Equivariant Brill–Noether theory for elliptic operators and super-rigidity of J–holomorphic maps

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Abstract

The space of Fredholm operators of fixed index is stratified by submanifolds according to the dimension of the kernel. We give sufficient conditions for a family of elliptic operators to intersect these strata transversely. The importance of our conditions is that they easily generalize to equivariant situations, such as those which arise from transversality questions for multiple covers of J–holomorphic maps. Using this abstract framework, we give a concise exposition of Wendl's progress towards establishing the super-rigidity conjecture.

1 Introduction

Let X and Y be two finite dimensional vector spaces. The space $\mathrm{Hom}(X, Y)$ is stratified by the submanifolds

$$\mathscr{L}_r \coloneqq \{L \in \mathrm{Hom}(X, Y) : \mathrm{rk}\, L = r\}$$

of codimension

$$\mathrm{codim}\, \mathscr{L}_r = (\dim X - r)(\dim Y - r).$$

This generalizes to infinite dimensions as follows. Let X and Y be two Banach spaces. The space of Fredholm operators from X to Y, denoted by $\mathscr{F}(X, Y)$, is stratified by the submanifolds

$$\mathscr{F}_{d,e} \coloneqq \{L \in \mathscr{F}(X, Y) : \dim\ker L = d \text{ and } \dim\mathrm{coker}\, L = e\} \tag{1.1}$$

of codimension

$$\mathrm{codim}\, \mathscr{F}_{d,e} = de. \tag{1.2}$$

In many geometric problems, especially in the study of moduli spaces in algebraic geometry, gauge theory, and symplectic topology, one is led to consider families of elliptic operators $D\colon \mathscr{P} \to \mathscr{F}(X, Y)$ parametrized by a Banach manifold $\mathscr{P}$ and to analyze the subsets $D^{-1}(\mathscr{F}_{d,e})$.

The archetypal example is *Brill–Noether theory* in algebraic geometry. Let Σ be a closed, connected Riemann surface. Denote by $\mathrm{Pic}(\Sigma)$ the Picard group of isomorphism classes of holomorphic line bundles $\mathscr{L} \to \Sigma$. Brill–Noether is concerned with the study of the subsets $G^r_d \subset \mathrm{Pic}(\Sigma)$, called the Brill–Noether loci, defined by

$$G^r_d \coloneqq \left\{[\mathscr{L}] \in \mathrm{Pic}(\Sigma) : \deg(\mathscr{L}) = d \text{ and } \dim H^0(\Sigma, \mathscr{L}) = r + 1\right\}.$$

The fundamental results of this theory deal with the questions of whether G^r_d is non-empty, smooth, and of the expected codimension.

This connects to the previous discussion as follows. Fix a Hermitian line bundle L of degree d. Denote the space of unitary connections on L by $\mathscr{A}(L)$. The complex gauge group $\mathscr{G}^{\mathbf{C}}(L)$ acts on $\mathscr{A}(L)$ and the quotient $\mathscr{A}(L)/\mathscr{G}^{\mathbf{C}}(L)$ is biholomorphic to $\mathrm{Pic}^d(\Sigma)$, the component of $\mathrm{Pic}(\Sigma)$ parametrizing holomorphic line bundles of degree d. Define the family of elliptic operators

$$\bar{\partial}\colon\ \mathscr{A}(L) \to \mathscr{F}(\Gamma(L), \Omega^{0,1}(\Sigma, L))$$

by assigning to every connection A the Dolbeault operator $\bar{\partial}_A = \nabla^{0,1}_A$. Set

$$\tilde{G}^r_d \coloneqq \bar{\partial}^{-1}(\mathscr{F}_{r+1,g-d+r}(X, Y)).$$

It follows from the Riemann–Roch Theorem and Hodge theory that the Brill–Noether loci can be described as the quotients

$$G^r_d = \tilde{G}^r_d/\mathscr{G}^{\mathbf{C}}(L).$$

If G^r_d is non-empty, then

$$\operatorname{codim} G^r_d = \operatorname{codim} \tilde{G}^r_d \leqslant (r + 1)(g - d + r).$$

This is an immediate consequence of the definition $\tilde{G}^r_d$ and (1.2). Ideally, every G^r_d is smooth of codimension $(r + 1)(g - d + r)$. This is not always true, but Gieseker [Gie82] proved that it holds for generic Σ; see also [EH83; Laz86]. For an extensive discussion of Brill–Noether theory we refer the reader to [ACGH85].

By analogy, for a general family of elliptic operators $D\colon\ \mathscr{P} \to \mathscr{F}(X, Y)$ we ask the following questions:

1. When are the subsets $D^{-1}(\mathscr{F}_{d,e})$ non-empty?
2. When are they smooth submanifolds of $\mathscr{P}$?
3. What are their codimensions?

Not much is known about (1), although index theory and the theory of spectral flow can yield partial results. A simple answer to questions (2) and (3) is that $D^{-1}(\mathscr{F}_{d,e})$ is smooth and of codimension de if the map D is transverse to $\mathscr{F}_{d,e}$. However, for many naturally occurring families of elliptic operators this condition does not hold. For example, if D is a family of elliptic operators over a manifold M and $\underline{V}$ is a local system, then the family $D^{\underline{V}}$ of the elliptic operators D twisted by $\underline{V}$ often is not transverse to $\mathscr{F}_{d,e}$ even if D is. Related issues arise for families of elliptic operators pulled back by a covering map $\pi\colon \tilde{M} \to M$. The purpose of this article is to give useful answers to questions (2) and (3) which apply to these *equivariant* situations. This theory is developed in Part I.

The issues discussed above are well-known to arise from multiple covers in the theory of J–holomorphic maps in symplectic topology. In fact, our motivation for writing this article came from trying to understand Wendl's progress towards establishing Bryan and Pandharipande's super-rigidity conjecture [Wen16]. In Part II we give a concise exposition of Wendl's work using the abstract framework developed here. A key difference between Wendl's approach and ours is the use of the language of local systems. We believe that this makes the proof easier to follow.

In future work we plan to study transversality for multiple covers of calibrated submanifolds in manifolds with special holonomy such as associative submanifolds in G_2–manifolds and special Lagrangians in Calabi–Yau 3–folds.

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Part I

Equivariant Brill–Theory Theory

Throughout this part, let (M, g) be a connected, oriented Riemannian manifold and let E and F be real vector bundles over M equipped with Euclidean metrics and metric connections. We fixed a point $x_0 \in M$. We assume that the injectivity radius of g is bounded below and that the Riemann curvature tensor R_g, the curvature tensors of the connections on E and F, as well as all of their derivatives are bounded. For $k \in \mathbf{N}_0$, we denote by $W^{k,2}\Gamma(E)$ and $W^{k,2}\Gamma(F)$ the Sobolev completion of the space of compactly supported sections of E and F with respect to the $W^{k,2}$–norm induced by the Euclidean metric and the connection on E and F, respectively. We set $L^2\Gamma(E) \coloneqq W^{0,2}\Gamma(E)$ and $L^2\Gamma(F) \coloneqq W^{0,2}\Gamma(F)$. Given two Banach spaces X and Y, we denote by $\mathscr{L}(X, Y)$ the Banach space of bounded linear operators from X to Y equipped with the operator norm.

2 Flexibility and Petri's condition

Definition 2.1. Let $k \in \mathbf{N}_0$. A **family of linear elliptic differential operators** of order k consists of a Banach manifold $\mathscr{P}$ and a smooth map

$$D\colon \mathscr{P} \to \mathscr{L}(W^{k,2}\Gamma(E), L^2\Gamma(F))$$

such that for every $p \in \mathscr{P}$ the operator $D_p := D(p)$ is the extension of a linear elliptic differential operator of order k with smooth coefficients which are bounded and all of whose derivatives are bounded.

Definition 2.2. Let $(D_p)_{p\in\mathscr{P}}$ be a family of linear elliptic differential operators. Given $d, e \in \mathbf{N}_0$, set

$$\mathscr{P}_{d,e} := \left\{p \in \mathscr{P} : \dim\ker D_p = d \text{ and } \dim\operatorname{coker} D_p = e\right\}.$$

We are interested in finding conditions under which $\mathscr{P}_{d,e}$ is a submanifold of $\mathscr{P}$. Since

$$\mathscr{P}_{d,e} = D^{-1}(\mathscr{F}_{d,e})$$

with $\mathscr{F}_{d,e}$ denoting the submanifold of $\mathscr{L}(W^{k,2}\Gamma(E), \Gamma(F))$ defined in (1.1), this is the case if the map D is transverse to $\mathscr{F}_{d,e}$. Let us describe what this means more concretely. If $p \in \mathscr{P}_{d,e}$, then D_p is a Fredholm operator and the normal space to $\mathscr{F}_{d,e}$ at D_p is

$$N_{D_p}\mathscr{F}_{d,e} = \operatorname{Hom}(\ker D_p, \operatorname{coker} D_p);$$

see, e.g., [Kos70, Section 1(b)] and Proposition 2.16.

Definition 2.3. Let $(D_p)_{p\in\mathscr{P}}$ be a family of linear elliptic differential operators. Let $p \in \mathscr{P}$. Denote by $\mathrm{d}_p D$ the derivative of the map D at p. Define $L_p\colon\ T_p\mathscr{P} \to \operatorname{Hom}(\ker D_p, \operatorname{coker} D_p)$ by

$$L_p(\hat{p})s := \mathrm{d}_p D(\hat{p})s \mod \operatorname{im} D_p$$

for $\hat{p} \in T_p\mathscr{P}$ and $s \in \ker D_p$.

L_p is the projection of $\mathrm{d}_p D$ on the normal space $N_{D_p}\mathscr{F}_{d,e}$. Therefore, D being transverse to $\mathscr{F}_{d,e}$ means that L_p is surjective for every $p \in \mathscr{P}_{d,e}$. In this case, the Regular Value Theorem guarantees that $\mathscr{P}_{d,e}$ is a submanifold of $\mathscr{P}$ of codimension

$$\dim\operatorname{Hom}(\ker D_p, \operatorname{coker} D_p) = de.$$

The task at hand is thus to find conditions which imply the surjectivity of L_p. For example, L_p is surjective if the evaluation map

$$\mathrm{ev}_p\colon\ \Gamma(\operatorname{Hom}(E,F)) \to \operatorname{Hom}(\ker D_p, \operatorname{coker} D_p) \tag{2.4}$$

satisfies the following two conditions:

1. The image of L_p contains the image of ev_p.
2. The evaluation map ev_p is surjective; equivalently, its adjoint ev_p^* is injective.

The following definitions introduce slight variations of these conditions.

Definition 2.5. A family of linear elliptic differential operators $(D_p)_{p\in\mathscr{P}}$ is called **flexible** if for every $p \in \mathscr{P}$ the following holds: for every $A \in \Gamma(\mathrm{Hom}(E, F))$ with compact support there is a $\hat{p} \in T_p\mathscr{P}$ such that

$$\mathrm{d}_p D(\hat{p})s = As \mod \operatorname{im} D_p \tag{2.6}$$

for every $s \in \ker D_p$.

The notion of flexibility is gauge invariant in the following sense.

Proposition 2.7. *Let $(D_p)_{p\in\mathscr{P}}$ and $(\tilde{D}_p)_{p\in\mathscr{P}}$ be families of linear elliptic differential operators. Let $\phi\colon \mathscr{P} \to \mathrm{Aut}(E)$ and $\psi\colon \mathscr{P} \to \mathrm{Aut}(F)$ be smooth families of gauge transformations of E and F parametrized by $\mathscr{P}$ which are bounded and all of whose derivatives are bounded. If $(D_p)_{p\in\mathscr{P}}$ and $(\tilde{D}_p)_{p\in\mathscr{P}}$ are related by*

$$\tilde{D}(p) = \phi(p) \circ D(p) \circ \psi(p),$$

then $(D_p)_{p\in\mathscr{P}}$ is flexible if and only if $(\tilde{D}_p)_{p\in\mathscr{P}}$ is flexible.

Proof. Since

$$\mathrm{d}_p\tilde{D}(\hat{p}) = \phi(p) \circ \mathrm{d}_p D(\hat{p}) \circ \psi(p) + \mathrm{d}_p\phi(\hat{p}) \circ \tilde{D}(p) + \tilde{D}(p) \circ \psi(p)^{-1} \circ \mathrm{d}_p\psi(\hat{p}),$$

for every $\hat{p} \in T_p\mathscr{P}$ and every $\tilde{s} \in \ker \tilde{D}(p)$ we have

$$\mathrm{d}_p\tilde{D}(\hat{p})\tilde{s} = \phi(p) \circ \mathrm{d}_p D(\hat{p}) \circ \psi(p)\tilde{s} \mod \operatorname{im} \tilde{D}_p.$$

Suppose that $(D_p)_{p\in\mathscr{P}}$ is flexible. Given $\tilde{A} \in \Gamma(\mathrm{Hom}(E, F))$ with compact support, let $\hat{p} \in T_p\mathscr{P}$ be such that (2.6) with $A = \phi(p)^{-1} \circ \tilde{A} \circ \psi(p)^{-1}$ holds for all $s \in \ker D_p$. Since $\ker D_p = \psi(p) \ker \tilde{D}_p$ and $\operatorname{im} \tilde{D}_p = \phi(p) \operatorname{im} D_p$, it follows that

$$\mathrm{d}_p\tilde{D}(\hat{p})\tilde{s} = \tilde{A}\tilde{s} \mod \operatorname{im} \tilde{D}_p$$

for every $\tilde{s} \in \ker \tilde{D}_p$. Therefore, $(\tilde{D}_p)_{p\in\mathscr{P}}$ is flexible as well. □

Remark 2.8. It is tempting to simplify Definition 2.5 and demand that for every $A \in \Gamma(\mathrm{Hom}(E, F))$ with compact support there is a $\hat{p} \in T_p\mathscr{P}$ such that

$$\mathrm{d}_p D(\hat{p}) = A.$$

If this holds, then we say that $(D_p)_{p\in\mathscr{P}}$ is **strongly flexible**. The disadvantage of strong flexibility is that it fails to be gauge invariant; that is: the analogue of Proposition 2.7 does not hold.

