneqq \mathrm{SO}(N\iota) \times_{\mathrm{SO}(4)} \mathrm{Sym}^k \mathbf{H} = (N\iota)^{\times k}/S_k.$$

Here $\mathbf{H} = \mathbf{R}^4$ is the space of quaternions and S_k is the symmetric group on k elements. To every normal vector field $n \in \Gamma(N\kappa)$ we assign a corresponding section $\tilde{n} \in \Gamma(\mathrm{Sym}^k N\iota)$ defined by

$$\tilde{n}(x) \coloneqq [n(\tilde{x}_1), \dots, n(\tilde{x}_k)]$$

with $\tilde{x}_1, \dots, \tilde{x}_k$ denoting the preimages of x with multiplicity. Given such a section $\tilde{n} \in \Gamma(\mathrm{Sym}^k N\iota)$, set

$$P_{\tilde{n}} \coloneqq \{(x, v) \in N\iota : v \in \tilde{n}(x)\}.$$

If $n \in \Gamma(N\kappa)$ is a normal vector field spanning $\ker F_\kappa$, then $P_{\tilde{n}}$ is a model for $\mathrm{im}(\tilde{\iota})$. In particular, $\mathrm{im}(\tilde{\iota})$ and $P_{\tilde{n}}$ are diffeomorphic in case they are smooth, which we conjecture be true generically if π is unbranched.

We can decompose $\mathrm{im}(\tilde{\iota})$ into components $P^1, \ldots, P^m$ such that P^j is an ℓ_j–fold cover of P and, for each $\tilde{x} \in P^j$ corresponding to $(x, v) \in P_{\tilde{n}}$, v appears in $\tilde{n}(x)$ with multiplicity k_j. Geometrically, $[\tilde{\iota}]$ represents

$$k_1 \cdot P^1 + \cdots + k_m \cdot P^m. \tag{4.5}$$

Clearly, we have

$$\sum_{j=1}^{m} \ell_j k_j = k. \tag{4.6}$$

Henceforth, let us assume that $\mathrm{im}(\tilde{\iota})$ is smooth. In the simplest case, we have $m = 1$ and $k_1 = k$. A moment's thought shows that in this case $\tilde{n}$ is a section of

$$\mathrm{Sym}^k_{\mathrm{reg}} N\iota := \Big\{(x, [v_1, \ldots, v_k]) \in \mathrm{Sym}^k N\iota : v_1, \ldots, v_k \text{ are pairwise distinct}\Big\},$$

the top stratum of $\mathrm{Sym}^k N\iota$. In general, $\tilde{n}$ will be a section of a stratum

$$\mathrm{Sym}^k_\lambda N\iota \subset \mathrm{Sym}^k N\iota$$

determined by λ, the partition of the natural number k given by (4.6). Each of the strata $\mathrm{Sym}^k_\lambda N\iota$ is a smooth fibre bundle, which is naturally equipped with a connection ∇ and and a Clifford multiplication γ on its vertical tangent bundle $V\,\mathrm{Sym}^k_\lambda N\iota$. These can be used to define a Fueter operator, which assigns to each section $\tilde{n} \in \Gamma(\mathrm{Sym}^k_\lambda N\iota)$ an element

$$\mathfrak{F}\tilde{n} \in \Gamma(\tilde{n}^* V\,\mathrm{Sym}^k_\lambda N\iota).$$

The condition that $n \in \Gamma(N\kappa)$ is in the kernel of F_κ means that

$$\mathfrak{F}\tilde{n} := \gamma(\nabla\tilde{n}) = 0;$$

that is, $\tilde{n}$ is a Fueter section of $\mathrm{Sym}^k_\lambda N\iota$.

The above discussion show that what causes $k_1 \cdot P^1 + \cdots + k_m \cdot P^m$ to collapse to $k \cdot \mathrm{im}(\tilde{\iota})$ is precisely a Fueter section $\tilde{n}$ of $\mathrm{Sym}^k_\lambda N\iota$. For simplicity, let us specialize to the case $m = 1$ and $k_1 = k$; that is:

- for $t < 0$, there are two embedded associative submanifolds $[\tilde{\iota}_t \colon \tilde{P} \longrightarrow Y]$ and $[\iota_t \colon P \longrightarrow Y]$ (of interest),

- as t tends to zero, $\tilde{\iota}_t$ converges to the associative immersion κ, the k–fold covering of ι_0, and then ceases to exist, and
- for $t > 0$, we only have the embedded associative submanifold $[\iota_t : P \to Y]$.

Extending the approach of Section 3.3, we would like to define weights w such that

$$w(\tilde{P}, \psi_{-t}) + w(k \cdot P, \psi_{-t}) = w(k \cdot P, \psi_{+t}) \tag{4.7}$$

for $0 < t \ll 1$. From the discussion in Section 3.3 we learn that $w(\tilde{P}, \psi_t)$ should be $\varepsilon(\tilde{P}, \psi_{-t}) \cdot \mathrm{SW}(\tilde{P})$ with $\varepsilon(P, \psi_{-t}) \in \{\pm 1\}$ as in Section 2.7 and $\mathrm{SW}(\tilde{P}) \in \mathbf{Z}$ being the Seiberg–Witten invariant of $\tilde{P}$. Thus (4.7) means that the weight $w(k \cdot P, \psi_t)$ must jump by $\pm\mathrm{SW}(\tilde{P})$ as t passes through zero.

We propose that $w(k \cdot P, \psi_t)$ should be defined as the signed count of solutions to the **$\mathrm{ADHM}_{1,k}$ Seiberg–Witten equation** on P. This is the Seiberg–Witten equation associated with the ADHM construction of $\mathrm{Sym}^k \mathbf{H}$. Unlike in the case of the classical Seiberg–Witten equation, compactness fails for the $\mathrm{ADHM}_{1,k}$ Seiberg–Witten equation. As a consequence, the number of solutions can jump as the geometric background varies. According to the **Haydys correspondence**, those jumps occur precisely when (possibly singular) Fueter sections of $\mathrm{Sym}^k N\iota$ appear. We will argue that in the above situation the jumps should be precisely by $\pm\mathrm{SW}(\tilde{P})$.

The next three sections are concerned with introducing the $\mathrm{ADHM}_{1,k}$ Seiberg–Witten equation, stating and proving the Haydys correspondence with stabilizers, and formally analyzing the failure of non-compactness for the $\mathrm{ADHM}_{1,k}$ Seiberg–Witten equation. After this discussion we will also explain what replaces (4.7), in general, and why defining w via the $\mathrm{ADHM}_{1,k}$ Seiberg–Witten equation should be consistent with that.

Remark 4.8. Of course, instead of weighted *count* embedded associatives with multiplicities, one should really try to define a Floer homology generalizing the discussion in Section 3.4.

Remark 4.9. We believe that this approach is also capable of dealing with branched covers. These should correspond to singular Fueter sections, that is, sections of $\mathrm{Sym}^k_\lambda N\iota$ defined outside a subset of codimension at most one (which corresponds to the branching locus) and extend a continuous section of the closure of $\mathrm{Sym}^k_\lambda N\iota$ in $\mathrm{Sym}^k N\iota$. It is known that singular Fueter sections appear in the compactifiction of Seiberg–Witten equations, cf. [DW17b].

5 The ADHM Seiberg–Witten equations

5.1 Seiberg–Witten equations in dimension three

We very briefly review how to associate a Seiberg–Witten equation to a quaternionic representation. More detailed discussions can be found in [Tau99; Pid04; Hay08; Sal13, Section 6; Nak15, Section 6(i)]; we follow [DW17a, Section 1] closely. The first ingredient is a choice of algebraic data.

Definition 5.1. A **quaternionic Hermitian vector space** is a real vector space S together with a linear map $\gamma\colon \operatorname{Im}\mathbf{H} \to \operatorname{End}(S)$ and an inner product $\langle\cdot,\cdot\rangle$, such that γ makes S into a left module over the quaternions $\mathbf{H} = \mathbf{R}\langle 1, i, j, k\rangle$, and i, j, k act by isometries. The **unitary symplectic group** $\mathrm{Sp}(S)$ is the subgroup of $\mathrm{GL}(S)$ preserving γ and $\langle\cdot,\cdot\rangle$. A **quaternionic representation** of a Lie group G on S is a homomorphism $\rho\colon G \to \mathrm{Sp}(S)$.

Let $\rho\colon G \to Sp(S)$ be a quaternionic representation. Denote by $\mathfrak{g}$ the Lie algebra of G. There is a canonical hyperkähler moment map $\mu\colon S \to (\mathfrak{g}\otimes \operatorname{Im}\mathbf{H})^*$ defined as follows. By slight abuse of notation denote by $\rho\colon \mathfrak{g} \to \mathfrak{sp}(S)$ the Lie algebra homomorphism induced by ρ. Combine ρ and γ into the map $\bar{\gamma}\colon \mathfrak{g}\otimes\operatorname{Im}\mathbf{H} \to \operatorname{End}(S)$ given by

$$\bar{\gamma}(\xi\otimes v)\Phi \coloneqq \rho(\xi)\gamma(v)\Phi.$$

The map $\bar{\gamma}$ takes values in the space of symmetric endomorphisms of S. Denote by $\bar{\gamma}^*\colon \operatorname{End}(S) \to (\mathfrak{g}\otimes\operatorname{Im}\mathbf{H})^*$ the adjoint of $\bar{\gamma}$. Define

$$\mu(\Phi) \coloneqq \frac{1}{2}\bar{\gamma}^*(\Phi\Phi^*).$$

Definition 5.2. The **canonical permuting action** $\theta\colon \mathrm{Sp}(1) \to \mathrm{O}(S)$ is defined by left-multiplication by unit quaternions. It satisfies

$$\theta(q)\gamma(v)\Phi = \gamma(\mathrm{Ad}(q)v)\theta(q)\Phi$$

for all $q \in \mathrm{Sp}(1) = \{q \in \mathbf{H} : |q| = 1\}$, $v \in \operatorname{Im}\mathbf{H}$, and $\Phi \in S$.

Definition 5.3. A set of **algebraic data** consists of:

- a quaternionic Hermitian vector space $(S, \gamma, \langle\cdot,\cdot\rangle)$,
- a compact, connected Lie group H, an injective homomorphism $\mathbf{Z}_2 \to Z(H)$, an Ad–invariant inner product on $\mathrm{Lie}(H)$,
- a closed, connected normal subgroup $G \triangleleft H$, and

- a quaternionic representation $\rho\colon H \to \mathrm{Sp}(S)$ such that $-1 \in \mathbf{Z}_2 \subset Z(H)$ acts as $-\mathrm{id}_S$.

Definition 5.4. Given a set of algebraic data, set

$$\hat{H} := (\mathrm{Sp}(1) \times H)/\mathbf{Z}_2, \quad K := H/G, \quad \text{and} \quad \hat{K} := (\mathrm{Sp}(1) \times K)/\mathbf{Z}_2.$$

The group K is called the **flavor group**.

Example 5.5. The $\mathrm{ADHM}_{r,k}$ Seiberg–Witten equation will arise by choosing

$$S = S_{r,k} := \mathrm{Hom}_{\mathbf{C}}(\mathbf{C}^r, \mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^k) \oplus \mathbf{H} \otimes_{\mathbf{R}} \mathfrak{u}(k)$$

with

$$G = \mathrm{U}(k) \triangleleft H = \mathrm{SU}(r) \times \mathrm{Sp}(1) \times \mathrm{U}(k)$$

where $\mathrm{SU}(r)$ acts on $\mathbf{C}^r$ in the obvious way, $\mathrm{U}(k)$ acts on $\mathbf{C}^k$ in the obvious way and on $\mathfrak{u}(k)$ by the adjoint representation, and $\mathrm{Sp}(1)$ acts on the first copy of $\mathbf{H}$ trivially and on the second copy by right-multiplication with the conjugate. The homomorphism $\mathbf{Z}_2 \to Z(H)$ is defined by $-1 \mapsto (\mathrm{id}_{\mathbf{C}^r}, -\mathrm{id}_{\mathbf{H}}, -\mathrm{id}_{\mathbf{C}^k})$. In particular,

$$\hat{H} = \mathrm{SU}(r) \times \mathrm{Spin}^{\mathrm{U}(k)}(4)$$

with

$$\mathrm{Spin}^{\mathrm{U}(k)}(n) := (\mathrm{Spin}(n) \times \mathrm{U}(k))/\mathbf{Z}_2.$$

Although notationally cumbersome, we usually prefer to think of $\hat{H}$ as

$$\hat{H} = \mathrm{SU}(r) \times \mathrm{Spin}^{\mathrm{U}(k)}(3) \times_{\mathrm{SO}(3)} \mathrm{SO}(4).$$

Here the second factor is the fiber product of $\mathrm{Spin}^{\mathrm{U}(k)}(3)$ with $\mathrm{SO}(4)$ with respect to the obvious homomorphism $\mathrm{Spin}^{\mathrm{U}(k)}(3) \to \mathrm{SO}(3)$ and the homomorphism $\mathrm{SO}(4)$ to $\mathrm{SO}(3)$ is given by the action on $\Lambda^+\mathbf{R}^4$.

After a set of algebraic data has been chosen, one still needs to fix the geometric data for which the Seiberg–Witten equation will be defined.

Definition 5.6. Let M be a closed, connected, oriented 3–manifold. A set of **geometric data** on M compatible with a set of algebraic data as in Definition 5.3 consists of:

- a Riemannian metric g on M,
- a principal $\hat{H}$–bundle $\hat{Q} \to M$ together with an isomorphism

$$\hat{Q} \times_{\hat{H}} \mathrm{SO}(3) \cong \mathrm{SO}(TM), \tag{5.7}$$

  and

- a connection B on the principal K–bundle

$$R := \hat{Q} \times_{\hat{H}} K.$$

Definition 5.8. Given a choice of geometric data, the **spinor bundle** and the **adjoint bundle** are the vector bundles[5]

$$\mathbf{S} := \hat{Q} \times_{\theta \times \rho} S \quad \text{and} \quad \mathfrak{g}_P := \hat{Q} \times_{\mathrm{Ad}} \mathfrak{g}.$$

Because of (5.7) the maps γ and μ induce maps

$$\gamma\colon T^*M \to \mathrm{End}(\mathbf{S}) \quad \text{and} \quad \mu\colon \mathbf{S} \to \Lambda^2 T^*M \otimes \mathfrak{g}_P.$$

Here we take μ to be the moment map corresponding to the action of $G \triangleleft H$.

Definition 5.9. Set

$$\mathscr{A}_B(\hat{Q}) := \left\{ A \in \mathscr{A}(\hat{Q}) : \begin{array}{l} A \text{ induces } B \text{ on } R \text{ and the} \\ \text{Levi-Civita connection on } TM \end{array} \right\}.$$

Any $A \in \mathscr{A}_B(\hat{Q})$ defines a covariant derivative $\nabla_A\colon \Gamma(\mathbf{S}) \to \Omega^1(M, \mathbf{S})$. The **Dirac operator** associated with A is the linear map $\not{D}_A\colon \Gamma(\mathbf{S}) \to \Gamma(\mathbf{S})$ defined by

$$\not{D}_A \Phi := \gamma(\nabla_A \Phi).$$

$\mathscr{A}_B(\hat{Q})$ is an affine space modeled on $\Omega^1(M, \mathfrak{g}_P)$ Denote by $\varpi\colon\ \mathrm{Ad}(\hat{Q}) \to \mathfrak{g}_P$ the projection induced by $\mathrm{Lie}(\hat{H}) \to \mathrm{Lie}(G)$.

Finally, we are in a position to define the Seiberg–Witten equation.

Definition 5.10. The **Seiberg–Witten equation** associated with the chosen algebraic and geometric data is the following system of partial differential equations for $(\Phi, A) \in \Gamma(\mathbf{S}) \times \mathscr{A}_B(\hat{Q})$:

$$\begin{aligned} \not{D}_A \Phi &= 0 \quad \text{and} \\ \varpi F_A &= \mu(\Phi). \end{aligned} \tag{5.11}$$

The Seiberg–Witten equation is invariant with respect to gauge transformations which preserve the flavor bundle R and $\mathrm{SO}(T^*M)$.

Definition 5.12. The **group of restricted gauge transformations** is

$$\mathscr{G}(P) := \left\{ u \in \mathscr{G}(\hat{Q}) : u \text{ acts trivially on } R \text{ and } \mathrm{SO}(TM) \right\}.$$

$\mathscr{G}(P)$ can be identified with the space of sections of $\hat{Q} \times_{\hat{H}} G$ with $\hat{H}$ acting on G via $[(q, h)] \cdot g = hgh^{-1}$.

[5]If $H = G \times K$, then there is a principal G–bundle $P \to M$ associated with $\hat{Q}$ and $\mathfrak{g}_P$ is the adjoint bundle of P. In general, P might not exist but traces of it remain, e.g., its adjoint bundle $\mathfrak{g}_P$ and its gauge group $\mathscr{G}(P)$.

5.2 Blown-up Seiberg–Witten equations

If $\mu^{-1}(0) = \{0\}$, then one proves in the same way as for the classical Seiberg–Witten equation that solutions of (5.11) obey a priori bounds on Φ. In many cases of interest $\mu^{-1}(0) \neq \{0\}$ and these a priori bounds fail. Anticipating this, we blow-up the Seiberg–Witten equation to force Φ to be bounded.

