Counting embedded curves in symplectic 6–manifolds

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Abstract

Based on computations of Pandharipande [Pan99], Zinger [Zin11] proved that the Gopakumar–Vafa BPS invariants $\mathrm{BPS}_{A,g}(X,\omega)$ for primitive Calabi–Yau classes and arbitrary Fano classes A on a symplectic 6–manifold (X,ω) agree with the signed count $n_{A,g}(X,\omega)$ of embedded J–holomorphic curves representing A and of genus g for a generic almost complex structure J compatible with ω. Zinger's proof of the invariance of $n_{A,g}$ is indirect, as it relies on Gromov–Witten theory. In this article we give a direct proof of the invariance of $n_{A,g}$. Furthermore, we prove that $n_{A,g}(X,\omega) = 0$ for $g \gg 1$, thus proving the Gopakumar–Vafa finiteness conjecture for primitive Calabi–Yau classes and arbitrary Fano classes.

1 Introduction

Are there invariants of symplectic manifolds which count embedded *pseudo-holomorphic curves?* Such counts can fail to be invariants for two reasons: (a) pseudo-holomorphic embeddings can degenerate to multiple covers, and (b) they can undergo bubbling and their domains can become singular. In the following we consider two situations in which both of these can be ruled out.

Let (X,ω) be a closed symplectic 6–manifold equipped with an almost complex structure J compatible with ω. Denote by $\mathscr{M}^\star_{A,g}(X,J)$ the moduli space of simple J–holomorphic maps of genus g representing a homology class $A \in H_2(X,\mathbf{Z})$. For a generic choice of J the moduli space $\mathscr{M}^\star_{A,g}(X,J)$ is an oriented smooth manifold of dimension

$$\dim \mathscr{M}^\star_{A,g}(X,J) = 2\langle c_1(X,\omega), A\rangle.$$

If A is a **Calabi–Yau class**, that is: $\langle c_1(X,\omega), A\rangle = 0$, then $\mathscr{M}^\star_{A,g}(X,J)$ is a finite set of signed points and can be counted. If A is primitive in $H_2(X,\mathbf{Z})$, then multiple cover phenomena can be ruled out, and it will be proved that this count defines an invariant $n_{A,g}(X,\omega)$. If A is a **Fano class**, that is: $\langle c_1(X,\omega), A\rangle > 0$, then $\mathscr{M}^\star_{A,g}(X,J)$ can be cut-down to a finite set of signed points by imposing incidence conditions governed by suitable cohomology classes $\gamma,\ldots,\gamma_\Lambda \in H^{\mathrm{even}}(X,\mathbf{Z})$. In this case multiple cover phenomena can be ruled out regardless of whether A is primitive or not, and it will be proved that counting the cut-down moduli space defines an invariant $n_{A,g}(X,\omega;\gamma_1,\ldots,\gamma_\Lambda)$.

These invariants are not new. They were considered by Zinger [Zin11, Theorem 1.5 and footnote 11] who proved that they agree with Gopakumar and Vafa's BPS invariants. The proof of the invariance of $n_{A,g}(X,\omega)$ and $n_{A,g}(X,\omega;\gamma_1,\dots,\gamma_\Lambda)$ in [Zin11] is indirect: it relies on these numbers satisfying the Gopakumar–Vafa formula and the invariance of Gromov–Witten invariants. The novelty in the present work is that we give a much simpler direct proof of invariance. Furthermore, we prove that the invariants vanish for g sufficiently large; thus establishing the Gopakumar–Vafa finiteness conjecture for primitive Calabi–Yau classes and arbitrary Fano classes.

1.1 Ghost components

The main technical result of this paper allows us to rule out, in certain situations, degenerations in which the limiting nodal pseudo-holomorphic map has a **ghost component**, that is: a component on which it is constant. The precise definitions used in the following statement are given in Section 2 and Section 3.

Theorem 1.1. *Let (X, g_∞, J_∞) be an almost Hermitian manifold and let $(J_k)_{k\in\mathbf{N}}$ be a sequence of almost complex structure on X converging to J_∞ in the C^1 topology. If $(u_k\colon (\Sigma_k, j_k) \to (X, J_k))_{k\in\mathbf{N}}$ is a sequence of pseudo-holomorphic maps which Gromov converges to the nodal J_∞–holomorphic map $u_\infty\colon (\Sigma_\infty, j_\infty, \nu_\infty) \to (X, J_\infty)$, then either*

1. *$(\Sigma_\infty, j_\infty, \nu_\infty)$ is smooth or*
2. *$(\Sigma_\infty, j_\infty, \nu_\infty)$ has at least one node and either*
    - *a. u_∞ has no ghost components or*
    - *b. u_∞ has at least one ghost component C and either*
        - *i. there is precisely one non-ghost component attached to C at a node $n \in C$ and $\mathrm{d}_{\nu_\infty(n)}u_\infty = 0$—that is: the corresponding node $\nu_\infty(n)$ in the non-ghost component is a critical point of u_∞—or*
        - *ii. there are at least two non-ghost components attached to C.*

Remark 1.2. Zinger [Zin09, Theorem 1.2] has analyzed in detail when a nodal pseudo-holomorphic map whose domain has arithmetic genus one appears as a Gromov limit of pseudo-holomorphic maps with smooth domain. Jingchen Niu's PhD thesis [Niu16] extends Zinger's analysis to genus two. Their results are based analyzing the obstruction map for Kuranishi model of a neighborhood of the limiting pseudo-holomorphic map. This idea goes back to Ionel [Ion98] and is also basis for the proof of Theorem 1.1 in Section 5. Recently, a different proof of a similar result for has appeared in the work of Ekholm and Shende [ES19, Lemma 4.9].

Given (X,ω) a symplectic manifold, denote by $\mathscr{J}(X,\omega)$ the set of almost complex structures J compatible with ω and denote by $\mathscr{J}_{\mathrm{emb}}(X,\omega)$ the subset of those J for which the following hold: (a) every there are no simple J–holomorphic maps of negative index, (b) every simple J–holomorphic map is an embedding, and (c) every two simple J–holomorphic maps of index zero either have disjoint images or are related by a reparametrization; see Definition 2.36. The complement of $\mathscr{J}_{\mathrm{emb}}(X,\omega)$ in $\mathscr{J}(X,\omega)$ has codimension two; in particular: $\mathscr{J}_{\mathrm{emb}}(X,\omega)$ is open and dense, and every path $(J_t)_{t\in[0,1]}$ in $\mathscr{J}(X,\omega)$ with end points in $\mathscr{J}_{\mathrm{emb}}(X,\omega)$ is homotopic relative end points to a path in $\mathscr{J}_{\mathrm{emb}}(X,\omega)$.

Theorem 1.3. *Let (X,ω) be a compact symplectic 6–manifold, let $(J_k)_{k\in\mathbf{N}}$ be a sequence of almost complex structures compatible with ω converging to J_∞, and let $(u_k\colon (\Sigma_k,j_k)\to(X,J_k))_{k\in\mathbf{N}}$ be a sequence of pseudo-holomorphic maps which Gromov converges to the nodal J_∞–holomorphic map $u_\infty\colon (\Sigma_\infty,j_\infty,\nu_\infty)\to(X,J_\infty)$. Set $A\coloneqq (u_\infty)_*[\Sigma_\infty]\in H_2(X,\mathbf{Z})$. If A is primitive, satisfies $\langle c_1(X,\omega),A\rangle=0$, and $J_\infty\in\mathscr{J}_{\mathrm{emb}}(X,\omega)$, then $(\Sigma_\infty,j_\infty,\nu_\infty)$ is smooth and u_∞ is an embedding.*

There is a variant of the definition $\mathscr{J}(X,\omega)$ adapted to pseudo-holomorphic maps with Λ marked points constrained by pseudo-cycles $f_1,\dots,f_\Lambda$. (See Appendix A for a review of the theory of pseudo-cycles.) The precise definition of this subspace $\mathscr{J}(X,\omega;f_1,\dots,f_\Lambda)$ is rather lengthy and deferred to Definition 2.44.

Theorem 1.4. *Let (X,ω) be a compact symplectic 6–manifold, let $(J_k)_{k\in\mathbf{N}}$ be a sequence of almost complex structures compatible with ω converging to J_∞, and let $(u_k\colon (\Sigma_k,j_k)\to(X,J_k))_{k\in\mathbf{N}}$ be a sequence of pseudo-holomorphic maps which Gromov converges to the nodal J_∞–holomorphic map $u_\infty\colon (\Sigma_\infty,j_\infty,\nu_\infty)\to(X,J_\infty)$. Set $A\coloneqq (u_\infty)_*[\Sigma_\infty]\in H_2(X,\mathbf{Z})$. Let $f_1,\dots,f_\Lambda$ be even-dimensional pseudo-cycles of positive codimension in general position. If*

1. $\operatorname{im} u_k\cap\operatorname{im} f_\lambda\neq\varnothing$ *for every* $\lambda=1,\dots,\Lambda$,
2. $2\langle c_1(X,\omega),A\rangle=\sum_{\lambda=1}^{\Lambda}(\operatorname{codim} f_\lambda-2)>0$, *and*
3. $J_\infty\in J_{\mathrm{emb}}(X,\omega;f_1,\dots,f_\Lambda)$.

then $(\Sigma_\infty,j_\infty,\nu_\infty)$ is smooth and u_∞ is an embedding with $\operatorname{im} u_\infty\cap\operatorname{im} f_\lambda\neq\varnothing$ *every* $\lambda=1,\dots,\Lambda$.

1.2 Embedded curve counts

Denote by $\mathscr{J}^\star_{\mathrm{emb}}(X,\omega)$ the subset of those $J\in\mathscr{J}_{\mathrm{emb}}(X,\omega)$ for which every simple J–holomorphic map is unobstructed; see Definition 2.36.

Theorem 1.5. *Let (X,ω) be a symplectic 6–manifold. Let $A\in H_2(X,\mathbf{Z})$ be a primitive class such that $\langle c_1(X,\omega),A\rangle=0$, and let $g\in\mathbf{N}_0$.*

1. *For every $J\in\mathscr{J}^\star_{\mathrm{emb}}(X,\omega)$ the moduli space $\mathscr{M}^\star_{A,g}(X,J)$ of simple J–holomorphic maps of genus g representing the class A is a compact oriented zero-dimensional manifold, and the signed count*

   $$n_{A,g}(X,\omega)\coloneqq \#\mathscr{M}^\star_{A,g}(X,J) \tag{1.6}$$

   is independent on the choice of $J\in\mathscr{J}_\star(X,\omega)$.
2. *There exists $g_0\in\mathbf{N}_0$, depending on (X,ω) and A, such that*

   $$n_{A,g}(X,\omega)=0\quad\textit{for every}\quad g\geqslant g_0.$$

Remark 1.7. In fact, $n_{A,g}(X,\omega)$ depends on ω only up to deformation.

Again, there is a variant $\mathscr{J}^{\star}_{\mathrm{emb}}(X,\omega;f_1,\ldots,f_\Lambda)$ of $\mathscr{J}^{\star}_{\mathrm{emb}}(X,\omega)$ adapted to pseudo-holomorphic maps with Λ marked points constrained by pseudo-cycles $f_1,\ldots,f_\Lambda$; see Definition 2.45.

Theorem 1.8. *Let (X,ω) be a symplectic 6–manifold, let $A \in H_2(X,\mathbf{Z})$, let $\gamma_1,\ldots,\gamma_\Lambda \in H^{\mathrm{even}}(X,\mathbf{Z})$ be such that $\deg(\gamma_\lambda) > 0$ and*

$$2\langle c_1(X,\omega),A\rangle = \sum_{\lambda=1}^{\Lambda}(\deg(\gamma_\lambda)-2) > 0,$$

and let $g \in \mathbf{N}_0$.

1. *Let $f_1,\ldots,f_\Lambda$ be pseudo-cycles in X which are Poincaré dual to $\gamma_1,\ldots,\gamma_\Lambda$ and in general position. For every $J \in \mathscr{J}^{\star}_{\mathrm{emb}}(X,\omega;f_1,\ldots,f_\Lambda)$ the moduli space $\mathscr{M}^{\star}_{A,g}(X,J;f_1,\ldots,f_\Lambda)$ of simple J–holomorphic maps of genus g representing the class A and intersecting $f_1,\ldots,f_\Lambda$ is a compact oriented zero-dimensional manifold, and the signed count*

   $$n_{A,g}(X,\omega;\gamma_1,\ldots,\gamma_\Lambda) \coloneqq \#\mathscr{M}^{\star}_{A,g}(X,J;f_1,\ldots,f_\Lambda) \tag{1.9}$$

   is independent on the choice of $f_1,\ldots,f_\Lambda$ and $J \in \mathscr{J}^{\star}_{\mathrm{emb}}(X,\omega;f_1,\ldots,f_\Lambda)$.

2. *There exists $g_0 \in \mathbf{N}_0$, depending on (X,ω), A, and $\gamma_1,\ldots,\gamma_\Lambda$, such that*

   $$n_{A,g}(X,\omega;\gamma_1,\ldots,\gamma_\Lambda) = 0 \quad \text{for all } g \geqslant g_0.$$

Remark 1.10. Remark 1.7 applies mutatis mutandis.

1.3 Gopakumar and Vafa's BPS invariants

Using ideas from M–theory, Gopakumar and Vafa [GV98a; GV98b] predicted that there exist integer invariants $\mathrm{BPS}_{A,g}(X,\omega)$ associated with every closed symplectic 6–manifold (X,ω) and a class $A \in H_2(X,\mathbf{Z})$ with $\langle c_1(X,\omega),A\rangle = 0$ and $g \in \mathbf{N}_0$, which count BPS states supported on embedded J–holomorphic curves representing A and of genus g. Gopakumar and Vafa did not give a direct mathematical definition of $\mathrm{BPS}_{A,g}(X,\omega)$; however, they conjectured that their invariants are related to the Gromov–Witten invariants $\mathrm{GW}_{A,g}(X,\omega)$ by the marvelous formula

$$\sum_{A}\sum_{g=0}^{\infty}\mathrm{GW}_{A,g}(X,\omega)\cdot t^{2g-2}q^{A} = \sum_{A}\sum_{g=0}^{\infty}\mathrm{BPS}_{A,g}(X,\omega)\cdot t^{2g-2}\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{\sin(kt/2)}{t/2}\right)^{2g-2}q^{kA} \tag{1.11}$$

with the sum taken over all non-zero Calabi–Yau classes A and, moreover, that $\mathrm{BPS}_{A,g}(X,\omega) = 0$ for $g \gg 1$.

In algebraic geometry, there are approaches to defining the BPS invariants for projective Calabi–Yau three-folds [HST01; PT09; PT10; KL12; MT18]. These satisfy the Gopakumar–Vafa formula (1.11) in some cases, but it is not currently known whether the formula holds in general.

An alternative approach is take (1.11) as the *definition* of $\mathrm{BPS}_{A,g}(X,\omega)$; see [BP01, Section 2]. This approach leads to the following conjecture.

Conjecture 1.12 (Gopakumar and Vafa [GV98a; GV98b]; see also [BP01, Conjecture 1.2]). *The numbers* $\mathrm{BPS}_{A,g}(X,\omega)$ *defined by* (1.11) *satisfy*

(integrality) $\mathrm{BPS}_{A,g}(X,\omega) \in \mathbf{Z}$, *and*

(finiteness) $\mathrm{BPS}_{A,g}(X,\omega) = 0$ *for* $g \gg 1$.

The Gopakumar–Vafa integrality conjecture has been proved by Ionel and Parker [IP18]. Zinger [Zin11, footnote 11] has proved that for primitive Calabi–Yau classes

$$\mathrm{BPS}_{A,g}(X,\omega) = n_{A,g}(X,\omega);$$

see also Appendix B. Therefore, Theorem 1.5 implies the following.

Corollary 1.13. *The Gopakumar–Vafa finiteness conjecture holds for primitive Calabi–Yau classes; that is: for every closed symplectic* 6*–manifold* (X,ω) *and every primitive Calabi–Yau class* $A \in H_2(X,\mathbf{Z})$ *there is a* $g_0(\omega,A)$ *such that for every* $g \geqslant g_0(\omega,A)$

$$\mathrm{BPS}_{A,g}(X,\omega) = 0.$$

The finiteness conjecture for general Calabi–Yau classes remains open, however. The genus bound in Corollary 1.13 is not effective; therefore, it is natural to ask the following.

Definition 1.14. Let (X,ω) be a closed symplectic 6–manifold and $A \in H_2(X,\mathbf{Z})$ a Calabi–Yau class. The **BPS Castelnuovo number** $\gamma_A(X,\omega)$ by

$$\gamma_A(X,\omega) \coloneqq \inf\bigl\{g \in \mathbf{Z} : \mathrm{BPS}_{A,g}(X,\omega) = 0\bigr\} \in \mathbf{N}_0 \cup \{\infty\}.$$

Question 1.15. Is there an effective bound on $\gamma_A(X,\omega)$ analogous to Castelnuovo's bound for the genus of an irreducible degree d curve in $\mathbf{P}^n$ [Cas89; ACGH85, Chapter III Section 2]?

There is an analogue of the Gopakumar–Vafa formula for Fano classes. Given $A \in H_2(X,\mathbf{Z})$, $g \in \mathbf{N}_0$, and $\gamma_1,\ldots,\gamma_\Lambda \in H^{\mathrm{even}}(X,\mathbf{Z})$ satisfying $\deg(\gamma_\lambda) > 0$ and

$$2\langle c_1(X,\omega),A\rangle = \sum_{\lambda=1}^{\Lambda}(\deg(\gamma_\lambda)-2) > 0, \tag{1.16}$$

denote by $\mathrm{GW}_{A,g}(X,\omega;\gamma_1,\ldots,\gamma_\Lambda)$ be the corresponding Gromov–Witten invariant. The analogue of (1.11) is

$$\begin{aligned}&\sum_{A}^{\infty}\sum_{g=0}^{\infty}\mathrm{GW}_{A,g}(X,\omega;\gamma_1,\ldots,\gamma_\Lambda)\cdot t^{2g-2}q^A\\ &\quad= \sum_{A}^{\infty}\sum_{g=0}^{\infty}\mathrm{BPS}_{A,g}(X,\omega;\gamma_1,\ldots,\gamma_\Lambda)\cdot t^{2g-2}\left(\frac{\sin(t/2)}{t/2}\right)^{2g-2+\langle c_1(X,\omega),A\rangle}q^A\end{aligned} \tag{1.17}$$

with the sum taken over all $A \in H_2(X, \mathbf{Z})$ satisfying (1.16). Zinger [Zin11, Theorem 1.5] has proved that

$$\mathrm{BPS}_{A,g}(X, \omega; \gamma_1, \ldots, \gamma_\Lambda) = n_{A,g}(X, \omega; \gamma_1, \ldots, \gamma_\Lambda);$$

thus establishing the analogue of the Gopakumar–Vafa integrality conjecture. Furthermore, Theorem 1.8 implies the following.

Corollary 1.18. *The analogue of the Gopakumar–Vafa finiteness conjecture holds for all Fano classes.*

Of course, there is an analogue of Question 1.15 in the Fano case.

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2 Nodal pseudo-holomorphic maps

This section reviews a few definitions and results regarding nodal pseudo-holomorphic maps.

2.1 Nodal manifolds

Definition 2.1. Let X be a manifold. A **nodal structure** on X is an involution ν whose fixed-point set has a discrete complement. The set of points not fixed by ν is called the **nodal set**. A **nodal manifold** is a manifold together with a nodal structure.

The quotient X/ν should be considered as the topological space underlying the nodal manifold (X, ν). The atlas of X induces a "nodal atlas" for X/ν consisting of "charts" mapping either to $\mathbf{R}^n$ or $\mathbf{R}^n \times \{0\} \cup \{0\} \times \mathbf{R}^n \subset \mathbf{R}^{2n}$. The nodes of X/ν are precisely the points mapping to $(0, 0) \in \mathbf{R}^{2n}$ in some chart or, equivalently, the images of the points in the nodal set.

Definition 2.2. Let (X_1, ν_1) and $(X_2; \nu_2)$ be nodal manifolds. A **nodal map** $f\colon (X_1, \nu_1) \to (X_2, \nu_2)$ is a smooth map $f\colon X_1 \to X_2$ such that

$$f \circ \nu_1 = \nu_2 \circ f.$$

Definition 2.3. Let (X, ν) be a nodal manifold. A **diffeomorphism** of (X, ν) is an element of

$$\mathrm{Diff}(X, \nu) \coloneqq \{\phi \in \mathrm{Diff}(X) : \phi \circ \nu = \nu \circ \phi\}.$$

Every manifold X canonically is a nodal manifold with $\nu = \mathrm{id}_X$ and a smooth map between manifolds, trivially, is a nodal map. In other words, the category of manifolds is a full subcategory of the category of nodal manifolds.

Definition 2.4. Let (X, ν) be a nodal manifold, let Y be a manifold, and let $f\colon (X, \nu) \to Y$ be a nodal map. For a vector bundle $E \to Y$, set

$$\Gamma(X, \nu; f^*E) := \{\xi \in \Gamma(X, f^*E) : \xi \circ \nu = \xi\}.$$

Remark 2.5. In the situation of the preceding definition, set $n := \dim X$ and let $p > n$. Given a Riemannian metric on X and a Euclidean metric on E, denote by $W^{1,p}\Gamma(X, f^*E)$ the completion of $\Gamma(X, f^*E)$ with respect to the $W^{1,p}$ norm. By Morrey's embedding theorem, $W^{1,p} \hookrightarrow C^{0,1-n/p}$. Therefore, the evaluations maps $\mathrm{ev}_x\colon \Gamma(X, f^*E) \to E_{f(x)}$ extend to $W^{1,p}\Gamma(X, f^*E)$ and

$$W^{1,p}\Gamma(X, \nu; f^*E) = \left\{\xi \in W^{1,p}\Gamma(X; f^*E) : \xi(\nu(x)) = \xi(x) \text{ for every } x \in X\right\}.$$

For $p < n$ it can be shown that the $W^{1,p}$ completion of $\Gamma(X, \nu; f^*E)$ agrees with the $W^{1,p}$ completion of $\Gamma(X; f^*E)$.

2.2 Nodal Riemann surfaces

Definition 2.6. A **nodal Riemann surface** is a Riemann surface (Σ, j) together with a nodal structure ν.

Definition 2.7. Let C be a complex analytic curve. A point of C is a **node** if it has a neighborhood which is isomorphic to a neighborhood of the point $(0, 0)$ in the curve

$$\{(z, w) \in \mathbf{C}^2 : zw = 0\}.$$

A **nodal curve** is a complex analytic curve all of whose points are either smooth or a node.

Let C be a nodal curve and denote by $\pi\colon \tilde{C} \to C$ its normalization. The complex analytic curve $\tilde{C}$ is smooth and, hence, equivalent to a closed Riemann surface (Σ, j). Since $\tilde{C}$ is obtained from C by replacing every node with a pair of points, Σ inherits a canonical nodal structure ν. This sets up an equivalence between complete, nodal curves C and closed, nodal Riemann surfaces (Σ, j, ν).

Definition 2.8. The **automorphism group** of a nodal Riemann surface (Σ, j, ν) is

$$\mathrm{Aut}(\Sigma, j, \nu) := \{\phi \in \mathrm{Diff}(\Sigma, \nu) : \phi_* j = j\}.$$

A nodal Riemann surface (Σ, j, ν) is **stable** if $\mathrm{Aut}(\Sigma, j, \nu)$ is finite.

Definition 2.9. Let (Σ, ν) be a nodal surface with nodal set S. The **arithmetic genus** of (Σ, ν) is

$$p_a(\Sigma, \nu) := 1 - \frac{1}{2}(\chi(\Sigma) - \#S). \tag{2.10}$$

Remark 2.11. If $(\tilde{\Sigma}, \tilde{\nu})$ denotes a nodal surface obtained from (Σ, ν) by attaching a 1–handle at some pairs of nodes $\{n, \nu(n)\}$, then

$$p_a(\Sigma, \nu) = p_a(\tilde{\Sigma}, \tilde{\nu}).$$

2.3 Nodal J–holomorphic maps

Throughout the next four subsections, let (X, J) be an almost complex manifold of dimension $2n$

Definition 2.12. A **nodal J–holomorphic map** $u\colon (\Sigma, j, \nu) \to (X, J)$ is a nodal Riemann surface (Σ, j, ν) together with a nodal map $u\colon (\Sigma, \nu) \to X$ which is J–holomorphic; that is:

$$\bar{\partial}_J(u, j) \coloneqq \frac{1}{2}(\mathrm{d}u + J(u) \circ \mathrm{d}u \circ j) = 0. \tag{2.13}$$

Definition 2.14. If $u\colon (\Sigma, j, \nu) \to (X, J)$ is a nodal J–holomorphic map and $\phi \in \mathrm{Diff}(\Sigma, \nu)$, then the **reparametrization** $\phi_* u \coloneqq u \circ \phi^{-1}\colon (\Sigma, \phi_* j, \nu) \to (X, J)$ is a nodal J–holomorphic map as well. The **automorphism group** of a nodal J–holomorphic map $u\colon (\Sigma, j, \nu) \to (X, J)$ is

$$\mathrm{Aut}(\Sigma, j, \nu, u) \coloneqq \{\phi \in \mathrm{Aut}(\Sigma, j, \nu) : u \circ \phi = u\}.$$

The map u is said to be **stable** if $\mathrm{Aut}(\Sigma, j, \nu, u)$ is finite.

Definition 2.15. Let $u\colon (\Sigma, j) \to (M, J)$ be a J–holomorphic map and let $\pi\colon (\tilde{\Sigma}, \tilde{j}) \to (\Sigma, j)$ be a holomorphic map of degree $\deg(\pi) \geqslant 2$. The composition $u \circ \pi\colon (\tilde{\Sigma}, \tilde{j}) \to (M, J)$ is said to be a **multiple cover of** u. A J–holomorphic map is **simple** if it is not constant and not a multiple cover.

2.4 Ghost components

Let $u\colon (\Sigma, j, \nu) \to (X, J)$ be a nodal J–holomorphic map.

Definition 2.16. Suppose $C \subset \Sigma$ is a union of connected components of Σ. Set

$$S_C^{\mathrm{int}} \coloneqq \{n \in S : n \in C \text{ and } \nu(n) \in C\} \quad \text{and} \quad S_C^{\mathrm{ext}} \coloneqq \{n \in S : n \in C \text{ and } \nu(n) \notin C\}$$

and denote by ν_C the nodal structure on C which agrees with ν_0 on S_C^{int} and the identity on the complement of S_C^{int}. Denote by $\check{C}$ the nodal curve associated with (C, j_0, ν_C).

Definition 2.17. A **ghost component** of u is a union C of connected components of Σ such that $u|_C$ is constant, $\check{C}$ is connected, and which is a maximal subset satisfying these properties.

The following two observations are important for the proof of Theorem 1.1.

Proposition 2.18. *If u is stable and C is a ghost component, then the marked nodal Riemann surface $(C, j, \nu_C, S_C^{\mathrm{ext}})$ is stable; that is: the subgroup $\mathrm{Aut}(\Sigma, j, \nu, S_C^{\mathrm{ext}})$ of $\mathrm{Aut}(\Sigma, j, \nu)$ fixing S_C^{ext} is finite.*

Definition 2.19. A nodal Riemann surface (Σ, j, ν) is **semi-stable** if every connected component of Σ either has genus at least one or genus zero and contains at least two nodes.

Proposition 2.20. *If (Σ, j, ν) is semi-stable, then the dualizing sheaf ω_C of the corresponding nodal curve C is base-point free.*

Proof. If C is smooth, then ω_C is the canonical bundle K_C. Since C is semi-stable, it cannot be $\mathbf{C}P^1$; therefore, K_C is base-point free.

In general, the dualizing sheaf of C is constructed as follows; see [ACGH11, p. 91]. Denote by $\pi \colon \Sigma \to C$ the normalization map. Denote by $\tilde{\omega}_C$ the subsheaf of $K_\Sigma(S)$ whose sections ζ satisfy

$$\operatorname{Res}_n \zeta + \operatorname{Res}_{\nu(n)} \zeta = 0 \tag{2.21}$$

for every $n \in S$. Here $\operatorname{Res}_n \eta$ denotes the residue of the meromorphic 1–form η at n. The dualizing sheaf ω_C then is

$$\omega_C = \pi_* \tilde{\omega}_C.$$

It follows directly from this description that ω_C is base-point free. □

2.5 Moduli spaces of nodal pseudo-holomorphic maps

Definition 2.22. Given $A \in H_2(X, \mathbf{Z})$ and $g \in \mathbf{N}_0$, the **moduli space of stable nodal J–holomorphic maps** representing A and of genus g is the set

$$\overline{\mathscr{M}}_{A,g}(X, J)$$

of equivalence classes of stable nodal J–holomorphic maps $u \colon (\Sigma, j, \nu) \to (X, J)$ up to reparametrization with

$$p_a(\Sigma, \nu) = g \quad \text{and} \quad u_*[\Sigma] = A.$$

The subset of $\overline{\mathscr{M}}_{A,g}(X, J)$ parametrizing simple J–holomorphic maps is denoted by

$$\mathscr{M}^\star_{A,g}(X, J).$$

At this stage, $\overline{\mathscr{M}}_{A,g}(X, J)$ is just a set. In Section 3.2, it will be equipped with the **Gromov topology**. This topology induces the C^∞ topology on $\mathscr{M}^\star_{A,g}(X, J)$.

Definition 2.23. Let (X, ω) be a symplectic manifold. Denote by $\mathscr{J}(X, \omega)$ the space of almost complex structure on X which are **compatible with** ω; that is:

$$g(\cdot, \cdot) \coloneqq \omega(\cdot, J\cdot)$$

defines a Riemannian metric on X. Equip $\mathscr{J}(X, \omega)$ with the C^∞ topology.