Flexibility (in fact, even strong flexibility) is not a rare condition and is usually easy to verify.

Definition 2.9. Let $D\colon \Gamma(E) \to \Gamma(F)$ be a linear elliptic differential operator. Denote its formal adjoint by $D^*\colon \Gamma(F) \to \Gamma(E)$. We say that D satisfies **Petri's condition** if the map

$$\ker D \otimes \ker D^* \to \Gamma(E \otimes F)$$

is injective.

Remark 2.10. In algebraic geometry, a Riemann surface Σ is said to satisfy Petri's condition if for every holomorphic line bundle $\mathscr{L} \to \Sigma$ the **Petri map**

$$H^0(\Sigma, \mathscr{L}) \otimes H^0(\Sigma, K_\Sigma \otimes \mathscr{L}^*) \to H^0(\Sigma, K_\Sigma) \tag{2.11}$$

is injective [ACGH85, Lemma 1.6, Chapter IV].

Petri's condition has the following important consequence.

Proposition 2.12. *Let $D\colon \Gamma(E) \to \Gamma(F)$ be a linear elliptic differential operator. Suppose that the extension of D to an operator $W^{k,2}\Gamma(E) \to L^2\Gamma(F)$ is Fredholm. Denote by*

$$m\colon \ \mathrm{Hom}(\ker D, \mathrm{coker}\, D) \to L^2\Gamma(\mathrm{Hom}(E, F))$$

the adjoint of the evaluation map (2.4). If D satisfies Petri's condition, then m is injective.

Proof. Denote by $D^*\colon \ \Gamma(F) \to \Gamma(E)$ the formal adjoint of D. Since D is Fredholm we can identify

$$\mathrm{coker}(D\colon \ W^{k,2}\Gamma(E) \to L^2\Gamma(F)) = \ker(D^*\colon \ L^2\Gamma(F) \to W^{-k,2}\Gamma(E)).$$

Set $n := \dim \ker D$. Let $s_1, \ldots, s_n$ be a L^2 orthonormal basis of $\ker D$. A computation shows that the map m is given by

$$m(B) = \sum_{i=1}^{n} \langle \cdot, s_i \rangle \, Bs_i. \tag{2.13}$$

If B is non-zero, then it follows from directly from Definition 2.9 that $m(B)$ cannot vanish identically. □

Petri's condition appears to be a subtle property and difficult to verify. However, there is a simple class of operators for which it holds almost trivially.

Proposition 2.14. *Every first order differential operator D over a manifold M of dimension one satisfies Petri's condition.*

Proof. By the uniqueness of solutions to ODEs, every element of $\ker D$ is determined by its value at any point $x \in M$. The same holds for D^*. This directly implies the assertion. □

Theorem 2.15. *Let $d, e \in \mathbf{N}_0$. If $(D_p)_{p\in\mathscr{P}}$ is a flexible family of linear elliptic differential operators satisfying Petri's condition, then $\mathscr{P}_{d,e} \subset \mathscr{P}$ is a smooth submanifold of codimension*

$$\mathrm{codim}\, \mathscr{P}_{d,e} = de.$$

The proof relies on the following observation.

Proposition 2.16. *For every $p \in \mathscr{P}_{d,e}$ there exits an open neighborhood $\mathscr{U}$ in $\mathscr{P}$ and a smooth map $Z\colon \mathscr{U} \to \operatorname{Hom}(\ker D_p, \operatorname{coker} D_p)$ such that*

$$\mathscr{P}_{d,e} \cap \mathscr{U} = Z^{-1}(0) \quad \textit{and} \quad \mathrm{d}_p Z = L_p.$$

Proof. Let $p_0 \in \mathscr{P}_{d,e}$. Pick a complement $\operatorname{coim} D_{p_0}$ of $\ker D_{p_0} \subset W^{k,2}\Gamma(E)$. Pick a lift of $\operatorname{coker} D_{p_0}$ to $L^2\Gamma(F)$. With respect to the splittings

$$W^{k,2}\Gamma(E) = \operatorname{coim} D_{p_0} \oplus \ker D_{p_0} \quad \text{and} \quad L^2\Gamma(F) = \operatorname{im} D_{p_0} \oplus \operatorname{coker} D_{p_0}$$

write D_p as

$$D_p = \begin{pmatrix} D_p^{11} & D_p^{12} \\ D_p^{21} & D_p^{22} \end{pmatrix}.$$

Let $\mathscr{U}$ be an open neighborhood of $p_0 \in \mathscr{P}$ such that D_p^{11} is invertible for every $p \in \mathscr{U}$. Define $Z\colon \mathscr{U} \to \operatorname{Hom}(\ker D_{p_0}, \operatorname{coker} D_{p_0})$ by

$$Z(p) \coloneqq D_p^{22} - D_p^{21}(D_p^{11})^{-1}D_p^{12}.$$

By direct inspection we see that $\mathrm{d}_{p_0} Z = L_{p_0}$. To see that $\mathscr{P}_{d,e} \cap \mathscr{U} = Z^{-1}(0)$ we compute that with

$$\Phi_p \coloneqq \begin{pmatrix} (D_p^{11})^{-1} & 0 \\ -D_p^{21}(D_p^{11})^{-1} & \mathrm{id} \end{pmatrix} \quad \text{and} \quad \Psi_p \coloneqq \begin{pmatrix} \mathrm{id} & -(D_p^{11})^{-1}D_p^{12} \\ 0 & \mathrm{id} \end{pmatrix}$$

we have

$$\Phi_p D_p \Psi_p = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & Z(p) \end{pmatrix}. \qquad \square$$

Proof of Theorem 2.15. In light of the above, the theorem will follow from the Regular Value Theorem applied to Z provided L_p is surjective for every $p \in \mathscr{P}_{d,e}$. Since $p \in \mathscr{P}_{d,e}$, D_p is Fredholm. Suppose $B \in \operatorname{Hom}(\ker D_p, \operatorname{coker} D_p) \cong \operatorname{Hom}(\ker D_p, \ker D_p^*)$ is perpendicular to $\operatorname{im} L_p$; that is

$$\langle B, L_p(\hat{p})\rangle = 0$$

for every $\hat{p} \in T_p\mathscr{P}$. Since $(D_p)_{p\in\mathscr{P}}$ is flexible, for every $A \in \Gamma(\operatorname{Hom}(E, F))$ with compact support

$$\langle m(B), A\rangle_{L^2} = 0.$$

Therefore, $m(B)$ vanishes. It follows from Proposition 2.12 that B vanishes. Therefore, L_p is surjective. $\square$

Remark 2.17. It should be pointed out that neither flexbility nor Petri's condition are necessary for the conclusion of Theorem 2.15 to hold. However, these conditions have the advantage that they can be easily adapted to the equivariant setting.

3 Pulling back and twisting

This section introduces two constructions which produce new linear elliptic operators from old ones: pulling back by a covering map and twisting by a Euclidean local system.

Definition 3.1. Let $\pi\colon \tilde{M} \to M$ be a covering map with $\tilde{M}$ connected. Let $D\colon \Gamma(E) \to \Gamma(F)$ be a linear differential operator. The **pullback** of D by π is the linear differential operator

$$\pi^*D\colon \Gamma(\pi^*E) \to \Gamma(\pi^*F)$$

characterized by

$$(\pi^*D)(\pi^*s) = \pi^*(Ds).$$

Definition 3.2. A **Euclidean local system** $\underline{V}$ is a Euclidean vector bundle $\underline{V}$ together with a flat metric connection. To each Euclidean local system we assign its **monodromy representation** $\mu\colon \pi_1(M, x_0) \to \mathrm{O}(V)$ with V denoting the fiber of $\underline{V}$ over x_0.

Remark 3.3. The map $\underline{V} \mapsto \mu$ induces a bijection between gauge equivalence classes of Euclidean local systems of rank r and equivalence classes of representations $\pi_1(M, x_0) \to \mathrm{O}(r)$ up to conjugation by $\mathrm{O}(r)$. For a more detailed discussion of local systems we refer the reader to [Dim04, Section 2.5; Voi07, Section 9.2.1].

Definition 3.4. Let $D\colon \Gamma(E) \to \Gamma(F)$ be a linear differential operator. Let $\underline{V}$ be a Euclidean local system on M. The **twist** of D by $\underline{V}$ is the linear differential operator

$$D^{\underline{V}}\colon \Gamma(E \otimes \underline{V}) \to \Gamma(F \otimes \underline{V})$$

characterized as follows: if U is a open subset M, $s \in \Gamma(U, E)$, and $f \in \Gamma(U, \underline{V})$ is constant, then

$$D^{\underline{V}}(s \otimes f) = (Ds) \otimes f.$$

The following shows that the pullback π^*D is equivalent to the twist $D^{\underline{V}}$ for a suitable choice of $\underline{V}$.

Proposition 3.5. *Let $D\colon \Gamma(E) \to \Gamma(F)$ be a linear differential operator. Let $\pi\colon \tilde{M} \to M$ be a covering map with $\tilde{M}$ connected. Fix $\tilde{x}_0 \in \tilde{M}$ with $\pi(\tilde{x}_0) = x_0$. Denote by*

$$C := \pi_*\pi_1(\tilde{M}, \tilde{x}_0) < \pi_1(M, x_0)$$

the characteristic subgroup of π and by

$$N := \bigcap_{g \in \pi_1(M, x_0)} gCg^{-1}$$

its normal core. Denote by $\underline{\mathbf{R}}$ the trivial rank one local system on $\tilde{M}$. Set

$$\underline{V} := \pi_*\underline{\mathbf{R}}.$$

The following hold:

1. *The monodromy representation of $\underline{V}$ factors through $G \coloneqq \pi_1(M, x_0)/N$.*

2. *There are isomorphisms $\pi_*\colon\ \Gamma(\pi^* E) \cong \Gamma(E \otimes \underline{V})$ and $\pi_*\colon\ \Gamma(\pi^* F) \cong \Gamma(F \otimes \underline{V})$ such that*

$$D^{\underline{V}} = \pi_* \circ \pi^* D \circ \pi_*^{-1}.$$

Remark 3.6. If π is a normal covering, then $C = N$ and $G = \pi_1(M, x_0)/N$ is its deck transformation group. If π has k sheets, then C has index k. Its normal core has index at most $k!$ by an elementary result known as Poincaré's Theorem. It follows from the observation that the kernel of the canonical homomorphism $\pi_1(M, x_0) \longrightarrow \mathrm{Bij}(G/C)$ is precisely N and $\mathrm{Bij}(G/C) \cong S_k$.

Proof of Proposition 3.5. The monodromy representation $\pi_1(M, x_0) \longrightarrow \mathrm{O}(V)$ of $\underline{V}$ is trivial on C; hence, it must factor through $G = \pi_1(M, x_0)/N$.

For every vector bundle $\tilde{G}$ over $\tilde{M}$ there is an isomorphism $\Gamma(\tilde{G}) \cong \Gamma(\pi_* \tilde{G})$. For every vector bundle G over $\tilde{M}$ there is an isomorphism

$$\pi_* \pi^* G \cong \pi_*(\pi^* G \otimes \underline{\mathbf{R}}) \cong G \otimes \pi_* \underline{\mathbf{R}} = G \otimes \underline{V}.$$

Denote the resulting isomorphism $\Gamma(\pi^* G) \cong \Gamma(G \otimes \underline{V})$ by π_*. For $s \in \Gamma(G)$ and $f \in C^\infty(\tilde{M})$ we have

$$\pi_*((\pi^* s) f) = s \otimes \pi_* f.$$

Let U be an open subset of M, $s \in \Gamma(U, E)$, and $f \in \Gamma(U, \underline{V})$. Suppose that f is constant. This is equivalent to the corresponding function $\tilde{f} \coloneqq (\pi_*)^{-1} f$ on $\tilde{U} \coloneqq \pi^{-1}(U)$ being locally constant. By the characterizing properties of $D^{\underline{V}}$ and $\pi^* D$ and since $\pi^* D$ is a differential operator, we have

$$D^{\underline{V}}(s \otimes f) = (Ds) \otimes f$$

and

$$\begin{aligned}
(\pi^* D)(\pi_*)^{-1}(s \otimes f) &= (\pi^* D)(\pi^* s \cdot \tilde{f}) \\
&= (\pi^*(Ds) \cdot \tilde{f}) \\
&= (\pi_*)^{-1}\left((Ds) \otimes f\right).
\end{aligned}$$

This proves that $D^{\underline{V}} = \pi_* \circ \pi^* D \circ \pi_*^{-1}$. □

4 Equivariant flexibility and the equivariant Petri condition

Twisting and pulling back lead to families of linear elliptic differential operators which fail to be flexible in the sense of Definition 2.5 (except for a few corner cases). In what follows we discuss variants of Theorem 2.15 which apply to such families of linear elliptic differential operators.

Throughout this section and the next, let G be the quotient of $\pi_1(M, x_0)$ by a finite index normal subgroup N. Denote by $\pi\colon \tilde{M} \to M$ the covering map with characteristic subgroup N. Denote by

$$V_1, \dots, V_m$$

the irreducible representations of G. Denote by $\underline{V}_i$ the local system associated with V_i and set

$$\mathbf{K}_i \coloneqq \mathrm{End}_G(V_i) \quad \text{and} \quad k_i \coloneqq \dim_{\mathbf{R}} \mathbf{K}_i.$$

Remark 4.1. The local system $\underline{V}_i$ carries a canonical $\mathbf{K}_i$–action. By Schur's Lemma, $\mathbf{K}_i$ is a real division algebra; hence, by Frobenius' Theorem, it is either $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$ and $k_i \in \{1, 2, 4\}$. Since $D_p^{\underline{V}_i}$ commutes with the action of $\mathbf{K}_i$, $\ker D_p^{\underline{V}_i}$ and $\operatorname{coker} D_p^{\underline{V}_i}$ are modules over $\mathbf{K}_i$.