Definition 5.13. The **blown-up Seiberg–Witten equation** is the following partial differential equation for $(\varepsilon, \Phi, A) \in [0, \infty) \times \Gamma(\mathbf{S}) \times \mathscr{A}_B(\hat{Q})$:

$$\begin{aligned} \not{D}_A \Phi &= 0, \\ \varepsilon^2 \varpi F_A &= \mu(\Phi), \quad \text{and} \\ \|\Phi\|_{L^2} &= 1. \end{aligned} \tag{5.14}$$

The **limiting Seiberg–Witten equation** is the following partial differential equation for $(\Phi, A) \in [0, \infty) \in \Gamma(\mathbf{S}) \times \mathscr{A}_B(\hat{Q})$:

$$\begin{aligned} \not{D}_A \Phi &= 0 \quad \text{and} \\ \mu(\Phi) &= 0 \end{aligned} \tag{5.15}$$

as well as $\|\Phi\|_{L^2} = 1$.

The phenomenon of Φ tending to infinity for (5.11) corresponds to ε tending to zero for (5.14) Formally, the compactifiction of the moduli space of solutions of (5.11) should thus be given by adding solution of the limiting equation. Taubes [Tau13a] and Haydys and Walpuski [HW15] proved that—up to allowing for codimension on singularities in the limiting solutions—this is true for the flat $\mathbf{PSL}(2, \mathbf{C})$–connections and the Seiberg–Witten equation with multiple spinors, which are particular instances of equation (5.11). Although one might initially hope that it is unnecessary to allow for singularities in solutions of the limiting equation, it has been shown in [DW17b] that this phenomenon cannot be avoided.

5.3 The ADHM$_{r,k}$ Seiberg–Witten equation

In a nutshell, the ADHM$_{r,k}$ Seiberg–Witten equation arises from the above construction by choosing the set of algebraic described in Example 5.5. Let us explain what this means in more detail.

Definition 5.16. Let M be an oriented Riemannian 3–manifold. A **spin$^{\mathrm{U}(k)}$–structure** on M is a principal $\mathrm{Spin}^{\mathrm{U}(k)}(3)$–bundle together with an isomorphism

$$\mathfrak{w} \times_{\mathrm{Spin}^{\mathrm{U}(k)}(3)} \mathrm{SO}(3) \cong \mathrm{SO}(TM). \tag{5.17}$$

The **spinor bundle** and the **adjoint bundle** associated with a $\mathrm{spin}^{\mathrm{U}(k)}$–structure $\mathfrak{w}$ are

$$W := \mathfrak{w} \times_{\mathrm{Spin}^{\mathrm{U}(k)}(3)} \mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^k \quad \text{and} \quad \mathfrak{g}_{\mathscr{H}} := \mathfrak{w} \times_{\mathrm{Spin}^{\mathrm{U}(k)}(3)} \mathfrak{u}(k)$$

respectively. The left multiplication by $\operatorname{Im} \mathbf{H}$ on $\mathbf{H} \otimes \mathbf{C}^k$ induces a **Clifford multiplication** $\gamma\colon\ TM \to \operatorname{End}(W)$.

A **spin connection** on $\mathfrak{w}$ is a is a connection A inducing the Levi-Civita connection on TM. Associated with each spin connection A there is a **Dirac operator** $\not{D}_A\colon\ \Gamma(W) \to \Gamma(W)$.

Denote by $\mathscr{A}^s(\mathfrak{w})$ the space of spin connections on $\mathfrak{w}$, by $\mathscr{G}^s(\mathfrak{w})$ the **restricted gauge group**, consisting of those gauge transformations which act act trivially on TM, and by $\varpi\colon\ \operatorname{Ad}(\mathfrak{w}) \to \mathfrak{g}_{\mathscr{H}}$ the map induced by the projection $\mathfrak{spin}^{\mathrm{U}(k)}(3) \to \mathfrak{u}(k)$.

Definition 5.18. Let M be an oriented 3–manifold. The **geometric data** needed to formulate the $\mathrm{ADHM}_{r,k}$ Seiberg–Witten equation are:

- a Riemannian metric g,
- a $\mathrm{spin}^{\mathrm{U}(k)}$–structure $\mathfrak{w}$,
- a Hermitian vector bundle E of rank r with a fixed trivialization $\Lambda^r E = \mathbf{C}$ and an $\mathrm{SU}(r)$–connection B,
- an oriented Euclidean vector bundle V of rank 4 together with an isomorphism

$$\mathrm{SO}(\Lambda^+ V) \cong \mathrm{SO}(TM) \tag{5.19}$$

  and a $\mathrm{SO}(4)$–connection C on V with respect to which this isomorphism is parallel.

Remark 5.20. If $\iota\colon\ P \to Y$ is an associative immersion, then the normal bundle $N\iota$ admits a natural isomorphism (5.19) by Proposition 2.12 and we can take C to be the connection induced by the Levi-Civita connection. In this context, the bundle E should be the restriction to P of a bundle on the ambient G_2–manifold to P and B should be the restriction of a G_2–instanton. Soon we will specialize to the case $r = 1$, in which both E and B are trivial.

The above data makes both $\operatorname{Hom}(E, W)$ and $V \otimes \mathfrak{g}_{\mathscr{H}}$ into Clifford bundles over M; hence, there are Dirac operators $\not{D}_{A,B}\colon\ \Gamma(\operatorname{Hom}(E,W)) \to \Gamma(\operatorname{Hom}(E,W))$ and $\not{D}_{A,C}\colon\ \Gamma(V \otimes \mathfrak{g}) \to \Gamma(V \otimes \mathfrak{g})$. The moment maps which will appear in the $\mathrm{ADHM}_{r,k}$ Seiberg–Witten equation can be understood as follows. If $\Psi \in \operatorname{Hom}(E,W)$, then $\Psi\Psi^* \in \operatorname{End}(W)$. Since $\Lambda^2 T^*M \otimes \mathfrak{g}_{\mathscr{H}}$ acts on W, there is an adjoint map $(\cdot)_0\colon\ \operatorname{End}(W) \to \Lambda^2 T^*M \otimes \mathfrak{g}_{\mathscr{H}}$. We define $\mu\colon\ \operatorname{Hom}(E,W) \to \Lambda^2 T^*M \otimes \mathfrak{g}_{\mathscr{H}}$ by

$$\mu(\Psi) := (\Psi\Psi^*)_0.$$

If $\xi \in V \otimes \mathfrak{g}$, then $[\xi \wedge \xi] \in V \otimes \mathfrak{g}_{\mathscr{H}}$. Denote its projection to $\Lambda^+ V \otimes \mathfrak{g}_{\mathscr{H}}$ by $[\xi \wedge \xi]^+$. Identifying $\Lambda^+ V \cong \Lambda^2 T^* M$ via (5.19), we define $\mu \colon V \otimes \mathfrak{g} \to \Lambda^2 T^* M \otimes \mathfrak{g}_{\mathscr{H}}$ by

$$\mu(\xi) := [\xi \wedge \xi]^+$$

Definition 5.21. Given a choice of geometric data as in Definition 5.18, the **ADHM$_{r,k}$ Seiberg–Witten equation** is the following partial differential equation for $(\Psi, \xi, A) \in \Gamma(\mathrm{Hom}(E, W)) \times \Gamma(V \otimes \mathfrak{g}_{\mathscr{H}}) \times \mathscr{A}^s(\mathfrak{w})$:

$$\begin{aligned} \not{D}_{A,B}\Psi &= 0, \\ \not{D}_{A,C}\xi &= 0, \quad \text{and} \\ \varpi F_A &= \mu(\Psi) + \mu(\xi). \end{aligned} \tag{5.22}$$

A solution of this equation is called an **ADHM$_{r,k}$ monopole**.

The corresponding **limiting ADHM$_{r,k}$ Seiberg–Witten equation** the following partial differential equation for $(\Psi, \xi, A) \in \Gamma(\mathrm{Hom}(E, W)) \times \Gamma(V \otimes \mathfrak{g}_{\mathscr{H}}) \times \mathscr{A}^s(\mathfrak{w})$

$$\begin{aligned} \not{D}_{A,B}\Psi &= 0, \\ \not{D}_{A,C}\xi &= 0, \quad \text{and} \\ \mu(\Psi) + \mu(\xi) &= 0. \end{aligned} \tag{5.23}$$

as well $\|(\Psi, \xi)\|_{L^2} = 1$.

The ADHM$_{r,k}$ Seiberg–Witten equation (5.22) is preserved by the action of the restricted gauge group $\mathscr{G}^s(\mathfrak{w})$.

Remark 5.24. Suppose that $r = k = 1$. A spin$^{\mathrm{U}(1)}$–structure is simply a spin^c–structure. A moment's thought shows that

$$\varpi F_A = \frac{1}{2} F_{\det A}.$$

Also, $\mathfrak{g}_{\mathscr{H}} = i\underline{\mathbf{R}}$; hence, $\not{D}_{A,C}$ is independent of A and $\mu(\xi) = 0$. The ADHM$_{1,1}$ Seiberg–Witten equation is thus simply

$$\not{D}_A \Psi = 0 \quad \text{and} \quad \frac{1}{2} F_{\det A} = \mu(\Psi),$$

the classical Seiberg–Witten equation (3.11) for (Ψ, A), together with the Dirac equation

$$\not{D}_C \xi = 0.$$

If $\iota \colon P \to Y$ is an associative immersion and $M = P$ and $V = N\iota$, then $\not{D}_C$ is essentially the Fueter operator F_ι defined in Definition 2.18. In particular, ξ must vanish if ι is unobstructed.

There is a variant of (5.22) in which ξ is taken to be a section of $V \otimes \mathfrak{g}^\circ_{\mathscr{H}}$ with $\mathfrak{g}^\circ_{\mathscr{H}}$ denoting the trace-free component of $\mathfrak{g}_{\mathscr{H}}$. For $r = k = 1$, this equation is identical to the classical Seiberg–Witten equation; however, working with this equation complicates the discussion in Section 7 even further.

6 The Haydys correspondence with stabilizers

Throughout this section we assume that algebraic data and geometric data as in Definition 5.3 and Definition 5.6 have been chosen. Denote by

$$X := S /\!\!/\!\!/ G = \mu^{-1}(0)/G$$

the **hyperkähler quotient** of X by G, and denote by $p\colon\ \mu^{-1}(0) \to X$ the canonical projection. The action of $\hat{H}$ on S induces an action of $\hat{K} = \hat{H}/G$ on X. Set

$$\mathbf{X} := \hat{R} \times_{\hat{K}} X.$$

If $\Phi \in \Gamma(\mathbf{S})$ satisfies $\mu(\Phi) = 0$, then

$$s := p \circ \Phi \in \Gamma(\mathbf{X}). \tag{6.1}$$

The Haydys correspondence [Hay12, Section 4.1] relates solutions of the limiting Seiberg–Witten equation (5.15) with certain sections of $\mathbf{X}$. The discussions of the Haydys correspondence available in the literature so far [Hay12, Section 4.1; DW17a, Section 3] assume that the action of G on $\mu^{-1}(0)$ is generically free. This hypothesis does not hold in Example 5.5 with $r = 1$, which leads to the $\mathrm{ADHM}_{1,k}$ Seiberg–Witten equation. This section is concerned with extending the Haydys correspondence to the case when G acts on $\mu^{-1}(0)$ with a non-trivial generic stabilizer.

6.1 Decomposition of hyperkähler quotients

Denote by $S_{\{e\}}$ the subset of S on which G acts freely. By [HKLR87, Section 3(D)], the quotient

$$(S_{\{e\}} \cap \mu^{-1}(0))/G$$

can be given the structure of hyperkähler manifold of dimension $4(\dim_{\mathbf{H}} S - \dim G)$ such that, for $\Phi \in S_{\{e\}} \cap \mu^{-1}(0)$,

$$p_*\colon\ (\rho(\mathfrak{g})\Phi)^{\perp} \cap T_\Phi \mu^{-1}(0) \to T_{[\Phi]}X \tag{6.2}$$

is a quaternionic isometry. If G acts on $\mu^{-1}(0)$ with trivial generic stabilizer (that is: $S_{\{e\}}$ is dense and open), then this makes an dense open subset of X into a hyperkähler manifold. In general, X can be decomposed as a union of hyperkähler manifolds according to orbit type as follows.

Definition 6.3. For $\Phi \in S$, denote by G_Φ the stabilizer of Φ in G. Let $T < G$ be a subgroup. Set

$$S_T := \{\Phi \in S : G_\Phi = T\} \quad \text{and} \quad S_{(T)} := \{\Phi \in S : gG_\Phi g^{-1} = T \text{ for some } g \in G\}.$$

Definition 6.4. Given a subgroup $T < G$, set

$$W_G(T) := N_G(T)/T.$$

Here $N_G(T)$ denotes the normalizer of T in G.

Remark 6.5. This notation is motivated by the example $S = \mathbf{H} \otimes \mathfrak{g}$, with G acting via the adjoint representation. In this case, the stabilizer T of a generic point in $\mu^{-1}(0)$ is a maximal torus and $W_G(T)$ is the Weyl group of G; cf. Section 7.1 for the case $G = \mathrm{U}(k)$.

Theorem 6.6 (Dancer and Swann [DS97, Theorem 2.1]; Sjamaar and Lerman [SL91], Nakajima [Nak94, Section 6]). *For each $T < G$, the quotient*

$$X_{(T)} := (\mu^{-1}(0) \cap S_{(T)})/G$$

is a hyperkähler manifold, and

$$X = \bigcup_{(T)} X_{(T)} \tag{6.7}$$

where (T) runs through all conjugacy classes of subgroups of G.[6] *More precisely, for each $T < G$:*

1. *S_T is a hyperkähler submanifold of S and $S_{(T)}$ is a submanifold of S.*
2. *We have*
$$(\mu^{-1}(0) \cap S_{(T)})/G = (\mu^{-1}(0) \cap S_T)/W_G(T).$$
3. *Denote by S_T^0 denotes the union of the components of S_T intersecting $\mu^{-1}(0)$. then $W_G(T)$ acts freely on S_T^0 and*
$$\mu(S_T^0) \subset (\mathfrak{w} \otimes \operatorname{Im} \mathbf{H})^*$$
with $\mathfrak{w} :=$ Lie$(W_G(T))$. In particular, the restriction of μ to S_T^0 induces a hyperkähler moment map on S_T^0 for the action of $W_G(T)$.
4. *$X_{(T)}$ can be given the structure of a hyperkähler manifold such that, for each $\Phi \in \mu^{-1}(0) \cap S_{(T)}$,*
$$p_* \colon (\rho(\mathfrak{g})\Phi)^\perp \cap \ker \mathrm{d}_\Phi\mu \cap T_\Phi S_{(T)} \to T_{[\Phi]} X_{(T)}$$
is a quaternionic isometry.

[6] There can be subgroups $T < G$ with $S_{(T)} \neq 0$, but $\mu^{-1}(0) \cap S_{(T)} = \varnothing$.

Proof. We recall Dancer and Swann's argument, since some aspects of it will play a role later on.

To prove (1), denote by

$$S^T := \{\Phi \in S : G_\Phi \supset T\}$$

the fixed-point set of the action of T. S^T is an $\mathbf{H}$–linear subspace of S and S_T is an open subset of S^T (by the Slice Theorem). Therefore, S_T is a hyperkähler submanifold of S. The group action induces a bijection

$$S_T \times_{W_G(T)} G/T \cong S_{(T)}, \quad [\Phi, gT] \mapsto \rho(g)\Phi.$$

This shows that $S_{(T)}$ is a submanifold of S. For future reference, we also observe that

$$T_\Phi S_T = S^T \quad \text{and} \quad T_\Phi S_{(T)} = S^T + \rho(\mathfrak{g})\Phi \cong \frac{S^T \oplus \rho(\mathfrak{g})\Phi}{\rho(\mathfrak{w})\Phi}. \tag{6.8}$$

The assertion made in (2) follows directly from the definitions.

To prove (3), observe that by the definition of S_T, the group $W_G(T) = N_G(T)/T$ acts freely on S_T. Since μ is G–equivariant, $\mu(S_T) \subset (\mathfrak{g}^*)^T \subset \mathfrak{n}^*$ with $\mathfrak{n} := \mathrm{Lie}(N_G(T))$. Let $\mathfrak{t} = \mathrm{Lie}(T)$. If $\Phi \in S_T$, then

$$\mathrm{d}_\Phi\mu \in \mathrm{Ann}_{\mathfrak{g}^*}\,\mathfrak{t} \otimes (\mathrm{Im}\,\mathbf{H})^*, \tag{6.9}$$

because, for $\xi \in \mathfrak{t}$, $v \in \mathrm{Im}\,\mathbf{H}$, and $\phi \in S$, we have

$$\langle (\mathrm{d}_\Phi\mu)\phi, \xi \otimes v\rangle = \langle \gamma(v)\rho(\xi)\Phi, \phi\rangle = 0;$$

Since $\mathfrak{w}^* = \mathfrak{n}^* \cap \mathrm{Ann}_{\mathfrak{g}_*}\,\mathfrak{t}$, we have $\mu(S_T^0) \subset (\mathfrak{w} \otimes \mathrm{Im}\,\mathbf{H})^*$. This proves (3).

Finally, we prove (4). Since

$$X_{(T)} = (\mu^{-1}(0) \cap S_{(T)})/G = (\mu^{-1}(0) \cap S_{(T)})/W_G(T) = S_T^0 /\!\!/\!\!/ W_G(T),$$

$X_{(T)}$ can be given a hyperkähler structure by the construction in [HKLR87, Section 3(D)]. If $\Phi \in S_T$, then

$$(\rho(\mathfrak{g})\Phi)^\perp \cap T_\Phi S_{(T)} = (\rho(\mathfrak{w})\Phi)^\perp \cap T_\Phi S_T$$

by (6.8); hence, by the discussion before Definition 6.3,

$$p_* \colon (\rho(\mathfrak{g})\Phi)^\perp \cap \ker \mathrm{d}_\Phi\mu \cap T_\Phi S_{(T)} = (\rho(\mathfrak{w})\Phi)^\perp \cap \ker \mathrm{d}_\Phi\mu \cap T_\Phi S_T \longrightarrow T_{[\Phi]}X_{(T)}$$

is a quaternionic isometry. This finishes the proof of (4). □

In general, this action of $\hat{K} = \hat{H}/G$ need not preserve the strata $X_{(T)}$. The following hypothesis, which holds for all the examples considered in this article, guarantees that the action of $\hat{H}$ on S preserves $S_{(T)}$ and that the action of $\hat{K}$ on X preserves $X_{(T)} \subset X$.