Definition 2.24. Given $A \in H_2(X, \mathbf{Z})$ and $g \in \mathbf{N}$, set

$$\overline{\mathscr{M}}_{A,g}(X, \omega) \coloneqq \coprod_{J \in \mathscr{J}(X,\omega)} \overline{\mathscr{M}}_{A,g}(X, J) \quad \text{and} \quad \mathscr{M}^\star_{A,g}(X, \omega) \coloneqq \coprod_{J \in \mathscr{J}(X,\omega)} \mathscr{M}^\star_{A,g}(X, J).$$

Denote by $\pi \colon \overline{\mathscr{M}}_{A,g}(X, \omega) \to \mathscr{J}(X, \omega)$ the canonical projection.

2.6 Linearization of the J–holomorphic map equation

Let $u\colon (\Sigma, j, \nu) \to (X, J)$ be a nodal J–holomorphic map. Let h be a Hermitian metric on (X, J) and let ∇ be a torsion-free connection on TX. Throughout the remainder of this article, let $p > 2$.

Definition 2.25. Given $\xi \in W^{1,p}\Gamma(\Sigma, \nu; u^*TX)$, set

$$u_\xi \coloneqq \exp_u(\xi)$$

and denote by $\Psi_\xi\colon\ L^p\Omega^{0,1}(\Sigma, u^*TX) \to L^p\Omega^{0,1}(\Sigma, u_\xi^*TX)$ the map induced by parallel transport along the geodesics $t \mapsto \exp_u(t\xi)$. Define $\mathfrak{F}_{u,j,\nu;J}\colon\ W^{1,p}\Gamma(\Sigma, \nu; u^*TX) \to L^p\Omega^{0,1}(\Sigma, u^*TX)$ by

$$\mathfrak{F}_{u,j,\nu;J}(\xi) \coloneqq \Psi_\xi^{-1}\bar{\partial}_J(u_\xi, j).$$

Definition 2.26. Define the linear operator $\mathfrak{d}_{u,j,\nu;J}\colon\ W^{1,p}\Gamma(\Sigma, \nu; u^*TX) \to L^p\Omega^{0,1}(\Sigma, u^*TX)$ by

$$\mathfrak{d}_{u,j,\nu;J}\xi \coloneqq \mathrm{d}_0\mathfrak{F}_{u,j,\nu;J}\xi = \frac{1}{2}\big(\nabla\xi + J(u)\circ(\nabla\xi)\circ j + (\nabla_\xi J)\circ \mathrm{d}u\circ j\big).$$

Remark 2.27. If u is J–holomorphic, then $\mathfrak{d}_{u,j,\nu;,J}$ does not depend on the choice of torsion-free connection ∇ on TX; see [MS12, Proposition 3.1.1].

The operator $\mathfrak{d}_{u,j,\nu;J}$ is the restriction to $W^{1,p}\Gamma(\Sigma, \nu; u^*TX)$ of the operator

$$\mathfrak{d}_{u,j;J}\colon\ W^{1,p}\Gamma(\Sigma, u^*TX) \to L^p\Omega^{0,1}(\Sigma, u^*TX)$$

given by the same formula. The former controls the deformation theory of u as a nodal J–holomorphic map from the nodal Riemann surface (Σ, j, ν) whereas the latter controls the deformation theory of u as a smooth J–holomorphic map from the smooth Riemann surface (Σ, j), ignoring the nodal structure.

Proposition 2.28. *The index of* $\mathfrak{d}_{u,j,\nu;J}$ *is given by*

$$\operatorname{index}\mathfrak{d}_{u,j,\nu;J} = 2\langle[\Sigma], u^*c_1(X,J)\rangle + 2n(1 - p_a(\Sigma,\nu)). \tag{2.29}$$

Proof. The inclusion

$$W^{1,p}\Gamma(\Sigma, \nu; u^*TX) \to W^{1,p}\Gamma(\Sigma, u^*TX).$$

has index $-n\#S$. By the Riemann–Roch Theorem,

$$\operatorname{index}\mathfrak{d}_{u,j;J} = 2\langle[\Sigma], u^*c_1(X,J)\rangle + n\chi(\Sigma).$$

These together with (2.10) imply the index formula. □

Remark 2.30. For our discussion in Section 5.7, which establishes the key technical result of this article, the following detailed description of the kernel and cokernel of $\mathfrak{d}_{u,j,\nu;J}$ will be important. Denote by

$$V_- \subset \bigoplus_{n\in S} T_{u(n)}X$$

the subspace of those $(v_n)_{n\in S}$ satisfying

$$v_{\nu(n)} = -v_n.$$

Define diff: $\ker \mathfrak{d}_{u,j;J} \longrightarrow V_-$ by

$$\operatorname{diff}\kappa \coloneqq (\kappa(n) - \kappa(\nu(n)))_{n\in S}.$$

Evidently,

$$\ker \mathfrak{d}_{u,j,\nu;J} = \ker \operatorname{diff}.$$

The map diff is induced by the analogously defined map $W^{1,p}(\Sigma, u^*TX) \longrightarrow V_-$ which fits in to the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & W^{1,p}(\Sigma,\nu;u^*TX) \arrow[r] \arrow[d, "\mathfrak{d}_{u,j,\nu;J}"] & W^{1,p}(\Sigma,u^*TX) \arrow[r] \arrow[d, "\mathfrak{d}_{u,j;J}"] & V_- \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L^p\Omega^{0,1}(\Sigma,u^*TX) \arrow[r, "="] & L^p\Omega^{0,1}(\Sigma,u^*TX) \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd}$$

Therefore, the Snake Lemma yields the short exact sequence

$$0 \longrightarrow \operatorname{coker}\operatorname{diff} \longrightarrow \operatorname{coker}\mathfrak{d}_{u,j,\nu;J} \longrightarrow \operatorname{coker}\mathfrak{d}_{u,j;J} \longrightarrow 0.$$

The dual sequence

$$0 \longrightarrow (\operatorname{coker}\mathfrak{d}_{u,j;J})^* \longrightarrow (\operatorname{coker}\mathfrak{d}_{u,j,\nu;J})^* \longrightarrow (\operatorname{coker}\operatorname{diff})^* \longrightarrow 0$$

can be understood as follows. Let $q \in (1,2)$ be such that $1/p + 1/q = 1$. The dual space $(\operatorname{coker}\mathfrak{d}_{u,j\nu;J})^*$ can be identified via the pairing between L^p and L^q with the space $\mathscr{H}$ consisting of those $\zeta \in L^q\Omega^{0,1}(\Sigma, u^*TX)$ which satisfy a distributional equation of the form

$$\mathfrak{d}^*_{u,j,J}\zeta = \sum_{n\in S} v_n\delta_n.$$

with $v = (v_n)_{n\in S} \in (\operatorname{im}\operatorname{diff})^\perp \cong (\operatorname{coker}\operatorname{diff})^*$ and δ_n denoting the Dirac δ distribution at n. The map $(\operatorname{coker}\mathfrak{d}_{u,j,\nu;J})^* \longrightarrow (\operatorname{coker}\operatorname{diff})^*$ maps ζ to v.

Definition 2.31. Define the map $\mathfrak{n}_{u,j,\nu;J}\colon W^{1,p}\Gamma(\Sigma,\nu;u^*TX) \longrightarrow L^p\Omega^{0,1}(\Sigma,u^*TX)$ by

$$\mathfrak{n}_{u,j,\nu;J}(\xi) \coloneqq \mathfrak{F}_{u,j,\nu;J}(\xi) - \bar\partial_J(u,j) - \mathfrak{d}_{u,j,\nu;J}\xi$$

Proposition 2.32 ([MS12, Proposition 3.5.3 and Remark 3.5.5]). *Denote by* $c_S > 0$ *an upper bound for the norm of the embedding* $W^{1,p}(\Sigma) \hookrightarrow C^{0,1-2/p}(\Sigma)$ *and let* $c_\xi > 0$. *For every* ξ_1, ξ_2 *with* $\|\xi_1\|_{W^{1,p}} \leqslant c_\xi$ *and* $\|\xi_2\|_{W^{1,p}} \leqslant c_\xi$

$$\|\mathfrak{n}_{u,j,\nu;J}(\xi_1) - \mathfrak{n}_{u,j,\nu;J}(\xi_2)\|_{L^p} \leqslant c(c_S, c_\xi, \|\mathrm{d}u\|_{L^p}) \cdot (\|\xi_1\|_{W^{1,p}} + \|\xi_2\|_{W^{1,p}}) \cdot \|\xi_1 - \xi_2\|_{W^{1,p}}.$$

So far, the complex structure j has been held fixed. Denote by $\mathscr{J}(\Sigma)$ the space of complex structures on Σ and by $\mathrm{Diff}_0(\Sigma, \nu)$ the group of diffeomorphism of Σ which are isotopic to the identity and commute with ν. Denote by

$$\mathscr{T} := \mathscr{J}(\Sigma)/\mathrm{Diff}_0(\Sigma, \nu)$$

the corresponding Teichmüller space. This is a complex manifold and for every $j \in \mathscr{J}(\Sigma)$ there is a **Teichmüller slice through** j; that is: an open neighborhood Δ of $0 \in \mathbf{C}^{\dim_{\mathbf{C}} \mathscr{T}}$ together with a $\mathrm{Aut}(\Sigma, j, \nu)$–equivariant map $\jmath\colon\ \Delta \to \mathscr{J}(\Sigma)$ such that $\jmath(0) = j$.

Definition 2.33. Trivializing the bundle over Δ whose fiber at $\sigma \in \Delta$ is $L^p\Omega^{0,1}(\Sigma, u^*TX)$ defined with respect to complex structure $\jmath(\sigma)$ gives rise to the map

$$\begin{aligned} W^{1,p}\Gamma(\Sigma, \nu; u^*TX) \times \Delta &\to L^p\Omega^{0,1}(\Sigma, u^*TX) \\ (\xi, \sigma) &\mapsto \mathscr{F}_{u,\jmath(\sigma),\nu;J}(\xi). \end{aligned} \tag{2.34}$$

Define $\mathrm{d}_{u,j}\bar{\partial}_{\nu;J}\colon\ W^{1,p}\Gamma(\Sigma, \nu; u^*TX) \oplus T_0\Delta \to L^p\Omega^{0,1}(\Sigma, u^*TX)$ to be the derivative of the map (2.34) at $(0, 0)$.

Definition 2.35. The **index** of u is

$$\begin{aligned} \mathrm{index}(u) &:= \mathrm{index}(\mathrm{d}_{u,j}\bar{\partial}_{\nu;J}) - \dim \mathfrak{aut}(\Sigma, j, \nu) \\ &= 2\langle[\Sigma], u^*c_1(X, J)\rangle + (2n - 3)(1 - p_a(\Sigma, \nu)). \end{aligned}$$

The map $u\colon\ (\Sigma, j, \nu) \to (M, J)$ is said to be **unobstructed** if $\mathrm{d}_{u,j}\bar{\partial}_{\nu;J}$ is surjective.

Henceforth, to simplify notation, we will often drop some or all of the subscripts j, ν, J from the maps defined above.

2.7 Transversality for simple maps

Throughout the remainder of this section, (X, ω) is a compact symplectic manifold of dimension $2n$ and we only consider pseudo-holomorphic maps from smooth Riemann surfaces.

Definition 2.36. Denote by $\mathscr{J}_{\mathrm{emb}}(M, \omega) \subset \mathscr{J}(M, \omega)$ the subspace of those almost complex structures compatible with ω for which the following hold:

1. there are no simple J–holomorphic maps with negative index,

2. every simple J–holomorphic map with $\operatorname{index}(u) < 2n-4$ is an embedding, and
3. every pair of simple J–holomorphic maps u_1, u_2 satisfying

$$\operatorname{index}(u_1) + \operatorname{index}(u_2) < 2n-4$$

   either have disjoint images or are related by a reparametrization.

Denote by $\mathscr{J}^\star_{\mathrm{emb}}(X,\omega) \subset \mathscr{J}_{\mathrm{emb}}(X,\omega)$ the subset of those J for which, moreover,

4. every simple J–holomorphic map is unobstructed.

Definition 2.37. Given $J_0, J_1 \in \mathscr{J}(X,\omega)$, denote by $\mathscr{J}(X,\omega;J_0,J_1)$ the space of smooth paths $(J_t)_{t\in[0,1]}$ in $\mathscr{J}(X,\omega)$ from J_0 and J_1. Given $J_0, J_1 \in \mathscr{J}^\star_{\mathrm{emb}}(X,\omega)$, denote by $\mathscr{J}^\star_{\mathrm{emb}}(X,\omega,J_0,J_1)$ subset of those $(J_t)_{t\in[0,1]} \in \mathscr{J}(X,\omega;J_0,J_1)$ such that for every $t \in [0,1]$:

1. $J_t \in \mathscr{J}_{\mathrm{emb}}(X,\omega)$ and
2. if $u\colon (\Sigma,j) \to (X,J_t)$ is a simple J_t–holomorphic map, then either:
    a. $\operatorname{coker} \mathrm{d}_{u,j}\bar\partial_{J_t} = \{0\}$ or
    b. $\dim\operatorname{coker} \mathrm{d}_{u,j}\bar\partial_{J_t} = 1$ and the map $\ker \mathrm{d}_{u,j}\bar\partial_{J_t} \to \operatorname{coker} \mathrm{d}_{u,j}\bar\partial_{J_t}$ defined by

$$\xi \mapsto \mathrm{pr}\left(\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=t} \mathrm{d}_{u,j}\bar\partial_{J_s}\xi\right),$$

    with $\mathrm{pr}\colon \Omega^{0,1}(\Sigma,u^*TX) \to \operatorname{coker} \mathrm{d}_{u,j}\bar\partial_{J_t}$ denoting the canonical projection, is surjective.

Proposition 2.38. *Let $A \in H_2(X,\mathbf{Z})$ and $g \in \mathbf{N}_0$.*

1. *For every $J \in \mathscr{J}^\star_{\mathrm{emb}}(X,\omega)$ the moduli space $\mathscr{M}^\star_{A,g}(X,J)$ is an oriented smooth manifold of dimension*

$$2\langle c_1(X,\omega), A\rangle + 2(n-3)(1-g).$$

2. *For every pair $J_0, J_1 \in \mathscr{J}^\star_{\mathrm{emb}}(X,\omega)$ and $(J_t)_{t\in[0,1]} \in \mathscr{J}^\star_{\mathrm{emb}}(X,\omega;J_0,J_1)$ the moduli space*

$$\mathscr{M}^\star_{A,g}\big(X,(J_t)_{t\in[0,1]}\big) := \coprod_{t\in[0,1]} \mathscr{M}^\star_{A,g}(X,J_t),$$

   is an oriented smooth manifold with boundary

$$\mathscr{M}^\star_{A,g}(X,J_1) \amalg -\mathscr{M}^\star_{A,g}(X,J_0).$$

This is a consequence of the Implicit Function Theorem; see [MS12, Theorem 3.1.6 and Theorem 3.1.7]. The orientation on the moduli spaces is obtained by trivalizing the determinant line bundle of the family of operators $\mathrm{d}_{u,j}\bar{\partial}_J$; see [MS12, Proof of Theorem 3.1.6, Remark 3.2.5, Appendix A.2]. If the moduli space is zero-dimensional, that is: a discrete set, then every $[u] \in \mathscr{M}^\star_{A,g}(X,J)$ is assigned a sign

$$\operatorname{sign}[u] \in \{+1,-1\}.$$

The signed count of $\mathscr{M}^\star_{A,g}(X,J)$ is then

$$\#\mathscr{M}^\star_{A,g}(X,J) \coloneqq \sum_{[u]\in\mathscr{M}^\star_{A,g}(X,J)} \operatorname{sign}[u].$$

Proposition 2.39.

1. *$\mathscr{J}^\star_{\mathrm{emb}}(X,\omega) \subset \mathscr{J}(X,\omega)$ is residual.*
2. *For every pair $J_0, J_1 \in \mathscr{J}^\star_{\mathrm{emb}}(X,\omega)$, $\mathscr{J}^\star_{\mathrm{emb}}(X,\omega;J_0,J_1) \subset \mathscr{J}(X,\omega;J_0,J_1)$ is residual.*

The proof is a standard application of the Sard–Smale theorem; cf. [OZ09, Theorem 1.2; IP18, Proposition A.4; MS12, Sections 3.2 and 6.3].

2.8 J–holomorphic maps with constraints

Definition 2.40. Let $\Lambda \in \mathbf{N}$. A **J–holomorphic map with Λ marked points** is a J–holomorphic map $u\colon (\Sigma,j) \to (M,J)$ together with Λ distinct labeled points $z_1,\dots,z_\Lambda \in \Sigma$.

The **reparametrization** of $(u;z_1,\dots,z_\Lambda)$ by $\phi \in \mathrm{Diff}(\Sigma)$ is the J–holomorphic map with Λ marked points $\phi_*(u;z_1,\dots,z_\Lambda) \coloneqq (u\circ\phi^{-1};\phi(z_1),\dots,\phi(z_\Lambda))$.

A J–holomorphic map $(u;z_1,\dots,z_\Lambda)$ with Λ marked points is said to be **simple** if u is simple.

Definition 2.41. Given $A \in H_2(X,\mathbf{Z})$, $g \in \mathbf{N}_0$, $\Lambda \in \mathbf{N}$, and $J \in \mathscr{J}(X,\omega)$, the **moduli space of simple J–holomorphic maps with Λ marked points** representing A and of genus g is the set

$$\mathscr{M}^\star_{A,g,\Lambda}(X,J)$$

of equivalence classes J–holomorphic maps $u\colon (\Sigma,j) \to (X,J)$ with Λ marked points $z_1,\dots,z_\Lambda$ up to reparametrization with

$$u_*[\Sigma] = A \quad \text{and} \quad g(\Sigma) = g.$$

Define the **evaluation map** $\mathrm{ev}\colon \mathscr{M}^\star_{A,g,\Lambda}(X,J) \to X^\Lambda$ by

$$\mathrm{ev}([u;z_1,\dots,z_\Lambda]) \coloneqq (u(z_1),\dots,u(z_\Lambda)).$$

Remark 2.42. Given two maps $f\colon X \to Z$ and $g\colon Y \to Z$, the **fiber product** is

$$X\,{}_f\!\times_g Y \coloneqq (f\times g)^{-1}(\Delta)$$

with $\Delta \subset Z\times Z$ denoting the diagonal. If X, Y, Z are smooth manifolds and f and g are transverse smooth maps, then $X\,{}_f\!\times_g Y$ is a submanifold of $X \times Y$ of dimension $\dim(X) + \dim(Y) - \dim(Z)$.

Let $(f_\lambda\colon V_\Lambda \to X)_{\lambda=1}^{\Lambda}$ be an Λ–tuple of pseudo-cycles in general position such that

$$\operatorname{codim}(f_\lambda) \coloneqq \dim X - \dim V_\lambda$$

is even and positive for every λ. The following discussion assumes some familiarity with the notions of a pseudo-cycle, pseudo-cycle cobordism, and pseudo-cycle transversality. In particular, we make use of the following facts, which are discussed in Appendix A: (a) For every $\lambda \in \{1,\dots,\Lambda\}$, there is a manifold V_λ^∂ of dimension $\dim(V_\lambda) - 2$ and a smooth map $f_\lambda^\partial\colon V_\lambda^\partial \to X$ whose image contains the pseudo-cycle boundary $\operatorname{bd}(f_\lambda)$. (b) A smooth map $g\colon M \to X$ is said to be transverse to the pseudo-cycle f_λ if it is transverse to both f_λ and f_λ^∂ in the usual sense. (c) For every $\Lambda \subset \{1,\dots,\Lambda\}$ the product $\prod_{\lambda\in I} f_\lambda$ is a pseudo-cycle and f_λ^∂ induce in a natural way a map from a smooth manifold whose image contains $\operatorname{bd}(\prod_{\lambda\in I} f_\lambda)$; see Proposition A.2.

In the following, $f_\lambda^\bullet\colon V_\lambda^\bullet \to X$ stands for either $f_\lambda\colon V_\lambda \to X$ or $f_\lambda^\partial\colon V_\lambda^\partial \to X$.

Definition 2.43. Given $A \in H_2(X,\mathbf{Z})$, $g \in \mathbf{N}_0$, and $J \in \mathscr{J}(X,\omega)$, set

$$\mathscr{M}^\star_{A,g}(X,J;f_1^\bullet,\dots,f_\Lambda^\bullet) \coloneqq \mathscr{M}^\star_{A,g,\Lambda}(X,J)\,{}_{\mathrm{ev}}{\times}_{f_1^\bullet\times\cdots\times f_\Lambda^\bullet}\, V_1^\bullet \times \cdots \times V_\Lambda^\bullet.$$

The **expected dimension** of $\mathscr{M}^\star_{A,g}(X,J;f_1,\dots,f_\Lambda)$ is

$$\operatorname{vdim} \mathscr{M}^\star_{A,g}(X,J;f_1,\dots,f_\Lambda) \coloneqq 2\langle c_1(X,\omega),A\rangle + (n-3)(2-2g) + \sum_{\lambda=1}^{\Lambda}(2-\operatorname{codim}(f_\lambda)).$$

The following are analogues of Definition 2.36 and Definition 2.37 in the setting of J–holomorphic maps with constraints.

Definition 2.44. Denote by $\mathscr{J}_{\mathrm{emb}}(X,\omega;f_1,\dots,f_\Lambda) \subset \mathscr{J}(X,\omega)$ the subset of those almost complex structures J compatible with ω for which the following conditions hold for every $A, A_1, A_2 \in H_2(X,\mathbf{Z})$, $g, g_1, g_2 \in \mathbf{N}_0$, $I, I_1, I_2 \subset \{1,\dots,\Lambda\}$ with $I_1 \cap I_2 = \varnothing$:

1. if $\operatorname{vdim} \mathscr{M}^\star_{A,g}\big(X,J;(f_\lambda^\bullet)_{\lambda\in I}\big) < 0$, then $\mathscr{M}^\star_{A,g}\big(X,J;(f_\lambda^\bullet)_{\lambda\in I}\big) = \varnothing$;

2. if $\operatorname{vdim} \mathscr{M}^\star_{A,g}\big(X,J;(f_\lambda^\bullet)_{\lambda\in I}\big) < 2n-4$, then every J–holomorphic map underlying an element of $\mathscr{M}^\star_{A,g}\big(X,J;(f_\lambda^\bullet)_{\lambda\in I}\big)$ is an embedding; and

3. if $\operatorname{vdim} \mathscr{M}^\star_{A_1,g_1}\big(X,J;(f_\lambda^\bullet)_{\lambda\in I_1}\big) + \operatorname{vdim} \mathscr{M}^\star_{A_2,g_2}\big(X,J;(f_\lambda^\bullet)_{\lambda\in I_2}\big) < 2n-4$, then every pair of every J–holomorphic maps underlying elements of $\mathscr{M}^\star_{A_1,g_1}\big(X,J;(f_\lambda^\bullet)_{\lambda\in I_1}\big)$ and $\mathscr{M}^\star_{A_2,g_2}\big(X,J;(f_\lambda^\bullet)_{\lambda\in I_2}\big)$ either have disjoint images or are related by a reparametrization.

Denote by $\mathscr{J}^\star_{\mathrm{emb}}(X,\omega;f_1,\dots,f_\Lambda)$ the subset of those elements of $\mathscr{J}_{\mathrm{emb}}(X,\omega;f_1,\dots,f_\Lambda)$ for which, moreover:

4. every simple J–holomorphic map is unobstructed and

5. for every $A \in H_2(X, \mathbf{Z})$, $g \in \mathbf{N}$, and $I \subset \{1, \dots, \Lambda\}$, the pseudo-cycle $\prod_{\lambda \in I} f_\lambda$ is transverse to $\mathrm{ev}\colon \mathscr{M}^\star_{A,g,|I|}(X, J) \to X^{|I|}$.

Definition 2.45. Given $J_0, J_1 \in \mathscr{J}^\star_{\mathrm{emb}}(X, \omega; f_1, \dots, f_\Lambda)$, denote by $\mathscr{J}^\star_{\mathrm{emb}}(X, \omega; f_1, \dots, f_\Lambda; J_0, J_1)$ the space of smooth paths $(J_t)_{t \in [0,1]}$ in $\mathscr{J}(X, \omega)$ from J_0 and J_1 such that for every $t \in [0, 1]$,

1. $J_t \in \mathscr{J}_{\mathrm{emb}}(X, \omega; f_1, \dots, f_\Lambda)$,

2. if $u\colon (\Sigma, j) \to (X, J_t)$ is a simple J_t–holomorphic map, then either:

   i. $\operatorname{coker} \mathrm{d}_{u,j} \bar{\partial}_{J_t} = \{0\}$ or
   ii. $\dim \operatorname{coker} \mathrm{d}_{u,j} \bar{\partial}_{J_t} = 1$ and the map $\ker \mathrm{d}_{u,j} \bar{\partial}_{J_t} \to \operatorname{coker} \mathrm{d}_{u,j} \bar{\partial}_{J_t}$ defined by

   $$\xi \mapsto \mathrm{pr}\left(\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=t} \mathrm{d}_{u,j} \bar{\partial}_{J_s} \xi \right),$$

   with $\mathrm{pr}\colon \Omega^{0,1}(\Sigma, u^*TX) \to \operatorname{coker} \mathrm{d}_{u,j} \bar{\partial}_{J_t}$ denoting the canonical projection, is surjective; in particular, for every $A \in H_2(X, \mathbf{Z})$, $g \in \mathbf{N}$, and $k \in \mathbf{N}$ the moduli space

   $$\mathscr{M}^\star_{A,g,k}\big(X, (J_t)_{t \in [0,1]}\big) \coloneqq \coprod_{t \in [0,1]} \mathscr{M}^\star_{A,g,k}(X, J_t)$$

   is an oriented smooth manifold with boundary $\mathscr{M}^\star_{A,g,k}(X, J_1) \amalg -\mathscr{M}^\star_{A,g,k}(X, J_0)$,

3. for every $A \in H_2(X, \mathbf{Z})$, $g \in \mathbf{N}$, and $I \subset \{1, \dots, \Lambda\}$ the pseudo-cycle $\prod_{\lambda \in I} f_\lambda$ is to the evaluation map $\mathrm{ev}\colon \mathscr{M}^\star_{A,g,|I|}\big(X, A; (J_t)_{t \in [0,1]}\big) \to X^{|I|}$.

The next two results are analogues of Proposition 2.38 and Proposition 2.39.

Proposition 2.46. *Let $A \in H_2(X, \mathbf{Z})$ and $g \in \mathbf{N}_0$.*

1. *For every $J \in \mathscr{J}^\star_{\mathrm{emb}}(X, \omega; f_1, \dots, f_\Lambda)$ the moduli space $\mathscr{M}^\star_{A,g}\big(X, J; f_1^\bullet, \dots, f_\Lambda^\bullet\big)$ is an oriented smooth manifold of dimension*

   $$\operatorname{vdim} \mathscr{M}^\star_{A,g}\big(X, J; f_1^\bullet, \dots, f_\Lambda^\bullet\big).$$

2. *For every pair $J_0, J_1 \in \mathscr{J}^\star_{\mathrm{emb}}(X, \omega; f_1, \dots, f_\Lambda)$ and $(J_t)_{t \in [0,1]} \in \mathscr{J}^\star_{\mathrm{emb}}(X, \omega; f_1, \dots, f_\Lambda; J_0, J_1)$ the moduli space*

   $$\mathscr{M}^\star_{A,g}\big(X, (J_t)_{t \in [0,1]}); f_1^\bullet, \dots, f_\Lambda^\bullet\big) \coloneqq \coprod_{t \in [0,1]} \mathscr{M}^\star_{A,g}(X, J_t; f_1^\bullet, \dots, f_\Lambda^\bullet)$$

   is an oriented smooth manifold with boundary

   $$\mathscr{M}^\star_{A,g}\big(X, J; f_1^\bullet, \dots, f_\Lambda^\bullet\big) \amalg -\mathscr{M}^\star_{A,g}\big(X, J; f_1^\bullet, \dots, f_\Lambda^\bullet\big).$$

Proposition 2.47.

1. $\mathscr{J}^{\star}_{\mathrm{emb}}(X,\omega;f_1,\dots,f_\Lambda)\subset\mathscr{J}(X,\omega)$ *is residual.*
2. *For every pair* $J_0,J_1\in\mathscr{J}^{\star}_{\mathrm{emb}}(X,\omega;f_1,\dots,f_\Lambda)$, $\mathscr{J}^{\star}_{\mathrm{emb}}(X,\omega;f_1,\dots,f_\Lambda;J_0,J_1)\subset\mathscr{J}(X,\omega;J_0,J_1)$ *is residual.*

Proof. The proof is a standard application of the Sard–Smale theorem [OZ09, Theorem 1.2], [IP18, Proposition A.4], [MS12, Sections 3.2 and 6.3] for details. However, we outline the proof of the first statement to explain how to account for the presence of the pseudo-cycles (f_λ); the proof of the second statement is almost identical.