Definition 4.2. A family of linear elliptic differential operators $(D_p)_{p\in\mathscr{P}}$ is called G**–equivariantly flexible** if for every $p \in \mathscr{P}$ the following holds: for every $A \in \Gamma(\mathrm{Hom}(E, F))$ with compact support there is a $\hat{p} \in T_p\mathscr{P}$ such that

$$\mathrm{d}_p\pi^*D(\hat{p})s = (\pi^*A)s \mod \operatorname{im} \pi^*D_p$$

for every $s \in \ker \pi^*D_p$.

Proposition 2.7 holds with flexible replaced by G–equivariantly flexible. If $(D_p)_{p\in\mathscr{P}}$ is strongly flexibly, then it is G–equivariantly flexible.

Definition 4.3. Let $D\colon \Gamma(E) \to \Gamma(F)$ be a linear elliptic differential operator. We say that D satisfies the G**–equivariant Petri condition** if π^*D satisfies Petri's condition.

Theorem 4.4. *Let $(D_p)_{p\in\mathscr{P}}$ be a G–equivariantly flexible family of linear elliptic differential operators satisfying the G–equivariant Petri condition. For every $\underline{d}, \underline{e} \in \mathbf{N}_0^m$ the subset*

$$\mathscr{P}^N_{\underline{d},\underline{e}} \coloneqq \left\{p \in \mathscr{P} : \dim_{\mathbf{K}_i} \ker D_p^{\underline{V}_i} = d_i \text{ and } \dim_{\mathbf{K}_i} \operatorname{coker} D_p^{\underline{V}_i} = e_i\right\} \subset \mathscr{P}$$

is a smooth submanifold of codimension

$$\operatorname{codim} \mathscr{P}^N_{\underline{d},\underline{e}} = \sum_{i=1}^m k_i d_i e_i.$$

The proof is given in Section 5. The following is an immediate consequence of Theorem 4.4.

Proposition 4.5. *Assume the situation of Theorem 4.4. Let $\underline{V}$ be a Euclidean local system whose monodromy representation factors through G. Let $\ell_1, \dots, \ell_m \in \mathbf{N}_0$ be such that*

$$\underline{V} \cong \bigoplus_{i=1}^m \underline{V}_i^{\oplus \ell_i}.$$

Given $d, e \in \mathbf{N}_0$, set

$$\mathfrak{m}^V_{d,e} := \left\{(\underline{d}, \underline{e}) \in \mathbf{N}_0^m \times \mathbf{N}_0^m : \sum_{i=1}^m \ell_i k_i d_i = d \text{ and } \sum_{i=1}^m \ell_i k_i e_i = e\right\} \quad \text{and}$$

$$\mathscr{P}^V_{d,e} := \left\{p \in \mathscr{P} : \dim\ker D^V_p = d \text{ and } \dim\operatorname{coker} D^V_p = e\right\}.$$

The following hold:

1. *If $\mathfrak{m}^V_{d,e}$ is empty, then so is $\mathscr{P}^V_{d,e}$.*
2. *$\mathscr{P}^V_{d,e} \subset \mathscr{P}$ is a disjoint finite union of submanifolds of codimension at least*

$$\min_{(\underline{d},\underline{e}) \in \mathfrak{m}^V_{d,e}} \sum_{i=1}^m k_i d_i e_i.$$

Proof assuming Theorem 4.4. Since

$$D^V_p = \bigoplus_{i=1}^m \left(D^{V_i}_p\right)^{\oplus \ell_i},$$

we have

$$\mathscr{P}^V_{d,e} = \coprod_{(\underline{d},\underline{e}) \in \mathfrak{m}^V_{d,e}} \mathscr{P}^N_{\underline{d},\underline{e}}.$$

The assertion thus follows from Theorem 4.4. ☐

5 Proof of Theorem 4.4

The representation associated to the local system

$$\underline{V} := \pi_* \underline{\mathbf{R}}$$

is the left regular representation: $\mathbf{R}[G] = \operatorname{Map}(G, \mathbf{R})$ with $(g \cdot f)(x) = f(g^{-1}x)$. Recall, the following classical result from representation theory.

Theorem 5.1. *Let G be a finite group. Let $V_1, \ldots, V_m$ be the irreducible representations of G. Denote by $\mathbf{K}_i := \operatorname{End}_G(V_i)$ the commuting algebra of V_i. The left regular representation of G can be decomposed into irreducible representations as follows*

$$\mathbf{R}[G] \cong \bigoplus_{j=1}^m V_i \otimes_{\mathbf{K}_i} V_i^*$$

with G acting on V_i through the irreducible representation and trivially on V_i^*. In particular, the multiplicity of V_i in $\mathbf{R}[G]$ is the dimension of V_i over $\mathbf{K}_i$. The right regular representation decomposes analogously with G acting on V_i^* trough the dual of the irreducible representation and trivially on V_i.

Proof. Although this result is classical, we provide a sketch of its proof. $\mathbf{R}[G]$ is the group algebra of G. A representation of G is nothing but an $\mathbf{R}[G]$–module. Maschke's Theorem says that for any finite group $\mathbf{R}[G]$ is a semisimple $\mathbf{R}$–algebra. Any finite-dimensional semisimple $\mathbf{R}$–algebra is a product of simple algebras. Wedderburn's Structure Theorem [Lan02, Chapter XVII Corollary 3.5] says that any simple $\mathbf{R}$–algebra A whose unique irreducible representation is denoted by V is isomorphic to $\mathrm{End}_{\mathbf{K}}(V) = V \otimes_{\mathbf{K}} V^*$ with $\mathbf{K} = \mathrm{End}_A(V)$. □

This has the following important consequence.

Proposition 5.2. *In the above situation the following hold:*

1. *The local system $\underline{V}$ decomposes as*

$$\underline{V} \cong \bigoplus_{i=1}^{m} \underline{V}_i \otimes_{\mathbf{K}_i} V_i^*. \tag{5.3}$$

2. *The space of sections of $\underline{V}$ is a G–representation and decomposes as*

$$\Gamma(\underline{V}) \cong \bigoplus_{i=1}^{m} \Gamma(\underline{V}_i) \otimes_{\mathbf{K}_i} V_i^* \tag{5.4}$$

with G acting on V_i^* through the contragredient[1] *of the irreducible representation.*

Proof. Given an open set $U \subset M$, set

$$\tilde{U} := \pi^{-1}(U) \subset \tilde{M}.$$

The restriction of π makes $\tilde{U}$ a principal G–bundle over U.

By construction for every such open set $U \subset M$ we have

$$\Gamma(U, \underline{V}) = C^\infty(\tilde{U}, \mathbf{R}).$$

Denote by $\tilde{U} \times_G G$ the quotient of $\tilde{U} \times G$ by the left action

$$g \cdot (x, h) = (xg^{-1}, gh).$$

It is obvious that $\tilde{U} \to \tilde{U} \times_G G, x \mapsto [x, 1]$ is an isomorphism of principal G–bundles. Denote by $C^\infty(\tilde{U}, \mathbf{R}[G])^G$ the set of smooth G–equivariant maps from $\tilde{U}$ to $\mathbf{R}[G]$. The G–equivariant exponential law asserts that

$$C^\infty(\tilde{U} \times_G G, \mathbf{R}) = C^\infty(\tilde{U}, \mathbf{R}[G])^G.$$

[1] Given a representation $\rho\colon G \to \mathrm{GL}(V)$, its contragredient is the representation $G \to \mathrm{GL}(V^*)$ given by $g \cdot v^* = v^* \circ \rho(g^{-1})$.

Putting everything together we obtain

$$\Gamma(U, \underline{V}) = C^\infty(\tilde{U}, \mathbf{R}[G])^G.$$

It follows from this identity that $\underline{V}$ is the vector bundle $\tilde{M} \times_G \mathbf{R}[G]$. Theorem 5.1 thus implies that

$$\begin{aligned}\underline{V} &= \tilde{M} \times_G \mathbf{R}[G] \\ &\cong \tilde{M} \times_G \left(\bigoplus_{i=1}^m V_i \otimes_{\mathbf{K}_i} V_i^* \right) \\ &= \bigoplus_{i=1}^m \underline{V}_i \otimes_{\mathbf{K}_i} V_i^*.\end{aligned}$$

This is the decomposition of $\underline{V}$ asserted in (1). It immediately implies the decomposition (5.4) of $\Gamma(\underline{V})$ as a vector space.

The deck transformation group G acts on $\tilde{M}$ on the right and thus on $C^\infty(\tilde{M}, \mathbf{R}) = \Gamma(\underline{V})$ on the left. Therefore, $\Gamma(\underline{V})$ is a G–representation. The right action of G on $\tilde{M}$ translates to the obvious right action of G on $\tilde{M} \times_G G$. Through the exponential law the induced left action on $\Gamma(\underline{V})$ corresponds to the inverse of the right action on $C^\infty(\tilde{M}, \mathbf{R}[G])^G$ induced by the right action on $\mathbf{R}[G]$. It follows from Theorem 5.1 that with respect to the decomposition (5.4) the latter action corresponds to the action of G on V_i^* via the contragredient of the irreducible representation. □

The following observation makes the proof of Theorem 4.4 ameanable to the method used to prove Theorem 2.15. Given $d, e \in \mathbf{N}_0$, set

$$\mathscr{P}^\pi_{d,e} := \left\{ p \in \mathscr{P} : \dim_\mathbf{R} \ker \pi^* D_p = d \text{ and } \dim_\mathbf{R} \operatorname{coker} \pi^* D_p = e \right\}.$$

Definition 5.5. Set $\ell_i := \dim_{\mathbf{K}_i} V_i$. Given $\underline{d} \in \mathbf{N}_0^m$, set

$$\sigma\underline{d} := \sum_{j=1}^m \ell_i k_i d_i.$$

Proposition 5.6. *Let $\underline{d}, \underline{e} \in \mathbf{N}_0^m$. Set $d := \sigma\underline{d}$ and $e := \sigma\underline{e}$. Then $\mathscr{P}^N_{\underline{d},\underline{e}}$ is an open and closed subset of $\mathscr{P}^\pi_{d,e}$.*

Proof. Proposition 3.5 and Proposition 5.2 provide G–equivariant isomorphisms

$$\Gamma(\pi^* E) \cong \bigoplus_{i=1}^m \Gamma(E \otimes \underline{V}_i) \otimes_{\mathbf{K}_i} V_i^* \quad \text{and} \quad \Gamma(\pi^* F) \cong \bigoplus_{i=1}^m \Gamma(F \otimes \underline{V}_i) \otimes_{\mathbf{K}_i} V_i^*. \tag{5.7}$$

With respect to these we have

$$\pi^* D_p = \bigoplus_{i=1}^m D_p^{\underline{V}_i} \otimes \mathrm{id}_{V_i^*}. \tag{5.8}$$

Therefore, $\mathscr{P}^N_{\underline{d},\underline{e}}$ is a subset of $\mathscr{P}^\pi_{d,e}$. In fact, we have

$$\mathscr{P}^\pi_{d,e} = \bigcup_{\substack{\sigma\underline{d}'=d\\ \sigma\underline{e}'=e}} \mathscr{P}^N_{\underline{d}',\underline{e}'}.$$

Let $p_0 \in \mathscr{P}^N_{\underline{d},\underline{e}}$. Choose a neighborhood $\mathscr{U}$ of p_0 such that for every $p \in \mathscr{U}$ and $i \in \{1,\dots,m\}$

$$\dim\ker D_p^{V_i} \leqslant \dim\ker D_{p_0}^{V_i}.$$

A point $p \in \mathscr{U}$ lies in $\mathscr{P}^\pi_{d,e}$ if and only if equality holds in all these inequalities and, therefore, $p \in \mathscr{P}^N_{\underline{d},\underline{e}}$. This proves that $\mathscr{P}^N_{\underline{d},\underline{e}}$ is open in $\mathscr{P}^\pi_{d,e}$. Applying the same reasoning to all other $\underline{d}'$ and $\underline{e}'$ with $\sigma\underline{d}' = d$ and $\sigma\underline{e}' = e$ proves that $\mathscr{P}^N_{\underline{d},\underline{e}}$ is also closed. □

Definition 5.9. Given $p \in \mathscr{P}$, define $L^\pi_p\colon\ T_p\mathscr{P} \to \mathrm{Hom}_G(\ker\pi^*D_p, \operatorname{coker}\pi^*D_p)$ by

$$L^\pi_p(\hat{p})s \coloneqq \mathrm{d}_p\pi^*D(\hat{p})s \mod \operatorname{im}\pi^*D_p$$

for $\hat{p} \in T_p\mathscr{P}$ and $s \in \ker\pi^*D_p$. The linear map L^π_p takes values in $\mathrm{Hom}_G(\ker\pi^*D_p, \operatorname{coker}\pi^*D_p)$ because π^*D_p is G–equivariant.