Hypothesis 6.10. *Given $T < G$, assume that, for all $h \in H$, there is a $g \in G$ such that*

$$hTh^{-1} = gTg^{-1}.$$

Proposition 6.11. *If Hypothesis 6.10 holds for $T < G$, then the action of $\hat{H}$ on S preserves the submanifold $S_{(T)}$ and the action of $\hat{K}$ on X preserves $X_{(T)}$.* □

Proof. For $h \in H$ and $\Phi \in S_{(T)}$, we have $G_{\rho(h)\Phi} = hG_\Phi h^{-1} = hTh^{-1} = gTg^{-1}$ for some $g \in G$, so $\rho(h)\Phi \in S_{(T)}$ and the action of H preserves $S_{(T)}$. The action of $\mathrm{Sp}(1)$ commutes with that of H and so it also preserves $S_{(T)}$. We conclude that $S_{(T)}$ is preserved by the action of $\hat{H}$. Since $X_{(T)} = (\mu^{-1}(0) \cap S_{(T)})/G$, the action of $\hat{K}$ preserves $X_{(T)}$. □

Proposition 6.12. *For any $T < G$, $N_G(T)$ is a normal subgroup of $N_H(T)$, and the identity $K = H/G$ induces an injective homomorphism $N_H(T)/N_G(T) \hookrightarrow K$. If Hypothesis 6.10 holds for $T < G$, then this map is an isomorphism*

$$N_H(T)/N_G(T) \cong K.$$

Proof. If $g \in N_G(T)$ and $h \in N_H(T)$, then $\tilde{g} := hgh^{-1} \in G$ since $G \triangleleft H$; hence, $\tilde{g} \in N_G(T)$. Since $N_H(T) \cap G = N_G(T)$, we have an injective homomorphism $N_H(T)/N_G(T) \hookrightarrow K$.

Assuming Hypothesis 6.10 and given $k = hG \in K$, there is a $g \in G$ such that

$$hTh^{-1} = gTg^{-1}.$$

It follows that $\tilde{h} := g^{-1}h \in N_H(T)$ and $\tilde{h}G = k$; hence, $N_H(T)/N_G(T) \hookrightarrow K$ is an isomorphism. □

Assuming Hypothesis 6.10 for $T < G$, we can define fiber bundles over M whose fibers are the strata $S_{(T)}$ and $X_{(T)}$:

$$\mathbf{S}_{(T)} := \hat{Q} \times_{\hat{H}} S_{(T)} \quad \text{and} \quad \mathbf{X}_{(T)} := \hat{R} \times_{\hat{K}} X_{(T)}.$$

If it holds for all $T < G$ with non-empty S_T, we decompose $\mathbf{S}$ and $\mathbf{X}$ as

$$\mathbf{S} = \bigcup_{(T)} \mathbf{S}_{(T)} \quad \text{and} \quad \mathbf{X}_{(T)} = \bigcup_{(T)} \mathbf{X}_{(T)}.$$

6.2 Lifting sections of $\mathbf{X}_{(T)}$

For the remainder of this section we will assume Hypothesis 6.10 for $T < G$. The first part of the Haydys correspondence is concerned with the questions:

When can a section $s \in \Gamma(\mathbf{X}_{(T)})$ can be lifted a section of $\Phi \in \Gamma(\mathbf{S}_{(T)})$ with $\mu(\Phi) = 0$ for some choice of $\hat{Q}$?

and

To what extend is the principal $\hat{H}$–bundle $\hat{Q}$ determined by s?

Proposition 6.13. *If $\Phi \in \Gamma(\mathbf{S}_{(T)})$, then*

$$\hat{Q}^\circ = \hat{Q}^\circ_\Phi := \left\{q \in \hat{Q} : \Phi(q) \in S_T\right\}^{7}$$

is a principal $N_{\hat{H}}(T)$–bundle over M whose associated principal $\hat{H}$–bundle is isomorphic to $\hat{Q}$. Moreover, the stabilizer of Φ in $\mathscr{G}(P) = \Gamma(\hat{Q} \times_{\hat{H}} G)$ is

$$\Gamma(\hat{Q}^\circ \times_{N_{\hat{H}}(T)} T) \subset \mathscr{G}(P),$$

and the kernel of $\rho(\cdot)\Phi \colon \mathfrak{g}_P \to \mathbf{S}$ is

$$\mathfrak{t}_P := \hat{Q}^\circ \times_{N_{\hat{H}}(T)} \mathrm{Lie}(T) \subset \mathfrak{g}_P. \tag{6.14}$$

Proof. If $\Phi \in S_T$, $\hat{h} = [(q,h)] \in \hat{H} = (\mathrm{Sp}(1) \times H)/\mathbf{Z}_2$ and $\Psi := \theta(q)\rho(h)\Phi$, then

$$G_\Psi = hG_\Phi h^{-1} = hTh^{-1}.^{8}$$

Therefore, $\Psi \in S_T$ if and only if $\hat{h} \in N_{\hat{H}}(T) = (\mathrm{Sp}(1) \times N_H(T))/\mathbf{Z}_2$. Moreover, for each $\Phi \in S_{(T)}$ there is a $g \in G \subset \hat{H}$ such that $\rho(g)\Phi(q) \in S_T$. This implies that $\hat{Q}^\circ$ is a principal $N_{\hat{H}}(T)$–bundle.

The isomorphism $\hat{Q}^\circ \times_{N_{\hat{H}}(T)} \hat{H} \cong \hat{Q}$ is given by $[(\hat{q}, \hat{h})] \mapsto \hat{q} \cdot \hat{h}$. In particular,

$$\mathscr{G}(P) \cong \Gamma(\hat{Q}^\circ \times_{N_{\hat{H}}(T)} G)$$

where $N_{\hat{H}}(T)$ acts on G by conjugation. The last two assertions follow from the fact that, for every $q \in \hat{Q}^\circ$, the G–stabilizer of $\Phi(q)$ is T. □

Definition 6.15. Given any $\Phi \in \Gamma(\mathbf{S}_{(T)})$, the **Weyl group bundle** associated with Φ is

$$\hat{Q}^\diamond = \hat{Q}^\diamond_\Phi := \hat{Q}^\circ_\Phi / T.$$

[7] Here we think of Φ as a $\hat{H}$–equivariant map $\Phi \colon \hat{Q} \to S$.

[8] Hypothesis 6.10 ensures that $hTh^{-1} \subset G$.

Proposition 6.16. *Suppose that two choices of geometric data have been made such that $\hat{R}_1 = \hat{R}_2$. Suppose that $\Phi_i \in \Gamma(\mathbf{S}_{\hat{Q}_i,(T)})$ satisfy $\mu(\Phi_i) = 0$. Denote by $\hat{Q}_i^\diamond$ the associated Weyl group bundles.*

If $p \circ \Phi_1 = p \circ \Phi_2 \in \Gamma(\mathbf{X}_{(T)})$, then there is an isomorphism $\hat{Q}_1^\diamond \cong \hat{Q}_2^\diamond$ compatible with the isomorphism

$$\hat{Q}_1^\diamond / W_G(T) \cong \hat{R}_1 = \hat{R}_2 \cong \hat{Q}_2^\diamond / W_G(T).$$

Remark 6.17. The principal $N_G(T)$–bundles $\hat{Q}_1^\circ$ and $\hat{Q}_2^\circ$ need not be isomorphic.

Proof of Proposition 6.16. Since $\hat{Q}_i/G \cong \hat{R}_i$, we have $\hat{Q}_i^\circ/N_G(T) \cong \hat{R}_i$. The sections Φ_i restrict to $N_G(T)$–equivariant maps $\Phi_i^\circ \colon \hat{Q}_i^\circ \to \mu^{-1}(0) \cap S_T$, which in turn induce $W_G(T)$–equivariant maps $\Phi_i^\diamond \colon \hat{Q}_i^\diamond = \hat{Q}_i^\circ/T \to \mu^{-1}(0) \cap S_T$. The resulting commutative diagrams

$$\begin{tikzcd}
\hat{Q}_i^\diamond \arrow[r, "\Phi_i^\diamond"] \arrow[d, "q_i^\diamond"] & \mu^{-1}(0) \cap S_T \arrow[d, "p"] \\
\hat{R}_i \arrow[r, "s"] & X_{(T)}
\end{tikzcd}$$

are pullback diagrams; hence, the assertion follows from the universal property of pullbacks. □

Proposition 6.18. *Let $\hat{R}$ be a principal $\hat{K}$–bundle. Given $s \in \Gamma(\mathbf{X}_{(T)})$, there exists a principal $W_{\hat{H}}(T)$–bundle $\hat{Q}^\diamond$ together with an isomorphism*

$$\hat{Q}^\diamond / W_G(T) \cong \hat{R}$$

and a section

$$\Phi^\diamond \in \Gamma(\hat{Q}^\diamond \times_{W_{\hat{H}}(T)} S_T)$$

satisfying

$$\mu(\Phi^\diamond) = 0 \quad \text{and} \quad p \circ \Phi^\diamond = s.$$

The section $\Phi^\diamond$ is unique up to the action of the restricted gauge group $\Gamma(\hat{Q}^\diamond \times_{W_{\hat{H}}(T)} W_G(T))$.

Proof. We can think of the section s as a $\hat{K}$–equivariant map $s \colon \hat{R} \to X_{(T)}$. The quotient map $p \colon \mu^{-1}(0) \cap S_T \to X_{(T)}$ defines a principal $W_G(T)$–bundle. Set

$$\begin{aligned}
\hat{Q}^\diamond &:= s^*(\mu^{-1}(0) \cap S_T) \\
&= \{(r, \Phi) \in R \times (\mu^{-1}(0) \cap S_T) : s(r) = W_G(T) \cdot \Phi\}
\end{aligned}$$

and denote by $\Phi^\diamond \colon \hat{Q}^\diamond \to \mu^{-1}(0) \cap S_T$ the projection to the second factor. The projection to the first factor $q^\diamond \colon \hat{Q}^\diamond \to R$ makes $\hat{Q}^\diamond$ into a principal $W_G(T)$–bundle over $\hat{R}$. We have the following diagram with the square being a pullback:

$$\begin{tikzcd}
\hat{Q}^\diamond \arrow[r, "\Phi^\diamond"] \arrow[d, "q^\diamond"] & \mu^{-1}(0) \cap S_T \arrow[d, "p"] \\
\hat{R} \arrow[r, "s"] \arrow[d] & X_{(T)} \\
M. &
\end{tikzcd}$$

$\hat{Q}^\diamond$ can be given the structure of a principal $W_{\hat{H}}(T)$–bundle over M as follows. By Proposition 6.12 we have a short exact sequence

$$0 \longrightarrow W_G(T) \longrightarrow W_{\hat{H}}(T) \xrightarrow{\pi} \hat{K} \longrightarrow 0.$$

Define an right-action of $W_{\hat{H}}(T)$ on $\hat{Q}^\diamond$ by

$$(r, \Phi) \cdot [\hat{h}] \coloneqq (r \cdot \pi([\hat{h}]), (\theta \times \rho)(\hat{h}^{-1})\Phi)$$

for $[\hat{h}] \in W_{\hat{H}}(T)$ and $(r, \Phi) \in \hat{Q}^\diamond$ and with θ as in Definition 5.2. A moment's thought shows that this action is free and

$$\hat{Q}^\diamond / W_{\hat{H}}(T) = (\hat{Q}^\diamond / W_G(T)) / \hat{K} = \hat{R} / \hat{K} = M.$$

Since s is $\hat{K}$–equivariant, $\Phi^\diamond$ is $W_{\hat{H}}(T)$–equivariant and thus defines the desired section. The assertion about the uniqueness of $\Phi^\diamond$ is clear. □

Proposition 6.19. *Assume the situation of Proposition 6.18. Suppose that $\hat{Q}^\circ$ is a principal $N_{\hat{H}}(T)$–bundle with an isomorphism*

$$\hat{Q}^\circ / T \cong \hat{Q}^\diamond;$$

that is: $\hat{Q}^\circ$ is a lift of the structure group from $W_{\hat{H}}(T)$ to $N_{\hat{H}}(T)$. Set

$$\hat{Q} \coloneqq \hat{Q}^\circ \times_{N_{\hat{H}}(T)} \hat{H}.$$

In this situation, there is a section Φ of $\mathrm{S}_{(T)} \coloneqq \hat{Q} \times_{\hat{H}} S_{(T)}$ satisfying

$$\mu(\Phi) = 0 \quad \text{and} \quad p \circ \Phi = s;$$

moreover, there is an isomorphism

$$\hat{Q}^\circ_\Phi \cong \hat{Q}^\circ.$$

Any other section satisfying these conditions is related to Φ by the action of $\mathscr{G}(P)$.

Proof. With $\Phi^\diamond$ as in Proposition 6.18 define $\Phi\colon \hat{Q} \to \mu^{-1}(0) \subset S$ by

$$\Phi([q,\hat{h}]) := (\theta \times \rho)(\hat{h}^{-1})\Phi^\diamond(qT).$$

This is well-defined because $\Phi^\diamond(qT)$ is T–invariant; moreover, Φ is manifestly $\hat{H}$–equivariant and, hence, defines the desired section. The assertion about the uniqueness of Φ is clear. □

To summarize the preceeding discussion and answer the questions raised at the beginning of this section:

1. s determines the Weyl group bundle $\hat{Q}^\diamond$ uniquely,
2. every s lifts to a section $\Phi^\diamond$ of $\hat{Q}^\diamond \times_{W_{\hat{H}}(T)} S_T$, and
3. if $\hat{Q}^\circ$ is a lift of the structure group of $\hat{Q}^\diamond$ from $W_{\hat{H}}(T)$ to $N_{\hat{H}}(T)$ and we set $\hat{Q} := \hat{Q}^\circ \times_{N_{\hat{H}}(T)} \hat{H}$, then $\Phi^\diamond$ induces a section Φ of $\mathbf{S}_{(T)} = \hat{Q} \times_{\hat{H}} S_{(T)}$ lifting s.

6.3 Projecting the Dirac equation

The second part of the Haydys correspondence is concerned with the question

To what extend is the Dirac equation for a section $\Phi \in \Gamma(\mathbf{S}_{(T)})$ equivalent to a differential equation for $s := p \circ \Phi \in \Gamma(\mathbf{X}_{(T)})$?

Definition 6.20. The **vertical tangent bundle** of $\mathbf{X}_{(T)} \xrightarrow{\pi} M$ is

$$V\mathbf{X}_{(T)} := \hat{R} \times_{\hat{K}} TX_{(T)}.$$

The hyperkähler structure on $X_{(T)}$ induces a **Clifford multiplication**

$$\gamma\colon \pi^* \operatorname{Im} \mathbf{H} \to \operatorname{End}(V\mathbf{X}_{(T)}).$$

Given $B \in \mathscr{A}(\hat{R})$ we can assign to each $s \in \Gamma(\mathbf{S})$ its covariant derivative $\nabla_B s \in \Omega^1(M, s^*V\mathbf{X})$. A section $s \in \Gamma(\mathbf{X})$ is called a **Fueter section** if it satisfies the **Fueter equation**

$$\mathfrak{F}(s) = \mathfrak{F}_B(s) := \gamma(\nabla_B s) = 0 \in \Gamma(s^*V\mathbf{X}_{(T)}). \tag{6.21}$$

The map $s \mapsto \mathfrak{F}(s)$ is called the **Fueter operator**.

Proposition 6.22. *Given $\Phi \in \Gamma(\mathbf{S}_{(T)})$ satisfying $\mu(\Phi) = 0$, set*

$$s := p \circ \Phi \in \Gamma(\mathbf{X}_{(T)}).$$

The following hold:

1. *$A \in \mathscr{A}_B(\hat{Q})$ satisfies $\not{D}_A\Phi = 0$ if and only if*

$$\mathfrak{F}_B(s) = 0 \quad \textit{and} \quad \nabla_A\Phi \perp \rho(\mathfrak{g}_P)\Phi. \tag{6.23}$$

2. *Let $\mathfrak{t}_P$ be as in (6.14). The space of connections*

$$\mathscr{A}_B^\Phi(\hat{Q}) := \left\{A \in \mathscr{A}_B(\hat{Q}) : \nabla_A\Phi \perp \rho(\mathfrak{g}_P)\Phi\right\} \tag{6.24}$$

is an affine space modeled on $\Omega^1(M, \mathfrak{t}_P)$ with $\mathfrak{t}_P$ as in (6.14). In particular, if $\mathfrak{F}_B(s) = 0$, there exists an $A \in \mathscr{A}_B(\hat{Q})$ such that $\not{D}_A\Phi = 0$; A is unique up to $\Omega^1(M, \mathfrak{t}_P)$.