Let (Σ,j) be a closed Riemann surface of genus g, and let $A\in H_2(X,\mathbf{Z})$. Denote by $W^{1,p}_{\mathrm{inj}}(\Sigma,X;A)$ the subset of $W^{1,p}(\Sigma,X)$ consisting of functions $u\colon\ \Sigma\to X$ which represent A and are **somewhere injective** in the sense that there exist $z_0\in\Sigma$ and $\delta>0$ such that for all $z\in\Sigma$

$$\mathrm{dist}_X(u(z_0),u(z))\geqslant\delta\,\mathrm{dist}_\Sigma(z_0,z).$$

A J–holomorphic map is somewhere injective if and only if it is simple [MS12, Proposition 2.5.1]. Given a slice $\mathscr{S}\subset\mathscr{J}(\Sigma)$ for the action of $\mathrm{Diff}_0(\Sigma)$ on $\mathscr{J}(\Sigma)$, set

$$\mathscr{X}=W^{1,p}_{\mathrm{inj}}(\Sigma,X;A)\times\mathscr{S}$$

and let $\mathscr{E}\to\mathscr{X}$ be a Banach vector bundle whose fiber over (u,j) is the space $L^p\Omega^{0,1}(\Sigma,u^*TX)$ defined using the complex structure j.

Let $s\colon\ \mathscr{J}(X,\omega)\times\mathscr{X}\to\mathscr{E}$ be a section given by $s(J,u,j)=\bar\partial_J(u,j)$. This section is transverse to the zero section and the moduli space $\mathscr{M}^{\star}_g(X,A)$ can be covered by a countable number of submanifolds of the form $s^{-1}(0)$, for different choices of (Σ,j). (This statement is the main part of Proposition 2.39, see [MS12, Section 3.2].)

Let I be an ordered subset of $\{1,\dots,\lambda\}$ and without loss of generality assume that $f^{\bullet}_\lambda=f_\lambda$ for all λ. Consider the map

$$\begin{aligned}S\colon\ \mathscr{J}(X,\omega)\times\mathscr{X}\times\prod_{\lambda\in I}(\Sigma\times V_\lambda)&\to\mathscr{E}\times\prod_{\lambda\in I}(X\times X)\\ S(J,u,j,(z_\lambda,x_\lambda)_{\lambda\in I})&\coloneqq(s(J,u,j),(u(z_\lambda),f_\lambda(x_\lambda))_{\lambda\in I}).\end{aligned}$$

Let $\Delta\subset X\times X$ be the diagonal. The result follows from the Sard–Smale theorem if we prove that S is transverse to the submanifold

$$\mathscr{X}\times\prod_{\lambda\in I}\Delta\subset\mathscr{E}\times\prod_{\lambda\in I}(X\times X),\tag{2.48}$$

where the embedding $\mathscr{X}\subset\mathscr{E}$ is the zero section. Since s is transverse to the zero section, it remains to see that the variations in the directions of u, (z_λ), and (x_λ) are transverse to $\prod_\lambda\Delta$ whenever $u(z_\lambda)=f_\lambda(x_\lambda)$ for all $\lambda\in I$.

Given $\lambda \in I$, let $I_\lambda \subset I$ be the subset of all those indices μ for which $z_\mu = z_\lambda$. We will show that the projection of the derivative

$$\mathrm{d}S = \mathrm{d}_{(J,u,j,(z_\lambda,x_\lambda)_{\lambda\in I})}S$$

on $\bigoplus_{\mu\in I_\lambda}(T_{u(z_\lambda)}X \oplus T_{u(z_\lambda)}X)$ is transverse to $\bigoplus_{\mu\in I_\lambda} T_{u(z_\lambda),u(z_\lambda)}\Delta$. Since I is a disjoint union of subsets of the form I_λ for some collection of indices λ, this will prove that $\mathrm{d}S$ is transverse to (2.48). Denote the said projection by $\mathrm{pr}_{I_\lambda}\mathrm{d}S$. For $\xi \in W^{1,p}(\Sigma, u^*TX)$ and $v_\mu \in T_{x_\mu}V_\mu$ for $\mu \in I_\lambda$,

$$\mathrm{pr}_{I_\lambda}\mathrm{d}S(\xi,(v_\mu)_{\mu\in I_\lambda}) = (\xi(z_\lambda), \mathrm{d}_{x_\mu}f_\mu \cdot v_\mu)_{\mu\in I_\lambda}. \tag{2.49}$$

The following two observations will complete the proof. First, the evaluation map $W^{1,p}(\Sigma, u^*TX) \to T_{u(z_\lambda)}X$ is surjective. In particular, if λ is the only element of I_λ, then $\mathrm{pr}_{I_\lambda}\mathrm{d}S$ is transverse to the subspace $T_{u(z_\lambda),u(z_\lambda)}\Delta \subset T_{u(z_\lambda)}X \oplus T_{u(z_\lambda)}X$. Second, if $|I_\lambda| \geqslant 2$, the point

$$(f_\mu(x_\mu))_{\mu\in I_\lambda} \in \prod_{\mu\in I_\lambda} X$$

lies on the diagonal $X \hookrightarrow \prod_{\mu\in I_\lambda} X$, and the map $\prod_{\mu\in I_\lambda} f_\mu \colon \prod_{\mu\in I_\lambda} V_\mu \to \prod_{\mu\in I_\lambda} X$ is transverse to this diagonal by the assumption that the pseudo-cycles $(f_\lambda)_{\lambda\in I}$ are in general position. It follows from these two observations and (2.49) that $\mathrm{pr}_{I_\lambda}\mathrm{d}S$ is transverse to $\bigoplus_{\mu\in I_\lambda} T_{u(z_\lambda),u(z_\lambda)}\Delta$. □

The following will be important for relating moduli spaces defined using cobordant pseudocycles. Let $F\colon W \to X$ be a cobordism between two pseudo-cycles f_1^0 and f_1^1 in X, with $F^\partial\colon W^\partial \to X$ such that $\mathbf{d}(F) \subset \operatorname{im} F^\partial$. In what follows, $F^\bullet$ denotes either F or F^∂. Let $f_2, \dots, f_\Lambda$ be pseudo-cycles in X such that $F, f_2, \dots, f_\Lambda$ are in general position, as in Definition A.4.

Given $J \in \mathscr{J}(M,\omega)$ and an ordered subset $I \subset \{2,\dots,\Lambda\}$, set

$$\mathscr{M}^\star_{A,g}(X,J;F^\bullet,(f^\bullet_\lambda)_{\lambda\in I}) \coloneqq \mathscr{M}^\star_{A,g,|I|+1}(X,J)\;{}_{\mathrm{ev}}{\times}_{F^\bullet\times\prod_{\lambda\in I}f^\bullet_\lambda}\; W^\bullet \times \prod_{\lambda\in I} V^\bullet_\lambda.$$

Definition 2.50. Let

$$\mathscr{J}^\star_{\mathrm{emb}}(X,\omega;F,f_2,\dots,f_\Lambda) \subset \mathscr{J}^\star_{\mathrm{emb}}(X,\omega;f_1^0,f_2,\dots,f_\Lambda) \cap \mathscr{J}^\star_{\mathrm{emb}}(X,\omega;f_1^1,f_2,\dots,f_\Lambda),$$

be the subset of those J for which the following conditions hold for every $A, A_1, A_2 \in H_2(X,\mathbf{Z})$, $g, g_1, g_2 \in \mathbf{N}_0$, $I, I_1, I_2 \subset \{2,\dots,\Lambda\}$ with $I_1 \cap I_2 = \varnothing$:

1. if $\operatorname{vdim}\mathscr{M}^\star_{A,g}\big(X,J;F^\bullet,(f^\bullet_\lambda)_{\lambda\in I}\big) < 2n-4$, then every J–holomorphic map underlying an element of $\mathscr{M}^\star_{A,g}\big(X,J;F^\bullet,(f^\bullet_\lambda)_{\lambda\in I}\big)$ is an embedding; and
2. if $\operatorname{vdim}\mathscr{M}^\star_{A_1,g_1}\big(X,J;F^\bullet,(f^\bullet_\lambda)_{\lambda\in I_1}\big) + \operatorname{vdim}\mathscr{M}^\star_{A_2,g_2}\big(X,J;(f^\bullet_\lambda)_{\lambda\in I_2}\big) < 2n-4$, then every pair of every J–holomorphic maps underlying elements of

   $$\mathscr{M}^\star_{A_1,g_1}\big(X,J;F^\bullet,(f^\bullet_\lambda)_{\lambda\in I_1}\big) \quad\text{and}\quad \mathscr{M}^\star_{A_2,g_2}\big(X,J;F^\bullet,(f^\bullet_\lambda)_{\lambda\in I_2}\big)$$

   either have disjoint images or are related by a reparametrization.

3. for every $A \in H_2(X, \mathbf{Z})$, $g \in \mathbf{N}$, and $I \subset \{2, \ldots, \Lambda\}$, the pseudo-cycle $F \times \prod_{\lambda \in I} f_\lambda$ is transverse as pseudocycle with boundary to $\mathrm{ev}\colon \mathscr{M}^\star_{A,g,|I|+1}(X, J) \to X^{|I|+1}$ in the sense of Definition A.3.

It follows from this definition that for every $J \in \mathscr{J}^\star_{\mathrm{emb}}(X, \omega; F, f_2, \ldots, f_\Lambda)$, $F \times f_2 \times \ldots \times f_\Lambda$ is transverse as pseudo-cycle cobordism to $\mathrm{ev}\colon \mathscr{M}^\star_{A,g,\Lambda}(X, J) \to X^\Lambda$. In this case, $\mathscr{M}^\star_{A,g}(X, J; F, f_2, \ldots, f_\Lambda)$ is an oriented cobordism from $\mathscr{M}^\star_{A,g}(X, J; f_1^0, f_2, \ldots, f_\Lambda)$ to $\mathscr{M}^\star_{A,g}(X, J; f_1^1, f_2, \ldots, f_\Lambda)$.

The proof of Proposition 2.47 can be easily adapted to show the following.

Proposition 2.51. *$\mathscr{J}^\star_{\mathrm{emb}}(X, \omega; F, f_2, \ldots, f_\Lambda)$ is residual in $\mathscr{J}(X, \omega)$.*

3 Gromov compactness

3.1 Deformations of nodal Riemann surfaces

Definition 3.1. Let $\mathscr{X}$ and A be complex manifolds and let $\pi\colon \mathscr{X} \to A$ be a holomorphic map. Set $n := \dim_{\mathbf{C}} A$ and suppose that $\dim_{\mathbf{C}} \mathscr{X} = n + 1$. A critical point $x \in \mathscr{X}$ of π is called **nodal** if there are holomorphic coordinates at x and holomorphic coordinates at $\pi(x)$ with respect to which

$$\pi(z, w, t_2, \ldots, t_n) = (zw, t_2, \ldots, t_n).$$

A **nodal family** is a surjective, proper, holomorphic map $\pi\colon \mathscr{X} \to A$ between complex manifolds of dimension $\dim_{\mathbf{C}} \mathscr{X} = \dim_{\mathbf{C}} A + 1$ such that every critical point of π is nodal. The **fiber** over $a \in A$ is the nodal Riemann surface (Σ, j, ν) associated with the nodal curve $\pi^{-1}(a)$. Henceforth, we engage in the abuse of notation to identify $\pi^{-1}(a)$ and (Σ, j, ν).

Definition 3.2. Let (Σ, j, ν) be a nodal Riemann surface. A **deformation** of (Σ, j, ν) is a nodal family $\pi\colon \mathscr{X} \to A$, together with a base-point $\star \in A$, and a nodal, biholomorphic map $\iota\colon (\Sigma, j, \nu) \to \pi^{-1}(\star)$.

Definition 3.3. Let (Σ, j, ν) be a nodal Riemann surface and let $(\pi\colon \mathscr{X} \to A, \star, \iota)$ and $(\rho\colon \mathscr{Y} \to B, \dagger, \kappa)$ be two deformations of (Σ, j, ν). A pair of holomorphic maps $\Phi\colon \mathscr{X} \to \mathscr{Y}$ and $\phi\colon A \to B$ forms a **morphism** $(\Phi, \phi)\colon (\rho, \star, \iota) \to (\mathscr{Y}, \dagger, \kappa)$ of deformations if

$$\phi(\star) = \dagger, \quad \rho \circ \Phi = \phi \circ \pi, \quad \Phi \circ \iota = \kappa$$

and for every $a \in A$ the restriction $\Phi\colon \pi^{-1}(a) \to \rho^{-1}(\phi(a))$ induces a nodal, biholomorphic map.

Definition 3.4. A deformation $(\rho\colon \mathscr{Y} \to B, \dagger, \kappa)$ of (Σ, j, ν) is **(uni)versal** if for every deformation $(\pi\colon \mathscr{X} \to A, \star, \iota)$ of (Σ, j, ν) there exists an open neighborhood U of $\star \in A$ and a (unique) morphism of deformations $(\pi\colon \pi^{-1}(U) \to U, \star, \iota) \to (\rho, \dagger, \kappa)$.

A nodal Riemann surface (Σ, j, ν) admits a universal deformation if and only if it is stable [DM69; ACGH11, Chapter XI Theorem 4.3; RS06, Theorem A]. However, every nodal Riemann surface (Σ, j, ν) admits a versal deformation; this will be discussed in detail in Section 4.

Definition 3.5. Let $(\pi\colon \mathscr{X} \to A, \star, \iota)$ be a deformation of a nodal Riemann surface (Σ, j, ν). Denote by S the nodal set of ν. A **framing** of $(\pi, \star, \iota)$ is a smooth embedding embedding $\Psi\colon (\Sigma\backslash S)\times A \to \mathscr{X}$ such that

$$\pi \circ \Psi = \mathrm{pr}_A \quad \text{and} \quad \Psi(\cdot, \star) = \iota.$$

3.2 The Gromov topology

Let X be a manifold and denote by $\mathscr{H}(X)$ the set of almost Hermitian structures (h, J) on X equipped with the C^∞ topology. The following defines a topology on

$$\overline{\mathscr{M}}_{A,g}(X) := \coprod_{(h,J)\in\mathscr{H}(X)} \overline{\mathscr{M}}_{A,g}(X, J)$$

Definition 3.6. Let (X, J, h) be an almost Hermitian manifold. Let (Σ, j, ν) be a closed, nodal Riemann surface. The **energy** of a nodal map $u\colon (\Sigma, \nu) \to X$ is

$$E(u) := \frac{1}{2}\int_\Sigma |\mathrm{d}u|^2 \,\mathrm{vol}.$$

Implicit in this definition is a choice of Riemannian metric in the conformal class determined by j. The right-hand side, however, is independent of this choice.

Definition 3.7. Let $(J_0, h_0) \in \mathscr{H}(X)$. Let $[u_0\colon (\Sigma_0, j_0, \nu_0) \to (M, J_0)] \in \overline{\mathscr{M}}_{A,g}(X, J_0)$, let $(\pi\colon \mathscr{X} \to A, \star, \iota)$ be a versal deformation of (Σ_0, j_0, ν_0), let Ψ be a framing of $(\pi, \star, \iota)$, let $\varepsilon > 0$. let $U_0 \subset C^\infty(\Sigma_0\backslash S, X)$ be an open neighborhood of $u_\infty|_{\Sigma_0\backslash S}$ in the C^∞_{loc} topology, and let $U_{\mathscr{H}}$ be an open neighborhood of (h_0, J_0) in $\mathscr{H}(X)$. Define

$$\mathscr{U}(u_0, \varepsilon, U_0, U_{\mathscr{H}}) \subset \overline{\mathscr{M}}_{A,g}(X)$$

to be the subset of the equivalences classes nodal J–holomorphic maps $u\colon (\Sigma, j, \nu) \to (M, J)$ satisfying the following:

1. $(g, J) \in U_{\mathscr{H}}$,
2. $|E(u) - E(u_0)| < \varepsilon$,
3. $(\Sigma, j, \nu) = \pi^{-1}(a)$ for some $a \in A$, and
4. $\tilde{u} := u \circ \Psi(\cdot, a) \in U_0$,

The **Gromov topology** on $\overline{\mathscr{M}}_{A,g}(X)$ is the coarsest topology with respect to which every subset of the form $\mathscr{U}(u_0, \varepsilon, U_0, U_J)$ is open.

The Gromov topology is metrizable and thus completely characterized by its notion of convergence, which is immediately seen to agree with the following on the level of nodal maps.

Definition 3.8. Let (X, J_∞, h_∞) be an almost Hermitian manifold and let $(J_k, h_k)_{k\in\mathbf{N}}$ be a sequence of almost Hermitian structures on X converging to (J_∞, h_∞) in the C^∞ topology. For every $k \in \mathbf{N}\cup\{\infty\}$ let $u_k\colon (\Sigma_k, j_k, \nu_k) \to (X, J_k)$ be a nodal J_k–holomorphic map. Denote by S the nodal set of $(\Sigma_\infty, \nu_\infty)$. The sequence $(u_k, j_k)_{k\in\mathbf{N}}$ **Gromov converges** to (u_∞, j_∞) if

1. $\lim_{k\to\infty} E(u_k) = E(u_\infty)$ and
2. there exist
   (a) a deformation $(\pi\colon \mathscr{X} \to A, a_\infty, \iota_\infty)$ of $(\Sigma_\infty, j_\infty, \nu_\infty)$ together with a framing Ψ,
   (b) a sequence $(a_k)_{k\in\mathbf{N}}$ in A converging to a_∞, and
   (c) for every $n \in \mathbf{N}$ a nodal, biholomorphic map $\iota_k\colon (\Sigma_k, j_k, \nu_k) \to \pi^{-1}(a_k)$

   such that the sequence of maps

$$\tilde{u}_k \coloneqq u_k \circ \iota_k^{-1} \circ \Psi(\cdot, a_k) \circ \iota_\infty\colon \Sigma_\infty\backslash S \to X$$

   converges to $u_\infty|_{\Sigma_\infty\backslash S}$ in the C^∞_{loc} topology.

Remark 3.9. If $(\pi, \star, \iota)$ is a versal deformation of $(\Sigma_\infty, j_\infty, \nu_\infty)$ and Ψ is a framing of this deformation, then for every sequence (u_k, j_k) which Gromov converges to (u_∞, j_∞) the deformation in Definition 3.8 can be assumed to be $(\pi, \star, \iota)$ and the framing can be assumed to be Ψ. This is an almost immediate consequence of the definition of a versal deformation.

Theorem 3.10 (Gromov [Gro85]; see also [PW93; Ye94; Hum97; MS12, Chapters 4 and 5]). *Let (X, J_∞, h_∞) be a closed almost Hermitian manifold and let $(J_k, h_k)_{k\in\mathbf{N}}$ be a sequence of almost Hermitian structures on X converging to (J_∞, j_∞) in the C^∞ topology. For every $k \in \mathbf{N}$ let*

$$u_k\colon (\Sigma_k, j_k, \nu_k) \to (X, J_k)$$

be a stable nodal J–holomorphic map. If

$$\limsup_{k\to\infty} \#\pi_0(\Sigma_k) < \infty, \quad \limsup_{k\to\infty} p_a(\Sigma_k, \nu_k) < \infty, \quad \textit{and} \quad \limsup_{k\to\infty} E(u_k) < \infty,$$

then there exists a stable nodal J_∞–holomorphic map $u_\infty\colon (\Sigma_\infty, j_\infty, \nu_\infty) \to (X, J_\infty)$ and a subsequence of $(u_k, j_k)_{k\in\mathbf{N}}$ which Gromov converges to (u_∞, j_∞).

Henceforth, let (X, ω) be a symplectic manifold. The set $\mathscr{J}(X, \omega)$ of almost complex structures compatible with ω injects into $\mathscr{H}(X)$.

Proposition 3.11 (Gromov [Gro85]; see also [MS12, Lemma 2.2.1]). *Let (X, ω) be a symplectic manifold and $J \in \mathscr{J}(X, \omega)$. Let (Σ, ν, j) be a closed, nodal Riemann surface. For every nodal map $u\colon (\Sigma, \nu) \to X$*

$$E(u) \geqslant \langle u^*[\omega], [\Sigma]\rangle,$$

and the equality holds if and only if u is J–holomorphic.

Set

$$\overline{\mathscr{M}}_{A,g}(X,\omega) := \coprod_{J\in\mathscr{J}(X,\omega)} \overline{\mathscr{M}}_{A,g}(X,J).$$

By the above energy identity, in the symplectic context, Theorem 3.10 is equivalent to the map

$$\pi\colon \overline{\mathscr{M}}_{A,g}(X,\omega) \to \mathscr{J}(X,\omega)$$

being proper.

3.3 Behavior near the vanishing cycles

The results of this subsection will be important for proving the surjectivity of the gluing construction in Section 5.6. Assume the situation of Definition 3.8. By condition (1) for every $\delta > 0$ there exists a $K \in \mathbf{N}_0$ and $r > 0$ such that for every $k \geqslant K$

$$E\big(u_k|_{N_k^r}\big) \leqslant \delta$$

with

$$N_k^r := \Sigma_k \backslash \{\Psi(z,a_k) : z \in \Sigma_0 \text{ with } d(z,S) \geqslant r\}. \tag{3.12}$$

The subset N_k^r can be partitioned into regions $N_{k,n}^r$ corresponding to the nodes $n \in S$. If n is not smoothed out in Σ_k, then the corresponding region is biholomorphic to

$$B_1(0) \amalg B_1(0)$$

with ν_k identifying the origins. If n is smoothed out in Σ_k, then the corresponding region is biholomorphic to

$$S^1 \times (-L_k, L_k)$$

with $\lim_{k\to\infty} L_k = \infty$.

The behavior of J–holomorphic maps from such domains and with small energy can be understood quite well through the following two results.

Lemma 3.13 ([MS12, Lemma 4.3.1]). *There is a constant $\delta = \delta(X,g,J) > 0$ such that for every $r > 0$ the following holds. If $u\colon (B_{2r}(0), i) \to (X,J)$ is a J–holomorphic map with*

$$E(u) \leqslant \delta,$$

then

$$\|\mathrm{d}u\|_{L^\infty(B_r(0))} \leqslant cr^{-1}E(u)^{1/2}.$$

Lemma 3.14 ([MS12, Lemma 4.7.3]). *For every $\mu \in (0,1)$ there are constants $\delta = \delta(X,g,J,\mu) > 0$ and $c = c(\mu) > 0$ such that for every $L > 0$ the following holds. If $u\colon (S^1 \times (-L,L), j_{\mathrm{cyl}}) \to (X,J)$ is a J–holomorphic map with*

$$E(u) \leqslant \delta,$$

then for every $\ell \in (0,L)$

$$E\big(u|_{S^1\times(L+\ell,L-\ell)}\big) \leqslant ce^{-2\mu(L-\ell)}E(u).$$

and for every $\theta \in S^1$ and $\ell \in [-L+1, L-1]$

$$|\mathrm{d}u|(\theta,\ell) \leqslant ce^{-\mu(L-|\ell|)}E(u)^{1/2}.$$

Proof. The first assertion is [MS12, Lemma 4.7.3]. The second assertion follows from the first by Lemma 3.13. □

The following is an important consequence of the previous two lemmas.

Proposition 3.15. *Let $(u_k\colon (\Sigma_k, j_k, \nu_k) \to (X, J_k))_{k\in\mathbf{N}}$ be a sequence of nodal pseudo-holomorphic maps which Gromov converges to $u_\infty\colon (\Sigma_\infty, j_\infty, \nu_\infty) \to (X, J_\infty)$. Denote by S the nodal set of $(\Sigma_\infty, \nu_\infty)$ and let N_k^r be as in (3.12). For every $\delta > 0$ there are $r > 0$ and $K \in \mathbf{N}$ such that for every $k \geqslant K$ and $n \in S$*

$$u_k(N_{k,n}^r) \subset B_\delta(u_\infty(n));$$

in particular, provided δ is sufficiently small,

$$(u_k)_*[\Sigma_k] = (u_\infty)_*[\Sigma_\infty].$$

4 Versal deformations of nodal Riemann surfaces

The purpose of this section is to construct a versal deformation of a nodal Riemann surface in a rather explicit manner.

4.1 Deformations of nodal curves

Let us briefly review parts of the deformation theory of nodal curves in the complex analytic category. For further details and proofs we refer the reader to [ACGH11, Chapter XI Section 3]. A thorough discussion of deformation theory in the algebraic category can be found in [Har10].

Definition 4.1. Let C be a nodal curve. A **deformation** of C is a proper flat morphism $\pi\colon \mathscr{X} \to A$ between analytic spaces such that every fiber of π a nodal curve, a base-point $\star \in A$, and an isomorphism $\iota\colon C \to \pi^{-1}(\star)$.

Proposition 4.2. *Every nodal family $\pi\colon \mathscr{X} \to A$ is flat. In particular, a deformation of a nodal Riemann surface (Σ, j, ν) is also a deformation of the associated nodal curve C.*

Definition 4.3. Let C be a nodal Riemann surface and let $(\pi\colon \mathscr{X} \to A, \star, \iota)$ and $(\rho\colon \mathscr{Y} \to B, \dagger, \kappa)$ be two deformations of C. A pair of analytic maps $\Phi\colon \mathscr{X} \to \mathscr{Y}$ and $\phi\colon A \to B$ forms a **morphism** $(\Phi, \phi)\colon (\rho, \star, \iota) \to (\mathscr{Y}, \dagger, \kappa)$ of deformations if

$$\phi(\star) = \dagger, \quad \rho \circ \Phi = \phi \circ \pi, \quad \Phi \circ \iota = \kappa,$$

and for every $a \in A$ the restriction $\Phi\colon \pi^{-1}(a) \to \rho^{-1}(\phi(a))$ induces an analytic isomorphism.

Definition 4.4. A deformation $(\rho\colon \mathscr{Y} \to B, \dagger, \kappa)$ of C is **(uni)versal** if for every deformation $(\pi\colon \mathscr{X} \to A, \star, \iota)$ of (Σ, j, ν) there exists an open neighborhood U of $\star \in A$ and a (unique) morphism of deformations $(\pi\colon \pi^{-1}(U) \to A, \star, \iota) \to (\rho, \dagger, \kappa)$.

Definition 4.5. Denote by $\mathbf{C}[\varepsilon]/\varepsilon^2$ the ring of dual numbers and set $D := \operatorname{Spec}\bigl(\mathbf{C}[\varepsilon]/\varepsilon^2\bigr)$. A **first order deformation** is a deformation over D.

Let C be a nodal curve. Every first order deformation $(\pi\colon \mathscr{X} \to D, 0, \iota)$ of C induces a short exact sequence

$$0 \to \mathscr{O}_C \cong \pi^*\Omega^1_D \to \Omega^1_{\mathscr{X}} \otimes \mathscr{O}_C \xrightarrow{\iota^*} \Omega^1_C \to 0.$$

The extension class $\delta \in \operatorname{Ext}^1(\Omega^1_C, \mathscr{O}_C)$ of this sequence depends on the first order deformation only up to isomorphism of deformations. Indeed, two first order deformation of C are isomorphic if and only if they yield the same extension class δ.

Definition 4.6. Let C be a nodal curve and let $(\pi\colon \mathscr{X} \to A, \star, \iota)$ be a deformation of C. Every $v \in T_\star A$ corresponds to an analytic map $\phi\colon D \to A$ mapping 0 to $\star$. The pullback of $(\pi, \star, \iota)$ via ϕ is a first order deformation. Denote by $\delta(v) \in \operatorname{Ext}^1(\Omega^1_C, \mathscr{O}_C)$ the corresponding extension class. The map $\delta\colon T_\star A \to \operatorname{Ext}^1(\Omega^1_C, \mathscr{O}_C)$ thus defined is called the **Kodaira–Spencer map**.

It is instructive to analyze $\operatorname{Ext}^1(\Omega^1_C, \mathscr{O}_C)$ more closely. The local-to-global Ext spectral sequence yields a short exact sequence

$$0 \to H^1(C, \mathscr{H}\mathrm{om}(\Omega^1_C, \mathscr{O}_C)) \to \operatorname{Ext}^1(\Omega^1_C, \mathscr{O}_C) \to H^0(C, \mathscr{E}\mathrm{xt}^1(\Omega^1_C, \mathscr{O}_C)) \to 0.$$

This can be interpreted in terms of the normalization $\pi\colon \tilde{C} \to C$ as follows. Denote by S the set of nodes of C and set $\tilde{S} := \pi^{-1}(S)$. It can be shown that

$$\mathscr{H}\mathrm{om}(\Omega^1_C, \mathscr{O}_C) = \pi_*\mathscr{T}_{\tilde{C}}(-\tilde{S}); \quad \text{hence:} \quad H^1(C, \mathscr{H}\mathrm{om}(\Omega^1_C, \mathscr{O}_C)) = H^1(\tilde{C}, \mathscr{T}_{\tilde{C}}(-\tilde{S})).$$

The space $H^1(\tilde{C}, \mathscr{T}_{\tilde{C}}(-\tilde{S}))$ parametrizes the deformations of the marked curve $(\tilde{C}, \tilde{S})$. The sheaf $\mathscr{E}\mathrm{xt}^1(\Omega^1_C, \mathscr{O}_C)$ is supported on the nodes of C; that is:

$$\mathscr{E}\mathrm{xt}^1(\Omega^1_C, \mathscr{O}_C) = \bigoplus_{n \in S} \operatorname{Ext}^1(\Omega^1_{C,n}, \mathscr{O}_{C,n}).$$

For every $n \in \tilde{S}$ and $\{n_1, n_2\} = \pi^{-1}(n)$

$$\operatorname{Ext}^1(\Omega^1_{C,n}, \mathscr{O}_{C,n}) = T_{n_1}\tilde{C} \otimes T_{n_2}\tilde{C}.$$

By considering the deformation $\{zw = \varepsilon\}$ of the node $\{zw = 0\}$, the space $\mathrm{Ext}^1(\Omega^1_{C,n}, \mathscr{O}_{C,n})$ can be seen to parametrize smoothings of the node n. The above discussion show that to first order all deformations of C arise from smoothing nodes and deforming its normalization while fixing the points mapping to the nodes. In the following we construct a deformation of C which induces all of these deformations to first order.