Proposition 5.10. *For every $p \in \mathscr{P}^N_{\underline{d},\underline{e}}$ there exits an open neighborhood $\mathscr{U}$ in $\mathscr{P}$ and a smooth map $Z\colon\ \mathscr{U} \to \mathrm{Hom}_G(\ker\pi^*D_p, \operatorname{coker}\pi^*D_p)$ such that*

$$\mathscr{P}_{d,e} \cap \mathscr{U} = Z^{-1}(0) \quad \textit{and} \quad \mathrm{d}_pZ = L^\pi_p.$$

Proof. The proof is almost identical to that of Proposition 2.16. Since $(\pi^*D_p)_{p\in\mathscr{P}}$ is a family of G–equivariant operators and the splittings

$$\Gamma(\pi^*E) = \operatorname{coim}\pi^*D_p \oplus \ker\pi^*D_p \quad \text{and} \quad \Gamma(\pi^*F) = \operatorname{im}\pi^*D_p \oplus \operatorname{coker}\pi^*D_p$$

can be chosen G–invariant, the map Z takes values in $\mathrm{Hom}_G(\ker\pi^*D_p, \operatorname{coker}\pi^*D_p)$. □

Proposition 5.11. *Let $\underline{d}, \underline{e} \in \mathbf{N}_0^m$. If $p \in \mathscr{P}^N_{\underline{d},\underline{e}}$, then*

$$\mathrm{Hom}_G(\ker\pi^*D_p, \operatorname{coker}\pi^*D_p) \cong \bigoplus_{i=1}^m \mathrm{Hom}_{\mathbf{K}_i}(\ker D_p^{V_i}, \operatorname{coker} D_p^{V_i}). \tag{5.12}$$

In particular,

$$\dim\mathrm{Hom}_G(\ker\pi^*D_p, \operatorname{coker}\pi^*D_p) = \sum_{i=1}^m k_i d_i e_i.$$

Proof. The group G acts on $\Gamma(\pi^*E)$ and $\Gamma(\pi^*F)$ by deck transformations, and on V_i^* through the contragredient of the irreducible representation. The G–equivariant isomorphisms (5.7) induce G–equivariant isomorphisms

$$\ker \pi^* D_p \cong \bigoplus_{i=1}^m \ker D_p^{V_i} \otimes_{\mathbf{K}_i} V_i^* \quad \text{and} \quad \operatorname{coker} \pi^* D_p \cong \bigoplus_{i=1}^m \operatorname{coker} D_p^{V_i} \otimes_{\mathbf{K}_i} V_i^*.$$

It follows that

$$\begin{aligned}
&\operatorname{Hom}_G(\ker \pi^* D_p, \pi^* \operatorname{coker} D_p^*) \\
&\quad \cong \bigoplus_{i,j=1}^m \operatorname{Hom}_G(\ker D_p^{V_i} \otimes_{\mathbf{K}_i} V_i^*, \operatorname{coker} D_p^{V_j} \otimes_{\mathbf{K}_j} V_j^*) \\
&\quad \cong \bigoplus_{i,j=1}^m \left(\ker D_p^{V_i}\right)^* \otimes_{\mathbf{K}_i} \operatorname{Hom}_G(V_i^*, V_j^*) \otimes_{\mathbf{K}_j} \operatorname{coker} D_p^{V_j} \\
&\quad \cong \bigoplus_{i=1}^m \operatorname{Hom}_{\mathbf{K}_i}(\ker D_p^{V_i}, \operatorname{coker} D_p^{V_i}).
\end{aligned}$$

Here we used Schur's lemma; that is: $\operatorname{Hom}_G(V_i^*, V_j^*)$ vanishes if $i \neq j$ and is equal to $\mathbf{K}_i$ if $i = j$. □

At this stage, all that remains to establish Theorem 4.4 is prove that for every $p \in \mathscr{P}^N_{\underline{d},\underline{e}}$ the linear map $L_p^\pi \colon T_p\mathscr{P} \to \operatorname{Hom}_G(\ker \pi^* D_p, \operatorname{coker} \pi^* D_p)$ is surjective. Suppose that an element B of

$$\operatorname{Hom}_G(\ker \pi^* D_p, \operatorname{coker} \pi^* D_p) \cong \operatorname{Hom}_G(\ker \pi^* D_p, \ker \pi^* D_p^*)$$

is perpendicular to $\operatorname{im} L_p^\pi$; that is:

$$\langle B, L_p^\pi(\hat{p}) \rangle = 0$$

for every $\hat{p} \in T_p\mathscr{P}$. Since $(D_p)_{p \in \mathscr{P}}$ is G–equivariantly flexible, for every $A \in \Gamma(\operatorname{Hom}(E, F))$ with compact support.

$$\langle m(B), \pi^* A \rangle_{L^2} = 0.$$

The map m is G–equivariant. Therefore, $m(B)$ is an element of

$$\Gamma(\operatorname{Hom}(\pi^* E, \pi^* F))^G = \pi^* \Gamma(\operatorname{Hom}(E, F));$$

that is, $m(B)$ is the pullback of a section of $\operatorname{Hom}(E, F)$. Consequently, $m(B)$ vanishes. D_p satisfies the G–equivariant Petri condition; that is, $\pi^* D_p$ satisfies Petri's condition. Therefore, it follows from Proposition 2.12 that B vanishes. This shows that L_p^π is surjective. □

Remark 5.13. By construction, $\pi^* D$ maps $\mathscr{P}$ into the space of G–equivariant bounded linear maps $\mathscr{L}(X, Y)^G$. If $p \in \mathscr{P}^\pi_{d,e}$, then the fiber of normal bundle of $\mathscr{F}^G_{d,e} \subset \mathscr{L}(X, Y)^G$ at $\pi^* D_p$ is $\operatorname{Hom}(\ker \pi^* D_p, \operatorname{coker} \pi^* D_p)^G$. The above shows that the image of $\pi^* D$ intersects $\mathscr{F}^G_{d,e}$ transversely under the hypotheses of Theorem 4.4

6 Self-adjoint operators

The theory developed here is not needed for

Definition 6.1. Let $k \in \mathbf{N}_0$. A **family of self-adjoint linear elliptic differential operators** of order k consists of a Banach manifold $\mathscr{P}$ and a smooth map

$$D\colon \mathscr{P} \to \mathscr{L}(W^{k,2}\Gamma(E), L^2\Gamma(E))$$

such that for every $p \in \mathscr{P}$ the operator $D_p \coloneqq D(p)$ is the extension of a self-adjoint linear elliptic differential operator of order k with smooth coefficients which are bounded and all of whose derivatives are bounded.

The theory developed in the earlier sections cannot be applied to D as above, because families of self-adjoint operators necessarily fail to be flexible in the sense of Definition 2.5 (except for a few corner cases). Moreover, requiring Petri's condition for self-adjoint operators is too strong. In this section, we adapt the notions of flexibility and Petri's condition to the case of self-adjoint operators so that an analogue of Theorem 2.15 holds.

Given a Euclidean vector space W, we denote by $\mathrm{Sym}(W)$ the space of self-adjoint endomorphisms of W and by S^2W the second symmetric power of W. Analogously, given a Euclidean vector bundle E, we denote by $\mathrm{Sym}(E)$ the bundle of self-adjoint endomorphisms of E and by S^2E the second symmetric power of E.

Definition 6.2. A family of self-adjoint linear elliptic differential operators $(D_p)_{p\in\mathscr{P}}$ is called **self-adjoint flexible** if for every $p \in \mathscr{P}$ the following holds: for every $A \in \Gamma(\mathrm{Sym}(E))$ with compact support there is a $\hat{p} \in T_p\mathscr{P}$ such that

$$\mathrm{d}_p D(\hat{p})s = As \mod \operatorname{im} D_p$$

for every $s \in \ker D_p$.

Definition 6.3. Let $D\colon \Gamma(E) \to \Gamma(E)$ be a self-adjoint linear elliptic differential operator with finite-dimensional L^2 kernel. We say that D satisfies the **self-adjoint Petri condition** if the map

$$S^2 \ker D \to \Gamma(S^2E)$$

is injective.

Theorem 6.4. *Let $(D_p)_{p\in\mathscr{P}}$ be a self-adjoint flexible family of self-adjoint linear elliptic differential operators satisfying the self-adjoint Petri condition Given $d \in \mathbf{N}_0$, set*

$$\mathscr{P}_d \coloneqq \left\{p \in \mathscr{P} : \dim\ker D_p = d\right\}.$$

Then $\mathscr{P}_d \subset \mathscr{P}$ is a smooth submanifold of codimension

$$\operatorname{codim} \mathscr{P}_d = \binom{d+1}{2}.$$

Proof. The proof is almost identical to that of Theorem 2.15. Let $p \in \mathscr{P}_d$. Denote by Π_p the orthogonal projection onto $\ker D_p$. Define $L_p\colon T_p\mathscr{P} \to \mathrm{Sym}(\ker D_p)$ by

$$L_p(\hat{p})s := \Pi_p \mathrm{d}_p D(\hat{p})s$$

for $\hat{p} \in T_p\mathscr{P}$ and $s \in \ker D_p$. There exits an open neighborhood $\mathscr{U}$ of p in $\mathscr{P}$ and a smooth map $Z\colon \mathscr{U} \to \mathrm{Sym}(\ker D_p)$ such that

$$\mathscr{P}_{d,e} \cap \mathscr{U} = Z^{-1}(0) \quad \text{and} \quad \mathrm{d}_p Z = L_p.$$

It follows as in the proof of Theorem 2.15 that L_p is surjective at p. Therefore, $\mathscr{P}_d$ is a submanifold of codimension $\binom{d+1}{2}$. □

There also is an analogue of the theory discussed in Section 4. Throughout the remainder of this section, we let G, N, $\pi\colon \tilde{M} \to M$, V_i, $\underline{V}_i$, $\mathbf{K}_i$, and k_i be as in Section 4.

Definition 6.5. A family of self-adjoint linear elliptic differential operators $(D_p)_{p\in\mathscr{P}}$ is called G**–equivariantly self-adjoint flexible** if for every $p \in \mathscr{P}$ the following holds: for every $A \in \Gamma(\mathrm{Sym}(E))$ with compact support there is a $\hat{p} \in T_p\mathscr{P}$ such that

$$\mathrm{d}_p\pi^*D(\hat{p})s = (\pi^*A)s \mod \mathrm{im}\,\pi^*D_p$$

for every $s \in \ker \pi^*D_p$.

Definition 6.6. Let $D\colon \Gamma(E) \to \Gamma(F)$ be a self-adjoint linear elliptic differential operator. We say that D satisfies the G**–equivariant self-adjoint Petri condition** if π^*D satisfies the self-adjoint Petri condition.

Theorem 6.7. *Let $(D_p)_{p\in\mathscr{P}}$ be a G–equivariantly self-adjoint flexible family of self-adjoint linear elliptic differential operators satisfying the G–equivariant self-adjoint Petri condition. Given $\underline{d} \in \mathbf{N}_0^m$, set*

$$\mathscr{P}^N_{\underline{d}} := \left\{p \in \mathscr{P} : \dim_{\mathbf{K}_i} \ker D_p^{\underline{V}_i} = d_i\right\}.$$

Then $\mathscr{P}^N_{\underline{d},\underline{e}} \subset \mathscr{P}$ is a smooth submanifold of codimension

$$\mathrm{codim}\,\mathscr{P}^N_{\underline{d},\underline{e}} = \sum_{i=1}^m d_i + k_i\binom{d_i}{2}.$$

Sketch of proof. The proof is almost identical to that of Theorem 4.4. The key differences are that L_p^π, defined in Definition 5.9, now takes values in $\mathrm{Sym}(\ker \pi^*D_p)^G$ and the isomorphism from Proposition 5.11 is replaced by

$$(\mathrm{Sym}(\ker \pi^*D_p))^G \cong \bigoplus_{i=1}^m \mathrm{Sym}_{\mathbf{K}_i}(\ker D_p^{\underline{V}_i}).$$

Having made these two adaptations the remainder of the proof follows the argument in Section 5 closely. The codimension formulae follow from

$$\dim \mathrm{Sym}_{\mathbf{K}}(\mathbf{K}^d) = d + k\binom{d}{2}.$$

with $k \coloneqq \dim_{\mathbf{R}} \mathbf{K}$. □

Theorem 6.7 has the following consequence.

Proposition 6.8. *Assume the situation of Theorem 6.7 Let $\underline{V}$ be a Euclidean local system whose monodromy representation factors through $G \coloneqq \pi_1(M, x_0)/N$. Let $\ell_1, \ldots, \ell_m \in \mathbf{N}_0$ be such that*

$$\pi_* \underline{\mathbf{R}} \cong \bigoplus_{i=1}^{m} \underline{V}_i^{\oplus \ell_i}.$$

Given $d \in \mathbf{N}_0$, set

$$\mathfrak{m}_d^{\underline{V}} \coloneqq \left\{ \underline{d} \in \mathbf{N}_0^m : \sum_{i=1}^{m} \ell_i k_i d_i = d \right\} \quad \text{and}$$
$$\mathscr{P}_d^{\underline{V}} \coloneqq \left\{ p \in \mathscr{P} : \dim \ker D_p^{\underline{V}} = d \right\}.$$

The following hold:

1. *If $\mathfrak{m}_d^{\underline{V}}$ is empty, then so is $\mathscr{P}_d^{\underline{V}}$.*
2. *$\mathscr{P}_d^{\underline{V}} \subset \mathscr{P}$ is a disjoint finite union of submanifolds of codimension at least*

$$\min_{\underline{d} \in \mathfrak{m}_{d,e}^{\underline{V}}} \sum_{i=1}^{m} d_i + k_i \binom{d_i}{2}.$$

Part II
Application to super-rigidity

7 Bryan and Pandharipande's super-rigidity conjecture

We begin by recalling the notion of super-rigidity as defined by Eftekhary [Eft16, Section 1] and Wendl [Wen16, Section 2.1]. Throughout, let (M, J, g) be an almost Hermitian $2n$–manifold.

Definition 7.1. A **J–holomorphic map** $u\colon (\Sigma, j) \to (M, J)$ is a pair consisting of a closed, connected Riemann surface (Σ, j) and a smooth map $u\colon \Sigma \to M$ satisfying the non-linear Cauchy–Riemann equation

$$\bar{\partial}_J(u, j) := \frac{1}{2}(\mathrm{d}u + J(u) \circ \mathrm{d}u \circ j) = 0. \tag{7.2}$$

Definition 7.3. Let $u\colon (\Sigma, j) \to (M, J)$ be a J–holomorphic map. Let $\phi \in \mathrm{Diff}(\Sigma)$ be a diffeomorphism. The **reparametrization** of u by ϕ is the J–holomorphic map $u \circ \phi^{-1}\colon (\Sigma, \phi_* j) \to (M, J)$.