3. *Any connection $A \in \mathscr{A}_B^\Phi(\hat{Q})$ reduces to a connection on $\hat{Q}^\circ$. Conversely, any connection on $\hat{Q}^\circ$ induces a connection in $\mathscr{A}_B^\Phi(\hat{Q})$.*

4. *The subbundle $\mathfrak{t}_P \subset \mathfrak{g}_P$ is parallel with respect to any $A \in \mathscr{A}_B^\Phi(\hat{Q})$.*

Proof. We prove (1). If $\not{D}_A\Phi = 0$, then it follows from $p_*(\nabla_A\Phi) = \nabla_B s$ that $\mathfrak{F}_B(s) = 0$. Let (e^1, e^2, e^3) be an orthonormal basis of T_x^*M. The equations $\not{D}_A\Phi = 0$ and $\nabla_A\mu(\Phi) = 0$ can be written as

$$\nabla_{A,e_i}\Phi = -\varepsilon_{ij}{}^k\gamma(e^j)\nabla_{A,e_k}\Phi \quad \text{and} \quad \langle\gamma(e^j)\nabla_{A,e_k}\Phi, \rho(\xi)\Phi\rangle = 0$$

for all $\xi \in \mathfrak{g}_{P,x}$. This proves that $\nabla_A\Phi \perp \rho(\mathfrak{g}_P)\Phi$. By Theorem 6.6(4), (6.23) implies $\not{D}_A\Phi = 0$

We prove (2). If $A \in \mathscr{A}_B^\Phi(\hat{Q})$ and $a \in \Omega^1(M, \mathfrak{g}_P)$ are such that $A + a \in \mathscr{A}_B^\Phi(\hat{Q})$, then

$$\rho(a)\Phi \perp \rho(\mathfrak{g}_P)\Phi;$$

hence, $\rho(a)\Phi = 0$ and it follows that $a \in \Omega^1(M, \mathfrak{t}_P)$ by Proposition 6.13. It remains to show that $\mathscr{A}_B^\Phi(\hat{Q})$ is non-empty. To see this, note that if $A \in \mathscr{A}_B^\Phi$, then one can find $a \in \Omega^1(M, \mathfrak{g}_P)$ such that $\nabla_A\Phi + \rho(a)\Phi$ is perpendicular to $\rho(\mathfrak{g}_P)\Phi$.

We prove (3). If $A \in \mathscr{A}_B^\Phi(\hat{Q})$ and H_A denote its horizontal distribution, then we need to show that for $q \in \hat{Q}^\circ$, $H_{A,q} \subset T_q\hat{Q}^\circ$. This, however, is an immediate consequence of the definitions of $\mathscr{A}_B^\Phi(\hat{Q})$ and $\hat{Q}^\circ$.

We prove (4). Suppose $\tau \in \Gamma(\mathfrak{t}_P)$, that is, $\rho(\tau)\Phi = 0$. Differentiating this identity along v yields

$$\rho(\nabla_{A,v}\tau)\Phi = -\rho(\tau)\nabla_{A,v}\Phi;$$

Set $\sigma = \nabla_{A,v}\tau$. We need to show that $\rho(\sigma)\Phi = 0$. We compute

$$\begin{aligned}|\rho(\sigma)\Phi|^2 &= -\langle\rho(\sigma)\Phi, \rho(\tau)\nabla_{A,v}\Phi\rangle\\ &= \langle\rho(\tau)\rho(\sigma)\Phi, \nabla_{A,v}\Phi\rangle\\ &= \langle\rho([\tau,\sigma])\Phi, \nabla_{A,v}\Phi\rangle = 0\end{aligned}$$

because $\nabla_A\Phi \perp \rho(\mathfrak{g}_P)\Phi$. □

To summarize:

1. The Dirac equation $\not{D}_A\Phi = 0$ implies the Fueter equation $\mathfrak{F}_B s = 0$.
2. Given a solution s of the Fueter equation and $\hat{Q}^\circ$ as at the end of the last subsection, there is a connection $A \in \mathscr{A}_B(\hat{Q})$ such that the lift Φ satisfies $\not{D}_A\Phi = 0$.
3. A is unique up to $\Omega^1(M, \mathfrak{t}_P)$ with $\mathfrak{t}_P$ as in (6.14).

7 Degenerations of $\mathrm{ADHM}_{1,k}$ monopoles

In this section we apply the general framework of the Haydys correspondence with stabilizers to analyze solutions of the limiting $\mathrm{ADHM}_{1,k}$ Seiberg–Witten equation (5.23). This leads us to a conjecture regarding the non-compactness phenomenon for the $\mathrm{ADHM}_{1,k}$ Seiberg–Witten equation.

7.1 The ADHM construction of $\mathrm{Sym}^k \mathbf{H}$

Identifying $\mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^r = \mathrm{Hom}_{\mathbf{C}}(\mathbf{C}^k, \mathbf{H})$, we can write the quaternionic vector space S from Example 5.5 with $r = 1$ as

$$S = \mathrm{Hom}_{\mathbf{C}}(\mathbf{C}^k, \mathbf{H}) \oplus \mathbf{H} \otimes_{\mathbf{R}} \mathfrak{u}(k).$$

The group $\mathrm{U}(k)$ acts on S via

$$\rho(g)(\Psi, \xi) := (\Psi g^{-1}, \mathrm{Ad}(g)\xi)$$

preserving the hyperkähler structure. As a first step towards understanding the Haydys correspondence for the $\mathrm{ADHM}_{1,k}$ Seiberg–Witten equation we will determine the hyperkähler quotient $S /\!\!/\!\!/ \mathrm{U}(k)$ and its decomposition into hyperkähler manifolds described in Theorem 6.6.

Definition 7.1. A **partition** of $k \in \mathbf{N}$ is a non-increasing sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \ldots)$ which sums to k. The **length** of a partition is

$$|\lambda| := \min\{n \in \mathbf{N} : \lambda_n = 0\} - 1.$$

With each partition λ we associate the groups

$$G_\lambda := \left\{\sigma \in S_{|\lambda|} : \lambda_{\sigma(n)} = \lambda_n \text{ for all } n \in \{1 \ldots, |\lambda|\}\right\}$$

and

$$T_\lambda := \prod_{n=1}^{|\lambda|} \mathrm{U}(\lambda_n) \subset \mathrm{U}(k).$$

For each partition λ of k, consider the generalized diagonal

$$\Delta_{|\lambda|} = \{v_1, \ldots, v_{|\lambda|} \in \mathbf{H}^{|\lambda|} : v_i = v_j \text{ for some } i \neq j\}$$

There is an embedding $(\mathbf{H}^{|\lambda|} \backslash \Delta_{|\lambda|})/G_\lambda \hookrightarrow \operatorname{Sym}^k \mathbf{H}$ defined by

$$[v_1, \ldots, v_{|\lambda|}] \mapsto [\underbrace{v_1, \ldots, v_1}_{\lambda_1 \text{ times}}, \cdots, \underbrace{v_{|\lambda|}, \ldots, v_{|\lambda|}}_{\lambda_{|\lambda|} \text{ times}}].$$

The image of this inclusion is denoted by $\operatorname{Sym}^k_\lambda \mathbf{H}$.

Theorem 7.2 (Nakajima [Nak99, Proposition 2.9]). *We have*

$$S /\!\!/\!\!/ G = \bigcup_\lambda S_{T_\lambda} /\!\!/\!\!/ W_{U(k)}(T_\lambda) = \bigcup_\lambda \operatorname{Sym}^k_\lambda \mathbf{H} = \operatorname{Sym}^k \mathbf{H}.$$

Here we take the union over all partitions λ of k.

Proposition 7.3. *The canonical moment map $\mu\colon S \to (\mathfrak{u}(k) \otimes \operatorname{Im}\mathbf{H})^*$ for the action $\rho\colon \mathrm{U}(k) \to \operatorname{Sp}(S)$ is given by*

$$\mu(\Psi, \xi) := \mu(\Psi) + \mu(\xi)$$

with[9]

$$\begin{aligned}\mu(\Psi) &:= \frac{1}{2}\big((\Psi^* i\Psi) \otimes i + (\Psi^* j\Psi) \otimes j + (\Psi^* k\Psi) \otimes k\big) \quad \textit{and} \\ \mu(\xi) &:= ([\xi_0, \xi_1] + [\xi_2, \xi_3]) \otimes i \\ &\quad + ([\xi_0, \xi_2] + [\xi_3, \xi_1]) \otimes j \\ &\quad + ([\xi_0, \xi_3] + [\xi_1, \xi_2]) \otimes k.\end{aligned}$$

Proof. We can compute the moment maps for the action of $\mathrm{U}(k)$ on $\operatorname{Hom}(\mathbf{C}^k, \mathbf{H})$ and $\mathbf{H} \otimes \mathfrak{u}(k)$ separately. If $v = v_1 i + v_2 j + v_3 k \in \operatorname{Im}\mathbf{H}$ and $\eta \in \mathfrak{u}(k)$, then

$$2\langle \mu(\Psi), v \otimes \eta\rangle = \langle \Psi, \gamma(v)\rho(\eta)\Psi\rangle = -\langle \Psi, \gamma(v)\Psi \circ \eta\rangle = \langle \Psi^*\gamma(v)\Psi, \eta\rangle$$

and

$$\begin{aligned}2\langle \mu(\xi), v \otimes \eta\rangle &= \langle \xi, \gamma(v)\rho(\eta)\xi\rangle \\ &= v_1\big(-\langle \xi_0, [\eta, \xi_1]\rangle + \langle \xi_1, [\eta, \xi_0]\rangle - \langle \xi_2, [\eta, \xi_3]\rangle + \langle \xi_3, [\eta, \xi_2]\rangle\big) \\ &\quad + v_2\big(-\langle \xi_0, [\eta, \xi_2]\rangle + \langle \xi_1, [\eta, \xi_3]\rangle + \langle \xi_2, [\eta, \xi_0]\rangle - \langle \xi_3, [\eta, \xi_2]\rangle\big) \\ &\quad + v_3\big(-\langle \xi_0, [\eta, \xi_3]\rangle - \langle \xi_1, [\eta, \xi_2]\rangle - \langle \xi_2, [\eta, \xi_1]\rangle + \langle \xi_3, [\eta, \xi_0]\rangle\big) \\ &= 2v_1\langle [\xi_0, \xi_1] + [\xi_2, \xi_3], \eta\rangle \\ &\quad + 2v_2\langle [\xi_0, \xi_2] + [\xi_3, \xi_1], \eta\rangle \\ &\quad + 2v_3\langle [\xi_0, \xi_3] + [\xi_1, \xi_2], \eta\rangle\end{aligned}$$

using that $\langle \xi, [\eta, \zeta]\rangle = -\langle [\xi, \zeta], \eta\rangle$ for $\xi, \eta, \zeta \in \mathfrak{u}(k)$. ∎

[9] We identify $(\mathfrak{u}(k) \otimes \operatorname{Im}\mathbf{H})^* = \mathfrak{u}(k) \otimes \operatorname{Im}\mathbf{H}$.

The key to proving Theorem 7.2 is the following result.

Proposition 7.4. *If $\mu(\Psi, \xi) = 0$, then $\Psi = 0$.*

One can derive this result using Geometric Invariant Theory [Nak99, Section 2.2]. We provide a proof in Appendix A, which is essentially follows Nakajima's reasoning but avoids the use of GIT and comparison results between GIT and Kähler quotients.

It follows from Proposition 7.4 that

$$S /\!\!/\!\!/ \mathrm{U}(k) = \mathbf{H} \otimes \mathfrak{g} /\!\!/\!\!/ \mathrm{U}(k).$$

The latter can be computed in a straight-forward fashion using the following observation.

Proposition 7.5. *We have*

$$|\mu(\xi)|^2 = \frac{1}{2} \sum_{\alpha,\beta=0}^{3} |[\xi_\alpha, \xi_\beta]|^2.$$

Proof. A direct computation shows that

$$|\mu(\xi)|^2 - \frac{1}{2} \sum_{\alpha,\beta=0}^{3} |[\xi_\alpha, \xi_\beta]|^2 = -2\langle \xi_0, [\xi_1, [\xi_2, \xi_3]] + [\xi_2, [\xi_3, \xi_1]] + [\xi_3, [\xi_1, \xi_2]]\rangle.$$

This expression vanishes by the Jacobi Identity. □

Proof of Theorem 7.2. From Proposition 7.4 and Proposition 7.5 it follows that we have $\mu(\Psi, \xi) = 0$ if and only if $\Psi = 0$ and $\xi \in \mathbf{H} \otimes \mathfrak{t}$ for some maximal torus $\mathfrak{t} \subset \mathfrak{u}(k)$. Therefore, for a fixed maximal torus $T \subset \mathrm{U}(k)$ and $\mathfrak{t} := \mathrm{Lie}(T)$,

$$S /\!\!/\!\!/ G = (\mathbf{H} \otimes \mathfrak{t})/W_{\mathrm{U}(k)}(T) \cong \mathbf{H}^k / S_k = \mathrm{Sym}^k \mathbf{H},$$

using that the Weyl group of $\mathrm{U}(k)$ is the permutation group S_k.

The map $S /\!\!/\!\!/ G \to \mathrm{Sym}^k \mathbf{H}$ can be described more directly as the joint spectrum. Since $\mu(\xi) = 0$ implies $[\xi, \xi] = 0 \in \Lambda^2 \mathbf{H} \otimes \mathfrak{g}$, we can find a basis $e_1, \dots, e_k$ of $\mathbf{C}^k$ and elements $v_1, \dots, v_k \in \mathbf{H}$ such that

$$\xi(e_i) = v_i \otimes e_i.$$

Up to ordering, the v_i are independent of the choice of basis e_i. The isomorphism $S /\!\!/\!\!/ G \to \mathrm{Sym}^k \mathbf{H}$ is the map

$$\xi \mapsto \mathrm{spec}(\xi) := \{v_1, \dots, v_k\}.$$

From this description the decomposition of $\mathrm{Sym}^k \mathbf{H}$ into its strata $\mathrm{Sym}^k_\lambda \mathbf{H}$ is clear. □

The following result, which can be viewed as the linearization of Proposition 7.5, will play in important role later.

Proposition 7.6. *Denote by* $R_\xi \colon \mathfrak{u}(k) \to \mathbf{H} \otimes \mathfrak{u}(k)$ *the linearization of the action of* $\mathrm{U}(k)$ *on* $\mathbf{H} \otimes \mathfrak{u}(k)$ *at* ξ *and by* $R_\xi^* \colon \mathbf{H} \otimes \mathfrak{u}(k) \to \mathfrak{u}(k)$ *its adjoint. If* $\mu(\xi) = 0$*, then*

$$|(\mathrm{d}_\xi \mu)\boldsymbol{\eta}|^2 + \frac{1}{2}|R_\xi^* \boldsymbol{\eta}|^2 = \sum_{\alpha \neq \beta = 0}^{3} |[\xi_\alpha, \boldsymbol{\eta}_\beta]|^2 + \frac{1}{2}\sum_{\alpha=0}^{3} |[\xi_\alpha, \boldsymbol{\eta}_\alpha]|^2.$$

Proof. If $\mu(\xi) = 0$, then on the one hand

$$|\mu(\xi + t\boldsymbol{\eta})|^2 = t^2 |(\mathrm{d}_\xi \mu)\boldsymbol{\eta}|^2 + O(t^3);$$

while on the other hand

$$\begin{aligned}
|\mu(\xi + t\boldsymbol{\eta})|^2 &= \frac{1}{2}\sum_{\alpha,\beta=0}^{3} |[\xi_\alpha + t\boldsymbol{\eta}_\alpha, \xi_\beta + t\boldsymbol{\eta}_\beta]|^2 \\
&= \frac{1}{2}\sum_{\alpha,\beta=0}^{3} |[\xi_\alpha, t\boldsymbol{\eta}_\beta] + [t\boldsymbol{\eta}_\alpha, \xi_\beta]|^2 + O(t^3) \\
&= t^2 \sum_{\alpha \neq \beta = 0}^{3} |[\xi_\alpha, \boldsymbol{\eta}_\beta]|^2 + \langle [\xi_\alpha, \boldsymbol{\eta}_\beta], [\boldsymbol{\eta}_\alpha, \xi_\beta] \rangle + O(t^3).
\end{aligned}$$

We also have

$$\begin{aligned}
|R_\xi^* \boldsymbol{\eta}|^2 &= \left| \sum_{\alpha=0}^{3} [\xi_\alpha, \boldsymbol{\eta}_\alpha] \right|^2 \\
&= 2\sum_{\alpha \neq \beta = 0}^{3} \langle [\xi_\alpha, \boldsymbol{\eta}_\alpha], [\xi_\beta, \boldsymbol{\eta}_\beta] \rangle + \sum_{\alpha=0}^{3} |[\xi_\alpha, \boldsymbol{\eta}_\alpha]|^2.
\end{aligned}$$

By the Jacobi identity

$$\begin{aligned}
\langle [\xi_\alpha, \boldsymbol{\eta}_\beta], [\boldsymbol{\eta}_\alpha, \xi_\beta] \rangle &= -\langle \boldsymbol{\eta}_\beta, [\xi_\alpha, [\boldsymbol{\eta}_\alpha, \xi_\beta]] \rangle \\
&= \langle \boldsymbol{\eta}_\beta, [\boldsymbol{\eta}_\alpha, [\xi_\beta, \xi_\alpha]] \rangle + \langle \boldsymbol{\eta}_\beta, [\xi_\beta, [\xi_\alpha, \boldsymbol{\eta}_\alpha]] \rangle \\
&= \langle \boldsymbol{\eta}_\beta, [\xi_\beta, [\xi_\alpha, \boldsymbol{\eta}_\alpha]] \rangle \\
&= -\langle [\xi_\beta, \boldsymbol{\eta}_\beta], [\xi_\alpha, \boldsymbol{\eta}_\alpha] \rangle.
\end{aligned}$$

Putting everything together, yields the asserted identity. ◻

7.2 The Haydys correspondence for the ADHM$_{1,k}$ Seiberg–Witten equation

Assume the situation of Section 5.3; that is, $\mathfrak{w}$ is a spin$^{\mathrm{U}(k)}$–structure on M with spinor bundle W and adjoint bundle $\mathfrak{g}_{\mathscr{H}}$, and V is a Dirac bundle of rank 4 over M with connection C. The **limiting ADHM$_{1,k}$ Seiberg–Witten equation** for $(\Psi, \xi, A) \in \Gamma(W) \times \Gamma(V \otimes \mathfrak{g}_{\mathscr{H}}) \times \mathscr{A}^s(\mathfrak{w})$ is

$$\begin{aligned} \not{D}_A \Psi &= 0, \\ \not{D}_{A,C} \xi &= 0, \quad \text{and} \\ \mu(\Psi) + \mu(\xi) &= 0 \end{aligned} \tag{7.7}$$

as well as $\|(\Psi, \xi)\|_{L^2} = 1$. It is an immediate consequence of Proposition 7.4 that

$$\Psi = 0;$$

and it follows from the discussion at the beginning of Section 6 and Theorem 7.2 that (Ψ, ξ) induces a section $\tilde{n}$ of the bundle of hyperkähler quotients $\mathbf{X} = \mathrm{Sym}^k V$; thus $\tilde{n} \in \Gamma(\mathrm{Sym}^k V)$. Let us assume that, in fact,

$$\tilde{n} \in \Gamma(\mathrm{Sym}^k_\lambda V),$$

for some partition λ of k, as in Section 4.3. In the notation of Section 6, $\mathrm{Sym}^k_\lambda V = \mathbf{X}_{(T_\lambda)}$. By Proposition 6.22, $\tilde{n}$ satisfies the Fueter equation.