4.2 Smoothing nodal Riemann surfaces

Let (Σ_0, j_0, ν_0) be a closed, nodal Riemann surface with nodal set S. Let g_0 be a Riemannian metric on Σ_0 in the conformal class determined by j_0 and such that there is a constant $R_0 > 0$ such that for every $n \in S$ the restriction of g_0 to $B_{4R_0}(n)$ is flat and for every $n_1, n_2 \in S$ the balls $B_{4R_0}(n_1)$ and $B_{4R_0}(n_2)$ are disjoint. For every $n \in S$ define the holomorphic charts $\phi_n \colon B_{4R_0}(n) \subset T_n\Sigma_0 \to \Sigma_0$ by

$$\phi_n(v) \coloneqq \exp_n(v)$$

and define $r_n \colon \Sigma_0 \to [0, \infty)$ by

$$r_n(z) \coloneqq \max\{d(n, z), 4R_0\}.$$

Given a pair of complex vector spaces V and W, denote by $\sigma \colon V \otimes W \to W \otimes V$ the isomorphism defined by $\sigma(v \otimes w) \coloneqq w \otimes v$.

Definition 4.7. A **smoothing parameter** for (Σ_0, j_0, ν_0) is an element

$$\tau = (\tau_n)_{n\in S} \in \prod_{n\in S} T_n\Sigma_0 \otimes_{\mathbf{C}} T_{\nu(n)}\Sigma_0$$

such that for every $n \in S$

$$\tau_{\nu(n)} = \sigma(\tau_n) \quad \text{and} \quad |\tau_n| < R_0^2.$$

Given a smoothing parameter τ, for every $n \in S$ set

$$\varepsilon_n \coloneqq |\tau_n| \quad \text{and} \quad \hat{\tau}_n \coloneqq \tau_n/|\tau_n| \quad \text{if} \quad \varepsilon_n \neq 0;$$

furthermore, set

$$\varepsilon \coloneqq \max\{\varepsilon_n : n \in S\}.$$

Henceforth, let $\tau = (\tau_n)_{n\in S}$ be a smoothing parameter for (Σ_0, j_0, ν_0).

Definition 4.8. Set

$$A_\tau \coloneqq \{w \in \Sigma_0 : \varepsilon_n/R_0 < r_n(w) < R_0 \text{ for some } n \in S \text{ with } \varepsilon_n \neq 0\}$$

and denote by $\iota_\tau \colon A_\tau \to A_\tau$ the biholomorphic map characterized by

$$\phi^{-1}_{\nu(n)} \circ \iota_\tau \circ \phi_n(v) \otimes v = \tau_n$$

for every $n \in S$ and $v \in T_n\Sigma_0$ with $\varepsilon_n/R_0 < |v| < R_0$.

Definition 4.9. Consider the Riemann surface with boundary

$$\Sigma_\tau^\circ := \left\{z \in \Sigma_0 : r_n(z) \geqslant \varepsilon_n^{1/2} \text{ for every } n \in S\right\}.$$

Denote by $\sim_\tau$ the equivalence relation on Σ_τ° generated by identifying the boundary components via ι_τ. The quotient

$$\Sigma_\tau := \Sigma_\tau^\circ / {\sim_\tau}$$

is a closed surface. The restrictions of the complex structure j_0 and the nodal structure ν_0 to Σ_τ° descends to a complex structure j_τ and a nodal structure ν_τ on Σ_τ. The nodal Riemann surface $(\Sigma_\tau, j_\tau, \nu_\tau)$ is called the **partial smoothing** of (Σ_0, j_0, ν_0) associated with τ.

Remark 4.10. The above construction smooths out every node with $\varepsilon_n > 0$. In particular, if all of the ε_n are positive, then ν_τ is the trivial nodal structure and $(\Sigma_\tau, j_\tau, \nu_\tau)$ is simply the Riemann surface (Σ_τ, j_τ).

Definition 4.11. Denote by Δ the space of smoothing parameters for (Σ_0, j_0, ν_0). Set

$$\mathscr{X} := \left\{(z,\tau) \in \Sigma_0 \times \Delta : z \in \Sigma_\tau^\circ\right\}/{\sim}$$

with $(z_1, \tau_1) \sim (z_2, \tau_2)$ if and only if $\tau_1 = \tau_2$ and $z_1 \sim_{\tau_1} z_2$ or $z_1, z_2 \in S$, $\nu(z_1) = z_2$, and $\varepsilon_{z_1} = \varepsilon_{z_2} = 0$. Denote by $\pi\colon \mathscr{X} \to \Delta$ the canonical projection.

Proposition 4.12. *$\mathscr{X}$ is a smooth manifold and the complex structure on $\Sigma_0 \times \Delta$ induces a complex structure on $\mathscr{X}$ such that π is a nodal family and for every $\tau \in \Delta$ the canonical map $\Sigma_\tau \to \pi^{-1}(\tau)$ induces a nodal, biholomorphic map $\iota_\tau\colon (\Sigma_\tau, j_\tau, \nu_\tau) \to \pi^{-1}(\tau)$.*

Proof. It suffices to consider the local model of a node $C_0 := \{(z,w) \in \mathbf{C}^2 : zw = 0\}$. $\tilde{\mathscr{X}} := \{(z,w,\tau) \in \mathbf{C}^2 \times \mathbf{C} : zw = \tau\}$ is a complex manifold. The map $\tilde{\pi}\colon \tilde{\mathscr{X}} \to \mathbf{C}$ defined by $\tilde{\pi}(z,w,\tau) := zw$ has only nodal critical points and its fiber over 0 is C_0. The nodal Riemann surface associated with C_0 is $\Sigma_0 = \mathbf{C} \amalg \mathbf{C}$ with the complex structure i on both components and the nodal structure which interchanges the origins of the components. The partial smoothing defined in Definition 4.9 is

$$\Sigma_\tau = \left(\left\{z \in \mathbf{C} : |z| \geqslant |\tau|^{1/2}\right\} \amalg \left\{w \in \mathbf{C} : |w| \geqslant |\tau|^{1/2}\right\}\right) \Big/ \sim_\tau .$$

The map $\Phi\colon \mathscr{X} \to \tilde{\mathscr{X}}$ defined by $\Phi([z],\tau) := (z, \tau/z, \tau)$ and $\Phi([w],\tau) := (\tau/z, z, \tau)$ is biholomorphic. This implies the assertion. ☐

4.3 Construction of a versal deformation

Let (Σ_0, j_0, ν_0) be a nodal Riemann surface with nodal set S. The construction from the previous subsection can be generalized to take into account deformations of the complex structure on (Σ_0, j_0, ν_0) away from S.

Denote by $\mathscr{J}(\Sigma_0)$ the space of almost complex structures on Σ_0 and by $\mathrm{Diff}_0(\Sigma_0, \nu_0)$ the group of diffeomorphism of Σ_0 which are isotopic to the identity and commute with ν_0. Denote by

$$\mathscr{T} := \mathscr{J}(\Sigma_0)/\mathrm{Diff}_0(\Sigma_0, \nu_0)$$

the corresponding Teichmüller space. This is a complex manifold and there is an open neighborhood Δ_1 of $0 \in \mathbf{C}^{\dim_{\mathbf{C}} \mathscr{T}}$ together with a map $j \colon \Delta_1 \to \mathscr{J}(\Sigma_0)$ such that

1. $j(0) = j_0$,
2. for every $\sigma \in \Delta_1$ the almost complex structure $j(\sigma)$ agrees with j_0 in some neighborhood U of S, and
3. the map $[j] \colon \Delta_1 \to \mathscr{T}$ is an embedding.

For every $\sigma \in \Delta_1$ set

$$\Sigma_{\sigma,0} := \Sigma_0, \quad j_{\sigma,0} := j(\sigma), \quad \nu_{\sigma,0} := \nu_0.$$

Choose a family of metric $(g_{\sigma,0})_{\sigma\in\Delta_1}$ whose restriction to the neighborhood U of S is independent of σ and such that for every $\sigma \in \Delta_1$ in the conformal class determined by $j_{\sigma,0}$. Let $R_0 > 0$ be such that the conditions at the beginning of Section 4.2 hold for every $\sigma \in \Delta_1$ and $B_{4R_0}(S) \subset U$.

Denote by Δ_2 the space of elements

$$\tau = (\tau_n)_{n\in S} \in \prod_{n\in S} T_n\Sigma_0 \otimes_{\mathbf{C}} T_{\nu(n)}\Sigma_0$$

such that for every $n \in S$

$$\tau_{\nu(n)} = \sigma(\tau_n) \quad \text{and} \quad |\tau_n| < R_0^2$$

Definition 4.13. Set $\Delta := \Delta_1 \times \Delta_2$. Set

$$\mathscr{X} := \left\{(z; \sigma, \tau) \in \Sigma_0 \times \Delta_1 \times \Delta_2 : z \in \Sigma_{\sigma,\tau}^{\circ}\right\}/\sim$$

with $(z_1; \sigma_1, \tau_1) \sim (z_2; \sigma_2, \tau_2)$ if and only if $\sigma_1 = \sigma_2$, $\tau_1 = \tau_2$ and $z_1 \sim_{\tau_1} z_2$ or $\tau_1 = \tau_2$ and $z_1 \sim_{\tau_1} z_2$ or $z_1, z_2 \in S$, $\nu(z_1) = z_2$, and $\varepsilon_{z_1} = \varepsilon_{z_2} = 0$. Denote by $\pi \colon \mathscr{X} \to \Delta$ the canonical projection.

Proposition 4.14. *$\mathscr{X}$ is a smooth manifold and the complex structure on $\Sigma_0 \times \Delta$ induces a complex structure on $\mathscr{X}$ such that π is a nodal family and for every $(\sigma, \tau) \in \Delta$ the canonical map $\Sigma_{\sigma,\tau} \to \pi^{-1}(\sigma, \tau)$ induces a nodal, biholomorphic map $\iota_{\sigma,\tau} \colon (\Sigma_{\sigma,\tau}, j_{\sigma,\tau}, \nu_{\sigma,\tau}) \to \pi^{-1}(\sigma, \tau)$.*

Theorem 4.15 (cf. [ACGH11, Chapter XI Theorem 3.17 and Section 4]). *Set $\star := (0, 0)$ and $\iota := \iota_{0,0}$. The deformation $(\pi, \star, \iota)$ of (Σ_0, j_0, ν_0) is versal.*

Proof. Denote by C the nodal curve associated with (Σ_0, j_0, ν_0). It is proved in [ACGH11, Chapter XI Theorem 3.17] that the Kodaira–Spencer map $\delta \colon T_0\Delta_1 \times T_0\Delta_2 \to \mathrm{Ext}^1(\Omega_C^1, \mathscr{O}_C)$ is an isomorphism. This implies that the deformation is versal. Indeed, C has some versal family $(\rho \colon \mathscr{Y} \to B, \dagger, \kappa)$ for which the Kodaira–Spencer map is an isomorphism. Therefore, after possibly shrinking Δ, there exists a morphism of deformations $(\Phi, \phi) \colon (\pi, \star, \iota) \to (\rho, \dagger, \kappa)$. Since both Kodaira–Spencer maps are isomorphism, after possibly shrinking Δ, ϕ is a holomorphic embedding. Therefore, after possibly shrinking both Δ and A, both deformations become isomorphic. □

To define a framing on the deformation $(\pi, \star, \iota)$, choose an increasing, smooth function $\eta\colon [0,2] \to [1,2]$ such that

$$\eta(0) = 1 \quad \text{and} \quad \eta(r) = r \quad \text{for every} \quad 3/2 \leqslant r \leqslant 2.$$

Definition 4.16. Define the framing $\Psi\colon \Sigma_0 \backslash S \times \Delta \to \mathscr{X}$ of $(\pi, \star, \iota)$ by

$$\Psi(z;\sigma,\tau) \coloneqq \begin{cases} \left(\phi_n\left(\eta\left(r_n(z)/\varepsilon_n^{1/2}\right) \cdot \frac{\phi_n^{-1}(z)}{r_n(z)/\varepsilon_n^{1/2}}\right); \sigma, \tau\right) & \text{if } r_n(z) \leqslant 2\varepsilon_n^{1/2} \text{ for some } n \in S \\ (z;\sigma,\tau) & \text{otherwise.} \end{cases}$$

Remark 4.17. Let $(\sigma, \tau) \in \Delta$ and $r \in (2\varepsilon^{1/2}, R_0)$. Set

$$\Sigma_0^r \coloneqq \{z \in \Sigma_0 : r_n(z) \geqslant r \text{ for every } n \in S\}.$$

Denote by

$$N_{\sigma,\tau}^r \coloneqq \Sigma_{\sigma,\tau} \backslash \Psi(\Sigma_0^r \times \{(\sigma,\tau)\})$$

the part of $\Sigma_{\sigma,\tau}$ not covered by Σ_0^r under the framing, cf. Section 3.3. By construction,

$$N_{\sigma,\tau}^r = \bigcup_{n \in S} N_{\sigma,\tau;n}^r$$

with

$$N_{\sigma,\tau;n}^r = N_{\sigma,\tau;\nu(n)}^r \coloneqq \left\{z \in \Sigma_{\sigma,\tau}^\circ : r_n(z) < r \text{ or } r_{\nu(n)}(z) < r\right\}/\sim_\tau .$$

If $\varepsilon_n = 0$, then $N_{\sigma,\tau;n}^r$ is biholomorphic to

$$B_r(0) \amalg B_r(0)$$

and the nodal structure $\nu_{\sigma,\tau}$ identifies the two origins. If $\varepsilon_n \neq 0$, then $N_{\sigma,\tau;n}^r$ is biholomorphic to

$$\{z \in \mathbf{C} : \varepsilon_n/r < |z| < r\} \cong S^1 \times (-\log(r\varepsilon_n^{-1/2}), \log(r\varepsilon_n^{-1/2})).$$

5 Smoothing nodal J–holomorphic maps

The purpose of this section is to prove Theorem 1.1. The strategy is to construct a Kuranishi model for a Gromov neighborhood of u_∞ and analyze the obstruction map. This idea goes back to Ionel [Ion98] and has been Zinger [Zin09] and Niu [Niu16] to give a sharp compactness results for genus one and two pseudo-holomorphic maps.

Throughout this section, fix a smooth function $\chi\colon [0,\infty) \to [0,1]$ with

$$\chi|_{[0,1]} = 1 \quad \text{and} \quad \chi|_{[2,\infty)} = 0$$

and, moreover, $p \in (2, \infty)$.

5.1 Riemannian metrics on smoothings

Definition 5.1. Let (Σ_0, j_0, ν_0) be a nodal Riemann surface with nodal set S. Denote by g_0 a Riemannian metric on Σ_0 as at the beginning of Section 4.2. Given a smoothing parameter τ, let Σ_τ° be as in Definition 4.9, and define the Riemannian metric g_τ° on Σ_τ° by

$$g_\tau^\circ \coloneqq g_0 + \sum_{n\in S} \chi\left(\frac{r_n}{2\varepsilon_n^{1/2}}\right) \cdot \left(\varepsilon_n \cdot (\phi_n)_*\left(r^{-2}\mathrm{d}r \otimes \mathrm{d}r + \theta \otimes \theta\right) - g_0\right)$$

with r denoting the distance from origin in $T_n\Sigma_0 \cong \mathbf{C}$ and $\theta = -\mathrm{d}r \circ j_0$. Since the Riemannian metric $r^{-2}\mathrm{d}r \otimes \mathrm{d}r + \theta \otimes \theta$ on $\mathbf{C}^*$ is invariant under the involution $z \mapsto \varepsilon/z$, g_τ° descends to a Riemannian metric g_τ on Σ_τ.

Proposition 5.2. *There is a constant $c > 1$ such that for every nodal Riemann surface and every smoothing parameter τ*

$$c^{-1}g_0 < g_\tau^\circ < cg_0.$$

Proof. Let $n \in S$. On the annulus $\left\{z \in \Sigma_0 : \varepsilon_n^{1/2} \leqslant r_n(z) \leqslant 4\varepsilon_n^{1/2}\right\}$,

$$\phi_n^* g = \mathrm{d}r \otimes \mathrm{d}r + r^2\theta \otimes \theta$$

and, therefore,

$$g_\tau^\circ = \left(F_{\varepsilon_n} \circ r_n\right) \cdot g_0 \quad \text{with} \quad F_{\varepsilon_n}(r) \coloneqq 1 + \chi\left(\frac{r}{2\varepsilon_n^{1/2}}\right) \cdot \left(\varepsilon_n r^{-2} - 1\right).$$

This implies the assertion because $c^{-1} < F_{\varepsilon_n}(r) < c$ for $\varepsilon_n^{1/2} \leqslant r \leqslant 4\varepsilon_n^{1/2}$. □

Henceforth, the L^p and $W^{1,p}$ norms of all sections and differential forms on Σ_τ are understood with respect to the metric g_τ. However, the above proposition will be often implicitly used to bound these norms by estimating various expressions with respect to g_0 over the corresponding region in Σ_0.

5.2 Approximate smoothing of nodal J–holomorphic maps

Throughout the next four sections, let (X, g, J) be an almost Hermitian manifold, let $c_u > 0$, let $u_0 \colon (\Sigma_0, j_0, \nu_0) \to (X, J)$ be a nodal J–holomorphic map, and let τ be a smoothing parameter. Furthermore, choose g_0 and R_0 as at the beginning of Section 4.2.

Definition 5.3. For every point $x \in X$, denote by $\tilde{U}_x \subset T_xX$ the region within the cut locus and set $U_x \coloneqq \exp_x(\tilde{U}_x)$ and $\frac{1}{2}U_x \coloneqq \exp_x(\frac{1}{2}\tilde{U}_x)$. The map $\exp_x \colon \tilde{U}_x \to U_x$ is a diffeomorphism and its inverse is denoted by $\exp_x^{-1} \colon U_x \to \tilde{U}_x$.

Furthermore, we assume the following.

Hypothesis 5.4. *The map* u_0 *and* $R_0 > 0$ *satisfy*

$$\|u_0\|_{C^2} \leqslant c_u \quad \textit{and} \quad u_0(B_{4R_0}(n)) \subset U_{u_0(n)} \quad \textit{for every} \quad n \in S.$$

Convention 5.5. Henceforth, constants may depend on p, (Σ_0, j_0, ν_0), (X, g, J), c_u, and R_0, but not on τ.

Definition 5.6. For $n \in S$ define $\chi_\tau^n\colon \Sigma_\tau \to [0,1]$ by

$$\chi_\tau^n(z) \coloneqq \chi\left(\frac{r_n(z)}{R_0}\right).$$

Define $\tilde{u}_\tau^\circ\colon \Sigma_\tau^\circ \to X$ by

$$\tilde{u}_\tau^\circ(z) \coloneqq \begin{cases} \exp_{u_0(n)}\bigl(\exp_{u_0(n)}^{-1} \circ u_0(z) + \chi_\tau^n(z) \cdot \exp_{u_0(n)}^{-1} \circ u_0(\iota_\tau(z))\bigr) & \text{if } r_n(z) \leqslant 2R_0 \\ u_0(z) & \text{otherwise.} \end{cases}$$

Since $u_0(\nu_0(n)) = u_0(n)$, the restriction of $\tilde{u}_\tau^\circ$ to

$$\left\{z \in \Sigma_\tau^\circ : r_n(z) \leqslant R \text{ for some } n \in S\right\}$$

is invariant under ι_τ. Therefore, $\tilde{u}_\tau^\circ$ descends to a smooth map

$$\tilde{u}_\tau\colon \Sigma_\tau \to X.$$

This map is called the **approximate smoothing** of u associated with τ.

Remark 5.7. This construction differs from that found, for example, in [MS12, Section 10.2; Par16, Section B.3] in which the approximate smoothing is constant in the middle of the neck region. The above construction is very similar to that in [Gou09, Section 2.1]. It leads to a smaller error term and significantly simplifies the discussions in Section 5.7. Morally, this section analyzes how the interaction between the different components of u_0 affects whether u_0 can be smoothed or not. The constructions in [MS12, Section 10.2; Par16, Section B.3] make it difficult to see these interactions.

Proposition 5.8. *The map* $\tilde{u}_\tau$ *satisfies*

$$\|\bar{\partial}_J(\tilde{u}_\tau, j_\tau)\|_{L^p} \leqslant c\|\bar{\partial}_J(u_0, j_0)\|_{L^p} + c\varepsilon^{\frac{1}{2}+\frac{1}{p}}. \tag{5.9}$$

For the proof of this result and for future reference let us observe that for every $k \geqslant 1$

$$\left(\int_{\varepsilon_n^{1/2} \leqslant r_n \leqslant 2R_0} r_n^{-kp}\right)^{\frac{1}{p}} \leqslant \left(\frac{2\pi}{kp-2}\right)^{\frac{1}{p}} \varepsilon_n^{\frac{1}{p}-\frac{k}{2}}. \tag{5.10}$$

Proof of Proposition 5.8. The map $\tilde{u}_\tau^\circ$ agrees with u_0 in the region where $r_n \geqslant 2R_0$ for every $n \in S$. Therefore, it suffices to consider the regions where $r_n \leqslant 2R_0$ for some $n \in S$. To simplify notation, identify U_x with $\tilde{U}_x$ via $\exp_x$ for $x := u(n)$. Having made this identification, in such a region, $\tilde{u}_\tau^\circ$ is given by

$$\tilde{u}_\tau^\circ = u_0 + \chi_\tau^n \cdot u_0 \circ \iota_\tau .$$

Therefore,

$$\begin{aligned}\bar{\partial}_J(\tilde{u}_\tau^\circ, j_\tau) &= \frac{1}{2}\big(\mathrm{d}\tilde{u}_\tau^\circ + J(\tilde{u}_\tau^\circ) \circ \mathrm{d}\tilde{u}_\tau^\circ \circ j\big) \\ &= \underbrace{\bar{\partial}_J(u_0, j_0) + \chi_\tau^n \cdot \bar{\partial}_J(u_0 \circ \iota_\tau, j_0)}_{=:\mathrm{I}} \\ &\quad + \underbrace{\frac{1}{2}\big(J(\tilde{u}_\tau^\circ) - J(u_0)\big) \circ \mathrm{d}u_0 \circ j_0}_{=:\mathrm{II}_1} + \underbrace{\chi_\tau^n \cdot \frac{1}{2}\big(J(\tilde{u}_\tau^\circ) - J(u_0 \circ \iota_\tau)\big) \circ \mathrm{d}(u_0 \circ \iota_\tau) \circ j_0}_{=:\mathrm{II}_2} \\ &\quad + \underbrace{\bar{\partial}\chi_\tau^n \cdot u_0 \circ \iota_\tau}_{=:\mathrm{III}} .\end{aligned}$$

The term I is controlled by the L^p norm of $\bar{\partial}_J(u_0, j_0)$ over the regions of Σ_0 where $\varepsilon_n/2R_0 \leqslant r_n \leqslant 2R_0$ for some $n \in S$. By Taylor expansion at $v_0(n)$

$$|\mathrm{II}_1| \leqslant c\|J\|_{C^1} \cdot |u_0 \circ \iota_\tau| \cdot |\mathrm{d}u_0| \leqslant c\varepsilon_n/r_n$$

and

$$\begin{aligned}|\mathrm{II}_2| &\leqslant c\|J\|_{C^1} \cdot \big(|u_0| + (1-\chi_\tau^n) \cdot |u_0 \circ \iota|\big) \cdot |\mathrm{d}(u_0 \circ \iota_\tau)| \\ &\leqslant c \cdot \big(r_n + (1-\chi_\tau^n)\varepsilon_n/r_n\big) \cdot \varepsilon_n/r_n^2 \leqslant c\varepsilon_n/r_n.\end{aligned}$$

On Σ_τ°, by definition, $r_n \geqslant \varepsilon_n^{1/2}$. Therefore and by (5.10),

$$\|\mathrm{II}_1 + \mathrm{II}_2\|_{L^p} \leqslant c\varepsilon_n^{\frac{1}{2}+\frac{1}{p}} .$$

The term III is supported in the region where $R_0 \leqslant r_n \leqslant 2R_0$ and satisfies $|\mathrm{III}| \leqslant c\varepsilon_n$. □

5.3 Fusing nodal vector fields

The fusing operator, introduced below, assigns to every vector field along u_0 a vector field along $\tilde{u}_\tau$ which agrees with u_0 outside the gluing region. This construction makes use of the following local trivializations of TX.

Definition 5.11. For every $x \in X$ and $y \in U_x$ define an isomorphism $\Phi_y = \Phi_y^x \colon T_xX \to T_yX$ by

$$\Phi_y^x(v) := \mathrm{d}_{\exp_x^{-1}(y)} \exp_x(v)$$

As y varies in U_x, these maps define a trivialization $\Phi = \Phi^x \colon U_x \times T_xX \to TX|_{U_x}$.

Definition 5.12. Define $\mathrm{fuse}_\tau^\circ \colon W^{1,p}\Gamma(\Sigma_0, \nu_0; u_0^*TX) \to W^{1,p}\Gamma(\Sigma_\tau^\circ, \nu_0; (\tilde{u}_\tau^\circ)^*TX)$ by

$$\mathrm{fuse}_\tau^\circ(\xi)(z) := \begin{cases} \Phi_{\tilde{u}_\tau([z])}\Big(\Phi_{u_0(z)}^{-1}\xi(z) + \chi_\tau^n(z)\cdot(\Phi_{u_0\circ\iota_\tau(z)}^{-1}\xi(\iota_\tau(z)) - \Phi_{u_0}^{-1}\xi(n))\Big) & \text{if } r_n(z) \leqslant 2R_0 \\ \xi(z) & \text{otherwise.} \end{cases}$$

In the above formula, $\Phi = \Phi^x$ with $x = u_0(n)$. For every $n \in S$ the restriction of $\mathrm{fuse}_\tau^\circ(\xi)$ to

$$\big\{z \in \Sigma_\tau^\circ : r_n(z) \leqslant R_0 \text{ for some } n \in S\big\}$$

is invariant under ι_τ. Therefore, fuse_τ° induces a map

$$\mathrm{fuse}_\tau \colon W^{1,p}\Gamma(\Sigma_0, \nu_0; u_0^*TX) \to W^{1,p}\Gamma(\Sigma_\tau, \nu_\tau; \tilde{u}_\tau^*TX).$$

The map fuse_τ should be thought of as the infinitesimal version of the approximate smoothing construction in Definition 5.6. The following is a counterpart of Proposition 5.8.

Proposition 5.13. *For every* $\xi \in W^{1,p}\Gamma(\Sigma_0, \nu_0; u_0^*TX)$

$$\|\mathfrak{d}_{\tilde{u}_\tau}\mathrm{fuse}_\tau(\xi)\|_{L^p} \leqslant c\|\mathfrak{d}_{u_0}\xi\|_{L^p} + c\sum_{n\in S}\left(\varepsilon_n^{\frac{1}{p}} + \varepsilon_n^{1-\frac{2}{p}}\right)\|\xi\|_{W^{1,p}}.$$

The proof requires the following results as a preparation.