Definition 7.4. Let $u\colon (\Sigma, j) \to (M, J)$ be a J–holomorphic map and let $\pi\colon (\tilde{\Sigma}, \tilde{j}) \to (\Sigma, j)$ be a holomorphic map of degree $\deg(\pi) \geqslant 2$. The composition $u \circ \pi\colon (\tilde{\Sigma}, \tilde{j}) \to (M, J)$ is said to be a **multiple cover of** u. A J–holomorphic map is **simple** if it is not constant and not a multiple cover.

Super-rigidity is a condition on the infinitesimal deformation theory of J–holomorphic maps up to reparametrization. We will have to briefly review this theory. Let $u\colon (\Sigma, j) \to (M, J)$ be a non-constant J–holomorphic map. Set

$$\mathrm{Aut}(\Sigma, j) := \{\phi \in \mathrm{Diff}(\Sigma) : \phi_* j = j\} \quad \text{and} \quad \mathfrak{aut}(\Sigma, j) := \{v \in \mathrm{Vect}(\Sigma) : \mathscr{L}_v j = 0\}.$$

Let $\mathscr{S}$ be an $\mathrm{Aut}(\Sigma, j)$–invariant slice of the Teichmüller space $\mathscr{T}(\Sigma)$ around j. Denote by

$$\mathrm{d}_{u,j}\bar{\partial}_J\colon \Gamma(u^*TM) \oplus T_j\mathscr{S} \to \Omega^{0,1}(u^*TM)$$

the linearization of $\bar{\partial}_J$ at (u, j) restricted to $C^\infty(\Sigma, M) \times \mathscr{S}$. The action of $\mathrm{Aut}(\Sigma, j)$ on $C^\infty(M) \times \mathscr{S}$ preserves $\bar{\partial}_J^{-1}(0)$. Consequently, there is an inclusion

$$\mathfrak{aut}(\Sigma, j) \hookrightarrow \ker \mathrm{d}_{u,j}\bar{\partial}_J.$$

The space of J–holomorphic maps up to reparametrization has virtual dimension

$$\operatorname{index} \mathrm{d}_{u,j}\bar{\partial}_J - \dim \mathfrak{aut}(\Sigma, j) = (n-3)\chi(\Sigma) + 2\langle [\Sigma], u^*c_1(M)\rangle;$$

see, e.g., [MS12, Section 3; Wen10, Theorem 0; IP18, Proposition 5.1].

Definition 7.5. The **index** of a J–holomorphic map $u\colon (\Sigma, j) \to (M, J)$ is

$$\operatorname{index}(u) := (n-3)\chi(\Sigma) + 2\langle [\Sigma], u^*c_1(M)\rangle. \tag{7.6}$$

The restriction of $\mathrm{d}_{u,j}\bar{\partial}_J$ to $\Gamma(u^*TM)$ is given by

$$\mathfrak{d}_{u,J}\xi = \frac{1}{2}\left(\nabla\xi + J \circ (\nabla\xi) \circ j + (\nabla_\xi J) \circ \mathrm{d}u \circ j\right) \tag{7.7}$$

for $\xi \in \Gamma(u^*TM)$. Here ∇ denotes any torsion-free connection on TM and also the induced connection on u^*TM. If (u, j) is a J–holomorphic map, then the right-hand side of (7.7) does not depend on the choice of ∇; see [MS12, Proposition 3.1.1]. The operator $\mathfrak{d}_{u,J}$ has the property that if $\xi \in \Gamma(T\Sigma)$, then $\mathfrak{d}_{u,J}(\mathrm{d}u(\xi))$ is a $(0,1)$–form taking values in $\mathrm{d}u(T\Sigma) \subset u^*TM$. If u is non-constant, then there is a unique complex subbundle

$$Tu \subset u^*TM$$

of rank one containing $\mathrm{d}u(T\Sigma)$; see [IS99, Section 1.3; Wen10, Section 3.3] and Appendix A. Since Tu agrees with $\mathrm{d}u(T\Sigma)$ outside finitely many points, $\mathfrak{d}_{u,J}$ maps $\Gamma(Tu)$ to $\Omega^{0,1}(Tu)$.

Definition 7.8. Let $u\colon (\Sigma, j) \to (M, J)$ be a non-constant J–holomorphic map. Set

$$Nu := u^*TM/Tu.$$

The **normal Cauchy–Riemann operator** associated with u is the linear map

$$\mathfrak{d}^N_{u,J}\colon \Gamma(Nu) \to \Omega^{0,1}(Nu)$$

induced by $\mathfrak{d}_{u,J}$.

If $\tilde{u} = u \circ \pi$ and u is an immersion, then $N\tilde{u} = \pi^* Nu$ and $Nu = u^*TM/T\Sigma$ is the normal bundle of the immersion u.

Proposition 7.9 ([IS99, Lemma 1.5.1; Wen10, Theorem 3]; see also Appendix A). *Let $u\colon (\Sigma, j) \to (M, J)$ be a non-constant J–holomorphic map. Denote by $Z(\mathrm{d}u)$ the number of critical points of u counted with multiplicity. The following hold:*

1. *There is a surjection*
$$\ker \mathrm{d}_{u,j}\bar{\partial}_J \twoheadrightarrow \ker \mathfrak{d}^N_{u,J}$$
whose kernel contains $\mathfrak{aut}(\Sigma, j)$ *and has dimension* $\dim \mathfrak{aut}(\Sigma, j) + 2Z(\mathrm{d}u)$.

2. *We have*
$$\operatorname{coker} \mathrm{d}_{u,j}\bar{\partial}_J \cong \operatorname{coker} \mathfrak{d}^N_{u,J}.$$

3. *We have*
$$\operatorname{index} \mathfrak{d}^N_{u,J} = \operatorname{index}(u) - 2Z(\mathrm{d}u) \leqslant \operatorname{index}(u).$$

Geometrically, the additional $2Z(\mathrm{d}u)$ dimensions correspond to deforming the location of the critical points of u without deforming its image $u(\Sigma)$.

Definition 7.10. A non-constant J–holomorphic map u is **rigid** if $\ker \mathfrak{d}^N_{u,J} = 0$.

A multiple cover $\tilde{u}$ of u may fail to be rigid, even if u itself is rigid.

Definition 7.11. A simple J–holomorphic map $u\colon (\Sigma, j) \to (M, J)$ is called **super-rigid** if it is rigid and all of its multiple covers are rigid.

If u is super-rigid, then it must have $\mathrm{index}(u) \leqslant 0$. Suppose that M admits a symplectic form ω. Bryan and Pandharipande [BP01, Section 1.2] conjectured that super-rigidity holds for every simple J–holomorphic map u with $\mathrm{index}(u) \leqslant 0$ provided J is a generic complex structure J compatible with ω.

Definition 7.12. An almost complex structure J is called **super-rigid** if the following hold:

1. Every simple J–holomorphic map of index zero is super-rigid.
2. Every simple J–holomorphic map has non-negative index.
3. Every simple J–holomorphic map of index zero is an embedding, and every two simple J–holomorphic maps of index zero either have disjoint images or are related by a reparametrization.

Remark 7.13. In dimension four, one should weaken (3) and require only that every simple J–holomorphic map of index zero is an immersion with transverse self-intersections, and that two such maps are either transverse to one another or are related by reparametrization. However, we will only be concerned with dimension at least six.

Definition 7.14. Let (M, ω) be a symplectic manifold. Denote by $\mathscr{J}(\omega)$ the separable Banach manifold[2] of almost complex structures on M compatible with ω. Denote by $\mathscr{J}_\diamond(\omega)$ the subset of those almost complex structures $J \in \mathscr{J}(\omega)$ which are super-rigid.

Definition 7.15. Let X be a topological space. A subset $A \subset X$ is called **residual** if it is the intersection of countably many dense open subsets.

Conjecture 7.16 (Bryan and Pandharipande). *Let (M, ω) be a symplectic manifold. If $\dim M \geqslant 6$, then $\mathscr{J}_\diamond(\omega) \subset \mathscr{J}(\omega)$ is a residual subset.*

This conjecture remains open. However, Wendl [Wen16] has made substantial progress towards proving it.

Definition 7.17. Let (M, ω) be a symplectic manifold. We denote by $\mathscr{J}_P(\omega)$ the interior of the set of those almost complex structures $J \in \mathscr{J}(\omega)$ satisfying the following: for every non-constant J–holomorphic map $u\colon (\Sigma, j) \to (M, J)$ of the normal Cauchy–Riemann operator $\mathfrak{d}_{u,J}^N$ satisfies Petri's condition. Set

$$\mathscr{J}_{\diamond,P}(\omega) \coloneqq \mathscr{J}_\diamond(\omega) \cap \mathscr{J}_P(\omega).$$

Theorem 7.18 (Wendl [Wen16]). *Let (M, ω) be a symplectic manifold of dimension at least six. If $\dim M \geqslant 6$, then $\mathscr{J}_{\diamond,P}(\omega) \subset \mathscr{J}_P(\omega)$ is a residual subset.*

The remainder of this part of the article is concerned with the proof of Theorem 7.18 as well as the proof of Theorem 15.2 which deals with the failure of super-rigidity along paths of almost complex structures. Throughout the next seven sections, (M, ω) is a symplectic manifold of dimension $2n \geqslant 6$.

[2]The cognisant reader will know that the space almost complex structures compatible with ω is naturally a Fréchet manifold. To obtain a separable Banach we work with Floer's C^∞_ε topology; see [Flo88, Section 5; MS12, Remark 3.2.7].

8 The universal moduli space of simple J–holomorphic maps

Let us recall some well-known facts about the moduli space of simple J–holomorphic maps.

Definition 8.1. Let $k \in \mathbf{Z}$. Denote by $\mathscr{M}_k(\omega)$ the space of pairs $(J;[u,j])$ consisting of:

- an almost complex structure $J \in \mathscr{J}_P(\omega)$, and
- an equivalence class of simple J–holomorphic maps $u\colon (\Sigma, j) \to (M, J)$ of index k up to reparametrization by $\mathrm{Diff}(\Sigma)$.

Theorem 8.2 ([Wen10, Theorem 0; IP18, Proposition 5.1]). *Let $k \in \mathbf{Z}$. $\mathscr{M}_k(\omega)$ is a separable Banach manifold. The projection map $\Pi\colon \mathscr{M}_k(\omega) \to \mathscr{J}_P(\omega)$ is a Fredholm map of index k.*

Proposition 8.3. *There is a residual subset $\mathscr{J}_{\geqslant 0}(\omega) \subset \mathscr{J}_P(\omega)$ such that for every $J \in \mathscr{J}_{\geqslant 0}(\omega)$ and every simple J–holomorphic map $u\colon (\Sigma, j) \to (M, J)$ we have $\mathrm{index}(u) \geqslant 0$.*

This is an immediate consequence of the following fact, which will be used throughout this article.

Proposition 8.4. *Let X and Y be separable Banach manifolds and let $f\colon X \to Y$ be a Fredholm map of index i. If $W \subset X$ is a submanifold of codimension at least $i+1$, then $Y \backslash f(W)$ is residual.*

Proof. Although this is well-known, let us explain the proof. Let $N_xW = T_xX/T_xW$ be the normal space to W at $x \in W$. There are short exact sequences

$$0 \to \ker(\mathrm{d}_x f)/\ker(\mathrm{d}_x f|_W) \to N_xW \to \mathrm{im}(\mathrm{d}_x f)/\mathrm{im}(\mathrm{d}_x f|_W) \to 0$$

and

$$0 \to \mathrm{im}(\mathrm{d}_x f)/\mathrm{im}(\mathrm{d}_x f|_W) \to \mathrm{coker}(\mathrm{d}_x f|_W) \to \mathrm{coker}(\mathrm{d}_x f) \to 0.$$

It follows that $f|_W$ is a Fredholm map of index

$$\mathrm{index}(\mathrm{d}_x f|_W) = \mathrm{index}(\mathrm{d}_x f) - \dim N_xW,$$

which is at most -1. In particular, $\dim\mathrm{coker}(\mathrm{d}_x f|_W) > 0$ for every $x \in W$. Therefore, every $y \in f(W)$ is critical for $f|_W$. However, by the Sard–Smale Theorem [Sma65] the set of critical values of $f|_W$ is meager, that is, its complement is residual. □

Definition 8.5. Denote by $\mathscr{M}_0^{\amalg}(\omega)$ the universal moduli space of simple, possibly disconnected J–holomorphic maps of index zero. Set

$$\mathscr{W}_{\not\subset}(\omega) := \left\{(J;[u,j]) \in \mathscr{M}_0^{\amalg}(\omega) : u \text{ is not an embedding}\right\}.$$

Theorem 8.6 (Oh and Zhu [OZ09, Theorem 1.1] and Ionel and Parker [IP18, Proposition A.4]). *$\mathscr{W}_{\not\subset}(\omega) \subset \mathscr{M}_0^{\amalg}(\omega)$ has codimension at least $2(n-2)$.*

9 Strong flexibility

The following result will imply that the flexibility assumptions of Theorem 2.15 and Theorem 4.4 are satisfied.

For two complex vector spaces V and W, denote by $\overline{\mathrm{Hom}}_{\mathbf{C}}(V, W)$ the space of $\mathbf{C}$–anti-linear homomorphisms from V to W, and similarly for vector bundles. In particular, if $E \longrightarrow \Sigma$ is a complex vector bundle, then $\Lambda^{0,1}T^*\Sigma \otimes E = \overline{\mathrm{Hom}}_{\mathbf{C}}(T\Sigma, E)$.