The following summarizes the Haydys correspondence for (7.7). On first reading, the reader might want to assume that $\lambda = (1, \dots, 1)$, the partition yielding the top stratum of $\mathrm{Sym}^k \mathbf{H}$, since this simplifies the situation noticeably. For $j = 1, \dots, m$, denote by k_j the j–th largest positive number appearing in the partition λ and by ℓ_j the multiplicity with which it appears. The reader should keep in mind that

$$W_{\hat{H}}(T_\lambda) = \left(\prod_{j=1}^m S_{\ell_j} \right) \times \mathrm{SO}(4)$$

with S_{ℓ_j} being the permutation groups, and

$$\begin{aligned} N_{\hat{H}}(T_\lambda) &= \mathrm{Spin}(4) \times_{\mathbf{Z}_2} \left(\prod_{j=1}^m S_{\ell_j} \ltimes \mathrm{U}(k_j)^{\ell_j} \right) \\ &= \left(\mathrm{Spin}(3) \times_{\mathbf{Z}_2} \left(\prod_{j=1}^m S_{\ell_j} \ltimes \mathrm{U}(k_j)^{\ell_j} \right) \right) \times_{\mathrm{SO}(3)} \mathrm{SO}(4) \end{aligned}$$

with $\hat{H}$ as in Example 5.5 and with T_λ as in Definition 7.1.

Proposition 7.8. *Given $\tilde{n} \in \Gamma(\mathrm{Sym}^k_\lambda V)$, set*

$$\tilde{M} := \{(x, v) \in \mathrm{Tot}\, V : v \in \tilde{n}(x)\}$$

and denote by $\pi\colon \tilde{M} \to M$ by the projection map.

1. *The map π is a $|\lambda|$–fold unbranched cover of M. Moreover, we can decompose $\tilde{M}$ into components $\tilde{M}_1, \dots, \tilde{M}_m$ such that $\pi_j := \pi|_{\tilde{M}_j}$ restricts to a ℓ_j–fold cover on $\tilde{M}_j$.*

2. *Denote by $Q^\diamond$ the principal $\prod_{j=1}^m S_{\ell_j}$–bundle over M whose fibre over x is*

$$Q^\diamond_x = \prod_{j=1}^m \mathrm{Bij}\big(\{1, \dots, \ell_j\}, \pi_j^{-1}(x)\big).$$

   The Weyl group bundle associated with $\tilde{n}$ is

$$\hat{Q}^\diamond = Q^\diamond \times \mathrm{SO}(V).$$

3. *The choice of a $\mathrm{N}_{\hat{H}}(T_\lambda)$–bundle $\hat{Q}^\circ$ lifting $\hat{Q}^\diamond$ is equivalent to the choice of a $\mathrm{spin}^{\mathrm{U}(k_j)}$–structure $\mathfrak{w}_j$ on $\tilde{M}_j$ for each $j = 1, \dots, m$.*

4. *Given a $\mathrm{spin}^{\mathrm{U}(k_j)}$–structure $\mathfrak{w}_j$ on $\tilde{M}_j$ for each $j = 1, \dots, m$, denote by (Ψ, ξ) the lift of $\tilde{n}$. The space of connections $\mathscr{A}_C^{\Psi,\xi}(\hat{Q})$, defined in (6.24), is identified with the space*

$$\prod_{j=1}^m \mathscr{A}^s(\mathfrak{w}_j)$$

   and $\mathfrak{t}_P$, defined in (6.14), is identified with the sum of the push-forward bundles

$$\bigoplus_{j=1}^m (\pi_j)_* \mathfrak{g}_{\mathscr{H}_j}.$$

In particular, a solution (Ψ, ξ, A) of (7.7) with $(\Psi, \xi) \in \Gamma(\mathrm{S}_{(T_\lambda)})$ is equivalent to a Fueter section $\tilde{n} \in \Gamma(\mathrm{Sym}^k_\lambda V)$ together with a $\mathrm{spin}^{\mathrm{U}(k_j)}$–structure $\mathfrak{w}_j$ on $\tilde{M}_j$ and a spin connection A_j on $\mathfrak{w}_j$ for each $j = 1, \dots, m$.

Remark 7.9. If $\lambda = (1, \dots, 1)$, then $m = 1$ and $\mathfrak{w}_1$ is simply a spin^c–structure on $\tilde{M}$.

Proof. This essentially follows from the discussion in Section 6.2 and Section 6.3 together with Theorem 7.2; however, (3) and (4) might require some explanation. Given a $\mathrm{spin}^{\mathrm{U}(k_j)}(3)$–structure $\mathfrak{w}_j$ on $\tilde{M}_j$ for each $j = 1, \dots, m$, denote by $\tilde{\mathfrak{w}}_j$ the corresponding $\mathrm{spin}^{\mathrm{U}(k_j)}(4)$–structure on $\pi_j^* V$. The principal $N_{\hat{H}}(T_\lambda)$–bundle $\hat{Q}^\circ$ with fibre over x given by

$$\hat{Q}^\circ_x = \prod_{j=1}^{m} \Big\{ (f, g_1, \ldots, g_{\ell_j}) \in \mathrm{Bij}\big(\{1, \ldots, \ell_j\}, \pi_j^{-1}(x)\big) \times \tilde{\mathfrak{w}}_j^{\ell_j} : g_i \in (\tilde{\mathfrak{w}}_j)_{f(i)} \Big\}$$

lifts $\hat{Q}^\diamond$. Conversely, given principal $N_{\hat{H}}(T_\lambda)$–bundle $\hat{Q}^\circ$ lifting $\hat{Q}^\diamond$ its pullback to $\tilde{M}_j$ contains a principal $\mathrm{Spin}^{\mathrm{U}(k_j)}(4)$–bundle $\tilde{\mathfrak{w}}_j$ which yields a $\mathrm{spin}^{\mathrm{U}(k_j)}$–structure on $\pi_j^* V$ and thus on $\tilde{M}_j$. With this discussion in mind (4) becomes apparent. □

7.3 Formal expansion around limiting solutions

Proposition 7.8(4) imposes very weak conditions on the connection $A \in \mathscr{A}^s(\mathfrak{w})$ for solutions of the limiting equation (7.7). To determine candidates for further constrains let $(\Psi_0 = 0, \xi_0, A_0)$ be a solution of (7.7) with $\tilde{n} \in \Gamma(\mathrm{Sym}^k_\lambda V)$ for some partition λ of k, and suppose

$$\Psi_\varepsilon = \sum_{i=1}^{\infty} \varepsilon^i \Psi_i, \quad \xi_\varepsilon = \sum_{i=0}^{\infty} \varepsilon^i \xi_i, \quad \text{and} \quad A_\varepsilon = A_0 + \sum_{i=i}^{\infty} \varepsilon^i a_i$$

is a formal power series solution of

$$\begin{aligned} \not{D}_{A_\varepsilon} \Psi_\varepsilon &= 0, \\ \not{D}_{A_\varepsilon, C} \xi_\varepsilon &= 0, \quad \text{and} \\ \varepsilon^2 \varpi F_{A_\varepsilon} &= \mu(\xi_\varepsilon). \end{aligned} \tag{7.10}$$

Moreover, we can assume the gauge fixing condition $\xi_1 \perp \rho(\mathfrak{g}_P)\xi_0$, that is,

$$R^*_{\xi_0} \xi_1 = 0$$

in the notation of Proposition 7.6. From Proposition 7.8, we know that

$$\xi_0 = (\xi_{0,1}, \cdots, \xi_{0,m}) \in \Gamma(V \otimes \mathfrak{t}_P) \quad \text{with} \quad \mathfrak{t}_P = \bigoplus_{j=1}^{m} (\pi_j)_* \mathfrak{g}_{\mathscr{H}_j}$$

and A_0 arises from spin connections $A_{0,j} \in \mathscr{A}^s(\mathfrak{w}_j)$. Here $\mathfrak{g}_{\mathscr{H}_j}$ denotes the adjoint bundle of the $\mathrm{spin}^{\mathrm{U}(k_j)}$–structure $\mathfrak{w}_j$.

The coefficient of ε on the right-hand side of the second equation of (7.10) must vanish; hence,

$$(\mathrm{d}_{\xi_0} \mu) \xi_1 = 0.$$

By Proposition 7.6 it follows that $[\xi_0 \wedge \xi_1] = 0$ and therefore

$$\mu(\xi_1) \in \Omega^2(M, [\mathfrak{t}_P, \mathfrak{t}_P])$$

by the following self-evident observation combined with Theorem 7.2.

Proposition 7.11. *If $\xi_0, \xi_1 \in \mathbf{H} \otimes \mathfrak{g}$, $[\xi_0 \wedge \xi_1] = 0$, and the stabilizer of $\xi_0 \in \mathrm{U}(k)$ is precisely $T_\lambda = \prod_{n=1}^{|\lambda|} \mathrm{U}(\lambda_n)$, then $\xi_1 \in \mathbf{H} \otimes \mathfrak{t}_\lambda$ with $\mathfrak{t}_\lambda = \bigoplus_{n=1}^{|\lambda|} \mathfrak{u}(\lambda_n)$. In particular,*

$$[\xi_1 \wedge \xi_1] \in \mathbf{H} \otimes [\mathfrak{t}_\lambda, \mathfrak{t}_\lambda] \subset \mathbf{H} \otimes \mathfrak{t}_\lambda.$$

Remark 7.12. If $\lambda = (1, \ldots, 1)$, then $[\mathfrak{t}_P, \mathfrak{t}_P] = 0$; cf. Remark 5.24.

The third equation in (7.10) to order ε^2 is thus equivalent to

$$\varpi F_{A_0} = \mu(\xi_1) + (\mathrm{d}_{\xi_0}\mu)\xi_2 + \mu(\Psi_1). \tag{7.13}$$

In terms of the spin connections $A_{0,j} \in \mathscr{A}^s(\mathfrak{w}_j)$, we have

$$\varpi F_{A_0} = \bigoplus_{j=1}^{m} (\pi_j)_* \varpi F_{A_{0,j}} \in \Omega^2(M, \mathfrak{t}_P).$$

By (6.9), we have

$$(\mathrm{d}_{\xi_0}\mu)\xi_2 \in \Omega^2(M, \mathfrak{t}_P^\perp).$$

Thus, if we denote by $\mu_\parallel(\Psi_1)$ the component of $\mu(\Psi_1)$ in $\mathfrak{t}_P$ and by $\mu_\perp(\Psi_1)$ the component of $\mu(\Psi_1)$ in $\mathfrak{t}_P^\perp \subset \mathfrak{g}_P$, then (7.13) is equivalent to

$$\begin{aligned} \varpi F_{A_0} &= \mu(\xi_1) + \mu_\parallel(\Psi_1) \quad \text{and} \\ (\mathrm{d}_{\xi_0}\mu)\xi_2 &= -\mu_\perp(\Psi_1). \end{aligned} \tag{7.14}$$

Since $\mathfrak{t}_P$ is parallel with respect to A_0 and $V \otimes \mathfrak{t}_P$ is perpendicular to $\bar{\gamma}(T^*M \otimes \mathfrak{g}_P)\xi_0$, the first and the second equation of (7.10) to order ε are equivalent to

$$\begin{aligned} \not{D}_{A_0}\Psi_1 &= 0, \\ \not{D}_{A_0,C}\xi_1 &= 0, \quad \text{and} \\ \gamma(a_1)\xi_0 &= 0. \end{aligned} \tag{7.15}$$

Let $\tilde{\Psi}_{1,j} \in \Gamma(W_j)$ and $\tilde{\xi}_j \in \Gamma(V \otimes \mathfrak{g}_{\mathscr{H}_j})$ be such that

$$\Psi_1 = \bigoplus_{j=1}^{m} (\pi_j)_* \tilde{\Psi}_{1,j} \quad \text{and} \quad \xi_1 = \bigoplus_{j=1}^{m} (\pi_j)_* \tilde{\xi}_{1,j}.$$

The first equation of (7.14) and the first two equations of (7.15) are precisely equivalent to the ADHM_{1,k_j} Seiberg–Witten equation

$$\begin{aligned} \not{D}_{A_{0,j}}\tilde{\Psi}_{1,j} &= 0, \\ \not{D}_{A_{0,j},C}\xi_{1,j} &= 0, \quad \text{and} \\ \varpi F_{A_{0,j}} &= \mu(\tilde{\Psi}_{1,j}) + \mu(\tilde{\xi}_{1,j}) \end{aligned}$$

on $\tilde{M}_j$ for $j = 1, \ldots, m$.

7.4 A compactness conjecture for $\mathrm{ADHM}_{1,k}$ monopoles

The discussion in the preceding sections together with known compactness results for Seiberg–Witten equations [Tau13a; Tau13b; HW15; Tau16; Tau17] lead us to the following conjecture.

Conjecture 7.16. *Let $(\varepsilon_i, \Psi_i, \xi_i, A_i)$ be a sequence of solutions of the blown-up $\mathrm{ADHM}_{1,k}$ Seiberg–Witten equation*

$$\begin{aligned} \not{D}_{A_i}\Psi_i &= 0, \\ \not{D}_{A_i,C}\xi_i &= 0, \\ \varepsilon_i^2 \varpi F_{A_i} &= \mu(\Psi_i) + \mu(\xi_i), \quad \text{and} \\ \|(\Psi_i, \xi_i)\|_{L^2} &= 1 \end{aligned}$$

with $\varepsilon_i \to 0$. After passing to a subsequence the following hold:

1. *There is a closed subset $Z \subset M$ of Hausdorff dimension at most one, such that outside of Z and up to gauge transformations (Ψ_i, ξ_i, A_i) converges to a limit $(0, \xi_0^\infty, A_0^\infty)$ and $\varepsilon_i^{-1}(\Psi_i, \xi_i - \xi_0^\infty)$ converges to a limit $(\Psi_1^\infty, \xi_1^\infty)$.*
2. *The triple $(0, \xi^\infty, A^\infty)$ is a solution of the limiting $\mathrm{ADHM}_{1,k}$ Seiberg–Witten equation (7.7).*
3. *There is a section $\tilde{n} \in \Gamma(M\backslash Z, \mathrm{Sym}^k_\lambda V)$ for some partition λ of k induced by ξ_0^∞. The section $\tilde{n}$ extends to to a continuous section of $\mathrm{Sym}^k V$ on all of M.*
4. *Denote by $\tilde{M}\backslash\tilde{Z}$ the unbranched cover of $M\backslash Z$ induced by $\tilde{n}$. If k_j, $\tilde{M}_j\backslash\tilde{Z}_j$, $\mathfrak{w}_j$ are as in Proposition 7.8 and $A_{0,j} \in \mathscr{A}^s(\mathfrak{w}_j)$ denote the spin connections giving rise to A_0^∞, and $\tilde{\Psi}_{1,j}$ and $\tilde{\xi}_{1,j}$ are such that*

   $$\Psi_1^\infty = \bigoplus_{j=1}^m (\pi_j)_*\tilde{\Psi}_{1,j} \quad \text{and} \quad \xi_1^\infty = \bigoplus_{j=1}^m (\pi_j)_*\tilde{\xi}_{1,j},$$

   then, for each $j = 1, \dots, m$, $(\tilde{\Psi}_{1,j}, \tilde{\xi}_{1,j}, A_{0,j})$ is a solution of the ADHM_{1,k_j} Seiberg–Witten equation on $\tilde{M}_j\backslash\tilde{Z}_j$.

Remark 7.17. The reader should observe that while $\tilde{M}_j$ in $\tilde{M}_j\backslash\tilde{Z}_j$ does exists, it need not be a smooth manifold.