Proposition 5.14. *For every* $n \in S$ *and* $\xi \in W^{1,p}\Gamma(\Sigma_0, \nu; u_0^*TX)$

$$\|\mathrm{d}\chi_\tau^n \cdot (\xi\circ\iota_\tau - \xi(n))\|_{L^p} \leqslant c\varepsilon_n^{1-\frac{2}{p}}\|\xi\|_{W^{1,p}}.$$

Proof. The term $\mathrm{d}\chi_\tau^n$ is supported in the region where $R_0 \leqslant r_n \leqslant 2R_0$ and satisfies $|\mathrm{d}\chi_\tau^n| \leqslant c$. Morrey's embedding theorem asserts that $W^{1,p} \hookrightarrow C^{0,1-2/p}$. Hence,

$$|\xi\circ\iota_\tau(z) - \xi(n)| \leqslant c(\varepsilon_n/r_n(z))^{1-2/p}\|\xi\|_{W^{1,p}}.$$

This implies the asserted inequality. □

Proposition 5.15. *Let* $U \subset \Sigma_0$ *be an open subset. Let* $u_1, u_2 \colon U \to U_x$ *and set*

$$v := \exp_x^{-1}\circ u_2 - \exp_x^{-1}\circ u_1.$$

For every $\xi \in C^\infty(U, T_xX)$

$$\left|(\Phi_{u_1}\circ\mathfrak{d}_{u_1}\circ\Phi_{u_1}^{-1} - \Phi_{u_2}\circ\mathfrak{d}_{u_2}\circ\Phi_{u_2}^{-1})\xi\right| \leqslant c(|v||\mathrm{d}\xi| + |\mathrm{d}v||\xi| + |\mathrm{d}u_1||\xi||v|).$$

Proof. To simplify notation, identify U_x with $\tilde{U}_x$ via $\exp_x$. Having made this identification, Φ becomes the identity map and $v = u_2 - u_1$. Therefore,

$$\begin{aligned}\mathfrak{d}_{u_1}\xi - \mathfrak{d}_{u_2}\xi &= \frac{1}{2}(J(u_1) - J(u_2)) \circ \nabla\xi \circ j \\ &\quad + \frac{1}{2}\big((\nabla_\xi J)(u_1) - (\nabla_\xi J)(u_2)\big) \circ \mathrm{d}u_1 \circ j \\ &\quad + \frac{1}{2}(\nabla_\xi J)(u_2) \circ (\mathrm{d}u_1 - \mathrm{d}u_2) \circ j.\end{aligned}$$

This implies the asserted inequality. □

Proof of Proposition 5.13. Outside the regions where $r_n \leqslant 2R_0$ for some $n \in S$ the operators $\mathfrak{d}_{u_0}$ and $\mathfrak{d}_{\tilde{u}_\tau}$ agree. Within such a region and with the usual identifications

$$\mathfrak{d}_{\tilde{u}_\tau^\circ}\mathrm{fuse}_\tau^\circ(\xi) = \mathfrak{d}_{u_0}\xi + \underbrace{(\mathfrak{d}_{\tilde{u}_\tau^\circ} - \mathfrak{d}_{u_0})\xi}_{=:\mathrm{I}} + \underbrace{\bar{\partial}\chi_\tau^n \cdot (\xi \circ \iota_\tau - \xi(n))}_{=:\mathrm{II}} + \underbrace{\chi_\tau^n \cdot \mathfrak{d}_{\tilde{u}_\tau^\circ}(\xi \circ \iota_\tau)}_{=:\mathrm{III}_1} - \underbrace{\chi_\tau^n \cdot \mathfrak{d}_{\tilde{u}_\tau^\circ}\xi(n)}_{=:\mathrm{III}_2}.$$

The difference $v \coloneqq \tilde{u}_\tau^\circ - u_0 = \chi_\tau^n \cdot u_0 \circ \iota_\tau$ satisfies

$$\begin{aligned}|v| &\leqslant c\varepsilon_n/r_n \leqslant c\varepsilon_n^{1/2} \quad \text{and} \\ |\mathrm{d}v| &\leqslant |\mathrm{d}\chi_\tau^n \cdot u_0 \circ \iota_\tau| + |\chi_\tau^n \mathrm{d}(u_0 \circ \iota_\tau)| \leqslant c\varepsilon_n/r_n^2.\end{aligned}$$

Therefore, by Proposition 5.15 and (5.10),

$$\|\mathrm{I}\|_{L^p} \leqslant c\varepsilon_n^{\frac{1}{p}}\|\xi\|_{W^{1,p}}.$$

By Proposition 5.14,

$$\|\mathrm{II}\|_{L^p} \leqslant c\varepsilon_n^{1-\frac{2}{p}}\|\xi\|_{W^{1,p}}.$$

The terms III_1 and III_2 can be written as

$$\mathrm{III}_1 = \chi_\tau^n \cdot \big(\mathfrak{d}_{\tilde{u}_\tau^\circ}\xi\big) \circ \iota_\tau + \chi_\tau^n \cdot \big(\mathfrak{d}_{\tilde{u}_\tau^\circ} - \mathfrak{d}_{\tilde{u}_\tau^\circ \circ \iota_\tau}\big)(\xi \circ \iota_\tau)$$

and

$$\mathrm{III}_2 = \chi_\tau^n \cdot \big(\mathfrak{d}_{\tilde{u}_\tau^\circ} - \mathfrak{d}_{u(n)}\big)\xi(n).$$

Again, by Proposition 5.15 and (5.10),

$$\|\mathrm{III}_1 + \mathrm{III}_2\|_{L^p} \leqslant \|\mathfrak{d}_{u_0}\xi\|_{L^p} + c\varepsilon_n^{\frac{1}{p}}\|\xi\|_{W^{1,p}}.$$

□

5.4 Construction of right inverses

Throughout this subsection, let $\mathscr{O} \subset L^p\Omega^{0,1}(\Sigma_0, u_0^*TX)$ be a finite dimensional subspace such that

$$\operatorname{im}\mathfrak{d}_{u_0} + \mathscr{O} = L^p\Omega^{0,1}(\Sigma_0, u_0^*TX). \tag{5.16}$$

In particular, $\mathscr{O} \cong \operatorname{coker}\mathfrak{d}_{u_0}$.

Definition 5.17. Define $\mathrm{pull}_\tau\colon L^p\Omega^{0,1}(\Sigma_0, u_0^*TX) \to L^p\Omega^{0,1}(\Sigma_\tau, \tilde{u}_\tau^*TX)$ by

$$\mathrm{pull}_\tau(\eta)([z]) := \begin{cases} \Phi_{\tilde{u}_\tau([z])}\Phi_{u_0(z)}^{-1}\eta(z) & \text{if } \varepsilon_n^{1/2} \leqslant r_n(z) \leqslant 2R_0 \\ \eta(z) & \text{otherwise.} \end{cases}$$

Definition 5.18. Define $\bar{\mathfrak{d}}_{u_0}\colon W^{1,p}\Gamma(\Sigma_0, \nu_0; u_0^*TX) \oplus \mathscr{O} \to L^p\Omega^{0,1}(\Sigma_0, u_0^*TX)$ by

$$\bar{\mathfrak{d}}_{u_0}(\xi, o) := \mathfrak{d}_{u_0}\xi + o.$$

Define $\bar{\mathfrak{d}}_{\tilde{u}_\tau}\colon W^{1,p}\Gamma(\Sigma_\tau, \nu_\tau; \tilde{u}_\tau^*TX) \oplus \mathscr{O} \to L^p\Omega^{0,1}(\Sigma_\tau, \tilde{u}_\tau^*TX)$ by

$$\bar{\mathfrak{d}}_{\tilde{u}_\tau}(\xi, o) := \mathfrak{d}_{\tilde{u}_\tau}\xi + \mathrm{pull}_\tau(o).$$

By construction, $\bar{\mathfrak{d}}_{u_0}$ is surjective and, hence, has a right inverse $\mathfrak{r}_{u_0}\colon L^p\Omega^{0,1}(\Sigma_0, u_0^*TX) \to W^{1,p}\Gamma(\Sigma_0, \nu_0; u_0^*TX) \oplus \mathscr{O}$ of $\bar{\mathfrak{d}}_{u_0}$. Henceforth, fix a choice of $\mathfrak{r}_{u_0}$. The purpose of this subsection is to construct a right inverse $\mathfrak{r}_{\tilde{u}_\tau}$ to $\bar{\mathfrak{d}}_{\tilde{u}_\tau}$ for sufficiently small ε.

Definition 5.19. Define $\mathrm{push}_\tau\colon L^p\Omega^{0,1}(\Sigma_\tau, \tilde{u}_\tau^*TX) \to L^p\Omega^{0,1}(\Sigma_0, u_0^*TX)$ by

$$\mathrm{push}_\tau(\eta)(z) := \begin{cases} 0 & \text{if } r_n(z) < \varepsilon_n^{1/2} \\ \Phi_{u_0(z)}\Phi_{\tilde{u}_\tau([z])}^{-1}\eta([z]) & \text{if } \varepsilon_n^{1/2} \leqslant r_n(z) \leqslant 2R_0 \\ \eta([z]) & \text{otherwise.} \end{cases}$$

Definition 5.20. Define $\tilde{\mathfrak{r}}_{\tilde{u}_\tau}\colon L^p\Omega^{0,1}(\Sigma_\tau, \tilde{u}_\tau^*TX) \to W^{1,p}\Gamma(\Sigma_\tau, \nu_\tau; \tilde{u}_\tau^*TX) \oplus \mathscr{O}$ by

$$\tilde{\mathfrak{r}}_{\tilde{u}_\tau} := (\mathrm{fuse}_\tau \oplus \mathrm{id}_{\mathscr{O}}) \circ \mathfrak{r}_{u_0} \circ \mathrm{push}_\tau.$$

Proposition 5.21. *The linear operator* $\tilde{\mathfrak{r}}_{\tilde{u}_\tau}$ *satisfies*

$$\left\|\bar{\mathfrak{d}}_{\tilde{u}_\tau} \circ \tilde{\mathfrak{r}}_{\tilde{u}_\tau} - \mathrm{id}\right\| \leqslant c \sum_{n \in S}\left(\varepsilon_n^{\frac{1}{p}} + \varepsilon_n^{1-\frac{2}{p}}\right)\|\mathfrak{r}_{u_0}\| \quad \textit{and} \tag{5.22}$$

$$\|\tilde{\mathfrak{r}}_{\tilde{u}_\tau}\| \leqslant c\|\mathfrak{r}_{u_0}\|.$$

Proof. The maps push_τ and pull_τ are bounded by a constant independent τ and, by Proposition 5.14, so is fuse_τ. This implies the estimate on $\|\tilde{\mathfrak{r}}_{\tilde{u}_\tau}\|$.

Let $\eta \in L^p\Omega^{0,1}(\Sigma_\tau, \tilde{u}_\tau^* TX)$. To prove (5.22), we estimate $\left\|\bar{\mathfrak{d}}_{\tilde{u}_\tau}\tilde{\mathfrak{r}}_{\tilde{u}_\tau}\eta - \eta\right\|_{L^p}$ as follows. Set

$$(\xi, o) := \mathfrak{r}_{u_0} \circ \mathrm{push}_\tau(\eta).$$

Since

$$\mathfrak{d}_{u_0}\xi + o = \mathrm{push}_\tau(\eta),$$

by Proposition 5.15 applied to $\tilde{u}_\tau^\circ$ and u_0 and using (5.10),

$$\begin{aligned}\|\mathfrak{d}_{\tilde{u}_\tau^\circ}\xi + o - \eta\|_{L^p} &\leqslant c\varepsilon^{\frac{1}{p}}\|\xi\|_{W^{1,p}}\\ &\leqslant c\varepsilon^{\frac{1}{p}}\|\mathfrak{r}_{u_0}\|\|\eta\|_{L^p}.\end{aligned}$$

Therefore, it remains to estimate

$$\mathfrak{d}_{\tilde{u}_\tau^\circ}\big(\chi_\tau^n\cdot(\xi\circ\iota_\tau - \xi(n))\big) = \underbrace{\bar{\partial}\chi_\tau^n\cdot(\xi\circ\iota_\tau - \xi(n))}_{=:\mathrm{I}} + \underbrace{\chi_\tau^n\cdot\mathfrak{d}_{\tilde{u}_\tau^\circ}(\xi\circ\iota_\tau)}_{=:\mathrm{II}} - \underbrace{\chi_\tau^n\cdot\mathfrak{d}_{\tilde{u}_\tau^\circ}\xi(n)}_{=:\mathrm{III}}. \tag{5.23}$$

By Proposition 5.14,

$$\|\mathrm{I}\|_{L^p} \leqslant c\varepsilon^{1-\frac{2}{p}}\|\mathfrak{r}_{u_0}\|\|\eta\|_{L^p}.$$

To estimate the second term, observe that in the region where $R_0 \leqslant r_n \leqslant 2R_0$

$$\mathfrak{d}_{u_0\circ\iota_\tau}(\xi\circ\iota_\tau) = \iota_\tau^*(\mathfrak{d}_{u_0}\xi) = \iota_\tau^*(\mathrm{push}_\tau(\eta)) = 0.$$

To understand the last identity, observe that $r_n(\iota_\tau(z)) = \varepsilon_n r_{\nu(n)}^{-1}(z)$ and $\mathrm{push}_\tau(\eta)$ is defined to vanish in the region of Σ_0 where $r_n \leqslant \varepsilon_n^{1/2}$. Thus, by Proposition 5.15 applied to $\tilde{u}_\tau$ and $u_0\circ\iota_\tau$,

$$\|\mathrm{II}\|_{L^p} \leqslant c\varepsilon^{\frac{1}{2p}}\|\mathfrak{r}_{u_0}\|\|\eta\|_{L^p}.$$

The vector field $\xi(n)$ is constant with respect the chosen trivialization. Since the operator $\mathfrak{d}_{u_0(n)}$ associated with the constant map agrees with the standard $\bar{\partial}$–operator,

$$\mathfrak{d}_{u_0(n)}\xi(n) = 0.$$

Therefore, by Proposition 5.15 applied to $\tilde{u}_\tau$ and the constant map $u(n)$,

$$\|\mathrm{III}\|_{L^p} \leqslant c\varepsilon^{\frac{1}{2p}}\|\mathfrak{r}_{u_0}\|\|\eta\|_{L^p}.$$ ☐

Throughout the remainder of this subsection, suppose the following.

Hypothesis 5.24. *The smoothing parameter τ is such that the right-hand side of (5.22) at most $1/2$.*

Definition 5.25. Define the **right inverse** $\mathfrak{r}_{\tilde{u}_\tau}\colon L^p\Omega^{0,1}(\Sigma_\tau, \tilde{u}_\tau^*TX) \to W^{1,p}\Gamma(\Sigma_\tau, \nu_\tau; \tilde{u}_\tau^*TX) \oplus \mathscr{O}$ associated with $\mathfrak{r}_{u_0}$ by

$$\mathfrak{r}_{\tilde{u}_\tau} := \tilde{\mathfrak{r}}_{\tilde{u}_\tau}\Big(\bar{\mathfrak{d}}_{\tilde{u}_\tau}\tilde{\mathfrak{r}}_{\tilde{u}_\tau}\Big)^{-1} = \tilde{\mathfrak{r}}_{\tilde{u}_\tau}\sum_{k=0}^{\infty}\Big(\mathrm{id} - \bar{\mathfrak{d}}_{\tilde{u}_\tau}\tilde{\mathfrak{r}}_{\tilde{u}_\tau}\Big)^k.$$

Proposition 5.26. *The right inverse* $\mathfrak{r}_{\tilde{u}_\tau}\colon L^p\Omega^{0,1}(\Sigma_\tau, \tilde{u}_\tau^*TX) \to W^{1,p}\Gamma(\Sigma_\tau, \nu_\tau; \tilde{u}_\tau^*TX) \oplus \mathscr{O}$ *satisfies*

$$\bar{\mathfrak{d}}_{u_\tau}\mathfrak{r}_{\tilde{u}_\tau} = \mathrm{id} \quad \textit{and}$$
$$\|\mathfrak{r}_{\tilde{u}_\tau}\| \leqslant c\|\mathfrak{r}_{u_0}\|;$$

furthermore,

$$\operatorname{im}\mathfrak{r}_{\tilde{u}_\tau} = \operatorname{im}\tilde{\mathfrak{r}}_{\tilde{u}_\tau}.$$

5.5 Complements of the image of $\mathfrak{r}_{\tilde{u}_\tau}$

Proposition 5.27. *Given* $c_f > 0$ *there is a constant* $\delta = \delta(c_f) > 0$ *such that the following holds. If* τ *satisfies* $\varepsilon < \delta$ *and* $K \subset W^{1,p}\Gamma(\Sigma_0, \nu_0; u_0^*TX)$ *is a subspace with* $\dim K = \dim\ker\bar{\mathfrak{d}}_{u_0}$ *and such that for every* $\kappa \in K$

$$\|\mathfrak{d}_{u_0}\kappa\|_{L^p} \leqslant \delta\|\kappa\|_{W^{1,p}} \quad \textit{and} \quad \|\kappa\|_{W^{1,p}} \leqslant c_f\|\mathrm{fuse}_\tau(\kappa)\|_{W^{1,p}},$$

then every $(\xi, o) \in W^{1,p}\Gamma(\Sigma_\tau, \nu_\tau; \tilde{u}_\tau^*TX) \oplus \mathscr{O}$ *can be uniquely written as*

$$(\xi, o) = \mathfrak{r}_{\tilde{u}_\tau}\eta + (\kappa, 0)$$

with $\eta \in L^p\Omega^{0,1}(\Sigma_\tau, \tilde{u}_\tau^*TX)$ *and* $\kappa \in K$*; moreover,*

$$\|\eta\|_{L^p} + \|\kappa\|_{W^{1,p}} \leqslant c(c_f)(\|\xi\|_{W^{1,p}} + |o|).$$

Proof. Because $\mathfrak{r}_{\tilde{u}_\tau}$ and $\mathrm{fuse}_\tau|_K$ are injective and given the hypothesis on $\mathrm{fuse}_\tau|_K$, it suffices to show that $W^{1,p}\Gamma(\Sigma_\tau, \nu_\tau; \tilde{u}_\tau^*TX) \oplus \mathscr{O}$ is the direct sum of $\operatorname{im}(\mathfrak{r}_{\tilde{u}_\tau})$ and $\operatorname{im}(\mathrm{fuse}_\tau|_K) \oplus 0$.

By the index formula (2.29), Remark 2.11, and Proposition 3.15,

$$\begin{aligned}\operatorname{index}\mathfrak{d}_{u_0} &= 2\langle(u_0^*c_1(X,J), [\Sigma_0]\rangle + 2n(1 - p_a(\Sigma_0, \nu_0)) \\ &= 2\langle(\tilde{u}_\tau^*c_1(X,J), [\Sigma_\tau]\rangle + 2n(1 - p_a(\Sigma_\tau, \nu_\tau)) \\ &= \operatorname{index}\mathfrak{d}_{\tilde{u}_\tau}.\end{aligned}$$

Therefore and because $\bar{\mathfrak{d}}_{u_0}$ is surjective and $\mathfrak{r}_{\tilde{u}_\tau}$ is injective,

$$\operatorname{codim}\operatorname{im}(\mathfrak{r}_{\tilde{u}_\tau}) = \operatorname{index}\bar{\mathfrak{d}}_{\tilde{u}_\tau} = \operatorname{index}\bar{\mathfrak{d}}_{u_0} = \dim\ker\bar{\mathfrak{d}}_{u_0}.$$

Hence, it remains to prove that $\operatorname{im}(\mathfrak{r}_{\tilde{u}_\tau})$ and $\operatorname{im}(\mathrm{fuse}_\tau|_K) \oplus 0$ intersect trivially.

Suppose that $\eta \in L^p\Omega^{0,1}(\Sigma_{\sigma,\tau}, \tilde{u}_\tau^*TX)$ and $\kappa \in K$ satisfy

$$\mathfrak{r}_{\tilde{u}_\tau}(\eta) = (\mathrm{fuse}_\tau(\kappa), 0).$$

By Proposition 5.13 as well as the hypothesis on fuse_τ and for sufficiently small δ,

$$\begin{aligned}
\|\eta\|_{L^p} &= \|\mathfrak{d}_{\tilde{u}_\tau}\mathrm{fuse}_\tau(\kappa)\|_{L^p} \\
&\leqslant c\left[\delta + \#S \cdot \left(\delta^{\frac{1}{p}+\delta^{1-\frac{2}{p}}}\right)\right]\|\kappa\|_{W^{1,p}} \\
&\leqslant cc_f\left[\delta + \#S \cdot \left(\delta^{\frac{1}{p}} + \delta^{1-\frac{2}{p}}\right)\right]\|\eta\|_{L^p} \\
&\leqslant \frac{1}{2}\|\eta\|_{L^p}.
\end{aligned}$$

Therefore, η vanishes. □

5.6 Kuranishi model for a neighborhood of nodal maps

Throughout, let (Σ_0, j_0, ν_0) be a nodal Riemann surface with nodal set S, let (X, g, J_0) be an almost Hermitian manifold, and let $u_0 \colon (\Sigma_0, j_0, \nu_0) \to (X, J_0)$ be a nodal J_0–holomorphic map. Let $(\pi \colon \mathscr{X} \to \Delta, \star = (0,0), \iota)$ be the versal deformation of (Σ_0, j_0, ν_0) constructed in Section 4.3 with fibers

$$(\Sigma_{\sigma,\tau}, j_{\sigma,\tau}, \nu_{\sigma,\tau}) = \pi^{-1}(\sigma, \tau).$$

Let $\delta_{\mathscr{J}} > 0$ and let

$$\mathscr{U} \subset \left\{J \in \mathscr{J}(X) : \|J - J_0\|_{C^1} < \delta_{\mathscr{J}}\right\}$$

be such that for every $k \in N$

$$\sup_{J \in \mathscr{U}} \|J - J_0\|_{C^k} < \infty.$$

In the upcoming discussion we may implicitly shrink Δ and $\delta_{\mathscr{J}}$, in order to ensure that Hypothesis 5.4 and Hypothesis 5.24 hold and various expressions involving $|\sigma|$, $\varepsilon := \max\{\varepsilon_n : n \in S\}$ with $\varepsilon_n := |\tau_n|$, and $\|J - J_0\|_{C^1}$ are sufficiently small.

The purpose of this subsection is to analyze whether u_0 can be slightly deformed to a J–holomorphic map $u_{\sigma,\tau} \colon (\Sigma_{\sigma,\tau}, j_{\sigma,\tau}, \nu_{\sigma,\tau}) \to (X, J)$ with $J \in \mathscr{U}$; more precisely: we construct a Kuranishi model for a Gromov neighborhood of u_0 in the space of nodal J–holomorphic maps with $J \in \mathscr{U}$.

To facilitate the discussion in Section 5.7 (and although it makes the present discussion somewhat more awkward than it needs to be) this construction proceeds in two steps. Choose a partition

$$S = S_1 \amalg S_2 \quad \text{with} \quad \nu_0(S_1) = S_1 \quad \text{and} \quad \nu_0(S_2) = S_2$$

and write every smoothing parameter τ as

$$\tau = (\tau_1, \tau_2) \quad \text{with} \quad \tau_1 = (\tau_{1,n})_{n \in S_1} \quad \text{and} \quad \tau_2 = (\tau_{2,n})_{n \in S_2}.$$

The first step of our construction varies σ and τ_1 but $\tau_2 = 0$ is fixed. The second step holds σ and τ_1 fixed and varies τ_2.

Denote by $u_{\sigma,0} \colon \Sigma_{\sigma,0} \to X$ the smooth map underlying u_0. Denote by

$$\mathfrak{d}_{u_0;J_0} \colon W^{1,p}\Gamma(\Sigma_0, \nu_0; u_0^*TX) \to L^p\Omega^{0,1}(\Sigma_0, u_0^*TX)$$

the linear operator associated with u_0 defined in Definition 2.26. Let

$$\mathscr{O} \subset \Omega^{0,1}(\Sigma_0, u_0^*TX) \subset L^p\Omega^{0,1}(\Sigma_0, u_0^*TX)$$

be a lift of $\operatorname{coker} \mathfrak{d}_{u_0;J_0}$; that is: $\dim \mathscr{O} = \dim \operatorname{coker} \mathfrak{d}_{u_0}$ and (5.16) holds. (The canonical choice is $\mathscr{O} = \ker \mathfrak{d}_{u_0;J_0}^*$, but this choice is not always most convenient.) Trivialize the bundle over $\Delta_1 \times \mathscr{U}$ whose fiber over $(\sigma, J) \in \Delta_1 \times \mathscr{U}$ is $\Omega^{0,1}(\Sigma_{\sigma,0}, u_{\sigma,0}^*TX)$ with the $(0,1)$–part taken with respect to $j_{\sigma,0}$ and J. This identifies $\Omega^{0,1}(\Sigma_0, u_0^*TX)$ and $\Omega^{0,1}(\Sigma_{\sigma,0}, u_{\sigma,0}^*TX)$ and thus exhibits $\mathscr{O}$ as a subset of $L^p\Omega^{0,1}(\Sigma_{\sigma,0}, u_{\sigma,0}^*TX)$ for which (5.16) holds for $\mathfrak{d}_{\tilde{u}_{\sigma,0};J}$ instead of $\mathfrak{d}_{u_0}$. Define

$$\bar{\mathfrak{d}}_{\tilde{u}_{\sigma,\tau_1,0};J} \colon W^{1,p}\Gamma(\Sigma_{\sigma,\tau_1,0}, \nu_{\sigma,0}; \tilde{u}_{\sigma,\tau_1,0}^*TX) \oplus \mathscr{O} \to L^p\Omega^{0,1}(\Sigma_{\sigma,\tau_1,0}, \tilde{u}_{\sigma,\tau_1,0}^*TX)$$

as in Definition 5.18. The construction in Section 5.4 yields a right inverse

$$\mathfrak{r}_{\tilde{u}_{\sigma,\tau_1,0};J} \colon L^p\Omega^{0,1}(\Sigma_{\sigma,\tau_1,0}, \tilde{u}_{\sigma,\tau_1,0}^*TX) \to W^{1,p}\Gamma(\Sigma_{\sigma,\tau_1,0}, \nu_{\sigma,\tau_1,0}; \tilde{u}_{\sigma,\tau_1,0}^*TX) \oplus \mathscr{O}.$$

of $\bar{\mathfrak{d}}_{\tilde{u}_{\sigma,\tau_1,0};J}$.

Proposition 5.28. *There are constants $\delta_\kappa, \Lambda > 0$ such that for every $(\sigma, \tau_1, 0) \in \Delta$ and $\kappa \in \ker \mathfrak{d}_{u_0}$ with $|\kappa| < \delta_\kappa$ there exists a unique pair*

$$(\xi(\sigma,\tau_1;J;\kappa), o(\sigma,\tau_1;J;\kappa)) \in \operatorname{im} \mathfrak{r}_{\tilde{u}_{\sigma,\tau_1,0};J} \subset W^{1,p}\Gamma(\Sigma_{\sigma,\tau_1,0}, \nu_{\sigma,\tau_1,0}; \tilde{u}_{\sigma,\tau_1,0}^*TX) \oplus \mathscr{O}$$

with

$$\|\xi(\sigma,\tau_1;J;\kappa)\|_{W^{1,p}} + |o(\sigma,\tau_1;J;\kappa)| \leqslant \Lambda$$

satisfying

$$\mathfrak{F}_{\tilde{u}_{\sigma,\tau_1,0};J}(\operatorname{fuse}_{\tau_1,0}\kappa + \xi(\sigma,\tau_1;J;\kappa)) + \operatorname{pull}_{\tau_1,0}(o(\sigma,\tau_1;J;\kappa)) = 0, \tag{5.29}$$

with $\mathfrak{F}$ as in Definition 2.25. Furthermore,

$$\|\xi(\sigma,\tau_1;J;\kappa)\|_{W^{1,p}} + |o(\sigma,\tau_1;J;\kappa)| \leqslant c\big(|\sigma| + |\tau_1|^{\frac{1}{2}+\frac{1}{p}} + \|J - J_0\|_{C^0} + |\kappa|\big). \tag{5.30}$$

Proof. Since $\mathfrak{r}_{\tilde{u}_{\sigma,\tau_1,0};J}$ is injective, (5.29) is equivalent to the fixed-point equation

$$\eta = \mathbf{F}(\eta) \coloneqq \eta - \mathfrak{F}_{\tilde{u}_{\sigma,\tau_1,0};J}\big(\operatorname{fuse}_{\tau_1,0}\kappa + \mathfrak{r}_{\tilde{u}_{\sigma,\tau_1,0};J}\eta\big).$$

By Proposition 2.32,

$$\mathbf{F}(\eta) = -\bar{\partial}_J(\tilde{u}_{\sigma,\tau_1,0}, j_{\sigma,\tau_1,0}) - \mathfrak{d}_{\tilde{u}_{\sigma,\tau_1,0};J}\operatorname{fuse}_{\tau_1,0}\kappa - \mathfrak{n}_{\tilde{u}_{\sigma,\tau_1,0};J}\big(\operatorname{fuse}_{\tau_1,0}\kappa + \operatorname{pr}_1\mathfrak{r}_{\tilde{u}_{\sigma,\tau_1,0};J}\eta\big).$$

Here pr_1 denotes the projection to the first summand of $W^{1,p}\Gamma(\Sigma_{\sigma,\tau_1,0},\nu_{\sigma,\tau_1,0};\tilde{u}^*_{\sigma,\tau_1,0}TX)\oplus\mathscr{O}$.

By Proposition 5.8 and Proposition 2.32,

$$\|\mathbf{F}(0)\|_{L^p}\leqslant c\big(|\sigma|+\|J-J_0\|_{C^0}+|\kappa|+|\kappa|^2\big).$$

Moreover, by Proposition 5.26 and Proposition 2.32,

$$\|\mathbf{F}(\eta_1)-\mathbf{F}(\eta_2)\|_{L^p}\leqslant c(|\kappa|+\|\eta_1\|_{L^p}+\|\eta_2\|_{L^p})\|\eta_1-\eta_2\|_{L^p}.$$

Therefore, provided δ_κ is sufficiently small, there is an $R>0$ such that $\|\mathbf{F}(0)\|_{L^p}\leqslant R/2$ and for every $\eta_1,\eta_2\in\bar{B}_R(0)\subset L^p\Omega^{0,1}(\Sigma_{\sigma,\tau},\tilde{u}^*_{\sigma,\tau}TX)$

$$\|\mathbf{F}(\eta_1)-\mathbf{F}(\eta_2)\|_{L^p}\leqslant\frac{1}{2}\|\eta_1-\eta_2\|_{L^p}.$$

This shows that $\mathbf{F}$ maps $\bar{B}_R(0)$ into $\bar{B}_R(0)$ and $\mathbf{F}\colon\ \bar{B}_R(0)\to\bar{B}_R(0)$ is a contraction. Thus, the first assertion follows from the Banach fixed point theorem. The second follows from the above and Proposition 5.8. □

This completes the first step. The second step is analogous with u_0 being replaced by the maps obtained from Proposition 5.28. For $(\sigma,\tau)\in\Delta$ and $\kappa\in\ker\mathfrak{d}_{u_0}$ with $\|\kappa\|_{W^{1,p}}<\delta_\kappa$ set

$$u_{\sigma,\tau_1,0;J;\kappa}\coloneqq\exp_{\tilde{u}_{\sigma,\tau_1,0}}(\mathrm{fuse}_{\tau_1,0}\kappa+\xi(\sigma,\tau_1;J;\kappa))\quad\text{and}\quad\tilde{u}_{\sigma,\tau;J;\kappa}\coloneqq\widetilde{\big(u_{\sigma,\tau_1,0;J;\kappa}\big)_\tau};$$

that is: $\tilde{u}_{\sigma,\tau;J;\kappa}$ is obtained from $u_{\sigma,\tau_1,0;J;\kappa}$ by the construction in Definition 5.6.