Proposition 9.1 ([Wen16, Lemma 6.1]). *Let $J \in \mathscr{J}(\omega)$. Let $u\colon (\Sigma, j) \longrightarrow (M, J)$ be a simple J–holomorphic map. Consider the set of* ***embedded points***

$$U := \{x \in \Sigma : u^{-1}(u(x)) = \{x\} \textit{ and } \mathrm{d}_x u \neq 0\}.$$

For every

$$A \in \Gamma(\mathrm{Hom}(Nu, \overline{\mathrm{Hom}}_{\mathbf{C}}(T\Sigma, Nu))$$

with support in U there exists a 1–parameter family $(J_t)_{t \in \mathbf{R}} \subset \mathscr{J}(\omega)$ such that:

1. *u is J–holomorphic with respect to all J_t, and*
2. *$\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \mathfrak{d}^N_{u,J_t} = A$.*

Proof. The tangent space to $\mathscr{J}(\omega)$ at J is given by

$$T_J\mathscr{J} = \left\{\hat{J} \in \Gamma(\mathrm{End}(TM)) : \hat{J}J + J\hat{J} = 0 \text{ and } \omega(\hat{J}\cdot, \cdot) + \omega(\cdot, \hat{J}\cdot) = 0\right\}.$$

This means that $T_J\mathscr{J}$ consists of anti-linear endomorphisms which are skew-adjoint with respect to ω. For $x \in U$, we can write $T_xM = T_x\Sigma \oplus N_x\Sigma$. Given $\hat{j} \in \overline{\mathrm{Hom}}_{\mathbf{C}}(T\Sigma, Nu)$, denote by $\hat{j}^\dagger$ its adjoint with respect to ω and set

$$\hat{J} := \begin{pmatrix} 0 & -\hat{j}^\dagger \\ \hat{j} & 0 \end{pmatrix}.$$

By construction $\hat{J}J + J\hat{J} = 0$ and $\omega(\hat{J}\cdot, \cdot) + \omega(\cdot, \hat{J}\cdot) = 0$.

Given $A \in \Gamma(\mathrm{Hom}(Nu, \overline{\mathrm{Hom}}_{\mathbf{C}}(T\Sigma, Nu))$ with support in U, pick $(J_t)_{t \in \mathbf{R}} \subset \mathscr{J}(\omega)$ such that $J_t|_{u(\Sigma)} = J$ for every t and such that for every $\xi \in \Gamma(Nu)$ we have

$$\frac{1}{2}\nabla_\xi \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} J_t = \begin{pmatrix} 0 & (A(\xi)j)^\dagger \\ -A(\xi)j & 0 \end{pmatrix}.$$

By construction u is J–holomorphic with respect to all J_t. It follows from (7.7) that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathfrak{d}^N_{u,J_t} = A. \qquad \square$$

10 Rigidity of unbranched covers

As a warm-up, let us explain how to prove that for a generic $J \in \mathscr{J}(\omega)$ all J–holomorphic maps of the form $u \circ \pi$ with $u\colon (\Sigma, j) \to (M, J)$ a simple J–holomorphic map with $\operatorname{index}(u) = 0$ and $\pi\colon (\tilde{\Sigma}, \tilde{j}) \to (\Sigma, j)$ an unbranched holomorphic covering map are rigid.

Definition 10.1. Let Σ and $\tilde{\Sigma}$ be closed, connected surfaces. Let $\pi\colon \tilde{\Sigma} \to \Sigma$ be a covering map. Denote by $\mathscr{M}_0^\Sigma(\omega)$ the component of $\mathscr{M}_0(\omega)$ consisting of those pairs $(J; [u, j])$ with the domain of u being Σ. Set

$$\mathscr{W}_\pi(\omega) \coloneqq \left\{(J; [u, j]) \in \mathscr{M}_0^\Sigma(\omega) : \pi^* u \text{ is not rigid}\right\}.$$

Proposition 10.2. *$\mathscr{W}_\pi(\omega) \subset \mathscr{M}_0^\Sigma(\omega) \backslash \mathscr{W}_{\not\subset}(\omega)$ has codimension at least one.*

Denote by $\Pi\colon \mathscr{M}_0^\Sigma(\omega) \to \mathscr{J}(\omega)$ the projection map. From Proposition 8.4 it follows that $\Pi(\mathscr{W}_\pi(\omega)) \subset \mathscr{J}_P(\omega)$ is meager, that is, its complement is residual. Since there are only countably many closed, connected surfaces and only countably many covering maps between closed, connected surfaces, the set of $J \in \mathscr{J}_P(\omega)$ such that any J–holomorphic map of the form $u \circ \pi$ as above is rigid is residual.

Proof of Proposition 10.2. Set

$$\mathscr{P} \coloneqq \mathscr{M}_0^\Sigma(\omega) \backslash \mathscr{W}_{\not\subset}(\omega).$$

Define Hilbert space bundles $\mathscr{X}$ and $\mathscr{Y}$ over $\mathscr{P}$ with

$$\mathscr{X}_{(J;[u,j])} = W^{1,2}\Gamma(Nu) \quad \text{and} \quad \mathscr{Y}_{(J;[u,j])} = L^2\Omega^{0,1}(\Sigma, Nu).$$

Define

$$\mathfrak{d}^N\colon \mathscr{P} \to \mathscr{L}(\mathscr{X}, \mathscr{Y})$$

by

$$\mathfrak{d}^N(J; [u, j]) \coloneqq \mathfrak{d}^N_{u,J}.$$

In Definition 2.1 we assumed E and F to be fixed. We can generalize this to E and F varying with $p \in \mathscr{P}$. In fact, this generalization is easily reduced to the situation considered in Definition 2.1 by locally trivializing the dependence of E and F on $p \in \mathscr{P}$. The equivariant notion of flexibility is not affected by changing the choice of local trivialization. Petri's condition is manifestly independent of the choice of trivialization. Having explained this, we will consider $(\mathfrak{d}_p)_{p \in \mathscr{P}}$ as a family of linear elliptic differential operators in this slightly generalized sense.

By Proposition 9.1, $(\mathfrak{d}_p)_{p \in \mathscr{P}}$ is strongly flexible and, therefore, G–equivariantly flexible for every G.

We have

$$i \coloneqq \operatorname{index} \mathfrak{d}^N_{u \circ \pi, J} = \deg(\pi) \cdot \operatorname{index} \mathfrak{d}^N_{u,J} \leqslant \deg(\pi) \cdot \operatorname{index}(u) = 0.$$

Therefore, it follows from Proposition 3.5 and Proposition 6.8 that

$$\mathscr{W}_\pi(\omega) = \bigcup_{d=1}^{\infty} \mathscr{P}_{d,d-i}^{\pi_*\underline{\mathbf{R}}}$$

has codimension at least one. □

There are two essential difficulties one needs to overcome to deal with holomorphic maps $\pi\colon (\tilde{\Sigma},\tilde{j}) \to (\Sigma, j)$ which are not covering maps:

1. Proposition 3.5 and Proposition 6.8 do not apply directly to branched covers.
2. These maps come in positive dimensional families whereas there are only countably many covering maps.

The upcoming three subsections will provide us with a framework to deal with both of these issues.

11 Cauchy–Riemann operators on punctured Riemann surfaces

The following discussion will allow us to remove isolated points from branched covers. In particular, it allows us to remove preimages of branch-points.

Set

$$D := \{z \in \mathbf{C} : |z| < 1\} \quad \text{and} \quad \dot{D} := D \backslash \{0\}.$$

Define a diffeomorphism $\psi\colon (0,\infty) \times S^1 \to \dot{D}$ by

$$\psi(t,\alpha) := e^{-(t+i\alpha)}.$$

Definition 11.1. Let Σ be a closed, connected Riemann surface. Let $Z \subset \Sigma$ be a finite set and set

$$\dot{\Sigma} := \Sigma \backslash Z.$$

For each $x \in Z$, let $\phi_x\colon D \to \Sigma$ be a chart such that $\phi_x(0) = x$ and $\phi_x(\dot{D}) \cap Z = \varnothing$. Choose a Riemannian metric g_{cyl} and a smooth function $\tau\colon \dot{\Sigma} \to [0,\infty)$ such that for every $x \in Z$

$$\psi^*\phi_x^* g_{\mathrm{cyl}} = (\mathrm{d}t)^2 + (\mathrm{d}\alpha)^2 \quad \text{and} \quad \tau \circ \phi_x \circ \psi(t,\alpha) = t.$$

Definition 11.2. Let $\dot{E}$ be a Euclidean vector bundle over $\dot{\Sigma}$ together with a metric connection. For any compactly supported section $s \in \Gamma(\dot{E})$ define

$$\|s\|_{L^2_{\mathrm{cyl}}}^2 := \int_{\dot{\Sigma}} |s|^2 \,\mathrm{vol}_{g_{\mathrm{cyl}}} \quad \text{and} \quad \|s\|_{W^{1,2}_{\mathrm{cyl}}}^2 := \|s\|_{L^2_{\mathrm{cyl}}}^2 + \|\nabla s\|_{L^2_{\mathrm{cyl}}}^2.$$

Denote by $W^{1,2}_{\mathrm{cyl}}(\dot E)$ the completion of the space of compactly supported sections of $\dot E$ with respect to $\|\cdot\|_{W^{1,2}_{\mathrm{cyl}}}$. Denote by $L^2_{\mathrm{cyl}}(\dot E)$ the completion of the space of compactly supported sections of $\dot E$ with respect to $\|\cdot\|_{L^2_{\mathrm{cyl}}}$.

We employ the following convention: for a weight function w and a normed space of sections X (such as $L^2_{\mathrm{cyl}}(\dot E)$ or $W^{1,2}_{\mathrm{cyl}}(\dot E)$) we denote by wX the space of sections of the form ws for $s \in X$ equipped with the norm $\|ws\| \coloneqq \|s\|_X$.

Proposition 11.3. *Let E be a Hermitian vector bundle over Σ together with a compatible connection. Let $\mathfrak{d}_E \colon \Gamma(E) \to \Omega^{0,1}(\Sigma, E)$ be a real Cauchy–Riemann operator. Set $\dot E \coloneqq E|_{\Sigma\setminus Z}$. Denote by $\mathfrak{d}_{\dot E} \colon \Gamma(\dot E) \to \Omega^{0,1}(\Sigma, \dot E)$ the restriction of $\mathfrak{d}_E$. The following hold:*

1. *For every $\delta \in \mathbf{R}$ the operator $\mathfrak{d}_{\dot E}$ extends to a bounded linear operator*
$$\mathfrak{d}_{\dot E,\delta} \colon e^{\delta\tau} W^{1,2}_{\mathrm{cyl}}\Gamma(\dot E) \to e^{\delta\tau} L^2\Omega^{0,1}_{\mathrm{cyl}}(\dot\Sigma, \dot E).$$
2. *The operator $\mathfrak{d}_{\dot E,\delta}$ is Fredholm if and only if $\delta \notin \mathbf{Z}$.*
3. *If $\delta \in (0,1)$, then*
$$\ker \mathfrak{d}_{\dot E,\delta} \cong \ker \mathfrak{d}_E \quad \textit{and} \quad \operatorname{coker} \mathfrak{d}_{\dot E,\delta} \cong \operatorname{coker} \mathfrak{d}_E.$$

Proof. For every $x \in Z$ identify $\phi_x(D)$ with D and choose a trivialization $E|_D \cong \underline{\mathbf{C}}^{\oplus r}$. In these trivializations we can write $\mathfrak{d}_E$ as $\bar\partial + a d\bar z$ with $a \in C^\infty(D, \mathrm{End}_{\mathbf{R}}(\mathbf{C}^r))$. Pulling back via ψ and identifying $\Omega^{0,1}((0,\infty)\times S^1, \mathbf{C}^{n-1})$ with $C^\infty((0,\infty)\times S^1, \mathbf{C}^{n-1})$, this becomes

$$\partial_t + i\partial_\alpha + \tilde a \tag{11.4}$$

with $\tilde a$ and all of its derivatives decaying exponentially as t goes to infinity. Since

$$\tau \circ \phi_x \circ \psi(t,\alpha) = t,$$

we have

$$\|e^{-\delta\tau}\mathfrak{d}_{\dot E} s\|_{L^2_{\mathrm{cyl}}} \lesssim \|e^{-\delta\tau} s\|_{W^{1,2}_{\mathrm{cyl}}}.$$

The above also proves that $\mathfrak{d}_{\dot E}$ is asymptotically translation invariant and thus the standard theory for such operators applies; see, e.g., [LM85; Don02, Section 3; HHN15, Section 2.1]. According to this theory, $\mathfrak{d}_{\dot E,\delta}$ is Fredholm if and only if $-\delta \notin \operatorname{spec}(i\partial_\alpha) = \mathbf{Z}$.

It remains to prove (3). Fix $\delta \in (0,1)$. If s is a smooth section of E over Σ, then

$$\|e^{-\delta\tau} s\|_{W^{1,2}_{\mathrm{cyl}}} < \infty.$$

Therefore, $\ker \mathfrak{d}_E \subset \ker \mathfrak{d}_{\dot E,\delta}$. Conversely, if $s \in \ker \mathfrak{d}_{\dot E,\delta}$, then it is smooth on $\dot\Sigma$, satisfies $\mathfrak{d}_E s = 0$ on $\dot\Sigma$, and around every puncture x obeys an estimate of the form

$$|s|(\phi_x(z)) \lesssim |z|^{-\delta}.$$

This implies that s is in L^2 and satisfies $\mathfrak{d}_E s = 0$ weakly on all of Σ. Therefore, s extends smoothly over Z and this extension lies in $\ker \mathfrak{d}_E$. This proves that $\ker \mathfrak{d}_E = \ker \mathfrak{d}_{\dot{E},\delta}$.