Remark 7.18. If $\Psi = 0$, $V = TM \oplus \mathbf{R}$ and $(a, \xi) \in \Omega^1(M, \mathfrak{g}_{\mathscr{H}}) \oplus \Omega^0(M, \mathfrak{g}_{\mathscr{H}}) = \Gamma(V \otimes \mathfrak{g}_{\mathscr{H}})$, then the $\mathrm{ADHM}_{1,k}$ Seiberg–Witten equation becomes the equation

$$\begin{aligned} F_{A+ia} - *[\xi, a] + *i\mathrm{d}_A\xi &= 0 \quad \text{and} \\ \mathrm{d}_A^* a &= 0 \end{aligned} \tag{7.19}$$

with

$$F_{A+ia} = F_A - \frac{1}{2}[a \wedge a] + i\mathrm{d}_A a.$$

If (a, ξ, A) is a solution of (7.19) and M is closed, then a simple integration by parts argument shows that $\mathrm{d}_A \xi = 0$; hence, $F_{A+ia} = 0$. That is, (7.19) is effectively the condition that condition that $A + ia$ is a flat $\mathrm{GL}_k(\mathbf{C})$–connection together with the moment map equation $\mathrm{d}_A^* a = 0$.

Conjecture 7.16 thus predicts that as limits of flat $\mathrm{GL}_k(\mathbf{C})$–connections we should see data consisting of a closed subset $Z \subset M$ of Hausdorff dimension at most one, $m \in \mathbf{N}$ and, for each $j = 1, \dots, m$, a ℓ_j–fold cover $\tilde{M}\backslash\tilde{Z}_j$ of $M\backslash Z$, and solutions of (7.19) on $\tilde{M}_j\backslash\tilde{Z}_j$ such that $\sum_{j=1}^m \ell_j k_j = k$.

8 A tentative proposal

Let ψ be a tamed, closed, definite 4–form, let P be a compact, connected, oriented 3–manifold, let $P \subset Y$ be an unobstructed associative embedding. Set

$$\mathfrak{M}^{1,k}(P, \psi) := \coprod \mathfrak{M}^{1,k}_{\mathfrak{w}}(P, \psi)$$

with the disjoint union taken over all spin$^{\mathrm{U}(k)}$–structures $\mathfrak{w}$ on P and

$$\mathfrak{M}^{1,k}_{\mathfrak{w}}(P, \psi) := \frac{\left\{(\Psi, \xi, A) \in \Gamma(W) \times \Gamma(NP \otimes \mathfrak{g}_{\mathscr{H}}) \times \mathscr{A}^s(\mathfrak{w}) : \begin{array}{c}(\Psi, \xi, A) \text{ satisfies } (5.22) \\ \text{with respect to } g_\psi|_P\end{array}\right\}}{\mathscr{G}^s(\mathfrak{w})}.$$

Ignoring issues to do with reducible solutions, one should be able to extract a number

$$w(kP, \psi) \in \mathbf{Z}$$

by counting $\mathfrak{M}^{1,k}(P, \psi)$, at least, for generic ψ and possibly after slightly perturbing the ADHM Seiberg–Witten equation (5.22). More generally, if P has connected components $P^1, \dots, P^m$ and $k_1, \dots, k_m \in \mathbf{N}$, we set

$$w(k_1 \cdot P^1 + \cdots + k_m \cdot P^m, \psi) := \prod_{j=1}^{k} w(k_j \cdot P^j, \psi).$$

For $k = 1$, this number is the Seiberg–Witten invariant $\mathrm{SW}(P) \in \mathbf{Z}$ mentioned in Section 3.3 For $k > 0$, this number should be independent of the choice of perturbation but it will depend on ψ. Assume the situation of Section 4.1; that is, we have

- a generic 1–parameter family of tamed, closed, definite 4–forms $(\psi_t)_{t\in(-T,T)}$,

- 1–parameter family of compact, connected, unobstructed embedded associative submanifolds $(P_t)_{t\in(-T,T)}$ with respect to $(\psi_t)_{t\in(-T,T)}$, and
- for $j = 1, \dots, m$, 1–parameter family of compact, connected, unobstructed embedded associative submanifold $(P_t^j)_{t\in(-T,0)}$ with respect to $(\psi_t)_{t\in(-T,0)}$ such that

  $$P_t^j \to \ell_j \cdot P_0$$

  as integral currents as t tends to zero for some $\ell_j \in \{2, 3, \dots\}$.

Given $k_1, \dots, k_m$, set

$$k \coloneqq \sum_{j=1}^{m} \ell_j k_j.$$

From the discussion in preceding three sections we expect that, for $0 < t \ll 1$,

$$w(k \cdot P_{-t}, \psi_{-t}) + w(k_1 \cdot P^1_{-t} + \dots + k_m \cdot P^m_{-t}, \psi_{-t}) = w(k \cdot P_{+t}, \psi_{+t}) \tag{8.1}$$

because Conjecture 7.16 suggests that as t passes through zero $w(k_1 \cdot P_0^1 + \dots + k_m \cdot P_0^1, \psi_0)$ $\mathrm{ADHM}_{1,k}$ monopoles on P_t degenerate and disappear (if counted with the correct sign).

Suppose that one can indeed define a weight w as above satisfying (8.1) as well as analogues of (3.9). Define

$$n_\beta(\psi) = \sum w(k_1 \cdot P^1 + \dots + k_m \cdot P^m, \psi) \tag{8.2}$$

with the summation ranging over all $m \in \mathbf{N}$, $k_1, \dots, k_m \in \mathbf{N}$ and all compact, connected, unobstructed embedded associative submanifolds $P^1, \dots, P^m \subset Y$ such that

$$\sum_{j=1}^{m} k_j [P^j] = \beta.$$

This number would be invariant under the transitions described in Section 3.1, Section 3.2, and Section 4.1.

From Section 3.3 we know that reducible solutions will prevent us from defining w in general. However, the above can serve as a first approximation. To deal with reducibles one likely has to develop $\mathrm{ADHM}_{1,k}$ analogues of Kronheimer and Mrowka's monopole homology and construct a enormous chain complex extending (3.29) which does depend on ψ but whose homology does not.

9 Counting holomorphic curves in Calabi–Yau 3–folds

Let Z be a Calabi–Yau 3–fold with Kähler form ω and holomorphic volume form Ω. The product $S^1 \times Z$ is naturally a G_2–manifold with the G_2–structure given by

$$\phi = dt \wedge \omega + \operatorname{Re}\Omega.$$

Every holomorphic curve $\Sigma \subset Z$ gives rise to an associative submanifold $S^1 \times \Sigma \subset S^1 \times Z$.

Proposition 9.1. *Let $\beta \in H_2(Z)$ be a homology class. Every associative submanifold in $S^1 \times Z$ representing a $[S^1] \times \beta$ is necessarily of the form $S^1 \times \Sigma$ with $\Sigma \subset Z$ a holomorphic curve.*

Proof. The argument in this proof goes back to Lewis [Lew98, Section 3.2]. Let $P \subset S^1 \times Z$ be an associative submanifold representing $[S^1] \times \beta$. Since

$$\phi|_P = (\mathrm{d}t \wedge \omega + \operatorname{Re}\Omega)|_P = \mathrm{vol}_P$$

there is a smooth function f on P such that

$$\mathrm{d}t \wedge \omega|_P = f\mathrm{vol}_P \quad \text{and} \quad \operatorname{Re}\Omega|_P = (1-f)\mathrm{vol}_P$$

By Wirtinger's inequality [Wir36], $f \leq 1$. We need to prove that $f = 1$, since this implies that ∂_t is tangent to P and, therefore, P is of the form $S^1 \times \Sigma$, with $\Sigma \subset Z$ calibrated by ω.

One the one hand we have

$$\int_P \mathrm{vol}_P = \langle [\phi], [P] \rangle = \langle [\mathrm{d}t \wedge \omega] + [\operatorname{Re}\omega], [S^1] \times \beta \rangle = \langle [\mathrm{d}t \wedge \omega], [S^1] \times \beta \rangle,$$

while on the other hand

$$\int_P f\mathrm{vol}_P = \int_P \mathrm{d}t \wedge \omega = \langle [\mathrm{d}t \wedge \omega], [P] \rangle = \langle [\mathrm{d}t \wedge \omega], [S^1] \times \beta \rangle.$$

It follows that f has mean-value 1 and thus $f = 1$ because $f \leq 1$. □

The deformation theory of the associative submanifold $S^1 \times \Sigma$ in $S^1 \times Z$ coincides with that of the holomorphic curve Σ in Y [CHNP15, Lemma 5.11]. In particular, the putative enumerative theory for associative submanifolds discussed in this paper should give rise to an enumerative theory for holomorphic curves in Calabi–Yau 3–folds. Algebraic geometry abounds in such theories and various interplays between them; see [PT14] for a beautiful exposition of this rich subject. Our approach is closer in spirit to the original differential-geometric proposal by Donaldson and Thomas [DT98]. We will argue, however, that it should recover a theory already known to algebraic geometers.

9.1 The Seiberg–Witten invariants of Riemann surfaces

In the naive approach of Section 3.3 each associative submanifold is counted with its total Seiberg–Witten invariant. The Seiberg–Witten equation (3.11) over the 3–manifold $M = S^1 \times \Sigma$, equipped with a product Riemannian metric, was studied extensively, e.g., by Morgan [Mor96], Mrowka, Ozsváth, and Yu [MOY97], and Muñoz and Wang [MW05]. The equation admits irreducible solutions only for the spin^c structures pulled-back from Σ. Such a spin^c structure corresponds to a Hermitian line bundle $L \to \Sigma$; the induced spinor bundle is $W = L \oplus T^*\Sigma^{0,1} \otimes L$. Up to gauge transformations, all irreducible solutions of the Seiberg–Witten equation are pulled-back from configurations on Σ satisfying the vortex-type equation for $(\psi_1, \bar{\psi}_2) \in \Gamma(L) \oplus \Omega^{0,1}(\Sigma, L)$ and $A \in \mathscr{A}(\det(W))$:

$$
\begin{aligned}
\bar{\partial}_A \psi_1 = 0, \quad \bar{\partial}_A^* \bar{\psi}_2 &= 0, \\
\langle \psi_1, \bar{\psi}_2 \rangle &= 0, \\
\frac{1}{2} i * F_A + |\psi_1|^2 - |\bar{\psi}_2|^2 &= 0.
\end{aligned}
\tag{9.2}
$$

Here $\langle \psi_1, \bar{\psi}_2 \rangle$ is the $(0,1)$–form obtained from pairing ψ_1 and $\bar{\psi}_2$ using the Hermitian inner product.

The second equation implies that either ψ_1 or $\bar{\psi}_2$ must vanish identically—which one, depends on the sign of the degree $2d := \langle c_1(W), \Sigma \rangle$. Supposing without loss of generality $d < 0$, it follows from integrating the third equation that $\psi_1 \neq 0$ and so $\bar{\psi}_2 = 0$, and the pair (A, ψ_1) defines a holomorphic line bundle of degree

$$\deg(L) = g - 1 + d$$

together with a non-zero holomorphic section; where to compute the degree of L, we have used that $\det(W) = L^2 \otimes K_\Sigma^{-1}$ where K_Σ is the canonical bundle of Σ. Such a pair corresponds to a degree $g - 1 + d$ effective divisor on Σ, the zero set of ψ_1 counted with multiplicities, which is just an element of the symmetric product $\mathrm{Sym}^{g-1+d} \Sigma$. In fact, the correspondence goes both ways.

Theorem 9.3 (Noguchi [Nog87], Bradlow [Bra90], and García-Prada [Gar93]). *If $g \geq 1$, the moduli space of solutions of (9.2) over Σ is homeomorphic to $\mathrm{Sym}^{g-1-|d|} \Sigma$ for $|d| \leq g - 1$ and empty otherwise. (For $|d| = 0$ the third equation is perturbed by a small non-zero multiple of the Kähler form.)*

Unless $|d| = g - 1$, the moduli space has positive dimension and in particular is not regular from the viewpoint of the deformation theory of the Seiberg–Witten equation. The cokernels of the linearization of the equation form a vector bundle over the moduli space. This obstruction bundle can be shown to be isomorphic, with its natural orientation, to the cotangent bundle of $\mathrm{Sym}^{g-1-|d|} \Sigma$ with the orientation induced from the complex structure. By general theory, the Seiberg–Witten invariant is given by the integral of the Euler class of the obstruction bundle over the moduli space; in this case, it is simply the signed Euler characteristic

$$(-1)^{g-1-|d|}\chi(\mathrm{Sym}^{g-1-|d|}\Sigma).$$

As a consequence, the total Seiberg–Witten invariant is

$$\begin{aligned}\mathrm{SW}(S^1\times\Sigma) &= \sum_d (-1)^{g-1-|d|}\chi(\mathrm{Sym}^{g-1-|d|}\Sigma)\\ &= \sum_d (-1)^{g-1+d}\chi(\mathrm{Sym}^{g-1+d}\Sigma).\end{aligned}$$

Here we can sum over all $d\in\mathbf{Z}$ since for $|d|>g-1$ we have $\chi(\mathrm{Sym}^{g-1-|d|}\Sigma)=0$, and the last equality follows from the symmetry $\chi(\mathrm{Sym}^{g-1-d}\Sigma)=\chi(\mathrm{Sym}^{g-1+d}\Sigma)$.

9.2 Rational curves and the Meng–Taubes invariant

We have so far ignored the borderline case $g=0$. Indeed, when $\Sigma=S^2$, the moduli space of irreducible solutions of (9.2) is empty for all choices of d. This is consistent with the general theory alluded to in Section 3.3: we have $b_1(S^1\times S^2)=1$ and, due to the appearance of reducible solutions, the total Seiberg–Witten invariant is defined only for 3–manifolds with $b_1\geq 2$. In full generality, this problem can be solved within the framework of Floer homology. However, if one considers only closed, oriented 3–manifolds with $b_1\geq 1$ there is also a middle ground approach due to Meng and Taubes [MT96]. For every such a 3–manifold M they define an invariant

$$\underline{\mathrm{SW}}(M)\in\mathbf{Z}[\![H]\!]/H.$$

Here H is the torsion-free part of $H^2(M,\mathbf{Z})$, $\mathbf{Z}[\![H]\!]$ is the set of $\mathbf{Z}$–valued functions on H, and H acts on $\mathbf{Z}[\![H]\!]$ by pull-back.

Remark 9.4. For $b_1>1$, the invariant is in fact in the ring $\mathbf{Z}[H]/H$, where $\mathbf{Z}[H]$ denotes the ring of functions with finite support, and for $b_1=1$, under the identification $H=\mathbf{Z}\langle q\rangle$ for a generator q, we have $(1+q)^2\underline{SW}(M)\in\mathbf{Z}[H]/H$. This is essentially a consequence of the compactness theorem for Seiberg–Witten monopoles.

The Meng–Taubes invariant takes a particularly simple form for $M=S^1\times\Sigma$. In this case, there is a distinguished spin^c structure, corresponding to the line bundle L being trivial, and the invariant can be naturally lifted to an element $\underline{\mathrm{SW}}(M)\in\mathbf{Z}[\![H]\!]$. Moreover, the support of $\underline{\mathrm{SW}}(M)$ is $\mathbf{Z}=H^2(\Sigma,\mathbf{Z})\subset H$, reflecting the fact that the Seiberg–Witten equation has solutions only for the spin^c structures pulled-back from Σ. Combining this with Remark 9.4, we see that $\underline{\mathrm{SW}}(M)$ can be interpreted as an element of the ring of formal Laurent series in a single variable q say, that is:

$$\underline{\mathrm{SW}}(M) \in \mathbf{Z}(\!(q)\!).$$

For $g \geq 1$, this is simply the Laurent polynomial whose coefficients are the non-zero Seiberg–Witten invariants:

$$\underline{\mathrm{SW}}(S^1 \times \Sigma) = \sum_d (-1)^{g-1+d} \chi(\mathrm{Sym}^{g-1+d} \Sigma) q^d \tag{9.5}$$

and we see that $\mathrm{SW}(S^1 \times \Sigma)$ is obtained by evaluating $\underline{\mathrm{SW}}(S^1 \times \Sigma)$ at $q = 1$. In fact, it is easy to see from the definition of the Meng–Taubes invariant that the same formula is true for $\Sigma = S^2$, although now the series has infinitely many non-zero terms. The point is that for a 3–manifold M with $b_1(M) = 1$, the Meng–Taubes invariant is defined using a perturbation of the Seiberg–Witten equation (3.11):

$$\begin{aligned} \not{D}_A \Psi &= 0 \quad \text{and} \\ \frac{1}{2} F_A &= \mu(\Psi) + \lambda\omega, \end{aligned} \tag{9.6}$$

where ω is a closed 2–form generating $H^2(M, \mathbf{R})$, and λ is a large number, depending on the chosen spin^c structure. For $M = S^1 \times S^2$, this amounts to perturbing the third equation in (9.2) by a large multiple of the Kähler form on S^2. As a result, the third equation can always be solved and for every $d > 0$ the moduli space of solutions of the perturbed equation consists of effective divisors of degree $d - 1$; these form the symmetric product $\mathrm{Sym}^{d-1} S^2 = \mathbf{C}P^{d-1}$ and the Meng–Taubes invariant of $S^1 \times S^2$ is given by (9.5) for $g = 0$. In particular, one cannot evaluate $\underline{\mathrm{SW}}(S^1 \times S^2)$ at $q = 1$ and is forced to work with the refined invariant.

9.3 A holomorphic Chern–Simons–Dirac functional

Like associative submanifolds, holomorphic curves in a Calabi–Yau 3–fold are critical points of a functional on the space of submanifolds.