Definition 5.31. Define $\mathrm{pull}_{\sigma,\tau_1,0;J;\kappa}\colon\ L^p\Omega^{0,1}(\Sigma_0,u_0^*TX)\to L^p\Omega^{0,1}(\Sigma_{\sigma,\tau_1,0},u^*_{\sigma,\tau_1,0;J;\kappa}TX)$ to be the composition of $\mathrm{pull}_{\tau_1,0}$ with the map induced by parallel transport along the geodesics

$$t\mapsto\exp_{\tilde{u}_{\sigma,\tau_1,0}}\big(t(\mathrm{fuse}_{\tau_1,0}\kappa+\xi(\sigma,\tau_1;J;\kappa))\big).$$

Furthermore, denote by $\mathrm{pull}_{\sigma,\tau;J;\kappa}\colon\ L^p\Omega^{0,1}(\Sigma_0,u_0^*TX)\to L^p\Omega^{0,1}(\Sigma_{\sigma,\tau},\tilde{u}^*_{\sigma,\tau;J;\kappa}TX)$ the composition of $\mathrm{pull}_{\sigma,\tau_1,0;J;\kappa}$ with $\mathrm{pull}_{\tau_2}\colon\ L^p\Omega^{0,1}(\Sigma_{\sigma,\tau_1,0},u^*_{\sigma,\tau_1,0;J;\kappa}TX)\to L^p\Omega^{0,1}(\Sigma_{\sigma,\tau},\tilde{u}^*_{\sigma,\tau;J;\kappa}TX)$ defined in Definition 5.17.

The subspace $\mathrm{pull}_{\sigma,\tau_1,0;J;\kappa}(\mathscr{O})$ satisfies (5.16) for $u_{\sigma,\tau_1,0;J;\kappa}$ instead of u_0. Define

$$\bar{\mathfrak{d}}_{\tilde{u}_{\sigma,\tau;J;\kappa}}\colon\ W^{1,p}\Gamma(\Sigma_{\sigma,\tau},\nu_{\sigma,\tau};\tilde{u}^*_{\sigma,\tau;J;\kappa}TX)\oplus\mathscr{O}\to L^p\Omega^{0,1}(\Sigma_{\sigma,\tau},\tilde{u}^*_{\sigma,\tau;J;\kappa}TX)$$

as in Definition 5.18. The construction in Section 5.4 yields a right inverse

$$\mathfrak{r}_{\tilde{u}_{\sigma,\tau;J;\kappa}}\colon\ L^p\Omega^{0,1}(\Sigma_{\sigma,\tau},\tilde{u}^*_{\sigma,\tau;J;\kappa}TX)\to W^{1,p}\Gamma(\Sigma_{\sigma,\tau},\nu_{\sigma,\tau};\tilde{u}^*_{\sigma,\tau;J;\kappa}TX)\oplus\mathscr{O}$$

of $\bar{\mathfrak{d}}_{\tilde{u}_{\sigma,\tau;J;\kappa}}$.

Proposition 5.32. *There are constants* $\delta_\kappa, \Lambda > 0$ *such that for every* $(\sigma, \tau; J) \in \Delta \times \mathscr{U}$ *and* $\kappa \in \ker \mathfrak{d}_{u_0}$ *with* $\|\kappa\|_{W^{1,p}} < \delta_\kappa$ *there exists a unique pair*

$$(\hat{\xi}(\sigma, \tau; J; \kappa), \hat{o}(\sigma, \tau; J; \kappa)) \in \operatorname{im} \mathfrak{r}_{\tilde{u}_{\sigma,\tau;\kappa};J} \subset W^{1,p}\Gamma(\Sigma_{\sigma,\tau}, \nu_{\sigma,\tau}; \tilde{u}^*_{\sigma,\tau;\kappa}TX) \oplus \mathscr{O}$$

with

$$\|\hat{\xi}(\sigma, \tau; J; \kappa)\|_{W^{1,p}} + |\hat{o}(\sigma, \tau; J; \kappa)| \leqslant \Lambda$$

satisfying

$$\mathfrak{F}_{\tilde{u}_{\sigma,\tau;J;\kappa}}(\hat{\xi}(\sigma, \tau; J; \kappa)) + \operatorname{pull}_{\sigma,\tau;J;\kappa}(o(\sigma, \tau_1; J; \kappa) + \hat{o}(\sigma, \tau; J; \kappa)) = 0. \tag{5.33}$$

Furthermore,

$$\|\hat{\xi}(\sigma, \tau; J; \kappa)\|_{W^{1,p}} + |\hat{o}(\sigma, \tau; J; \kappa)| \leqslant c\|\bar{\partial}_J(\tilde{u}_{\sigma,\tau;J;\kappa}, j_{\sigma,\tau}) + \operatorname{pull}_{\sigma,\tau;J;\kappa}(o(\sigma, \tau_1; J; \kappa))\|_{L^p}. \tag{5.34}$$

Proof. This is similar to the proof of Proposition 5.28. ☐

Definition 5.35. Set $\mathscr{I} := B_{\delta_\kappa}(0) \subset \ker \mathfrak{d}_{u_0}$. The **Kuranishi map** $\operatorname{ob}\colon \Delta \times \mathscr{U} \times \mathscr{I} \to \mathscr{O}$ is defined by

$$\operatorname{ob}(\sigma, \tau; J; \kappa) := o(\sigma, \tau_1; J; \kappa) + \hat{o}(\sigma, \tau; J; \kappa),$$

with o and $\hat{o}$ as in Proposition 5.28 and Proposition 5.32.

The upshot of the preceding discussion is that u_0 can be slightly deformed to a J–holomorphic map $u_{\sigma,\tau}\colon (\Sigma_{\sigma,\tau}, j_{\sigma,\tau}, \nu_{\sigma,\tau}) \to (X, J)$; if and only if there is a $\kappa \in \mathscr{I}$ with $\operatorname{ob}(\sigma, \tau; J; \kappa) = 0$. The following shows that this Kuranishi model indeed describes a Gromov neighborhood of $u_0\colon (\Sigma_0, j_0, \nu_0) \to (X, J_0)$.

Proposition 5.36. *Let* $(\sigma_k, \tau_k)_{k\in\mathbf{N}}$ *be a sequence in* Δ *converging to* $(0, 0)$ *and let* $(J_k)_{k\in\mathbf{N}}$ *be a sequence in* $\mathscr{U}$ *converging to* J_0. *If*

$$\left(u_k\colon (\Sigma_{\sigma_k,\tau_k}, j_{\sigma_k,\tau_k}, \nu_{\sigma_k,\tau_k}) \to (X, J_k)\right)_{k\in\mathbf{N}}$$

is a sequence of nodal pseudo-holomorphic maps which Gromov converges to $u_0\colon (\Sigma_0, j_0, \nu_0) \to (X, J_0)$ *then there is a* $K \in \mathbf{N}$ *such that for every* $k \geqslant K$ *there are* $\kappa_k \in \ker \mathfrak{d}_{u_0}$ *and* $(\xi_k, 0) \in \operatorname{im} \mathfrak{r}_{\tilde{u}_{\sigma_k,\tau_k;\kappa_k};J_k}$ *with*

$$u_k = \exp_{\tilde{u}_{\sigma_k,\tau_k;\kappa_k}}(\xi_k);$$

moreover,

$$\lim_{k\to\infty} |\kappa_k| = 0 \quad \textit{and} \quad \lim_{k\to\infty} \|\xi_k\|_{W^{1,p}} = 0.$$

In particular,

$$\operatorname{ob}(\sigma_k, \tau_k; J_k; \kappa_k) = 0.$$

The proof of this proposition relies on the following result.

Proposition 5.37. *Assume the situation of Proposition 5.36. There are $K \in \mathbf{N}$, $\delta_\kappa > 0$, and $c > 0$ such that for every $k \geqslant K$ and $\kappa \in \ker \mathfrak{d}_{u_0}$ with $\|\kappa\|_{W^{1,p}} < \delta_\kappa$ there is a $\zeta_{k;\kappa} \in \Gamma(\Sigma_{\sigma_k,\tau_k}, \tilde{u}^*_{\sigma_k,\tau_k;J_k;\kappa}TX)$ with*

$$u_k = \exp_{\tilde{u}_{\sigma_k,\tau_k;J_k;\kappa}}(\zeta_{k;\kappa});$$

moreover,

$$\limsup_{k\to\infty} \|\zeta_{k;\kappa}\|_{W^{1,p}} \leqslant c|\kappa|.$$

Proof. The proof has two steps: the construction of $\zeta_{k;\kappa}$ and the proof of the convergence statement.

Step 1. *There are $K \in \mathbf{N}$ and $\delta_\kappa > 0$ such that for every $k \geqslant K$, $\kappa \in \ker \mathfrak{d}_{u_0}$ with $\|\kappa\|_{W^{1,p}} < \delta_\kappa$, and $z \in \Sigma_{\sigma_k,\tau_k}$*

$$u_k(z) \in U_{\tilde{u}_{\sigma_k,\tau_k;J_k;\kappa}(z)};$$

in particular, there is a section $\zeta_{k;\kappa} \in \Gamma(\Sigma_{\sigma_k,\tau_k}, \tilde{u}^*_{\sigma_k,\tau_k;J_k}TX)$ given by

$$\zeta_{k;\kappa} := \exp^{-1}_{\tilde{u}_{\sigma_k,\tau_k;J_k;\kappa}} \circ u_k.$$

By (5.30), (5.34), and Proposition 5.8,

$$d(\tilde{u}_{\sigma_k,\tau_k;J_k;\kappa}, \tilde{u}_{\sigma_k,\tau_k;J_k;0}) \leqslant c\big(|\sigma_k| + \varepsilon_k^{\frac{1}{2}+\frac{1}{p}} + \|J_k - J_0\|_{C^0} + |\kappa|\big). \tag{5.38}$$

Therefore, it suffices to consider $\kappa = 0$ and prove that there exists a $K \in \mathbf{N}$ such that for every $k \geqslant K$

$$u_k(z) \in \frac{1}{2}U_{\tilde{u}_{\sigma_k,\tau_k;J_k;0}(z)}.$$

Using the framing Ψ from Definition 4.16, define $v_k \colon \Sigma_0\backslash S \to X$ and $\tilde{v}_{\kappa,k} \colon \Sigma_0\backslash S \to X$ by

$$\begin{aligned} v_k &:= u_k \circ \iota_k^{-1} \circ \Psi(\,\cdot\,;\sigma_k,\tau_k) \circ \iota_0 \quad \text{and} \\ \tilde{v}_k &:= \tilde{u}_{\sigma_k,\tau_k;J_k;0} \circ \iota_k^{-1} \circ \Psi(\,\cdot\,;\sigma_k,\tau_k) \circ \iota_0, \end{aligned}$$

cf. Definition 3.8. Both of the sequences $(v_k)_{k\in\mathbf{N}}$ and $(\tilde{v}_k)_{k\in\mathbf{N}}$ converge to $u_0 \colon \Sigma_0\backslash S \to X$ in the C^∞_{loc} topology—the former by Definition 3.8 and the latter by construction.

With the notation of Remark 4.17 for $r > 0$ and $n \in S$ set

$$N^r_{k,n} := N^r_{\sigma_k,\tau_k;n}.$$

Choose $r > 0$ as in Proposition 3.15 with $\delta := \frac{1}{8}\operatorname{inj}_g(X)$. By the preceding paragraph, the assertion holds for sufficiently large k and $z \notin N^r_{k,n}$. By Proposition 3.15 and by construction of $\tilde{u}_{\sigma,\tau}$, for sufficiently large k

$$u_k(N^r_{k,n}) \subset B_\delta(u_0(n)) \quad \text{and} \quad \tilde{u}_{\sigma_k,\tau_k;J_k;0}(N^r_{k,n}) \subset B_\delta(u_0(n));$$

hence, for every $z \in N^r_{k,n}$

$$u_k(z) \in \frac{1}{2}U_{\tilde{u}_{\sigma_k,\tau_k;J_k;0}(z)}.$$

Step 2. *There is a constant* $c > 0$ *such that the sections* $\zeta_{k;\kappa}$ *defined in the preceding step satisfy*

$$\limsup_{k\to\infty}\|\zeta_{k;\kappa}\|_{W^{1,p}} \leqslant c|\kappa|.$$

By (5.30), (5.34), and Proposition 5.8, we can restrict to $\kappa = 0$. Furthermore, it suffices to prove that for every $n \in S$

$$\lim_{s\downarrow 0}\limsup_{k\to\infty}\|\zeta_{k;0}\|_{W^{1,p}(N^s_{k,n})} = 0. \tag{5.39}$$

The case when n is not smoothed is straightforward. The framing extends to identify a neighborhood of n in Σ_{σ_k,τ_k} with a neighborhood of n in Σ_0. It follows from Lemma 3.13 and elliptic regularity that, on this subset, the maps u_k converge to u_0 in the C^∞_{loc} topology. Let us therefore assume that n is smoothed out; that is: $\varepsilon_{k;n} \neq 0$ for sufficiently large k.

Define $\rho_k \in C^\infty(N^r_{k;n}, T_{u_0(n)}X)$ and $\tilde\rho_k \in C^\infty(N^r_{k;n}, T_{u_0(n)}X)$ by

$$\rho_k \coloneqq \exp^{-1}_{u_0(n)} \circ u_k \quad\text{and}\quad \tilde\rho_k \coloneqq \exp^{-1}_{u_0(n)} \circ \tilde u_{\sigma_k,\tau_k;J_k;0}.$$

By construction,

$$\lim_{k\to\infty}\|\tilde\rho_k\|_{W^{1,p}} = 0.$$

Therefore, it suffices to prove that

$$\lim_{s\downarrow 0}\limsup_{k\to\infty}\|\rho_k\|_{W^{1,p}(N^s_{k,n})} = 0$$

As explained in Remark 4.17, the subset $N^r_{k;n}$ is biholomorphic to the cylinder

$$S^1 \times (-L_k, L_k) \quad\text{with}\quad L_k \coloneqq \log(r\varepsilon_{n;k}^{-1/2}).$$

Hence, ρ_k can be thought of as a map $\rho_k^{\mathrm{cyl}}\colon S^1 \times (-L_k, L_k) \to T_{u_0(n)}X$. More concretely, the canonical chart ϕ_n defines a holomorphic embedding

$$\phi_n\colon \{v \in T_n\Sigma_0 : \varepsilon_{n;k}^{1/2} \leqslant |v| < r\} \to N^r_{k;n}$$

which glues via ι_τ with the embedding $\phi_{\nu(n)}$ to a biholomorphic map

$$B_r(0)\backslash\bar B_{\varepsilon_{n;k}/r}(0) \cong N^r_{k;n}.$$

Choose identifications $T_n\Sigma_0 \cong \mathbf{C} \cong T_{\nu(n)}\Sigma_0$ such that $\iota_\tau(z) = \varepsilon_n/z$. The map ρ_k^{cyl} is then defined by

$$\rho_k^{\mathrm{cyl}}(\theta,\ell) \coloneqq \begin{cases}\rho_k \circ \phi_n\big(\varepsilon_{n;k}^{1/2}e^{\ell+i\theta}\big) & \text{if } t \geqslant 0\\ \rho_k \circ \phi_{\nu(n)}\big(\varepsilon_{n;k}^{1/2}e^{-\ell-i\theta}\big) & \text{if } t \leqslant 0.\end{cases}$$

Since u_k is J_k–holomorphic, ρ_k^{cyl} is $\exp^*_{u_0(n)}(J_k)$–holomorphic. Since the energy is conformally invariant,

$$E\big(\rho_k^{\mathrm{cyl}}\big) = E\big(u_{k|N^r_{k;n}}\big).$$

Choose $\mu \in (1 - 2/p, 1)$. By Lemma 3.14,

$$|\nabla\rho_k^{\mathrm{cyl}}(\theta,\ell)| \leqslant ce^{-\mu(L_k-|\ell|)}E\big(u_k|_{N^r_{k;n}}\big)^{1/2}$$

By the above and Proposition 5.2, for $z \in \Sigma^\circ_{\sigma,\tau}$ with $r_n(z) < r$

$$|\nabla\rho_k(z)| \leqslant cr^{-\mu}r_n(z)^{\mu-1}E\big(u_k|_{N^r_{k;n}}\big)^{1/2}.$$

There is a corresponding estimate with n replaced with $\nu(n)$. Hence,

$$\|\nabla\rho_k(z)\|^p_{L^p(N^s_{k;n})} \leqslant \frac{cr^{-\mu p}}{(\mu-1)p+2}s^{(\mu-1)p+2}E\big(u_k|_{N^r_{k;n}}\big)^{p/2}.$$

Since $(\mu-1)p+2 > 0$, the right-hand converges to zero as s converges to zero. ☐

Proof of Proposition 5.36. Let $k \geqslant K$ and $\kappa \in \ker\mathfrak{d}_{u_0}$ with $|\kappa| < \delta_\kappa$. Let $\zeta_{k;\kappa}$ be as in Proposition 5.37. By Proposition 5.27 the latter can be uniquely written as

$$\zeta_{k,\kappa} = \mathrm{fuse}_{\tau_k}(\lambda_{k;\kappa}) + \mathfrak{r}_{\tilde u_{\sigma_k,\tau_k;J_k;\kappa}}\eta_{k;\kappa} \quad \text{with} \quad \lambda_{k;\kappa} \in \ker\mathfrak{d}_{u_0}.$$

It remains to be proved that after possibly increasing K for every $k \geqslant K$ there exists a $\kappa \in \ker\mathfrak{d}_{u_0}$ with $|\kappa| < \delta_\kappa$ and $\lambda_{k;\kappa} = 0$. The following statement is a consequence of (5.38), (5.39), and the fact that $\mathfrak{r}_{\tilde u_{\sigma_k,\tau_k;J_k;\kappa}}$ depends smoothly on κ when interpreted as a family of operators on a fixed Banach space $L^p\Omega^{0,1}(\Sigma_{\sigma_k,\tau_k}, \tilde u^*_{\sigma_k,\tau_k;J_k;0}TX) \oplus \mathscr{O}$ using parallel transport along geodesics. If δ_κ is sufficiently small, then for every κ, u_k can be written in the form

$$u_k = \exp_{\tilde u_{\sigma_k,\tau_k;J_k,0}}\big(\mathrm{fuse}_{\tau_k}(\kappa + \lambda_{k;\kappa}) + \mathfrak{r}_{\tilde u_{\sigma_k,\tau_k;J_k;0}}\hat\eta_{k;\kappa} + \mathfrak{e}_{k;\kappa}\big),$$

with $\mathfrak{e}_{k;\kappa}$ satisfying $\limsup_{k\to\infty}\|\mathfrak{e}_{k;0}\|_{W^{1,p}} = 0$ and a quadratic estimate

$$\|\mathfrak{e}_{k;\kappa_1} - \mathfrak{e}_{k;\kappa_2}\|_{W^{1,p}} \leqslant c(|\kappa_1| + |\kappa_2|)|\kappa_1 - \kappa_2| \tag{5.40}$$

It follows from Proposition 5.27 that for $|\kappa| \leqslant \delta_\kappa$,

$$\kappa + \lambda_{k;\kappa} + \pi(\mathfrak{e}_{k;\kappa}) = 0,$$

where π denotes the projection on $\mathrm{fuse}_{\tau_k}(\ker\mathfrak{d}_{u_0})$ followed by $\mathrm{fuse}^{-1}_{\tau_k}$ (since fuse_{τ_k} is injective on $\ker\mathfrak{d}_{u_0}$ for k sufficiently large). Thus, the existence of a unique small κ such that $\lambda_{k;\kappa} = 0$ is a consequence of (5.40) and the Banach fixed point theorem applied to the map $\kappa \mapsto -\pi(\mathfrak{e}_{k;\kappa})$. ☐

5.7 The leading order term of the obstruction on ghost components

Assume the situation of Section 5.6. The purpose of this subsection is to analyze the leading order term of part of the obstruction map ob constructed in Section 5.6. This construction requires a choice of partition of S and a choice of lift $\mathscr{O} \subset L^p\Omega^{0,1}(\Sigma_0, u_0^*TX)$ of coker $\mathfrak{d}_{u_0}$. The following paragraphs introduce a particular choice tailored to the upcoming discussion.

Denote by

$$C_{\mathrm{ghost}} \subset \Sigma_0$$

a ghost component of u_0; see Section 2.4 for the definition of a ghost component and related notation. Set

$$C_\clubsuit := \Sigma_0 \backslash C_{\mathrm{ghost}},$$

and abbreviate

$$\nu_\clubsuit := \nu_{C_\clubsuit} \quad \text{and} \quad \nu_{\mathrm{ghost}} := \nu_{C_{\mathrm{ghost}}}.$$

Denote by

$$x_0 \in X$$

the constant value which u_0 takes on C_{ghost}.

The partition we choose is

$$S = S_1 \amalg S_2 \quad \text{with} \quad S_1 := S_{C_\clubsuit}^{\mathrm{int}} \amalg S_{C_{\mathrm{ghost}}}^{\mathrm{int}} \quad \text{and} \quad S_2 := S_{C_\clubsuit}^{\mathrm{ext}} \amalg S_{C_{\mathrm{ghost}}}^{\mathrm{ext}}.$$

The choice of $\mathscr{O}$ is slightly more involved. Denote by

$$W_\star^{1,p}\Gamma(C_\clubsuit, \nu_\clubsuit; u_0^*TX) \subset W^{1,p}\Gamma(C_\clubsuit, \nu_\clubsuit; u_0^*TX).$$

the subspace of those $\xi \in W^{1,p}\Gamma(C_\clubsuit, \nu_\clubsuit; u_0^*TX)$ such that the restriction of ξ to $S_{C_\clubsuit}^{\mathrm{ext}}$ is constant. Furthermore, denote the restriction of $\mathfrak{d}_{u_0|_{C_\clubsuit}}$ to this subspace by

$$\mathfrak{d}_{u_0,\star} \colon W_\star^{1,p}\Gamma(C_\clubsuit, \nu_\clubsuit; u_0^*TX) \to L^p\Omega^{0,1}(C_\clubsuit, u_0^*TX). \tag{5.41}$$

Let

$$\mathscr{O}_\star \subset \Omega^{0,1}(\Sigma_\clubsuit, u_0^*TX) \subset L^p\Omega^{0,1}(\Sigma_\clubsuit, u_0^*TX)$$

be a lift of coker $\mathfrak{d}_{u_0,\star}$ such that every $o \in \mathscr{O}_\star$ vanishes in a neighborhood of $S_{C_\clubsuit}^{\mathrm{ext}}$. Furthermore, let

$$\mathscr{O}_{\mathrm{ghost}} \subset \Omega^{0,1}(C_{\mathrm{ghost}}, \mathbf{C}) \otimes_{\mathbf{C}} T_{x_0}X \subset L^p\Omega^{0,1}(C_{\mathrm{ghost}}, u_0^*TX)$$

be a lift of coker$(\bar\partial \otimes_{\mathbf{C}} \mathbf{1})$.

Every $\xi \in W_\star^{1,p}\Gamma(C_\clubsuit, \nu_\clubsuit; u_0^*TX)$ can be extended to Σ_0 by making it take the same constant value on C_{ghost} which it takes onn $S_{C_\clubsuit}^{\mathrm{ext}}$. This defines an inclusion

$$W_\star^{1,p}\Gamma(C_\clubsuit, \nu_\clubsuit; u_0^*TX) \subset W^{1,p}\Gamma(\Sigma_0, \nu_0; u_0^*TX). \tag{5.42}$$

Furthermore, extension by zero defines inclusions

$$L^p\Omega^{0,1}(C_\clubsuit, u_0^*TX) \subset L^p\Omega^{0,1}(\Sigma_0, u_0^*TX) \quad \text{and} \quad L^p\Omega^{0,1}(C_{\mathrm{ghost}}, u_0^*TX) \subset L^p\Omega^{0,1}(\Sigma_0, u_0^*TX).$$

Set

$$\mathscr{O} \coloneqq \mathscr{O}_\star \oplus \mathscr{O}_{\mathrm{ghost}}.$$

Proposition 5.43. *The map (5.42) induces an isomorphism* $\ker \mathfrak{d}_{u_0,\star} \cong \ker \mathfrak{d}_{u_0}$ *and* $\mathscr{O}$ *is a lift of* $\operatorname{coker} \mathfrak{d}_{u_0}$.

Proof. Denote by $\nu_\amalg$ the nodal structure on Σ_0 which agrees with ν_0 on the complement of S_1 and is the identity on S_2. This nodal structure disconnects $C_\clubsuit$ and C_{ghost}. Denote by

$$\mathfrak{d}_{u_0,\amalg} \colon W^{1,p}\Gamma(\Sigma_0, \nu_\amalg; u_0^*TX) \to L^p\Omega^{0,1}(\Sigma_0, u^*TX)$$

the operator induced by $\mathfrak{d}_{u_0}$. Define V_- and $\operatorname{diff} \colon \ker \mathfrak{d}_{u_0,\amalg} \to V_-$ as in Remark 2.30 with S_2 instead of S. As is explained in Remark 2.30,

$$\ker \mathfrak{d}_{u_0} = \ker \operatorname{diff}$$

and there is a short exact sequence

$$0 \longrightarrow \operatorname{coker}\operatorname{diff} \longrightarrow \operatorname{coker}\mathfrak{d}_{u_0} \longrightarrow \operatorname{coker}\mathfrak{d}_{u_0,\amalg} \longrightarrow 0.$$

The domain and codomain of $\mathfrak{d}_{u_0,\amalg}$ decompose as

$$\begin{aligned} W^{1,p}\Gamma(\Sigma_0, \nu_\amalg; u_0^*TX) &= W^{1,p}\Gamma(C_\clubsuit, \nu_\clubsuit; u_0^*TX) \oplus W^{1,p}\Gamma(C_{\mathrm{ghost}}, \nu_{\mathrm{ghost}}; \mathbf{C}) \otimes_{\mathbf{C}} T_{x_0}X \quad \text{and} \\ L^p\Omega^{0,1}(\Sigma_0, u_0^*TX) &= L^p\Omega^{0,1}(C_\clubsuit, u_0^*TX) \oplus L^p\Omega^{0,1}(C_{\mathrm{ghost}}, \mathbf{C}) \otimes_{\mathbf{C}} T_{x_0}X. \end{aligned}$$

With respect to these decompositions

$$\mathfrak{d}_{u_0,\amalg} = \begin{pmatrix} \mathfrak{d}_{u_0,\clubsuit} & 0 \\ 0 & \bar\partial \otimes_{\mathbf{C}} \mathbf{1} \end{pmatrix}$$

with $\mathfrak{d}_{u_0,\clubsuit} \coloneqq \mathfrak{d}_{u_0|C_\clubsuit}$. Therefore,

$$\ker \mathfrak{d}_{u_0,\amalg} = \ker \mathfrak{d}_{u_0,\clubsuit} \oplus T_{x_0}X \quad \text{and} \quad \operatorname{coker}\mathfrak{d}_{u_0,\amalg} = \operatorname{coker}\mathfrak{d}_{u_0,\clubsuit} \oplus \operatorname{coker}(\bar\partial \otimes_{\mathbf{C}} \mathbf{1}).$$

The task at hand is to understand $\ker \mathfrak{d}_{u_0}$ and $\operatorname{coker}\mathfrak{d}_{u_0}$ in terms of the above.

Identifying

$$V_- = \operatorname{Map}(S_{C_\clubsuit}^{\mathrm{ext}}, T_{x_0}X)$$

the map $\operatorname{diff} \colon \ker \mathfrak{d}_{u_0,\clubsuit} \oplus T_{x_0}X \to V_-$ becomes

$$\operatorname{diff}(\kappa, v)(n) = \kappa(n) - v.$$

Set

$$V := V_- / T_{x_0} X$$

and denote by

$$\mathrm{ev}\colon \ \ker \mathfrak{d}_{u_0,\clubsuit} \to V$$

the evaluation map. The projection maps $\ker \mathfrak{d}_{u_0,\amalg} \to \ker \mathfrak{d}_{u_0,\clubsuit}$ and $V_- \to V$ induce isomorphisms

$$\ker \mathrm{diff} \cong \ker \mathrm{ev} \quad \text{and} \quad \operatorname{coker} \mathrm{diff} \cong \operatorname{coker} \mathrm{ev}.$$

Therefore,

$$\ker \mathfrak{d}_{u_0} \cong \ker \mathrm{ev}$$

and there is a short exact sequence

$$0 \longrightarrow \operatorname{coker} \mathrm{ev} \longrightarrow \operatorname{coker} \mathfrak{d}_{u_0} \longrightarrow \operatorname{coker} \mathfrak{d}_{u_0,\clubsuit} \oplus \operatorname{coker}(\bar\partial \otimes_{\mathbf{C}} \mathbf{1}) \longrightarrow 0.$$

The spaces $\ker \mathrm{ev}$ and $\operatorname{coker} \mathrm{ev}$ can be described in terms of the operator $\mathfrak{d}_{u_0,\star}$ given by (5.41). As in Remark 2.30,

$$\ker \mathfrak{d}_{u_0,\star} = \ker \mathrm{ev}$$

and there is a short exact sequence

$$0 \longrightarrow \operatorname{coker} \mathrm{ev} \longrightarrow \operatorname{coker} \mathfrak{d}_{u_0,\star} \longrightarrow \operatorname{coker} \mathfrak{d}_{u_0,\clubsuit} \longrightarrow 0.$$

The isomorphism $\ker \mathfrak{d}_{u_0,\star} \cong \ker \mathrm{ev} \cong \ker \mathfrak{d}_{u_0}$ is induced by (5.42) induces an isomorphism $\cong \ker \mathfrak{d}_{u_0}$. Furthermore, there is a short exact sequence

$$0 \longrightarrow \operatorname{coker} \mathfrak{d}_{u_0,\star} \longrightarrow \operatorname{coker} \mathfrak{d}_{u_0} \longrightarrow \operatorname{coker}(\bar\partial \otimes_{\mathbf{C}} \mathbf{1}) \longrightarrow 0.$$

This implies that $\mathscr{O}$ is a lift of $\operatorname{coker} \mathfrak{d}_{u_0}$. □

Construct the Kuranishi model as in Section 5.6 for the above choices of $S = S_1 \amalg S_2$ and $\mathscr{O}$.