The isomorphism of the cokernels follows by a similar argument. While the metric g_{cyl} does not extend to Σ, it is conformal to a metric g which does. Denote by $\mathfrak{d}_E^*$ the formal adjoint of $\mathfrak{d}_E$ with respect to g and by $\mathfrak{d}_{\dot{E}}^*$ the formal adjoint of $\mathfrak{d}_{\dot{E}}$ with respect to g_{cyl}. In fact, since the L^2 inner product on 1–forms on a Riemann surface depends only on the conformal class of the metric, we have

$$\mathfrak{d}_E^* = \mathfrak{d}_{\dot{E}}^*$$

on $(0,1)$–forms compactly supported in $\dot{\Sigma}$. The above reasoning shows that, for every $\delta \in \mathbf{R}$, $\mathfrak{d}_{\dot{E}}^*$ extends to a bounded linear operator $\mathfrak{d}_{\dot{E},\delta}^* \colon\ e^{\delta\tau} W^{1,2}_{\mathrm{cyl}} \Omega^{0,1}(\dot{\Sigma}, \dot{E}) \to e^{\delta\tau} L^2_{\mathrm{cyl}} \Gamma(\dot{E})$. Since $\mathfrak{d}_E$ and $\mathfrak{d}_{\dot{E},\delta}$ are Fredholm and by elliptic regularity, $\operatorname{coker} \mathfrak{d}_E \cong \ker \mathfrak{d}_E^*$ and $\operatorname{coker} \mathfrak{d}_{E,\delta} \cong \ker \mathfrak{d}_{\dot{E},-\delta}^*$. If $\alpha \in \ker \mathfrak{d}_E^*$, then $|\alpha|_{g_{\mathrm{cyl}}} \lesssim e^{-\tau}$; hence,

$$\|e^{\delta\tau}\alpha\|_{L^2_{\mathrm{cyl}}} < \infty.$$

It follows that $\alpha \in \ker \mathfrak{d}_{\dot{E},-\delta}^*$. Therefore, $\ker \mathfrak{d}_E^* \subset \ker \mathfrak{d}_{\dot{E},-\delta}^*$. Conversely, if $\alpha \in \ker \mathfrak{d}_{\dot{E},-\delta}^*$, then it is smooth on $\dot{\Sigma}$, satisfies $\mathfrak{d}_E^* \alpha = 0$ on $\dot{\Sigma}$, and

$$|\alpha|(\phi_x(z)) \lesssim |z|^{\delta - 1}.$$

The factor $|z|^{-1}$ arises from relating the norm of a 1–form in Euclidean coordinates to its norm in cylindrical coordinates. This implies that α is in L^2 and satisfies $\mathfrak{d}_E^* \alpha = 0$ weakly on all of Σ. Therefore, α extends smoothly over Z and this extension lies in $\ker \mathfrak{d}_E^*$. This proves that $\ker \mathfrak{d}_E^* = \ker \mathfrak{d}_{\dot{E},-\delta}^*$. □

Proposition 11.5. *Assume the situation of Proposition 11.3. Let $\underline{V}$ be a Euclidean local system on $\dot{\Sigma}$. For every $x \in Z$ denote by $\mu_x \in \mathrm{O}(V)$ the monodromy of $\underline{V}$ around x. The following hold:*

1. *For every $\delta \in \mathbf{R}$ the operator $\mathfrak{d}_{\dot{E}}^{\underline{V}}$ extends to a bounded linear operator*

$$\mathfrak{d}_{\dot{E},\delta}^{\underline{V}} \colon\ e^{\delta\tau} W^{1,2}_{\mathrm{cyl}} \Gamma(\dot{E} \otimes \underline{V}) \to e^{\delta\tau} L^2 \Omega^{0,1}_{\mathrm{cyl}}(\dot{\Sigma}, \dot{E} \otimes \underline{V}).$$

2. *The operator $\mathfrak{d}_{\dot{E},\delta}^{\underline{V}}$ is Fredholm if and only if for all $x \in Z$ we have $e^{-2\pi i \delta} \notin \operatorname{spec}(\mu_x)$.*

3. *Set*

$$\delta_0 := \min\{\delta \in (0,1] : e^{-2\pi i\delta} \in \operatorname{spec}(\mu_x) \textit{ for any } x \in Z\}.$$

If $\delta \in (0, \delta_0)$, then

$$\operatorname{index} \mathfrak{d}_{E,\delta}^{\underline{V}} = \operatorname{rk} \underline{V} \cdot \operatorname{index} \mathfrak{d}_E - \operatorname{rk} E \cdot \sum_{x \in Z} \dim(V/V^{\mu_x}).$$

Here V^{μ_x} denotes the μ_x–invariant subspace of V.

Proof. Arguing as in the proof of Proposition 11.3 we can write $\mathfrak{d}^{\underline{V}}_{\dot{E}}$ near $x \in Z$ as

$$\partial_t + i\nabla_\alpha^{\underline{V}_x} + \tilde{a}$$

acting on $E \otimes \underline{V}$. Here $\underline{V}_x$ denotes the local system over S^1 obtained by pulling back $\underline{V}$ via $\phi_x \circ \psi$ and then restricting to an S^1 factor in $(0, \infty) \times S^1$. This immediately implies (1). Since

$$\operatorname{spec}(i\nabla_\alpha^{\underline{V}_x}) = \left\{\lambda \in \mathbf{R} : e^{2\pi i\lambda} \in \operatorname{spec}(\mu_x)\right\},$$

assertion (2) follows as well.

Wendl [Wen16, Section 4] gives a proof of the index formula (3) using the Riemann–Roch theorem for punctured surfaces developed by Schwarz [Sch95, Section 3.3]. In order to keep the present article self-contained we provide a proof using Kawasaki's orbifold Riemann–Roch theorem [Kaw79].

For every $x \in Z$ the monodromy of $\underline{V}$ around x factors through a cyclic group $\mathbf{Z}_{\ell_x}$. Denote by $\hat{\Sigma}$ the orbifold whose underlying topological space is Σ and with orbifold points precisely at the points of Z and with isotropy group at x given by $\mathbf{Z}_{\ell_x}$. The local system $\underline{V}$ over $\dot{\Sigma}$ extends to a local system $\underline{\hat{V}}$ over $\hat{\Sigma}$. The reader will have no trouble to verify that Proposition 11.3(3) extends to Σ replaced with $\hat{\Sigma}$ and E replaced with $E \otimes \hat{V}$ provided we impose that $\delta \in (0, \delta_0)$. This implies that

$$\operatorname{index} \mathfrak{d}^{\underline{V}}_{\dot{E},\delta} = \operatorname{index} \mathfrak{d}_{E\otimes\underline{\hat{V}}}.$$

The latter agrees with the index of $\bar{\partial}_{E\otimes\underline{\hat{V}}}$.

The orbifold Riemann–Roch theorem asserts that

$$\operatorname{index} \bar{\partial}_{E\otimes\underline{\hat{V}}} = \int_{\Lambda\hat{\Sigma}} \operatorname{td}_{\Lambda\hat{\Sigma}}(T\Sigma)\operatorname{ch}_{\Lambda\hat{\Sigma}}(E \otimes \underline{\hat{V}}). \tag{11.6}$$

Here $\Lambda\hat{\Sigma}$ is the inertia orbifold. In the situation at hand it is given by

$$\Lambda\hat{\Sigma} = \hat{\Sigma} \sqcup \{(x, g) : x \in Z \text{ and } g \neq 1 \in \mathbf{Z}_{\ell_x}\}$$

with the points (x, g) being isolated and having isotropy group $\mathbf{Z}_{\ell_x}$. Furthermore, the differential form $\operatorname{td}_{\Lambda\hat{\Sigma}}(T\Sigma)\operatorname{ch}_{\Lambda\hat{\Sigma}}(E \otimes \underline{\hat{V}})$ agrees with the product of the usual Chern–Weil representatives of the Todd class and the Chern character on $\hat{\Sigma}$, and with

$$\frac{1}{\ell_x}\operatorname{td}_g(T_x\Sigma)\operatorname{ch}_g(E_x \otimes V)$$

on (x, g). Here td_g and ch_g denote the equivariant Todd class and the equivariant Chern character, respectively. In light of the above discussion the orbifold Riemann–Roch theorem becomes

$$\operatorname{index} \bar{\partial}_{E\otimes\underline{\hat{V}}} = \int_{\hat{\Sigma}} \operatorname{td}(T\Sigma)\operatorname{ch}(E \otimes \underline{\hat{V}}) + \operatorname{rk}(E) \sum_{x\in Z} \frac{1}{\ell_x} \sum_{g\neq 1\in \mathbf{Z}_{\ell_x}} \operatorname{td}_g(T_x\Sigma)\operatorname{ch}_g(V).$$

Since $\underline{\hat{V}}$ is flat, the first summand is

$$\begin{aligned}\operatorname{rk}\underline{V}\cdot\int_\Sigma \operatorname{td}(T\Sigma)\operatorname{ch}(E) &= \operatorname{rk}\underline{V}\cdot\operatorname{index}\bar\partial_E\\ &= \operatorname{rk}\underline{V}\cdot\operatorname{index}\mathfrak{d}_E.\end{aligned}$$

To evaluate the contribution of $x\in Z$ to the second summand observe that the index formula for the local system $\underline{V}$ restricted to the orbifold $[\{x\}/\mathbf{Z}_{\ell_x}]$ reads

$$\begin{aligned}\dim V^{\mu_x} &= \operatorname{index}\bar\partial_{\underline{\hat{V}}_x}\\ &= \int_{\{x\}}\operatorname{td}(T\{x\})\operatorname{ch}(\underline{\hat{V}}_x) + \frac{1}{\ell_x}\sum_{g\neq 1\in\mathbf{Z}_{\ell_x}}\operatorname{td}_g(T_x\Sigma)\operatorname{ch}_g(E_x\otimes V).\end{aligned}$$

The first term on the right-hand side is simply $\dim V$. Therefore,

$$\begin{aligned}\frac{1}{\ell_x}\sum_{g\neq 1\in\mathbf{Z}_{\ell_x}}\operatorname{td}_g(T_x\Sigma)\operatorname{ch}_g(E_x\otimes V) &= \dim V^{\mu_x}-\dim V\\ &= -\dim(V/V^{\mu_x}).\end{aligned}$$

This finishes the proof of the index formula. □

Remark 11.7. Instead of working with punctured Riemann surfaces we could also work with orbifold Riemann surfaces. Indeed, if $\pi\colon\tilde\Sigma\to\Sigma$ is a branched cover, then $\pi_*\underline{\mathbf{R}}$ is not a local system on Σ but it is a local system on an orbifold whose underlying topological space is Σ but which has non-trivial isotropy groups over the branching locus.

12 From branched covers to local systems

The following allows us to detect the failure of super-rigidity using local systems over punctured Riemann surfaces.

Proposition 12.1. *Let $u\colon(\Sigma,j)\to(M,J)$ be a non-constant J–holomorphic map. If $\pi\colon(\tilde\Sigma,\tilde j)\to(\Sigma,j)$ is a non-constant holomorphic map such that $u\circ\pi$ is not rigid, there exists a finite set $Z\subset\Sigma$ and an irreducible Euclidean local system $\underline{V}$ on $\dot\Sigma=\Sigma\backslash Z$ whose monodromy representation has kernel of finite index such that the following holds. Set $\dot u:=u|_{\dot\Sigma}$. For $\delta\in\mathbf{R}$ denote by*

$$\mathfrak{d}^{N,\underline{V}}_{\dot u,\delta}\colon W^{1,2}_{\mathrm{cyl}}\Gamma(N\dot u\otimes\underline{V})\to L^2_{\mathrm{cyl}}\Omega^{0,1}(\dot\Sigma,N\dot u\otimes\underline{V})$$

the extension of $\mathfrak{d}^{N,\underline{V}}_{\dot u}$. For $0<\delta\ll 1$ the operator $\mathfrak{d}^{N,\underline{V}}_{\dot u,\delta}$ has a non-trivial kernel.

Proof. Let Z be the branching locus of π. Set

$$\tilde{u} \coloneqq u \circ \pi, \quad \tilde{Z} \coloneqq \pi^{-1}(Z), \quad \dot{\tilde{\Sigma}} \coloneqq \tilde{\Sigma} \backslash \tilde{Z}, \quad \dot{\pi} \coloneqq \pi|_{\dot{\tilde{\Sigma}}}, \quad \text{and} \quad \dot{\tilde{u}} \coloneqq \tilde{u}|_{\dot{\tilde{\Sigma}}}.$$

For every $x \in \tilde{Z}$ choose charts $\tilde{\phi}_x \colon D \to \tilde{\Sigma}$ and $\phi_x \colon D \to \Sigma$ such that $\tilde{\phi}_x(0) = x$, $\tilde{\phi}_x(\dot{D}) \cap \tilde{Z} = \varnothing$, and

$$\pi(\tilde{\phi}_x(z)) = \phi_x(z^{r_x})$$

with $r_x \in \mathbf{N}_0$ denoting the ramification index of x. Let g_{cyl} be a Riemannian metric on $\dot{\Sigma}$ and let $\tau \colon \dot{\Sigma} \to [0, \infty)$ be a smooth function such that

$$\psi^* \phi_x^* g_{\mathrm{cyl}} = (\mathrm{d}t)^2 + (\mathrm{d}\alpha)^2 \quad \text{and} \quad \tau \circ \phi_x \circ \psi(t, \alpha) = t.$$

Set

$$\tilde{g}_{\mathrm{cyl}} \coloneqq \dot{\pi}^* g_{\mathrm{cyl}} \quad \text{and} \quad \tilde{\tau} \coloneqq t \circ \dot{\pi}.$$

These satisfy

$$\psi^* \tilde{\phi}_x^* \tilde{g}_{\mathrm{cyl}} = r_x \cdot \left((\mathrm{d}t)^2 + (\mathrm{d}\alpha)^2\right) \quad \text{and} \quad \tau \circ \phi_x \circ \psi(t, \alpha) = r_x \cdot t.$$

By Proposition 11.3 for $0 < \delta \leqslant \min\{1/r_x : x \in Z\}$ we have

$$\ker \mathfrak{d}_{\dot{\tilde{u}}, \delta} \cong \ker \mathfrak{d}_{\tilde{u}} \quad \text{and} \quad \operatorname{coker} \mathfrak{d}_{\dot{\tilde{u}}, \delta} \cong \operatorname{coker} \mathfrak{d}_{\tilde{u}}.$$

Set $\underline{V} \coloneqq \dot{\pi}_* \underline{\mathbf{R}}$. With respect to the isomorphisms

$$\dot{\pi}_* \colon \Gamma(N\dot{\tilde{u}}) \cong \Gamma(N\dot{u} \otimes \underline{V}) \quad \text{and} \quad \dot{\pi}_* \colon \Omega^{0,1}(\dot{\tilde{\Sigma}}, N\dot{\tilde{u}}) \cong L^2\Omega^{0,1}(\dot{\Sigma}, N\dot{u} \otimes \underline{V})$$

from Proposition 3.5, we have

$$\|\dot{\pi}_* s\|_{W^{1,2}_{\mathrm{cyl}}} = \|s\|_{W^{1,2}_{\mathrm{cyl}}} \quad \text{and} \quad \|\dot{\pi}^* s\|_{L^2_{\mathrm{cyl}}} = \|s\|_{L^2_{\mathrm{cyl}}}$$

as well as

$$\mathfrak{d}^{N,\underline{V}}_{\dot{u},\delta} = \dot{\pi}_* \circ \mathfrak{d}^N_{\tilde{u},\delta} \circ \dot{\pi}_*^{-1}. \tag{12.2}$$

It follows that if $\mathfrak{d}^N_{\tilde{u}}$ has a non-trivial kernel, then so does $\mathfrak{d}^{N,\underline{V}}_{\dot{u},\delta}$.