Definition 9.7. For a closed, oriented surface Σ, let $\mathrm{Imm}(\Sigma, Z)$ be the space of immersions from Σ to Z. Define a complex 1–form $\delta\mathfrak{L}^{\mathbf{C}} \in \Omega^1(\mathrm{Imm}(\Sigma, Z), \mathbf{C})$ by

$$\delta_\iota \mathfrak{L}^{\mathbf{C}}(n) := \int_\Sigma \iota^* i(n)\Omega,$$

for $\iota \in \mathrm{Imm}(\Sigma, Z)$ and $n \in T_\iota \mathrm{Imm}(\Sigma, Z) = \Gamma(\Sigma, \iota^* TZ)$.

Proposition 9.8.

1. *An immersion $\iota\colon \Sigma \to Z$ is holomorphic with respect to the conformal structure induced from $\iota^* g$ if and only if $\delta_\iota \mathfrak{L}^{\mathbf{C}} = 0$ and $\iota^*\omega$ is positive.*

2. *$\delta\mathfrak{L}^{\mathbf{C}}$ is* $\mathrm{Diff}_+(\Sigma)$*–invariant.*

3. *There is a* $\mathrm{Diff}_+(\Sigma)$*–equivariant covering space $\pi\colon \widetilde{\mathrm{Imm}}(\Sigma, Z) \to \mathrm{Imm}(\Sigma, Z)$ and a* $\mathrm{Diff}_+(\Sigma)$*–equivariant function $\tilde{\mathfrak{L}}^{\mathbf{C}}\colon \widetilde{\mathrm{Imm}}(\Sigma, Z) \to \mathbf{C}$ whose derivative is $\pi^*\delta\mathfrak{L}^{\mathbf{C}}$.*

Proof. This is an analogue of Proposition 2.16. To prove (1) we can use the dimensional reduction of the associator identity (2.10):

$$|\omega(u,v)|^2 + |i(u,v)\,\mathrm{Re}\,\Omega|^2 = |u \wedge v|^2$$

for all tangent vectors u, v to Z. The condition $\delta_\iota \mathfrak{L}^{\mathbf{C}} = 0$ is equivalent to $i(u,v)\,\mathrm{Re}\,\Omega = 0$ whenever u and v are tangent to $\mathrm{im}\,\iota$. On the other hand, by Wirtinger's inequality, the equality $\iota^*\omega = \mathrm{vol}_{\iota^* g}$ is equivalent to ι being holomorphic. The function $\tilde{\mathfrak{L}}^{\mathbf{C}}$ from (3) is proved by integrating Ω along 3–dimensional cycles bounding Σ. □

For a given complex curve Σ one can also consider a holomorphic analogue of the Chern–Simons–Dirac functional [Nak15, Section 7]. For a triple $(A, \psi_1, \psi_2) \in \mathscr{A}(\det(W)) \times \Gamma(L) \times \Omega^{1,0}(\Sigma, L^*)$ define

$$\mathrm{CSD}^{\mathbf{C}}(A, \psi_1, \psi_2) := \int_\Sigma \bar{\partial}_A \psi_1 \wedge \psi_2.$$

The functional is invariant under the action of the group of complex gauge transformations $C^\infty(\Sigma, \mathbf{C}^*)$ and its critical points are triples satisfying the first two equations of (9.2). Of course, the point of Theorem 9.3 is that the $C^\infty(\Sigma, \mathbf{C}^*)$–equivalence classes of such triples are in one-to-one correspondence with $C^\infty(\Sigma, \mathrm{U}(1))$–equivalence classes of solutions of the full equation (9.2) and so the critical locus of $\mathrm{CSD}^{\mathbf{C}}$ consists of effective divisors on Σ.

We can now combine the two complex functionals and, as in Section 3.5, for every coupling constant $\kappa > 0$ consider

$$\mathfrak{L}\mathrm{CSD}^{\mathbf{C}}_\kappa(\iota, A, \psi_1, \psi_2) = \mathfrak{L}^{\mathbf{C}}(\iota) + \kappa \mathrm{CSD}^{\mathbf{C}}(A, \psi_1, \psi_2).$$

Here we use the conformal structure on Σ induced from ι to write down the integral $\mathrm{CSD}^{\mathbf{C}}$. Moreover, we identify the bundles of $(0,1)$–forms as topological bundles for different conformal structures, in the same way that we identified the spinor bundles of different metrics before. The critical points of $\mathfrak{L}\mathrm{CSD}^{\mathbf{C}}_\kappa$ are solutions to a coupled system equation similar to (3.33) and (3.32). However, in this case the second equation of (9.2), obtained by varying A, implies that one of the sections ψ_1,

ψ_2 must vanish and so the term obtained by varying $\mathrm{CSD}^{\mathbf{C}}(A, \psi_1, \psi_2)$ with respect to the immersion ι

$$\int_\Sigma (\delta_\iota \bar{\partial}_A)\psi_1 \wedge \psi_2$$

also vanishes. As a result, for every κ the critical points of $\mathfrak{L}\mathrm{CSD}^{\mathbf{C}}_\kappa$ are pairs consisting of a holomorphic curve Σ and an effective divisor on Σ. This suggests that there should be an enumerative theory counting such pairs in a Calabi–Yau 3–fold.

Remark 9.9. The fact that, unlike in the setting of Section 3.5, the critical points of $\mathfrak{L}\mathrm{CSD}^{\mathbf{C}}_\kappa$ do not depend on κ seems to be a special feature of the Seiberg–Witten equation (9.2), and not a general phenomenon. In particular, we do not expect it to hold for more general Seiberg–Witten equations. In any case, in the limit $\kappa \to 0$ the critical point equation always decouples.

9.4 Stable pair invariants of Calabi–Yau 3–folds

A numerical invariant counting holomorphic curves in Calabi–Yau 3–folds together with points on them was introduced by Pandharipande and Thomas; see [PT14, Section $4\frac{1}{2}$] for a brief introduction and [PT09; PT10] for more technical accounts. Since the space of curves and points on them is not necessarily compact, one considers the larger moduli space of so-called **stable pairs**. Roughly speaking, such a pair consists of a coherent sheaf F on Z together with a section $s \in H^0(Z, F)$ which, thought of as a sheaf morphism $s\colon \mathscr{O}_Z \to F$, is surjective outside a zero-dimensional subset of Z. The sheaf is required to be supported on a (possibly singular and thickened) holomorphic curve $\Sigma \subset Z$.[10]

Example 9.10. The simplest examples arise when Σ is smooth and (F, s) is the pushforward of a pair $(\mathscr{L}, \psi)$ on Σ consisting of a holomorphic line bundle and a non-zero section. Conversely, all stable pairs whose (scheme-theoretic) support is a smooth, unobstructed curve are of this form [PT09, Section 4.2].

The topological invariants of a stable pair are the homology class $[\Sigma] \in H_2(Z)$ and the Euler characteristic $\chi(X, F) \in \mathbf{Z}$. For instance, in Example 9.10, with Σ of genus g, the Grothendieck–Riemann–Roch theorem gives us

$$\chi(X, F) = 1 - g + \deg(\mathscr{L}). \tag{9.11}$$

For every $\beta \in H_2(Z)$ and $d \in \mathbf{Z}$, Pandharipande and Thomas use virtual fundamental class techniques to define an integer $\mathrm{PT}_{d,\beta}$ which counts stable pairs with homology

[10] More precisely, F is pure of dimension one and s has zero-dimensional cokernel.

class β and Euler characteristic d. These numbers for different values of d can be conveniently packaged into the generating function

$$\mathrm{PT}_\beta = \sum_d \mathrm{PT}_{\beta,d} q^d.$$

For a holomorphic curve $\Sigma \subset Z$ with $[\Sigma] = \beta$, denote by $\mathrm{PT}_\Sigma(q)$ the contribution to $\mathrm{PT}_\beta(q)$ coming from stable pairs whose support is Σ. (It makes sense to talk about such a contribution even for non-isolated curves [PT10, Section 3.1].)

In the situation of Example 9.10, the moduli space of stable pairs with support on Σ and Euler characteristic d is simply the space of effective divisors whose degree, computed using (9.11), is $g-1+d$. From the deformation theory of such stable pairs one concludes that in this case,

$$\mathrm{PT}_\Sigma(q) = \sum_d (-1)^{g-1+d} \chi(\mathrm{Sym}^{g-1+d}\, \Sigma) q^d; \tag{9.12}$$

see [PT09, Equation (4.4)] for details. As a result, we obtain

Proposition 9.13. *If $\Sigma \subset Z$ is a smooth, unobstructed holomorphic curve, then*

$$\mathrm{PT}_\Sigma = \underline{\mathrm{SW}}(S^1 \times \Sigma).$$

Remark 9.14. From the 3–dimensional perspective, the symmetry between d and $-d$ is a special case of the involution in Seiberg–Witten theory induced from the involution on the space of spin^c structures [Mor96, Section 6.8]; from the 2–dimensional viewpoint, it is a manifestation of the Serre duality between $H^1(\mathscr{L})$ and $H^0(K_\Sigma \otimes \mathscr{L}^*)$.

Remark 9.15. The fact that the stable pair invariant is partitioned into an integers worth of invariants corresponding to the degrees of the spin^c structures on curves suggests that something similar could be true for associative submanifolds. However, unlike in the dimensionally reduced setting, where a spin^c corresponds in a natural way to an integer, for two distinct associatives P_1 and P_2 we are not aware of any way to relate the spin^c–structures on them.

In general, the stable pair invariant includes also more complicated contributions from singular and obstructed curves representing the given homology class. For irreducible classes, Pandharipande and Thomas proved that such a contribution is a finite sum of Laurent series of the form (9.12) [PT10, Theorem 3 and Section 3].

9.5 ADHM bundles over Riemann surfaces

The stable pair invariant includes also contributions from thickened curves. If a homology class $\beta \in H_2(Z, \mathbf{Z})$ is divisible by k and β/k is represented by a holomorphic curve $\Sigma \subset Z$, then there exist stable pairs having $k\Sigma$ as their scheme-theoretic support. Thinking of $S^1 \times k\Sigma$ as the multiple cover of the associative $S^1 \times \Sigma$ in $S^1 \times Z$, we are led by the discussion of Section 4.3 to the conclusion that the contribution of such a thickened curve should be in some way related to the solutions of the $\mathrm{ADHM}_{1,k}$ Seiberg–Witten equation on the 3–manifold $S^1 \times \Sigma$. We will argue that this is indeed the case.

Consider the more general $\mathrm{ADHM}_{r,k}$ Seiberg–Witten equation introduced in Section 5.3 under the following assumptions:

Hypothesis 9.16. *Let Σ be a closed Riemann surface and $M = S^1 \times \Sigma$ with the geometric data as in Definition 5.18 such that*

1. *g is a product Riemannian metric,*
2. *E and the connection B are pulled-back from Σ,*
3. *V and the connection C are pulled-back from a $\mathrm{U}(2)$–bundle with a connection on Σ such that $\Lambda^2_{\mathbf{C}} V \cong K_\Sigma$ as bundles with connections.*

Proposition 9.17. *If Hypothesis 9.16 holds and (Ψ, ξ, A) is an irreducible solution of the $\mathrm{ADHM}_{r,k}$ Seiberg–Witten equation (5.22), then the $\mathrm{spin}^{\mathrm{U}(k)}$–structure $\mathfrak{w}$ is pulled-back from a $\mathrm{spin}^{\mathrm{U}(k)}$–structure on Σ and (Ψ, ξ, A) is gauge-equivalent to a configuration pulled-back from Σ, unique up to gauge equivalence on Σ.*

This is a special case of [Doa17, Theorem 3.8]. In the situation of Proposition 9.17, equation (5.22) reduces to a non-abelian vortex equation on Σ. Recall that a choice of a $\mathrm{spin}^{\mathrm{U}(k)}$–structure on Σ is equivalent to a choice of a $\mathrm{U}(k)$–bundle $\mathscr{H} \to \Sigma$. Consequently, A can be seen as a connection on $\mathscr{H}$. The corresponding spinor bundles are

$$\mathfrak{g}_{\mathscr{H}} = \mathfrak{u}(\mathscr{H}) \quad \text{and} \quad W = \mathscr{H} \oplus T^*\Sigma^{0,1} \otimes \mathscr{H}.$$

Proposition 9.18. *Let (A, Ψ, ξ) be a configuration pulled-back from Σ. Under the splitting $W = \mathscr{H} \oplus T^*\Sigma^{0,1} \otimes \mathscr{H}$ we have $\Psi = (\psi_1, \psi_2^*)$ where*

$$\begin{aligned} \psi_1 &\in \Gamma(\Sigma, \mathrm{Hom}(E, \mathscr{H})), \\ \psi_2 &\in \Omega^{1,0}(\Sigma, \mathrm{Hom}(\mathscr{H}, E)), \\ \xi &\in \Gamma(\Sigma, V \otimes \mathrm{End}(\mathscr{H})). \end{aligned}$$

Equation (5.22) for (A, Ψ, ξ) is equivalent to

$$\begin{aligned} \bar{\partial}_{A,B}\psi_1 = 0, \quad \bar{\partial}_{A,B}\psi_2 = 0, \quad \bar{\partial}_{A,C}\xi &= 0, \\ [\xi \wedge \xi] + \psi_1\psi_2 &= 0, \\ i * F_A + [\xi \wedge \xi^*] + \psi_1\psi_1^* - *\psi_2^*\psi_2 &= 0. \end{aligned} \tag{9.19}$$

In the second equation we use the isomorphism $\Lambda^2_{\mathbf{C}} V \cong K_\Sigma$ so that the left-hand side is a section of $\Omega^{1,0}(\Sigma, \mathrm{End}(\mathscr{H}))$. In the third equation we contract V with V^* so that the left-hand side is a section of $i\mathfrak{u}(\mathscr{H})$.

This follows from [Doa17, Proposition 3.6, Remark 3.7] and the complex description (A.1) of the hyperkähler moment map appearing in the ADHM construction.

For $\tau \in \mathbf{R}$ and $\theta \in H^0(\Sigma, K_\Sigma)$ we consider a perturbation of (9.19)

$$
\begin{aligned}
\bar\partial_{A,B}\psi_1 = 0, \quad \bar\partial_{A,B}\psi_2 = 0, \quad \bar\partial_{A,C}\xi = 0,\\
[\xi \wedge \xi] + \psi_1\psi_2 = \theta \otimes \mathrm{id},\\
i * F_A + [\xi \wedge \xi^*] - \psi_1\psi_1^* + *\psi_2^*\psi_2 = \tau\, \mathrm{id}.
\end{aligned}
\tag{9.20}
$$

There is a Hitchin–Kobayashi correspondence between gauge-equivalence classes of solutions of (9.20) and isomorphism classes of certain holomorphic data on Σ. Let $\mathscr{E} = (E, \bar\partial_B)$ and $\mathscr{V} = (V, \bar\partial_C)$ be the holomorphic bundles induced from the unitary connections on E and V.

Definition 9.21. An **ADHM bundle** with respect to $(\mathscr{E}, \mathscr{V}, \theta)$ is a quadruple

$$(\mathscr{H}, \psi_1, \psi_2, \xi)$$

consisting of

- a rank k holomorphic vector bundle $\mathscr{H} \to \Sigma$,
- $\psi_1 \in H^0(\Sigma, \mathrm{Hom}(\mathscr{E}, \mathscr{H}))$,
- $\psi_2 \in H^0(\Sigma, K_\Sigma \otimes \mathrm{Hom}(\mathscr{H}, \mathscr{E}))$, and
- $\xi \in H^0(\Sigma, \mathscr{V} \otimes \mathrm{End}(\mathscr{H}))$

such that

$$[\xi \wedge \xi] + \psi_1\psi_2 = \theta \otimes \mathrm{id} \in H^0(\Sigma, K_\Sigma \otimes \mathrm{End}(\mathscr{H})).$$

Definition 9.22. For $\delta \in \mathbf{R}$, the δ**–slope** of an ADHM bundle $(\mathscr{H}, \psi_1, \psi_2, \xi)$ is

$$\mu_\delta(\mathscr{H}) := \frac{2\pi}{\mathrm{vol}(\Sigma)} \frac{\deg \mathscr{H}}{\mathrm{rank}\, \mathscr{H}} + \frac{\delta}{\mathrm{rank}\, \mathscr{H}}.$$

The **slope** of $\mathscr{H}$ is $\mu(\mathscr{H}) := \mu_0(\mathscr{H})$.

Definition 9.23. Let $\delta \in \mathbf{R}$. An ADHM bundle $(\mathscr{H}, \psi_1, \psi_2, \xi)$ is δ**–stable** if it satisfies the following conditions:

1. if $\delta > 0$, then $\psi_1 \neq 0$ and if $\delta < 0$, then $\psi_2 \neq 0$,
2. if $\mathscr{G} \subset \mathscr{H}$ is a proper ξ–invariant holomorphic subbundle such that $\mathrm{im}\, \psi_1 \subset \mathscr{G}$, then $\mu_\delta(\mathscr{G}) < \mu_\delta(\mathscr{H})$,
3. if $\mathscr{G} \subset \mathscr{H}$ is a proper ξ–invariant holomorphic subbundle such that $\mathscr{G} \subset \ker \psi_2$, then $\mu(\mathscr{G}) < \mu_\delta(\mathscr{H})$.

We say that $(\mathscr{H}, \psi_1, \psi_2, \xi)$ is **δ–polystable** if there exists a ξ–invariant decomposition $\mathscr{H} = \bigoplus_i \mathscr{G}_i \bigoplus_j \mathscr{I}_j$ such that:

1. $\mu_\delta(\mathscr{G}_i) = \mu_\delta(\mathscr{H})$ for every i and the restrictions of (ψ_1, ψ_2, ξ) to each $\mathscr{G}_i$ define a δ–stable ADHM bundle, and
2. $\mu(\mathscr{I}_j) = \mu_\delta(\mathscr{H})$ for every j, the restrictions of ψ_1, ψ_2 to each $\mathscr{I}_j$ are zero, and there exist no ξ–invariant proper subbundle $\mathscr{J} \subset \mathscr{I}_i$ with $\mu(\mathscr{J}) < \mu(\mathscr{I}_j)$.