As a final piece of preparation, let us make the following observation, which by Remark 2.30, in particular, gives an explicit description of $\mathscr{O}^*_{\mathrm{ghost}} = \operatorname{coker}(\bar\partial \otimes_{\mathbf{C}} \mathbf{1})^*$.

Proposition 5.44. *Let (C, ν) be a nodal Riemann surface with nodal set S. Denote the corresponding nodal curve by $\check{C}$ and its dualizing sheaf by $\omega_{\check{C}}$. Let $q \in (1,2)$ be such that $1/p + 1/q = 1$. The subspace*

$$\mathscr{H} \subset L^q\Omega^{0,1}(C, \mathbf{C})$$

of those $\bar\zeta$ which satisfy

$$\bar\partial^* \bar\zeta = \sum_{n \in S} f(n)\delta(n) \tag{5.45}$$

for some $f\colon S \to \mathbf{C}$ with $f \circ \nu = -f$ satisfies

$$\overline{\mathscr{H}} = H^0(\check{C}, \omega_{\check{C}}).$$

Proof. If $\check{C}$ is smooth, then $\check{C} = C$ and the dualizing sheaf ω_C is simply the canonical sheaf K_C. By the Kähler identities,

$$\begin{aligned}\overline{\mathscr{H}} &= \overline{\ker\bigl(\bar{\partial}^*\colon\ \Omega^{0,1}(C,\mathbf{C}) \to \Omega^0(C,\mathbf{C})\bigr)} \\ &= \overline{\ker\bigl(\partial\colon\ \Omega^{0,1}(C,\mathbf{C}) \to \Omega^{1,1}(C,\mathbf{C})\bigr)} \\ &\cong \ker\bigl(\bar{\partial}\colon\ \Omega^{1,0}(C,\mathbf{C}) \to \Omega^{1,1}(C,\mathbf{C})\bigr) \\ &\cong H^0(C, K_C).\end{aligned}$$

Recall from the proof of Proposition 2.20, that the dualizing sheaf of $\check{C}$ is constructed as follows; Denote by $\pi\colon\ C \to \check{C}$ the normalization map. Denote by $\tilde{\omega}_{\check{C}}$ the subsheaf of $K_C(S)$ whose sections ζ satisfy

$$\operatorname{Res}_n \zeta + \operatorname{Res}_{\nu(n)} \zeta = 0$$

for every $n \in S$, with $\operatorname{Res}_n \eta$ being the residue of the meromorphic 1–form η at n. The dualizing sheaf $\omega_{\check{C}}$ then is

$$\omega_{\check{C}} = \pi_* \tilde{\omega}_{\check{C}}.$$

Therefore, $H^0(\check{C}, \omega_{\check{C}}) = H^0(C, \tilde{\omega}_{\check{C}})$. By definition every $\zeta \in H^0(C, \tilde{\omega}_{\check{C}})$ is smooth away from S and blows-up at most like $1/\operatorname{dist}(n,\cdot)$ at n for $n \in S$; hence: $\zeta \in L^q\Omega^{0,1}(C,\mathbf{C})$. The residue condition amounts to (5.45). This shows that $H^0(\check{C}, \omega_{\check{C}}) \subset \mathscr{H}$. Conversely, by elliptic regularity every $\zeta \in \mathscr{H}$ defines an element of $H^0(\check{C}, \omega_{\check{C}})$. □

The following is the technical backbone of the proof of Theorem 1.1.

Lemma 5.46. *Denote by $\check{C}$ the nodal curve corresponding to $(C_{\mathrm{ghost}}, \nu_{\mathrm{ghost}})$. For every $L \in \mathbf{N}_0$ there is a constant $c > 0$ such that the obstruction map defined in Definition 5.35 satisfies the following. For every $(\sigma, \tau; J; \kappa) \in \Delta \times \mathscr{U} \times \mathscr{I}$, $\zeta \in H^0(\check{C}, \omega_{\check{C}})$, and $v \in T_{x_0}X$*

$$\bigl\langle \operatorname{pull}_{\sigma,\tau;J;\kappa}(\operatorname{ob}(\sigma,\tau;J;\kappa)), \operatorname{pull}_{\sigma,\tau;J;\kappa}(\bar{\zeta} \otimes_{\mathbf{C}} v)\bigr\rangle_{L^2} = \sum_{n \in S^{\mathrm{ext}}_{C_{\mathrm{ghost}}}} \pi\bigl\langle\bigl(\zeta \otimes_{\mathbf{C}} \mathrm{d}_{\nu_0(n)} u_{\sigma,\tau_1,0;J;\kappa}\bigr)(\tau_n), v\bigr\rangle + \mathfrak{e}$$

with

$$|\mathfrak{e}| \leqslant c\varepsilon_{\mathrm{ghost}}^{\frac{1}{2}} \sum_{n \in S^{\mathrm{ext}}_{C_{\mathrm{ghost}}}} \varepsilon_n$$

and

$$\varepsilon_{\mathrm{ghost}} \coloneqq \max\bigl\{\varepsilon_n : n \in S^{\mathrm{int}}_{C_{\mathrm{ghost}}} \cup S^{\mathrm{ext}}_{C_{\mathrm{ghost}}}\bigr\}.$$

Proof. The proof is based on analyzing the expression

$$0 = \langle \mathfrak{F}_{\tilde{u}_{\sigma,\tau;J;\kappa}}(\hat{\xi}(\sigma,\tau;J;\kappa)) + \operatorname{pull}_{\sigma,\tau;J;\kappa}(o(\sigma,\tau_1;J;\kappa) + \hat{o}(\sigma,\tau;J;\kappa)), \operatorname{pull}_{\sigma,\tau;J;\kappa}(\bar{\zeta} \otimes_{\mathbf{C}} v)\rangle_{L^2}$$

and the identity

$$\operatorname{ob}(\sigma,\tau;J;\kappa) \coloneqq o(\sigma,\tau_1;J;\kappa) + \hat{o}(\sigma,\tau;J;\kappa).$$

Step 1. *The vector field $\xi(\sigma, \tau_1; J; \kappa)$ is constant on C_{ghost} and $o(\sigma, \tau_1; J; \kappa)$ is supported on $C_\clubsuit$; in particular,*

$$\langle \mathrm{pull}_{\sigma,\tau;J;\kappa}(o(\sigma, \tau_1; J; \kappa)), \mathrm{pull}_{\sigma,\tau;J;\kappa}(\bar{\zeta} \otimes_{\mathbf{C}} v)\rangle_{L^2} = 0.$$

With only slight changes in notation the construction in Proposition 5.28 can be carried out for $u_0|_{C_\clubsuit}$ with $W^{1,p}_\star(C_\clubsuit, \nu_\clubsuit; u_0^*TX)$ instead of $W^{1,p}\Gamma(C_\clubsuit, \nu_\clubsuit; u_0^*TX)$ and with the choice of $\mathscr{O} = \mathscr{O}_\clubsuit$ and $S_2 = \varnothing$. For every $(\sigma, \tau_1, 0; J) \in \Delta \times \mathscr{U}$ and $\kappa \in \ker \mathfrak{d}_{u_0,\star}$ with $|\kappa| < \delta_\kappa$ denote by $\xi(\sigma, \tau_1; J; \kappa)$ and $o(\sigma, \tau_1; J; \kappa)$ the solution of (5.29) obtained in this way.

Henceforth, regard $\xi(\sigma, \tau_1; J; \kappa)$ as an element of $W^{1,p}\Gamma(\Sigma_0 \nu_0; u_0^*TX)$ and $o(\sigma, \tau_1; J; \kappa)$ as an element of $\mathscr{O}$. By construction these satisfy (5.29) for u_0 and with the choices of $\mathscr{O}$ and $S = S_1 \amalg S_2$ made in the discussion preceding Lemma 5.46. Therefore and since $\ker \mathfrak{d}_{u_0,\star} = \ker \mathfrak{d}_{u_0}$, $\xi(\sigma, \tau_1; J; \kappa)$ and $o(\sigma, \tau_1; J; \kappa)$ are precisely the output produced by Proposition 5.28. The first part of the assertion thus holds by construction.

Step 2. *The term*

$$\bar{\partial}_J(\tilde{u}_{\sigma,\tau;J;\kappa}, j_{\sigma,\tau}) + \mathrm{pull}_{\sigma,\tau;J;\kappa}(o(\sigma, \tau_1; J; \kappa))$$

is supported in the regions where $r_n \leqslant 2R_0$ for some $n \in S^{\mathrm{ext}}_{C_{\mathrm{ghost}}}$. Set $x_n \coloneqq u_{\sigma,\tau_1,0;J;\kappa}(n)$. Identifying U_{x_n} with $\tilde{U}_{x_n}$ via $\exp_{x_n}$ in the region where $r_n \leqslant 2R_0$ the error term can be written as

$$\bar{\partial}\chi^n_{\tau_2} \cdot u_{\sigma,\tau_1,0;J;\kappa} \circ \iota_{\tau_2} + \mathfrak{e}_1$$

with

$$|\bar{\partial}\chi^n_{\tau_2} \cdot u_{\sigma,\tau_1,0;J;\kappa} \circ \iota_{\tau_2}| \leqslant c\varepsilon_n \quad \textit{and} \quad |\mathfrak{e}_1| \leqslant c\varepsilon_n^2.$$

The proof is a refinement of that of Proposition 5.8. A priori, the error term $\bar{\partial}_J(\tilde{u}_{\sigma,\tau;J;\kappa}, j_{\sigma,\tau}) + \mathrm{pull}_{\sigma,\tau;J;\kappa}(o(\sigma, \tau_1; J; \kappa))$ is supported in the in the regions where $r_n \leqslant 2R_0$ for some $n \in S_2$. If $n \in S^{\mathrm{ext}}_{C_\clubsuit}$, then it is immediate from Definition 5.6 that $\tilde{u}_{\sigma,\tau;J;\kappa}$ agrees with $u_{\sigma,\tau_1,0;J;\kappa}$ in the region under consideration; hence, the error term vanishes. For $n \in S^{\mathrm{ext}}_{C_{\mathrm{ghost}}}$, in the region under consideration and with the identifications having been made,

$$\tilde{u}^\circ_{\sigma,\tau;J;\kappa} = \chi^n_{\tau_2} \cdot u_{\sigma,\tau_1,0;J;\kappa} \circ \iota_{\tau_2}.$$

Therefore,

$$\begin{aligned}\bar{\partial}_J(\tilde{u}^\circ_{\sigma,\tau;J;\kappa}, j_{\sigma,\tau}) = {} & \bar{\partial}\chi^n_{\tau_2} \cdot u_{\sigma,\tau_1,0;J;\kappa} \circ \iota_{\tau_2} \\ & + \underbrace{\chi^n_{\tau_2} \cdot \frac{1}{2}\big(J(\tilde{u}^\circ_{\sigma,\tau;J;\kappa}) - J(u_{\sigma,\tau_1,0;J;\kappa} \circ \iota_{\tau_2})\big) \circ \mathrm{d}(u_{\sigma,\tau_1,0;J;\kappa} \circ \iota_{\tau_2}) \circ j_{\sigma,\tau_1,0}}_{=:\mathrm{I}} \\ & + \underbrace{\chi^n_{\tau_2} \cdot \bar{\partial}_J(u_{\sigma,\tau_1,0;J;\kappa} \circ \iota_{\tau_2}, j_{\sigma,\tau_1,0})}_{=:\mathrm{II}}.\end{aligned}$$

The term I is supported in the region where $R_0 \leqslant r_n \leqslant 2R_0$. By Taylor expansion at $v_0(n)$, in this region

$$\begin{aligned} |u_{\sigma,\tau_1,0;J;\kappa} \circ \iota_{\tau_2}| &\leqslant c\varepsilon_n/r_n \quad \text{and} \\ |\mathrm{d}(u_{\sigma,\tau_1,0;J;\kappa} \circ \iota_{\tau_2})| &\leqslant c\varepsilon_n/r_n^2. \end{aligned}$$

Therefore,

$$|\mathrm{I}| \leqslant c\varepsilon_n^2 \quad \text{and} \quad |\bar\partial \chi_{\tau_2}^n \cdot u_{\sigma,\tau_1,0;J;\kappa} \circ \iota_{\tau_2}| \leqslant c\varepsilon_n.$$

Since ι_{τ_2} is holomorphic and $o(\sigma, \tau_1; \kappa)$ is defined by (5.29),

$$\mathrm{II} = \chi_{\tau_2}^n \cdot \iota_{\tau_2}^* \bar\partial_J(u_{\sigma,\tau_1,0;J;\kappa}, j_{\sigma,\tau_1,0}) = -\chi_{\tau_2}^n \cdot \iota_{\tau_2}^* o(\sigma, \tau_1; J; \kappa),$$

and thus II vanishes by our choice of $\mathscr{O}$.

Step 3. *For every* $n \in S_{C_{\text{ghost}}}^{\text{ext}}$

$$\langle \bar\partial \chi_{\tau_2}^n \cdot u_{\sigma,\tau_1,0;J;\kappa} \circ \iota_{\tau_2}, \operatorname{pull}_{\sigma,\tau;J;\kappa}(\bar\zeta \otimes_{\mathbf{C}} v)\rangle_{L^2} = -\pi\big\langle\big(\zeta \otimes_{\mathbf{C}} \mathrm{d}_{v_0(n)} u_{\sigma,\tau_1,0;J;\kappa}\big)(\tau_n), v\big\rangle + \mathfrak{e}_2$$

with

$$|\mathfrak{e}_2| \leqslant c\varepsilon_n^2|\zeta||v|.$$

To simplify the notation, we choose an identification $T_nC = \mathbf{C}$ and work in the canonical holomorphic coordinate z on C at n and the coordinate system at $v_0(n)$ with respect to which $w = \iota_{\tau_2}(z) = \varepsilon_n/z$. In particular, with respect to the induced identification $T_{v_0(n)}C = \mathbf{C}$ the gluing parameter is simply $\tau_n = \varepsilon_n \cdot 1 \otimes_{\mathbf{C}} 1$.

Since $u_{\sigma,\tau_1,0;J;\kappa}$ is J–holomorphic, by Taylor expansion,

$$u_{\sigma,\tau_1,0;J;\kappa}(\varepsilon_n/z) = \partial_w u_{\sigma,\tau_1,0;J;\kappa}(0) \cdot \varepsilon_n/z + \mathfrak{r} \quad \text{with} \quad |\mathfrak{r}| \leqslant c\varepsilon_n^2/|z|^2.$$

The term

$$\mathfrak{e}_2' \coloneqq \langle \bar\partial \chi_{\tau_2}^n \cdot \mathfrak{r}, \operatorname{pull}_{\sigma,\tau;J;\kappa}(\zeta \otimes_{\mathbf{C}} v)\rangle_{L^2}$$

satisfies

$$\left|\mathfrak{e}_2'\right| \leqslant c\varepsilon_n^2|\zeta||v|.$$

Since ζ is holomorphic,

$$\int_{S^1} \zeta(re^{i\alpha})\,\mathrm{d}\alpha = 2\pi \cdot \zeta(0).$$

Therefore,

$$\begin{aligned} \langle \bar\partial \chi_{\tau_2}^n \cdot z^{-1}, \bar\zeta\rangle_{L^2} &= \int_{R_0 \leqslant |z| \leqslant 2R_0} \frac{1}{2}\chi'\left(\frac{|z|}{R_0}\right)\frac{\zeta(z)}{R_0|z|}\,\mathrm{vol} \\ &= \int_{R_0}^{2R_0} \frac{1}{2}\chi'\left(\frac{r}{R_0}\right)\frac{1}{R_0} \cdot \left(\int_{S^1} \zeta(re^{i\alpha})\,\mathrm{d}\alpha\right)\mathrm{d}r \\ &= \int_{R_0}^{2R_0} \chi'\left(\frac{r}{R_0}\right)\frac{1}{R_0}\,\mathrm{d}r \cdot \pi\zeta(re^{i\alpha}). \end{aligned}$$

The integral evaluates to -1. Thus the assertion follows because the term $\langle \zeta(0) \cdot \partial_w u_{\sigma,\tau_1,0;J;\kappa}(0), v\rangle$ can be written in coordinate-free form as

$$\pi\Big\langle\big(\zeta \otimes_{\mathbf{C}} \mathrm{d}_{v_0(n)} u_{\sigma,\tau_1,0;J;\kappa}\big)(\tau_n), v\Big\rangle.$$

Step 4. *The term*

$$\mathfrak{e}_3 := \langle \mathfrak{d}_{\tilde{u}_{\sigma,\tau;J;\kappa}} \hat{\xi}(\sigma,\tau;J;\kappa) + \mathfrak{n}_{\tilde{u}_{\sigma,\tau;J;\kappa}}(\hat{\xi}(\sigma,\tau;J;\kappa)), \mathrm{pull}_{\sigma,\tau;J;\kappa}(\bar{\zeta} \otimes_{\mathbf{C}} v)\rangle_{L^2}$$

satisfies

$$|\mathfrak{e}_3| \leqslant c\varepsilon_{\mathrm{ghost}}^{\frac{1}{2}} \sum_{n \in S^{\mathrm{ext}}_{C_{\mathrm{ghost}}}} \varepsilon_n |\zeta||v|$$

By Step 2 and Proposition 5.32,

$$\|\hat{\xi}(\sigma,\tau;J;\kappa)\|_{W^{1,p}} \leqslant c\varepsilon_n.$$

This immediately implies that

$$\mathfrak{e}_3' := \langle \mathfrak{n}_{\tilde{u}_{\sigma,\tau;J;\kappa}}(\hat{\xi}(\sigma,\tau;J;\kappa)), \mathrm{pull}_{\sigma,\tau;J;\kappa}(\bar{\zeta} \otimes_{\mathbf{C}} v)\rangle_{L^2}$$

satisfies

$$|\mathfrak{e}_3'| \leqslant c \sum_{n \in S^{\mathrm{ext}}_{C_{\mathrm{ghost}}}} \varepsilon_n^2 |\zeta||v|.$$

It remains to estimate

$$\mathfrak{e}_3'' := \langle \mathfrak{d}_{\tilde{u}_{\sigma,\tau;J;\kappa}} \hat{\xi}(\sigma,\tau;J;\kappa), \mathrm{pull}_{\sigma,\tau;J;\kappa}(\bar{\zeta} \otimes_{\mathbf{C}} v)\rangle_{L^2}.$$

Set

$$C^{\circ}_{\sigma,\tau} := C_{\mathrm{ghost}} \cap \Sigma^{\circ}_{\sigma,\tau}.$$

Since κ and $\xi(\sigma,\tau_1;\kappa)$ are constant on C, Proposition 5.15 implies that the term

$$\mathfrak{d}_{\tilde{u}_{\sigma,\tau;J;\kappa}} \hat{\xi}(\sigma,\tau;J;\kappa) - \bar{\partial}\hat{\xi}(\sigma,\tau;J;\kappa),$$

defined over $C^{\circ}_{\sigma,\tau}$, is supported in the regions where $\varepsilon_n^{1/2} \leqslant r_n \leqslant 2R_0$ for some $n \in S^{\mathrm{ext}}_{C_{\mathrm{ghost}}}$ and satisfies

$$\begin{aligned}
&\sum_{n \in S^{\mathrm{ext}}_{C_{\mathrm{ghost}}}} \int_{\varepsilon_n^{1/2} \leqslant r_n \leqslant 2R_0} |\mathfrak{d}_{\tilde{u}_{\sigma,\tau;J;\kappa}} \hat{\xi}(\sigma,\tau;J;\kappa) - \bar{\partial}\hat{\xi}(\sigma,\tau;J;\kappa)| \\
&\quad\leqslant c \sum_{n \in S^{\mathrm{ext}}_{C_{\mathrm{ghost}}}} \int_{\varepsilon_n^{1/2} \leqslant r_n \leqslant 2R_0} |\hat{\xi}(\sigma,\tau;J;\kappa)||\nabla\hat{\xi}(\sigma,\tau;J;\kappa)| + |\hat{\xi}(\sigma,\tau;J;\kappa)|^2 \\
&\quad\leqslant c\|\hat{\xi}(\sigma,\tau;J;\kappa)\|^2_{W^{1,p}} \\
&\quad\leqslant c \sum_{n \in S^{\mathrm{ext}}_{C_{\mathrm{ghost}}}} \varepsilon_n^2.
\end{aligned}$$

Therefore,

$$|\mathfrak{e}_3''| \leqslant c \sum_{n \in S^{\mathrm{ext}}_{C_{\mathrm{ghost}}}} \varepsilon_n^2 + \left|\left\langle \bar{\partial}\hat{\xi}(\sigma,\tau;J;\kappa), \bar{\zeta} \otimes_{\mathbf{C}} v \right\rangle_{L^2(C^\circ_{\sigma,\tau})}\right|.$$

Since $\bar{\partial}^*\zeta = 0$ on $C^\circ_{\sigma,\tau}$, integration by parts yields

$$\begin{aligned}\left|\left\langle \bar{\partial}\hat{\xi}(\sigma,\tau;J;\kappa), \bar{\zeta} \otimes_{\mathbf{C}} v \right\rangle_{L^2(C^\circ_{\sigma,\tau})}\right| &\leqslant c\varepsilon_{\mathrm{ghost}}^{\frac{1}{2}} \sum_{n \in S^{\mathrm{ext}}_{C_{\mathrm{ghost}}}} \|\hat{\xi}(\sigma,\tau;J;\kappa)\|_{W^{1,p}} |\zeta||v| \\ &\leqslant c\varepsilon_{\mathrm{ghost}}^{\frac{1}{2}} \sum_{n \in S^{\mathrm{ext}}_{C_{\mathrm{ghost}}}} \varepsilon_n.\end{aligned}$$

Combining the above estimates yields the asserted estimate on $\mathfrak{e}_3$.

Step 5. *Conclusion of the proof.*

By Step 1, Step 2, Step 3, and Step 4 the term

$$\mathfrak{o} := \langle \mathrm{pull}_{\sigma,\tau;J;\kappa}(\mathrm{ob}(\sigma,\tau;J;\kappa)), \mathrm{pull}_{\sigma,\tau;J;\kappa}(\bar{\zeta} \otimes_{\mathbf{C}} v)\rangle_{L^2}$$

satisfies

$$\begin{aligned}\mathfrak{o} &= -\langle \mathfrak{F}_{\tilde{u}_{\sigma,\tau;J;\kappa}}(\hat{\xi}(\sigma,\tau;J;\kappa)), \mathrm{pull}_{\sigma,\tau;J;\kappa}(\bar{\zeta} \otimes_{\mathbf{C}} v)\rangle_{L^2} \\ &= \sum_{n \in S^{\mathrm{ext}}_{C_{\mathrm{ghost}}}} \pi\left\langle \left(\zeta \otimes_{\mathbf{C}} \mathrm{d}_{\nu_0(n)} u_{\sigma,\tau_1,0;J;\kappa}\right)(\tau_n), v\right\rangle + \mathfrak{e}_1 + \mathfrak{e}_2 + \mathfrak{e}_3\end{aligned}$$

with the error terms arising from the corresponding terms in the preceding steps. For example, $\mathfrak{e}_1$ arises from the L^2 inner product of the error term from Step 2 and $\mathrm{pull}_{\sigma,\tau;J;\kappa}(\bar{\zeta} \otimes_{\mathbf{C}} v)$. The preceding steps thus yield the asserted estimate on $\mathfrak{e} = \mathfrak{e}_1 + \mathfrak{e}_2 + \mathfrak{e}_3$. ☐

5.8 Proof of Theorem 1.1

Without loss of generality, $(\Sigma_k, j_k) = (\Sigma_{\sigma_k,\tau_k}, j_{\sigma_k,\tau_k})$ with $\tau_{k;n} \neq 0$ for every $k \in \mathbf{N}$ and $n \in S$; in particular, ν_{σ_k,τ_k} is the trivial nodal structure. By Proposition 5.37 there is a sequence $(\kappa_k)_{\kappa \in \mathbf{N}}$ in $\mathscr{I}$ corresponding to $(u_k)_{k \in \mathbf{N}}$.

Again, without loss of generality, $u_\infty \colon (\Sigma_\infty, j_\infty, \nu_\infty) \to (X, J_\infty)$ has at least one ghost component $(C_{\mathrm{ghost}}, \nu_{\mathrm{ghost}})$ with precisely one non-ghost component attached; that is, $S^{\mathrm{ext}}_{C_{\mathrm{ghost}}} = \{n\}$. Since u_∞ is stable, $(C_{\mathrm{ghost}}, \nu_{\mathrm{ghost}})$ becomes stable after adding n as a marked point. In particular, the nodal curve $\check{C}$ corresponding to $(C_{\mathrm{ghost}}, \nu_{\mathrm{ghost}})$ is semistable and, hence, $\omega_{\check{C}}$ is base-point free by Proposition 2.20. Therefore, there exists a $\zeta \in H^0(\check{C}, \omega_{\check{C}})$ with $\zeta(n) \neq 0$.

Since $\mathrm{ob}(\sigma_k, \tau_k; J_k; \kappa_k) = 0$, it follows from Lemma 5.46 that for ζ as above and every $v \in T_{x_0}X$ that

$$|\mathrm{d}_{\nu_\infty(n)} u_{\sigma,\tau_1,0;J;\kappa}(\tau_n)| \leqslant \varepsilon_{k;\mathrm{ghost}}^{1/2}.$$

Passing to the limit as k tends to ∞ yields that $\mathrm{d}_{\nu_\infty(n)} u_0 = 0$.

6 Calabi–Yau classes in symplectic 6–manifolds

6.1 Proof Theorem 1.3

Denote by $C_1, \ldots, C_I$ the connected components of Σ_∞ on which u_∞ is non-constant and set $u_\infty^i \coloneqq u_\infty|_{C_i}$ and $A_i \coloneqq (u_\infty^i)_*[C_i]$. By the index formula (2.29),

$$\sum_{i=1}^{I} \operatorname{index}(u_\infty^i) = \sum_{i=1}^{I} 2\langle c_1(X,\omega), A_i\rangle = 2\langle c_1(X,\omega), A\rangle = 0$$

Since $J_\infty \in \mathscr{J}_{\mathrm{emb}}(X,\omega)$, we have $\operatorname{index}(u_\infty^i) \geqslant 0$ for all i and so every u_∞^i has index zero. Consequently, the images of (the simple maps underlying) u_∞^i and u_∞^j either agree or are disjoint. However,

$$\operatorname{im} u_\infty = \bigcup_{i=1}^{I} \operatorname{im} u_\infty^i$$

is connected. Therefore and since A is primitive, $I = 1$ and u_∞^1 is simple and, hence, an embedding because $J \in \mathscr{J}_{\mathrm{emb}}(X,\omega)$. Given the above, it follows from Theorem 1.1 that $(\Sigma_\infty, j_\infty, \nu_\infty)$ is smooth.

6.2 Proof of Theorem 1.5

The proof of Theorem 1.5(1) is completely standard and straightforward. Nevertheless, let us spell it out. Let $J \in \mathscr{J}_{\mathrm{emb}}^\star(X,\omega)$. By Proposition 2.38 and Theorem 1.3, $\mathscr{M}_{A,g}^\star(X,J)$ is a compact oriented zero-dimensional manifold; that is: finite set of points with signs. The signed count

$$\#\mathscr{M}_{A,g}^\star(X,J)$$

is independent of the choice of J. To see this, let $J_0, J_1 \in \mathscr{J}_{\mathrm{emb}}^\star(X,\omega)$ and $(J_t)_{t\in[0,1]} \in \mathscr{J}_{\mathrm{emb}}^\star(X,\omega;J_0,J_1)$. By Proposition 2.38 and Theorem 1.3, $\mathscr{M}_{A,g}^\star\big(X,(J_t)_{t\in[0,1]}\big)$ is a compact oriented manifold with boundary

$$\mathscr{M}_{A,g}^\star(X,J_1) \amalg -\mathscr{M}_{A,g}^\star(X,J_0).$$

Therefore,

$$\#\mathscr{M}_{A,g}^\star(X,J_0) = \#\mathscr{M}_{A,g}^\star(X,J_0).$$

Theorem 1.5(2) follows from [DW18a, Theorem 1.6]. Indeed, the latter asserts that for every $J \in \mathscr{J}_\star(X,\omega)$ the set

$$\coprod_{g=0}^{\infty} \mathscr{M}_{A,g}^\star(X,J)$$

is a finite set. Therefore, there exists a $g_0 \in \mathbf{N}_0$ such that for every $g \geqslant g_0$ the moduli space $\mathscr{M}_{A,g}^\star(X,A;J)$ is empty; in particular, $n_{A,g}(X,\omega) = 0$ for $g \geqslant g_0$. □

7 Fano classes in symplectic 6–manifolds

The proofs in this section make use of definitions and results from Section 2.8.

7.1 Proof of Theorem 1.4

Denote by $\tilde{C}_1, \dots, \tilde{C}_I$ the connected components of Σ_∞ on which u_∞ is non-constant, set $\tilde{u}^i_\infty := u_\infty|_{\tilde{C}_i}$, denote by $u^i_\infty \colon C_i \to X$ the simple map underlying $\tilde{u}^i_\infty$, let $d_i \in \mathbf{N}$ be the degree of the covering map relating $\tilde{u}^i_\infty$ and u^i_∞, set $A_i := (u^i_\infty)_*[C_i]$, and set $g_i := g(C_i)$.