Proposition 9.24. *Let $\tau = \mu_\delta(\mathscr{H})$. If (A, ψ_1, ψ_2, ξ) is a solution of (9.20) and $\mathscr{H}$ is the holomorphic bundle induced from $(\mathscr{H}, \bar{\partial}_A)$, then $(\mathscr{H}, \psi_1, \psi_2, \xi)$ is a δ–polystable ADHM bundle. Conversely, every δ–polystable ADHM bundle arises in this way from a solution to (9.20) which is unique up to gauge equivalence.*

Proof. The difficult part is showing that every δ–polystable ADHM bundle admits a compatible unitary connection solving the third equation of (9.20), unique up to gauge equivalence. This is a special case of the Hitchin–Kobayashi correspondence for vortex equations associated with quiver representations [ÁG03], with the minor difference that the connections on the bundles E and V are fixed and not part of a solution. In particular, had $\mathscr{E}$ and $\mathscr{V}$ been holomorphically trivial, the result would be a direct application of [ÁG03, Theorem 31]. The necessary adjustment in the case when $\mathscr{E}$ and $\mathscr{V}$ are non-trivial but fixed is discussed, in a different but similar setting, in [BGM03].

For the sake of completeness, we prove the easier implication that equation (9.20) implies δ–polystability. Taking the trace of

$$i\Lambda F_A + [\xi \wedge \xi^*] + \psi_1\psi_1^* - \psi_2^*\psi_2 = \tau \,\mathrm{id} \tag{9.25}$$

and integrating over Σ, we obtain

$$2\pi \deg(\mathscr{H}) + \|\psi_1\|_{L^2}^2 - \|\psi_2\|_{L^2}^2 - \tau \,\mathrm{rank}(\mathscr{H})\mathrm{vol}(\Sigma) = 0, \tag{9.26}$$

which proves the first item of Definition 9.23.

Let $\mathscr{G} \subset \mathscr{H}$ be a proper ξ–invariant holomorphic subbundle. Denote the underlying complex vector bundle by G, then $\mathscr{H} = G \oplus G^\perp$. We have

$$A = \begin{pmatrix} A_1 & b \\ -b^* & A_2 \end{pmatrix},$$

where A_1 is a unitary connection on G compatible with the holomorphic structure $\mathscr{G}$, A_2 is a unitary connection on $G^\perp$ compatible with the holomorphic structure $\mathscr{H}/\mathscr{G}$, and $b \in \Omega^{0,1}(\mathrm{Hom}(G^\perp, G))$ is a $(0,1)$–form representing the extension class. The curvature splits, correspondingly, into

$$F_A = \begin{pmatrix} F_{A_1} - b \wedge b^* & \mathrm{d}b \\ -\mathrm{d}b^* & F_{A_2} - b^* \wedge b \end{pmatrix}.$$

Let $\pi\colon \mathscr{H} \to G$ be the orthogonal projection. We have

$$[\xi \wedge \xi^*] = \begin{pmatrix} [(\pi\xi) \wedge (\pi\xi)^*] & * \\ 0 & * \end{pmatrix}.$$

Suppose $\operatorname{im}\psi_1 \subset G$ so that $\pi\psi_1 = \psi_1$ Restrict (9.25) to G, take trace, and integrate over Σ to obtain

$$2\pi \deg(\mathscr{G}) + \|b\|^2_{L^2} + \|\psi_1\|^2_{L^2} - \|\pi^*\psi_2\|^2_{L^2} - \tau \operatorname{rank}(\mathscr{G})\operatorname{vol}(\Sigma) = 0.$$

Subtracting (9.26) and using $\|\pi^*\psi_2\|_{L^2} \leq \|\psi_2\|_{L^2}$ results in

$$\frac{2\pi}{\operatorname{vol}(\Sigma)} \deg(\mathscr{G}) - \tau \operatorname{rank}(\mathscr{G}) \leq \frac{2\pi}{\operatorname{vol}(\Sigma)} \deg(\mathscr{H}) - \tau \operatorname{rank}(\mathscr{H}),$$

which is equivalent to $\mu_\delta(\mathscr{G}) \leq \mu_\delta(\mathscr{H})$. The equality holds if and only if $b = 0$ and $\pi^*\psi_2 = \psi_2$. The first condition guarantees that $\mathscr{H} = \mathscr{G} \oplus (\mathscr{H}/\mathscr{G})$ as holomorphic vector bundles; the second, that $\psi_2 \in H^0(\Sigma, K_\Sigma \otimes \operatorname{Hom}(\mathscr{G}, \mathscr{E}))$. Thus, $(A_1, \psi_1, \psi_2, \xi)$ defines a solution of (9.20) with the underlying ADHM bundle $(\mathscr{G}, \psi_1, \psi_2, \xi)$ satisfying $\mu_\delta(\mathscr{G}) = \mu_\delta(\mathscr{H})$, and we can proceed by induction.

Suppose $G \subset \ker\psi_2$ so that $\pi^*\psi_2 = 0$. The same argument gives us

$$\frac{2\pi}{\operatorname{vol}(\Sigma)} \deg(\mathscr{G}) \leq \tau \operatorname{rank}(\mathscr{G}),$$

which is equivalent to $\mu(\mathscr{G}) \leq \mu_\delta(\mathscr{H})$. The equality holds if and only if $\mathscr{H} = \mathscr{G} \oplus (\mathscr{H}/\mathscr{G})$ as holomorphic vector bundles and $\psi_1 \in H^0(\Sigma, \operatorname{Hom}(\mathscr{E}, \mathscr{H}/\mathscr{G}))$. In this case, (9.20) implies that (A_1, ξ) solves a version of the Hitchin equation and so defines a polystable Higgs bundle with slope $\mu_\delta(\mathscr{H})$. □

Stable ADHM bundles on Riemann surfaces were studied extensively by Diaconescu [Dia12b; Dia12a] in the case when $n = 1$, $\mathscr{E}$ is a trivial line bundle, and $\mathscr{V}$ is the direct sum of two line bundles. Thus, we have a splitting $\xi = (\xi_1, \xi_2)$ and $[\xi \wedge \xi] = [\xi_1, \xi_2]$, so the holomorphic equation

$$[\xi \wedge \xi] + \psi_1\psi_2 = \theta \otimes \mathrm{id}$$

is preserved by the $\mathbf{C}^*$–action $t(\psi_1, \psi_2, \xi_1, \xi_2) = (t\psi_1, t^{-1}\psi_2, t\xi_1, t^{-1}\xi_2)$. Moreover, if the perturbing form θ is chosen to be zero, there is an additional $\mathbf{C}^*$–symmetry given by rescaling all the sections. Assuming that the stability parameter δ is sufficiently large, Diaconescu shows the fixed-point locus of the resulting $\mathbf{C}^* \times \mathbf{C}^*$–action on the moduli space of δ–stable ADHM bundles is compact. Furthermore, the moduli space is equipped with a $\mathbf{C}^* \times \mathbf{C}^*$–equivariant perfect obstruction theory which can be used to extract from it a numerical invariant via equivariant virtual integration.

This number is then shown to be equal to the **local stable pair invariant** of the non-compact Calabi–Yau 3–fold $\mathscr{V}$. This invariant counts, again in the equivariant and virtual sense, stable pairs whose scheme-theoretic support is the k–th thickening of the zero section $\Sigma \subset \mathscr{V}$. Here k is the rank of $\mathscr{H}$ so that the stable ADHM bundles in question correspond, by Proposition 9.24, to solutions to the $\mathrm{ADHM}_{1,k}$ Seiberg–Witten equation on $S^1 \times \Sigma$. This suggests that the relation between Seiberg–Witten monopoles and stable pairs discussed in the previous section could extend to the case of multiple covers.

9.6 Towards a numerical invariant

Due to the appearance of reducible solutions, one does not expect to be able to count solutions to the $\mathrm{ADHM}_{1,k}$ Seiberg–Witten equation on a general 3–manifold. Instead, the enumerative theory for associatives in tamed almost G_2–manifolds should incorporate a version of equivariant Floer homology, as explained in Section 3.4 and Section 8. However, the existence of the stable pair invariant and the discussion of the previous sections indicate that we can hope for a differential-geometric invariant counting pseudo-holomorphic curves in a symplectic Calabi–Yau 6–manifold Z which in the integrable case would recover the stable pair invariant.[11]

For a homology class $\beta \in H_2(Z, \mathbf{Z})$ the invariant would take values in the ring of Laurent series $\mathbf{Z}(\!(q)\!)$ and, in the first approximation, be defined by

$$n_\beta(Z) = \sum_{\Sigma^1, \dots, \Sigma^m} \prod_{j=1}^{m} \underline{\mathrm{SW}}_{1,k_j}(S^1 \times \Sigma^j)\,\mathrm{sign}(\Sigma^j).$$

Some explanation is in order:

1. The sum is taken over all collections of embedded, connected pseudo-holomorphic curves $\Sigma^1, \dots, \Sigma^m$ such that

   $$\sum_{j=1}^{m} k_j[\Sigma^j] = \beta.$$

   We assume here that we can choose a generic tamed almost-complex structure such that there are finitely many such curves and all of them are unobstructed.

2. $\mathrm{sign}(\Sigma) = \pm 1$ comes from an orientation on the moduli space of pseudo-holomorphic curves.

[11] Such an invariant should encode the same symplectic information as the Gromov–Witten invariants by the conjectural GW/PT correspondence [PT14, Sections $3\frac{1}{2}$ and $4\frac{1}{2}$].

3. $\underline{\mathrm{SW}}_{1,k}(S^1 \times \Sigma)$ is a generalization of the Meng–Taubes invariant defined using the moduli spaces of solutions to the $\mathrm{ADHM}_{1,k}$ Seiberg–Witten equation on $S^1 \times \Sigma^j$. Such an invariant is yet to be defined but if it exists, it should be naturally an element of $\mathbf{Z}(\!(q)\!)$ because of the identification of the set of the $\mathrm{spin}^{\mathrm{U}(k)}$ structures on Σ with the integers, as in Section 9.2.

4. We use here crucially that $b_1(S^1 \times \Sigma) \geq 1$; otherwise even the classical Meng–Taubes invariant $\underline{\mathrm{SW}}_{1,1}$ is ill-defined. For $k > 1$, the $\mathrm{ADHM}_{1,k}$ Seiberg–Witten equation, admits in general, reducible solutions: for example, flat connections or solutions to the $\mathrm{ADHM}_{1,k-1}$ Seiberg–Witten equation. A good feature of the dimensionally-reduced setting is that if the perturbing holomorphic 1–form θ in (9.20) is non-zero, then we automatically avoid reducible solutions. Indeed, a simple algebraic argument shows that in this case the triple (ξ, ψ_1, ψ_2) has trivial stabilizer in $\mathrm{U}(k)$ at every point where θ is non-zero.

A Proof of Proposition 7.4

For the proof it is convenient to write S as $\mathbf{C}^k \oplus j\mathbf{C}^k \oplus \mathrm{End}(\mathbf{C}^k) \oplus j\,\mathrm{End}(\mathbf{C}^k)$. A direct computation shows that with respect to this identification the moment map is given by

$$\mu(v, w, A^*, B) = \frac{1}{2}(vv^* - ww^* - [A, A^*] - [B, B^*]) + j(wv^* - [A, B]). \tag{A.1}$$

Therefore, if $(\Psi, \xi) = (v + jw, A^* + jB) \in \mu^{-1}(0)$, then

$$vv^* - ww^* = [A, A^*] + [B, B^*] \quad \text{and} \quad wv^* = [A, B]. \tag{A.2}$$

Set $T := [A, A^*] + [B, B^*]$. Taking traces and inner products with v and w, (A.2) implies

$$|v| = |w| =: \lambda, \quad \langle v, w\rangle = 0, \tag{A.3}$$

$$\langle Tv, v\rangle = \lambda^4, \quad \text{and} \quad \langle Tw, w\rangle = -\lambda^4. \tag{A.4}$$

Proposition A.5 ([Nak99, Lemma 2.8]). *Denote by V_1 the smallest subspace of $\mathbf{C}^k$ which contains w and is preserved by both A and B. We have $v \perp V_1$.*

Proof. Let C be a product of As and Bs. We need to show that $\langle v, Cw\rangle = 0$. The proof is by induction on k, the number of factors of C. If $k = 0$, then $C = \mathrm{id}$ and we have $\langle v, w\rangle = 0$ by (A.3).

By induction we can assume that $\langle v, \tilde{C}w\rangle = 0$ for all $\tilde{C}$ with fewer than k factors. If $C = C_l BAC_r$, then

$$\begin{aligned} Cw &= C_l BAC_r w = C_l ABC_r w - C_l [A, B] C_r w \\ &= C_l ABC_r w - C_l wv^* C_r w = C_l ABC_r w \end{aligned}$$

because $v^* C_r w = \langle v, C_r w\rangle$ and C_r has fewer than k factors. Henceforth, we can assume that $C = A^{k_1} B^{k_2}$. For such C, we have

$$\begin{aligned}
\langle v, A^{k_1} B^{k_2} w\rangle &= \mathrm{tr}(A^{k_1} B^{k_2} w v^*) = \mathrm{tr}(A^{k_1} B^{k_2} [A, B]) \\
&= \mathrm{tr}([A^{k_1} B^{k_2}, A] B) = \mathrm{tr}(A^{k_1} [B^{k_2}, A] B) \\
&= \sum_{\ell=0}^{k_2-1} \mathrm{tr}(A^{k_1} B^{\ell} [B, A] B^{k_2-\ell}) = \sum_{\ell=0}^{k_2-1} \mathrm{tr}(B^{k_2-\ell} A^{k_1} B^{\ell} [B, A]) \\
&= -\sum_{\ell=0}^{k_2-1} \langle v, B^{k_2-\ell} A^{k_1} B^{\ell} w\rangle = -k_2 \langle v, A^{k_1} B^{k_2} w\rangle.
\end{aligned}$$

This concludes the proof. □

As a warm up consider the case $k = 2$. If $\lambda > 0$, then

$$(w/\lambda, v/\lambda)$$

is an orthonormal basis for $\mathbf{C}^2$. With respect to this basis A and B are given by matrices of the form

$$A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix}.$$

Consequently, the first diagonal entry of $T = [A, A^*] + [B, B^*]$ is

$$T_{11} = |a_{12}|^2 + |b_{12}|^2 > 0.$$

However, since $\langle Tw, w\rangle = -\lambda^4$ according to (A.4), we have

$$T_{11} = -\lambda^2 < 0.$$

It follows that $\lambda = 0$; that is, $\Psi = v + jw = 0$.

In general, let V_1 be as in Proposition A.5 and set $V_2 := V_1^{\perp}$. With respect to the splitting $\mathbf{C}^k = V_1 \oplus V_2$, we have

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix}.$$

It follows from $wv^* = [A, B]$ and $v \in V_2$, that

$$[A_{11}, B_{11}] = [A, B]_{11} = 0;$$

Moreover, we have

$$T_{11} = ([A, A^*] + [B, B^*])_{11} = [A_{11}, A_{11}^*] + [B_{11}, B_{11}^*] + A_{12} A_{12}^* + B_{12} B_{12}^*;$$

hence,

$$[A_{11}, A_{11}^*] + [B_{11}, B_{11}^*] + A_{12}A_{12}^* + B_{12}B_{12}^* + ww^* = 0.$$

Thus $[A_{11}, A_{11}^*] + [B_{11}, B_{11}^*] \leq 0$. By Proposition A.6, it follows that $[A_{11}, A_{11}^*] = [B_{11}, B_{11}^*] = 0$. Since $A_{12}A_{12}^* + B_{12}B_{12}^* + ww^*$ is a sum of non-negative definite matrices, we must have $|w| = 0$; hence, $\Psi = v + jw = 0$ by (A.3).

Proposition A.6. *If* $[A, B] = 0$ *and* $[A, A^*] + [B, B^*] \leq 0$*, then* A *and* B *can be simultaneously diagonalized and* $[A, A^*] = [B, B^*] = 0$.

Proof. Since A and B commute, we can simultaneously upper triagonalize them; that is, after conjugating A and B with a unitary matrix we can assume that

$$A = \Lambda + U \quad \text{and} \quad B = M + V$$

where Λ, M are diagonal and U, V are strictly upper triangular. We have

$$[A, A^*] = [\Lambda, \Lambda^*] + [\Lambda, U^*] - [\Lambda^*, U] + [U, U^*].$$

The first term vanishes, and the second and third terms have vanishing diagonal entries. Writing $U = (u_{mn})$, the m–th diagonal of $[A, A^*]$ is

$$\sum_{n=1}^{k} |u_{mn}|^2 - |u_{nm}|^2;$$

and similarly for B with $V = (v_{mn})$.

The first diagonal entry of $[A, A^*] + [B, B^*]$ is

$$\sum_{n=1}^{k} |u_{1n}|^2 + |v_{1n}|^2.$$

Being non-positive, this term vanishes. The second diagonal entry is

$$\sum_{n=1}^{k} |u_{2n}|^2 + |v_{2n}|^2 - |u_{12}|^2 - |v_{12}|^2 = \sum_{n=1}^{k} |u_{2n}|^2 + |v_{2n}|^2$$

Being non-positive, this term vanishes as well. Repeating this argument eventually shows that $U = V = 0$. □

This completes the proof of Proposition 7.4. □

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