Denote by I_0 the subset of those $i \in \{1, \dots, I\}$ with $\langle c_1(X,\omega), A_i\rangle = 0$ and set $I_+ := \{1, \dots, I\}\backslash I_0$. Without loss of generality all of the pseudo-cycles f_λ have $\operatorname{codim}(f_\lambda) \geqslant 4$. For every $i \in I_+$ denote by Λ_i the subset of those $\lambda \in \{1, \dots, \Lambda\}$ such that

$$\operatorname{im} u^i_\infty \cap \overline{\operatorname{im} f_\lambda} \neq \varnothing. \tag{7.1}$$

Since $\mathscr{J}_{\mathrm{emb}}(X,\omega; f_1, \dots, f_\Lambda)$, for every $i \in I_0$ and $\lambda \in \{1, \dots, \Lambda\}$, $\operatorname{im} u^i_\infty \cap \overline{\operatorname{im} f_\lambda} = \varnothing$. Therefore and since $u_k(\Sigma_k)$ converges to $u_\infty(\Sigma_\infty)$ in the Hausdorff topology, for every $\lambda \in \{1, \dots, \Lambda\}$ there exists at least one $i \in I_+$ such that (7.1). For every $i \in \{1, \dots, I\}$ and $\lambda \in \Lambda_i$ set

$$f^i_\lambda := \begin{cases} f_\lambda & \text{if } \operatorname{im} u^i_\infty \cap \operatorname{im} f_\lambda \neq \varnothing \\ f^\partial_\lambda & \text{otherwise} \end{cases}$$

with f^∂_λ as in in Section 2.8; in particular: $\operatorname{codim} f_\lambda \leqslant \operatorname{codim} f^i_\lambda$ with equality if and only if $\operatorname{im} u^i_\infty \cap \operatorname{im} f_\lambda \neq \varnothing$. By definition, u^i_∞ represents an element of $\mathscr{M}^\star_{A,g}(X, J; (f^i_\lambda)_{\lambda\in\Lambda_i})$. Therefore and since $J \in \mathscr{J}_{\mathrm{emb}}(X,\omega, f_1, \dots, f_\Lambda)$,

$$2\langle c_1(X,\omega), A_i\rangle - \sum_{\lambda\in\Lambda_i} \bigl(\operatorname{codim}(f^i_\lambda) - 2\bigr) \geqslant 0.$$

On the one hand, multiplying by d_i and summing yields

$$\begin{aligned} \sum_{i\in I_+} \sum_{\lambda\in\Lambda_i} \bigl(\operatorname{codim}(f^i_\lambda) - 2\bigr) &\leqslant \sum_{i\in I_+} \sum_{\lambda\in\Lambda_i} d_i \bigl(\operatorname{codim}(f^i_\lambda) - 2\bigr) \\ &\leqslant \sum_{i=1}^{I} 2\langle c_1(X,\omega), d_i A_i\rangle \\ &= 2\langle c_1(X,\omega), A\rangle \\ &= \sum_{\lambda=1}^{\Lambda} (\operatorname{codim}(f_\lambda) - 2) \end{aligned}$$

On the other hand, by the preceding discussion, the reverse inequality also holds. Therefore, equality holds and this implies that

1. $d_i = 1$ for every $i \in I_+$,
2. $2\langle c_1(X,\omega), A_i\rangle = \sum_{\lambda\in\Lambda_i}\big(\operatorname{codim}(f^i_\lambda) - 2\big)$,
3. $f^i_\lambda = f_\lambda$, and
4. the subsets Λ_i are non-empty and pairwise disjoint.

This implies that for every $i \in I_+$ the map $\tilde{u}^i_\infty$ agrees with u^i_∞ and thus is simple; moreover, it is of index zero and its image intersects f_λ for every $\lambda \in \Lambda_i$. Furthermore, every f_λ intersects image of precisely one map u^i_∞ with $i \in I_+$. Therefore, the images of the maps u^i_∞ with $i \in I_+$ are pairwise disjoint.

Since $2\langle c_1(X,\omega), A\rangle > 0$, I_+ is non-empty. For $i \in I_+$ and $j \in I_0$ the images of u^i_∞ and u^j_∞ must also be disjoint, because once they intersect they have to agree but $A_i \neq A_j$. However,

$$\operatorname{im} u_\infty = \bigcup_{i=1}^{I} \operatorname{im} u^i_\infty$$

is connected. Therefore, if $I_0 \neq \varnothing$, then there is at least $i \in I_0$ and $j \in I_+$ such that the images of u^i_∞ and u^j_∞ intersect. The preceding discussion shows this to be impossible; hence: $I_0 = \varnothing$. Similarly, if I_+ were to contain more than one element, then there are $i, j \in I_+$ with such that the images of u^i_∞ and u^j_∞ intersect—which is impossible. Therefore, $I = 1$ and $\tilde{u}^1_\infty = u^1_\infty$ is an embedding.

Given the above, it follows from Theorem 1.1 that $(\Sigma_\infty, j_\infty, \nu_\infty)$ is smooth and $\operatorname{im} u_\infty \cap \operatorname{im} f_\lambda \neq \varnothing$ every $\lambda = 1, \dots, \Lambda$. □

7.2 Proof of Theorem 1.8

Given Gromov compactness, Theorem 1.4, and Proposition 2.46, the proof that $n_{A,g}(X,\omega;\gamma_1,\dots,\gamma_\Lambda)$ is well-defined and independent of the choice of J is identical to that of Theorem 1.5.

To prove that $n_{A,g}(X,\omega;\gamma_1,\dots,\gamma_\Lambda)$ is independent of the choice of pseudo-cycle representatives, suppose that f^0_1 and f^1_1 are two representatives of $\mathrm{PD}[\gamma_1]$ such that $f^i_1, f_2, \dots, f_\Lambda$ are in general position for $i = 0, 1$. Let $F\colon W \to X$ be a pseudo-cycle cobordism between f^0_1 and f^1_1 such that $F, f_2, \dots, f_\Lambda$ are in general position. Let J be an element of the set $\mathcal{J}^\star_{\mathrm{emb}}(X,\omega;F,f_2,\dots,f_\Lambda)$ defined in Definition 2.50, which exists by Proposition 2.51. It follows that $\mathcal{M}^\star_{A,g}(X,J;f^0_1,\dots,f_\Lambda)$ and $\mathcal{M}^\star_{A,g}(X,J;f^1_1,\dots,f_\Lambda)$ are finite sets of points with orientations and $\mathcal{M}^\star_{A,g}(X,J;F,f_2,\dots,f_\Lambda)$ is an oriented 1–dimensional cobordism between them. This cobordism is compact by Gromov's compactness and the argument used in the proof in Theorem 1.4. Thus,

$$\#\mathcal{M}^\star_{A,g}(X,J;f^0_1,\dots,f_\Lambda) = \#\mathcal{M}^\star_{A,g}(X,J;f^1_1,\dots,f_\Lambda).$$

To prove that $n_{A,g}(X,\omega;\gamma_1,\dots,\gamma_\Lambda) = 0$ for $g \gg 1$ we show that, given $J \in \mathcal{J}^\star_{\mathrm{emb}}(X,\omega,f_1,\dots,f_\Lambda)$, there are only finitely many distinct J–holomorphic curves in X representing A and intersecting $\operatorname{im} f_\lambda$, of arbitrary genus. Here a J**–holomorphic curve** is the image of a simple J–holomorphic map. The proof is a minor variation of the proof of [DW18a, Theorem 1.6]. Suppose, by contradiction, that there are infinitely many such curves C_k. Considering them as **J–holomorphic cycles**, we can pass to a subsequence which **converges geometrically** to a J–holomorphic cycle $C_\infty = \sum_{i=1}^{I} m_i C_\infty^i$, see [DW18a, Definition 4.1, Definition 4.2, Lemma 1.9]. Here $m_i > 0$ and each C_∞^i is a J–holomorphic curve. Geometric convergence implies that C_k converges to C_∞ in the Hausdorff distance and

$$\sum_{i=1}^{I} m_i [C_\infty^i] = A.$$

The argument from the proof Theorem 1.4 we shows that

1. $m_i = 1$ for all i,
2. C_∞ has only one connected component,
3. $[C_\infty] = A$,
4. C_∞ intersects all $\operatorname{im} f_\lambda$, and consequently
5. C_∞ is embedded and unobstructed by the condition $J \in \mathscr{J}^{\star}_{\mathrm{emb}}(X, \omega; f_1, \dots, f_\Lambda)$.

We will now adapt the rescaling argument from the proof of [DW18a, Proposition 5.1]—originally due to Taubes in the 4–dimensional setting [Tau96]—to the present situation. Let $N \to C_\infty$ be the normal bundle of C_∞ in X. Identify a neighborhood of C_∞ with a neighborhood of the zero section in N using the exponential map. For $k \gg 1$, C_k is contained in that neighborhood and by abuse of notation we will consider C_k as an $\exp^* J$–holomorphic curve in N and f_λ as maps to N.

Since C_k are distinct, $C_k \neq C_\infty$. For $\varepsilon > 0$ denote by $\sigma_\varepsilon \colon N \to N$ the map which rescales the fibers by ε. Let (ε_k) be a sequence of positive numbers converging to zero and such that the rescaled sequence

$$\tilde{C}_k \coloneqq (\sigma_{\varepsilon_k})^{-1}(C_k)$$

satisfies

$$d_H(\tilde{C}_k, C_\infty) = 1,$$

where d_H is the Hausdorff distance. The curves $\tilde{C}_k$ are J_k–holomorphic where $J_k \coloneqq \sigma_{\varepsilon_k}^* \exp^* J$. The sequence of rescaled almost complex structures (J_k) converges to an almost complex structure J_∞ which is tamed by a symplectic form [DW18a, Proposition 3.10]. In the same way as in the proof of [DW18a, Proposition 5.1] we conclude that the sequence $(\tilde{C}_k)$ converges geometrically to a J_∞–holomorphic cycle whose support is a union of J_∞–holomorphic curves $\tilde{C}_\infty \subset N$ satisfying

$$d_H(\tilde{C}_\infty, C_\infty) = 1.$$

Since $[\tilde{C}_k] = [C_\infty] = A$ for all k, and the bundle projection $\pi \colon N \to C_\infty$ is J_∞–holomorphic, π induces an isomorphism $\tilde{C}_\infty \cong C_\infty$. Let $\iota \colon C_\infty \to X$ be the inclusion map and $\mathfrak{d}_\iota$ is the deformation operator corresponding to ι, as in Definition 2.26. By [DW18a, Proposition 3.12], $\tilde{C}_\infty$ is the graph of a non-zero section $\xi \in \Gamma(C_\infty, N) \subset \Gamma(C_\infty, \iota^*TX)$ satisfying $\mathfrak{d}_\iota \xi = 0$

For every $\lambda = 1, \dots, \Lambda$, let V_λ be the domain of f_λ, and let $z_{\lambda,k} \in C_k$ and $x_{\lambda,k} \in V_\lambda$ be such that $z_{\lambda,k} = f_\lambda(x_{\lambda,k})$. After passing to a subsequence, we may assume that

$$\lim_{k\to\infty} z_{\lambda,k} = z_\lambda \in C_\infty \quad \text{and} \quad \lim_{k\to\infty} x_{\lambda,k} = x_\lambda \in V_\lambda,$$

and $z_\lambda = f_\lambda(x_\lambda)$. Set $\tilde{z}_{\lambda,k} := \sigma_{\varepsilon_k}^{-1}(z_{\lambda,k})$. After possibly passing to a further subsequence,

$$\lim_{k\to\infty} \tilde{z}_{\lambda,k} = \xi(z_\lambda). \tag{7.2}$$

Let $\mathrm{pr}_N \mathrm{d}_{x_\lambda} f_\lambda \colon T_{x_\lambda} V_\lambda \to N_{z_\lambda}$ be the projection of the derivative of f_λ at x_λ on $N_{z_\lambda} \subset T_{z_\lambda} X$. We will show that for every λ there exists $w_\lambda \in T_{x_\lambda} V_\lambda$ such that $\lim_{k\to\infty} \tilde{z}_{\lambda,k} = \mathrm{pr}_N \mathrm{d}_{x_\lambda} f_\lambda \cdot w_\lambda$.

Let g be the genus of C_∞, so that the embedding $\iota\colon C_\infty \to X$ corresponds to an element in $\mathscr{M}^\star_{A,g,\Lambda}(X,J)$. Since $J \in \mathscr{J}^\star_{\mathrm{emb}}(X,\omega,f_1,\dots,f_\Lambda)$, the following maps:

1. the derivative of $\mathrm{ev}_\Lambda \colon \mathscr{M}^\star_{A,g,\Lambda}(X,J) \to X^\Lambda$ at $[\iota, z_1, \dots, z_\Lambda]$, and
2. the derivative of $\prod_{\lambda=1}^\Lambda f_\lambda \colon \prod_{\lambda=1}^\Lambda V_\lambda \to X^\Lambda$ at $\prod_{\lambda=1}^\Lambda x_\lambda$

are transverse to each other. Since

$$\dim \mathscr{M}^\star_{A,g,\Lambda}(X,J) + \sum_{\lambda=1}^{\Lambda} \dim V_\lambda = \Lambda \dim X,$$

the images of these two maps intersect trivially. In particular, $\mathrm{pr}_N \mathrm{d}_{x_\lambda} f_\lambda$ is injective for every λ, as otherwise there would exist $v \in T_{z_\lambda} C_\infty$ and $w \in T_{x_\lambda} V_\lambda$ for some λ such that

$$\mathrm{d}_{z_\lambda} \iota \cdot v = \mathrm{d}_{x_\lambda} f_\lambda \cdot w,$$

violating the above transversality condition. Fix a trivialization of N in a neighborhood of z_λ and a chart centered at x_λ in V_λ. Denoting by pr_N the projection on the fiber N_{z_λ} in the given trivialization, the Taylor expansion gives us

$$\mathrm{pr}_N z_{\lambda,k} = \mathrm{pr}_N f_\lambda(x_{\lambda,k}) = \mathrm{pr}_N \mathrm{d}_{x_\lambda} f_\lambda (x_{\lambda,k} - x_\lambda) + O(|x_{\lambda,k} - x_\lambda|^2).$$

Since $\mathrm{pr}_N \mathrm{d}_{x_\lambda} f_\lambda$ is injective, there is a constant $c > 0$ such that

$$|x_{\lambda,k} - x_\lambda| \leqslant c |\mathrm{pr}_N z_{\lambda,k}| \leqslant c\varepsilon_k.$$

Thus, after passing to a subsequence, we may assume that the sequence $\varepsilon_k^{-1}(x_{\lambda,k} - x_\lambda)$ converges to a limit $w_\lambda \in T_{x_\lambda} V_\lambda$. By construction,

$$\lim_{k\to\infty} \tilde{z}_{\lambda,k} = \lim_{k\to\infty} \mathrm{pr}_N \tilde{z}_{\lambda,k} = \mathrm{pr}_N \mathrm{d}_{x_\lambda} f_\lambda \cdot w_\lambda.$$

Comparing this equation with (7.2), we see that for every λ there exists $v_\lambda \in T_{z_\lambda} C_\infty$ such that

$$\xi(z_\lambda) + \mathrm{d}_{z_\lambda} \iota \cdot v_\lambda = \mathrm{d}_{x_\lambda} f_\lambda \cdot w_\lambda.$$

Since $\xi \neq 0$, this violates the condition that the images of the maps (1) and (2) intersect trivially. The contradiction shows that the sequence (C_k) cannot exist. □

A Pseudocycles

Given a collection of homology classes, we are interested in counting J–holomorphic maps passing through cycles representing these classes. Since not every homology class is represented by a map from a manifold, it is convenient to use the language of pseudocycles. We briefly review the theory of pseudocycles below; for details, see [MS12, Section 6.5; Sch99; Kah01; Zin08].

Definition A.1.

1. A subset of a smooth manifold X is said to have **dimension at most** k if it is contained in the image of a smooth map from a smooth k–dimensional manifold.[1]
2. A k–**pseudocycle** is a smooth map $f\colon V \longrightarrow X$ from an oriented k–dimensional manifold V such that the closure $\overline{f(V)}$ is compact and the **boundary of** f, defined by
$$\mathrm{bd}(f) := \bigcap_{K\subset V \text{ compact}} \overline{f(V-K)},$$
has dimension at most $k-2$. We will use notation
$$\mathrm{codim}(f) := \dim(X) - \dim(V).$$
3. Two k–pseudocycles $f_i\colon V_i \longrightarrow X$, for $i = 0, 1$, are **cobordant** if there exists a smooth, oriented $(k+1)$–dimensional manifold with boundary W and a smooth map $F\colon W \longrightarrow X$ such that $\overline{F(W)}$ is compact, $\mathrm{bd}(F)$ has dimension at most $k-1$, and
$$\partial W = V_1 \sqcup -V_0 \quad \text{and} \quad F|_{V_1} = f_1, \quad F|_{V_0} = f_0.$$
4. Denote by $H_k^{\text{pseudo}}(X)$ the set of equivalence classes of k–pseudocycles up to cobordism. The disjoint union operation endows $H_k^{\text{pseudo}}(X)$ with the structure of an abelian group.

There is a natural isomorphism $H_*^{\text{pseudo}}(\cdot) \cong H_*(\cdot, \mathbf{Z})$ as functors from the category of smooth manifolds to the category of $\mathbf{Z}$–graded abelian groups [Sch99; Kah01; Zin08]. In what follows we will use this isomorphism to identify these two homology theories and represent any class in $H_*(X, \mathbf{Z})$ by a pseudocycle.

Proposition A.2. *If $f\colon V \longrightarrow X$ be a k–pseudocycle and $g\colon W \longrightarrow X$ is an l–pseudocycle, then $f \times g\colon V \times W \longrightarrow X$ is an $(k+l)$–pseudocycle.*

Proof. Since
$$\overline{(f\times g)(V\times W)} = \overline{f(V)} \times \overline{g(W)},$$
the set $\overline{(f\times g)(V\times W)}$ is compact. Moreover,
$$\mathrm{bd}(f\times g) = \left(\mathrm{bd}(f) \times \overline{g(W)}\right) \cup \left(\overline{f(V)} \times \mathrm{bd}(g)\right),$$
so $\mathrm{bd}(f\times g)$ can be covered by images of maps from manifolds of dimension $k+l-2$. □

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[1] All manifolds are assumed to be Hausdorff, paracompact, and without boundary unless said otherwise.

Definition A.3. Let M be a smooth manifold and let $g\colon M \to X$ be a smooth map. We say that a k–pseudocycle $f\colon V \to X$ is **transverse as a pseudocycle to g**[2] if

1. there exists a smooth manifold V^∂ of dimension $\dim V^\partial \leqslant \dim V - 2$ and a smooth map $f^\partial\colon V^\partial \to X$ such that $\operatorname{bd}(f) \subset \operatorname{im} f^\partial$,
2. f and f^∂ are transverse to g as smooth maps from manifolds.

If W is a manifold with boundary ∂W, we require additionally that f is transverse as a pseudocycle to $g|_{\partial W}\colon \partial W \to X$.

Similarly, if M is a manifold without boundary and $F\colon W \to X$ is a cobordism between two pseudocycles f_0 and f_1, we say that F is **transverse as a pseudocycle cobordism to g** if

1. there exists a smooth manifold with boundary W^∂ of dimension $\dim W^\partial \leqslant \dim W - 2$ and a smooth map $F^\partial\colon W^\partial \to X$ such that $\operatorname{bd}(F) \subset \operatorname{im} F^\partial$ and $\operatorname{bd}(f_i) \subset \operatorname{im} F^\partial|_{\partial W^\partial}$ for $i = 0, 1$,
2. F and F^∂ are transverse to g as smooth maps from manifolds with boundary.

Definition A.4. Let $(f_\lambda\colon V_\lambda \to X)$ be a collection of pseudocycles indexed by a finite ordered set I. We say that $(f_\lambda)_{\lambda\in I}$ are **in general position** if for every ordered subset $S \subset I$, the pseudocycle

$$\prod_{\lambda\in S} f_\lambda\colon \prod_{\lambda\in S} V_\lambda \to X^{|S|}$$

is transverse as a pseudocycle to the diagonal $X \hookrightarrow X^{|S|}$.

Similarly, if one of f_λ is a cobordism between pseudocycles, then so is $\prod_{\lambda\in S} f_\lambda$ and we require that it is transverse to the diagonal as pseudocycle cobordism.

Proposition A.5. *Given a finite collection of pseudocycles $(f_\lambda\colon V_\lambda \to X)_{\lambda\in I}$, the set*

$$\{(\phi_\lambda)_{\lambda\in I} \in \operatorname{Diff}(X)^{|I|} : (\phi_\lambda \circ f_\lambda)_{\lambda\in I} \text{ are in general position}\}$$

is residual in $\operatorname{Diff}(X)^{|I|}$.

Proof. The proof is similar to that of [MS12, Lemma 6.5.5]. Let us work with the group $\operatorname{Diff}_k(X)$ of C^k diffeomorphism for any integer $k \geqslant 1$; the corresponding statement for $\operatorname{Diff}(X)$ follows then using standard arguments [MS12, pp. 52–54, Remark 3.2.7]. A countable intersection of residual sets is residual, so, without loss of generality, consider the case $S = I$. Consider a map

$$\mathscr{F}\colon \operatorname{Diff}_k(X)^{|I|} \times \prod_{\lambda\in I} V_\lambda \to X^{|I|}$$
$$\mathscr{F}((\phi_\lambda)_{\lambda\in I}, (x_\lambda)_{\lambda\in I}) \coloneqq (\phi_\lambda \circ f_\lambda(x_\lambda))_{\lambda\in I}.$$

---

[2] McDuff and Salamon [MS12, Definition 6.5.10] use the term **weakly transverse**, which we prefer to avoid, regarding that this notion of transversality is stronger than the transversality of f and g as smooth maps in the usual sense.

Let $\Delta \subset X^{|I|}$ be the diagonal. If we show that $\mathscr{F}$ is transverse to Δ, it follows from the Sard–Smale theorem that for all $(\phi_\lambda)_{\lambda \in I}$ from a residual subset of $\mathrm{Diff}_k(X)$ the maps $\prod \phi_\lambda \circ f_\lambda$ is transverse to Δ. (The same argument can be applied to f_λ^∂ to conclude transversality as pseudocycles.) In fact, the derivative of $\mathscr{F}$ is surjective at every point $\mathbf{x} = ((\phi_\lambda)_{\lambda \in I}, (x_\lambda)_{\lambda \in I})$. Without loss of generality suppose that $\phi_\lambda = \mathrm{id}$ for all $\lambda \in I$. Let $\mathrm{Vect}_k(X)$ denote the space of C^k vector fields on X. Given

$$\boldsymbol{\xi} = (\xi_\lambda)_{\lambda \in I} \in \prod_{\lambda \in I} T_{\mathrm{id}} \,\mathrm{Diff}_k(X) = \prod_{\lambda \in I} \mathrm{Vect}_k(X),$$

we have

$$\mathrm{d}_{\mathbf{x}} \mathscr{F}(\boldsymbol{\xi}) = (\xi_\lambda(f_\lambda(x_\lambda)))_{\lambda \in I} \in \prod_{\lambda \in I} T_{f_\lambda(x_\lambda)} X.$$

Since for every $p \in X$ the evaluation map $\mathrm{Vect}(X) \longrightarrow T_p X$ is surjective, the map $\mathrm{d}_{\mathbf{x}} \mathscr{F}$ is surjective, which finishes the proof. □

B Proof of $n_{A,g} = \mathrm{BPS}_{A,g}$

In this section, we outline Zinger's proof that for a primitive Calabi–Yau class

$$n_{A,g}(X, \omega) = \mathrm{BPS}_{A,g}(X, \omega),$$

where $\mathrm{BPS}_{A,g}(X, \omega)$ is the Gopakumar–Vafa invariant defined in terms of the Gromov–Witten invariants via (1.11). We use the same notation as in the proof of Theorem 1.5.

Given $J \in \mathscr{J}^\star_{\mathrm{emb}}(X, \omega)$, every stable J–holomorphic map of arithmetic genus h factors through a J–holomorphic embedding from a smooth domain of genus $g \leqslant h$. In other words, every element of $\overline{\mathscr{M}}_{A,h}(X, J)$ is of the form $[u \circ \varphi]$ for some $[u] \in \mathscr{M}^\star_{A,g}(X, J)$ with $g \leqslant h$, and $[\varphi] \in \overline{\mathscr{M}}_{[\Sigma],h}(\Sigma, j)$. Here (Σ, j) is the domain of u. Denote by $(\tilde{\Sigma}, \tilde{\nu}, \tilde{j})$ the domain of φ. Given such J–holomorphic maps, let N be the normal bundle of $u(\Sigma)$, and let

$$\mathfrak{d}_u^N \colon\, W^{1,p}\Gamma(\Sigma, u^*N) \longrightarrow L^p\Omega^{0,1}(\Sigma, u^*N)$$

be the restriction of the operator $\mathfrak{d}_u = \mathfrak{d}_{u,j;J}$ to the subbundle $u^*N \subset u^*TX$ followed by the projection on $\tilde{u}^*N$. We similarly define

$$\mathfrak{d}_{\tilde{u}}^N \colon\, W^{1,p}\Gamma(\tilde{\Sigma}, \tilde{\nu}; \tilde{u}^*N) \longrightarrow L^p\Omega^{0,1}(\tilde{\Sigma}, \tilde{u}^*N).$$

The spaces $\operatorname{coker} \mathfrak{d}_{\tilde{u}}^N$, as φ varies, play an important role in computing the contribution of maps factoring through u to the Gromov–Witten invariant of (X, ω). In this case, there is a simple description of these spaces.

First, we will see that $\ker \mathfrak{d}_u^N = \{0\}$ and $\operatorname{coker} \mathfrak{d}_u^N = \{0\}$. Indeed, the Hermitian metric on u^*TX induced from X gives us a splitting $u^*TX = T\Sigma \oplus N_u$, with respect to which

$$\mathfrak{d}_u = \begin{pmatrix} \bar{\partial}_{T\Sigma} & * \\ 0 & \mathfrak{d}_u^N \end{pmatrix};$$

see, for example, [DW18b, Appendix A]. Since u is unobstructed, i.e. $\operatorname{coker} \mathfrak{d}_u = \{0\}$, and $\operatorname{index}(u) = 0$, we have $\ker \mathfrak{d}_u^N = \{0\}$ and $\operatorname{coker} \mathfrak{d}_u^N = \{0\}$.

Second, since $\varphi\colon (\tilde{\Sigma}, \tilde{\nu}, \tilde{j}) \longrightarrow (\Sigma, j)$ has degree one, $(\tilde{\Sigma}, \tilde{\nu}, \tilde{j})$ has a unique irreducible component which is mapped by φ biholomorphically to (Σ, j), and φ is constant on the other components. In particular, $\tilde{u}^*N$ is trivial over these components. It follows that $\ker \mathfrak{d}_{\tilde{u}}^N \cong \{0\}$ and $\operatorname{coker} \mathfrak{d}_{\tilde{u}}^N$ is the direct sum of the corresponding spaces for the standard $\bar{\partial}$–operator with values in the trivial bundle $\tilde{u}^*N$ over the components which are mapped to a point by φ.

In this situation, the following is a special instance of [Zin11, Theorem 1.2].

Proposition B.1.

1. *The family of vector spaces* $\operatorname{coker} \mathfrak{d}_{u\circ\varphi}^N$*, as* $[\tilde{\Sigma}, \tilde{\nu}, \tilde{j}, \varphi] \in \overline{\mathscr{M}}_{[\Sigma],h}(\Sigma, j)$ *varies, forms an oriented orbibundle* $\mathfrak{O}_h(\Sigma, j, u) \longrightarrow \overline{\mathscr{M}}_{[\Sigma],h}(\Sigma, j)$*, called the* ***obstruction bundle****.*
2. *Denoting by* $[\overline{\mathscr{M}}_{[\Sigma],h}(\Sigma, j)]^{vir}$ *the virtual fundamental class and by* $e(\mathfrak{O}_h(\Sigma, j, u))$ *the Euler class of the obstruction bundle, we have*

$$\mathrm{GW}_{A,h}(X, \omega) = \sum_{g=0}^{h} \sum_{[u]\in\mathscr{M}_{A,g}^{\star}(X,J)} \operatorname{sign}(\Sigma, j, u)\langle e(\mathfrak{O}_h(\Sigma, j, u)), [\overline{\mathscr{M}}_{[\Sigma],h}(\Sigma, j)]^{vir}\rangle.$$

Pandharipande [Pan99, Section 2.3] proved that for $g \coloneqq g(\Sigma)$,

$$\sum_{h=g}^{\infty} \langle e(\mathfrak{O}_h(\Sigma, j, u)), [\overline{\mathscr{M}}_{[\Sigma],h}(\Sigma, j)]^{\mathrm{vir}}\rangle t^{2h-2} = t^{2g-2}\left(\frac{\sin(t/2)}{t/2}\right)^{2g-2}$$

Therefore, after changing the order of summation $\sum_{h=0}^{\infty}\sum_{g=0}^{h} = \sum_{g=0}^{\infty}\sum_{h=g}^{\infty}$, we obtain

$$\sum_{h=0}^{\infty} \mathrm{GW}_{A,h}(X, \omega)t^{2h-2} = \sum_{g=0}^{\infty} n_{A,g}(X, \omega)t^{2g-2}\left(\frac{\sin(t/2)}{t/2}\right)^{2g-2}.$$

Since the numbers $\mathrm{BPS}_{A,g}(X, \omega)$ are uniquely determined by the Gopakumar–Vafa formula (1.11) [BP01, Section 2],t $n_{A,g}(X, \omega) = \mathrm{BPS}_{A,g}(X, \omega)$.

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