Decompose $\underline{V}$ into irreducible local systems

$$\underline{V} \cong \bigoplus_{i=1}^m \underline{V}_i^{\oplus \ell_i}.$$

The operator $\mathfrak{d}^{N,\underline{V}}_{\dot{u},\delta}$ decomposes accordingly as

$$\mathfrak{d}^{N,\underline{V}}_{\dot{u},\delta} = \bigoplus_{i=1}^m \left(\mathfrak{d}^{N,\underline{V}_i}_{\dot{u},\delta}\right)^{\oplus \ell_i}.$$

Consequently, if $\mathfrak{d}^N_{\tilde{u}}$ has a non-trivial kernel, then the same must hold for $\mathfrak{d}^{N,\underline{V}_i}_{\dot{u},\delta}$ for at least one $i \in \{1, \ldots, m\}$. □

13 Local systems over punctured Riemann surfaces

We determine the codimensions of the loci at which the phenomenon described in Proposition 12.1 occurs. The this the crucial ingredient in the proof of Theorem 7.18.

Definition 13.1. Let Σ be a closed, connected surface and let $Z \subset \Sigma$ be a finite subset. Let $N \triangleleft \pi_1(\Sigma \backslash Z, x_0)$ be a normal subgroup with finite index and set $G := \pi_1(\Sigma \backslash Z, x_0)/N$. Let $\mathbf{K}$ be either $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$. Denote by $\mathscr{ML}^{\Sigma,Z}_{\mathbf{K},N}(\omega)$ the set of equivalence classes of triples $(J; u, j; \underline{V})$ consisting of:

- an almost complex structure $J \in \mathscr{J}(\omega)$,
- a embedded J–holomorphic map $u\colon (\Sigma, j) \to (M, J)$ of index zero, and
- an irreducible Euclidean local system $\underline{V}$ on $\Sigma \backslash Z$ which does not extend across any $x \in Z$ and whose monodromy representation factors through G and has commuting algebra $\mathbf{K}$

subject to the constraint

$$J \in \mathscr{J}_P(\omega).$$

The equivalence relation on the triples $(J; u, j; \underline{V})$ is generated by reparametrization by $\mathrm{Diff}(\Sigma, Z) := \{\phi \in \mathrm{Diff}(\Sigma) : \phi|_Z = \mathrm{id}_Z\}$ and isomorphisms of local local systems. We set

$$\mathscr{ML}^{\Sigma,Z}_{\mathbf{K}}(\omega) := \coprod_N \mathscr{ML}^{\Sigma,Z}_{\mathbf{K},N}(\omega).$$

Set $r := \#Z$. The projection map $\mathscr{ML}^{\Sigma,Z}_{\mathbf{K}}(\omega) \to \mathscr{M}^{\Sigma}_0(\omega)$ given by $(J; [u, j; V]) \mapsto (J, [u, j])$ is a submersion with $2r$–dimensional fibers. In light of Proposition 12.1, the subsets defined in the following are responsible for the failure of super-rigidity.

Definition 13.2. In the situation of Definition 13.1, for every $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}\,\mathbf{H}\}$, N, and $d \in \mathbf{N}_0$ we define

$$\mathscr{W}^{\Sigma,Z}_{\mathbf{K},N,d}(\omega) := \left\{(J; [u, j; \underline{V}]) \in \mathscr{ML}^{\Sigma,Z}_{\mathbf{K},N} : \dim_{\mathbf{K}} \ker \mathfrak{d}^{N,\underline{V}}_{u,\delta} = d \text{ for } 0 < \delta \ll 1\right\},$$

and

$$\mathscr{W}^{\Sigma,Z}_{\mathbf{K},d}(\omega) := \coprod_N \mathscr{W}^{\Sigma,Z}_{\mathbf{K},N,d}(\omega).$$

As we will see shortly, the subsets defined in the following are the most typical cause of the failure of super-rigidity. They do not play a role in the proof of Theorem 7.18, but are of crucial importance in Section 15.

Definition 13.3. In the situation of Definition 13.1, denote by

$$\mathscr{W}^{\Sigma,Z}_{\mathbf{R},N,1,\bullet}(\omega) \subset \mathscr{W}^{\Sigma,Z}_{\mathbf{R},N,1}(\omega)$$

the subset consisting of all $(J; [u, j; \underline{V}])$ such that:

1. u is an immersion,

2. $\dim(V/V^{\mu_x}) = 1$ for every $x \in Z$, and

3. every other local system over $\Sigma\backslash Z$ whose monodromy representation factors through G extends to Σ.

Set

$$\mathscr{W}^{\Sigma,Z}_{\mathbf{R},1,\bullet}(\omega) := \coprod_N \mathscr{W}^{\Sigma,Z}_{\mathbf{R},N,1}(\omega).$$

Proposition 13.4. *Assume the situation of Definition 13.1. Set $k := \dim_{\mathbf{R}} \mathbf{K}$. For every $d \in \mathbf{N}_0$ the following hold:*

1. *The subset $\mathscr{W}^{\Sigma,Z}_{\mathbf{K},d}(\omega) \subset \mathscr{ML}^{\Sigma,Z}_{\mathbf{K}}(\omega)$ has codimension at least $k(d + (n-1)r)$.*

2. *The subset $\mathscr{W}^{\Sigma,Z}_{\mathbf{R},1,\bullet}(\omega) \subset \mathscr{ML}^{\Sigma,Z}_{\mathbf{R},N}(\omega)$ is smooth and has codimension $(1 + (n-1)r)$.*

3. *The subset $\mathscr{W}^{\Sigma,Z}_{\mathbf{R},1}(\omega)\backslash\mathscr{W}^{\Sigma,Z}_{\mathbf{R},1,\bullet}(\omega) \subset \mathscr{ML}^{\Sigma,Z}_{\mathbf{R},N}(\omega)$ has codimension greater than $(1 + (n-1)r)$.*

Proof. Set

$$\mathscr{P} := \mathscr{ML}^{\Sigma,Z}_{\mathbf{K},N}(\omega).$$

Define Hilbert space bundles $\mathscr{X}$ and $\mathscr{Y}$ over $\mathscr{P}$ with fibers

$$\mathscr{X}_{(J;[u,j;\underline{V}])} = e^{\delta\tau} W^{1,2}_{\mathrm{cyl}}\Gamma(N\dot{u}) \quad \text{and} \quad \mathscr{Y}_{(J;[u,j;\underline{V}])} = e^{\delta\tau} L^2\Omega^{0,1}_{\mathrm{cyl}}(\dot{\Sigma}, N\dot{u})$$

with $\dot{\Sigma} := \Sigma\backslash Z$ and $\dot{u} := u|_{\dot{Z}}$. Let $\mathscr{L}(\mathscr{X},\mathscr{Y})$ be a Banach space bundle over $\mathscr{P}$ whose fibers are the spaces of bounded linear operators between the fibers of $\mathscr{X}$ and $\mathscr{Y}$. Define the section $\mathfrak{d}^N \colon \mathscr{P} \to \mathscr{L}(\mathscr{X},\mathscr{Y})$ by

$$\mathfrak{d}^N_{(J;[u,j;\underline{V}])} := \mathfrak{d}^N_{\dot{u},\delta}.$$

In Definition 2.1 we assumed D to be a map into $\mathscr{L}(W^{k,2}\Gamma(E), L^2\Gamma(F))$. However, this can generalized to cover the situation at hand. In the situation above the Sobolev spaces are weighted and E and F depend on $p \in \mathscr{P}$. The weights are insubstantial because in light of the commutative diagram

$$\begin{array}{ccc} W^{1,2}_{\mathrm{cyl}}\Gamma(N\dot{u}) & \xrightarrow{e^{-\delta\tau}\mathfrak{d}^N_{\dot{u},\delta}e^{\delta\tau}} & L^2\Omega^{0,1}_{\mathrm{cyl}}(\dot{\Sigma}, N\dot{u}) \\ \downarrow e^{\delta\tau} & & \downarrow e^{\delta\tau} \\ e^{\delta\tau}W^{1,2}_{\mathrm{cyl}}\Gamma(N\dot{u}) & \xrightarrow{\mathfrak{d}^N_{\dot{u},\delta}} & e^{\delta\tau}L^2\Omega^{0,1}_{\mathrm{cyl}}(\dot{\Sigma}, N\dot{u}) \end{array}$$

we might as well work with $e^{-\delta\tau}\mathfrak{d}^N_{\dot{u},\delta}e^{\delta\tau}$ acting between unweighted spaces. As discussed in the proof of Proposition 10.2, the dependence of E and F on $p \in \mathscr{P}$ also is not an issue, because we can locally trivialize the dependence on $p \in \mathscr{P}$. The notion of equivariant flexibility is independent of the choice of trivialization. The Petri conditions are manifestly independent of the choice of trivialization. Having explained this, we will consider $(\mathfrak{d}_p^N)_{p\in\mathscr{P}}$ as a family of linear elliptic differential operators in this slightly generalized sense.

By Proposition 9.1, $(\mathfrak{d}_p^N)_{p\in\mathscr{P}}$ is strongly flexible and, therefore, G–equivariantly flexible for every G. By hypothesis, $\mathfrak{d}_p^N$ satisfies Petri's condition for every $p \in \mathscr{P}$. Therefore, we can apply Theorem 4.4.

Since $\underline{V}_1 = \underline{V}$ does not extend over any $x \in Z$, the subspaces $V_1^{\mu_x} \subset V_1$ defined in Proposition 11.5 must be non-trivial $\mathbf{K}_1$–linear subspaces. Because the normal bundle $N\dot{u}$ has rank $n-1$ and Z has precisely r elements, it follows from Proposition 11.5 (3) that

$$\operatorname{index}\mathfrak{d}_{u,\delta}^{N,\underline{V}_1} \leqslant -(n-1)k_1 r$$

and $\operatorname{index}\mathfrak{d}_{u,\delta}^{N,\underline{V}_i} \leqslant 0$ for $i \in \{2,\dots,m\}$. For every $(\underline{d},\underline{e}) \in \mathbf{N}_0^m \times \mathbf{N}_0^m$ with $\mathscr{P}_{\underline{d},\underline{e}} \neq \varnothing$ we must have

$$\operatorname{index}\mathfrak{d}_{u,\delta}^{N,\underline{V}_i} = k_i(d_i - e_i).$$

In particular,

$$e_1 \geqslant d_1 + (n-1)r$$

and $e_i \geqslant d_i$ for $i \in \{2,\dots,m\}$. Consequently, if $d_1 = d$, then

$$\operatorname{codim}\mathscr{P}_{\underline{d},\underline{e}} \geqslant \sum_{i=1}^m k_i d_i e_i \geqslant \sum_{i=1}^m k_i e_i \geqslant k(d + (n-1)r). \tag{13.5}$$

This directly implies (1). If $\mathbf{K} = \mathbf{R}$ and $d = 1$, then the inequality (13.5) is sharp precisely when:

1. u is an immersion,
2. $\dim(V/V^{\mu+x}) = 1$ for every $x \in Z$, and
3. $\underline{V}_i$ extends to Σ for every $i \in \{1,\dots,m\}$.

This implies (2) and (3). □

14 Proof of Theorem 7.18

We want to show that $\mathscr{J}_{\diamond,P}(\omega)$, the set of super-rigid almost complex structures, is residual in $\mathscr{J}(\omega)$. This will follow from combining the results of the preceding sections with Proposition 8.4.

For every $g, r \in \mathbf{N}_0$, fix a closed, connected surface Σ_g of genus g and a subset $Z_{g,r} \subset \Sigma_g$ with precisely r elements. Abbreviate

$$\mathscr{M}_0^g(\omega) := \mathscr{M}_0^{\Sigma_g}(\omega), \quad \mathscr{ML}_{\mathbf{K}}^{g,r}(\omega) := \mathscr{ML}_{\mathbf{K}}^{\Sigma_g,Z_{g,r}}(\omega), \quad \text{and} \quad \mathscr{W}_{\mathbf{K},d}^{g,r}(\omega) := \mathscr{W}_{\mathbf{K},d}^{\Sigma_g,Z_{g,r}}(\omega).$$

Denote by $\Pi\colon \mathcal{M}_0^{\amalg}(\omega) \to \mathcal{J}_P(\omega)$ and $\Pi\colon \mathcal{ML}^g_{\mathrm{K},r} \to \mathcal{J}_P(\omega)$ the projections to $\mathcal{J}_P(\omega)$. These are Fredholm maps of index 0 and $2r$, respectively.

Recall from Definition 7.12 that super-rigidity consists of three conditions (1), (2), and (3). Set

$$\mathcal{W}_{\geqslant 1}(\omega) := \bigcup_{\substack{g,r,d\in\mathbf{N}_0\\ \mathrm{K}\in\{\mathbf{R},\mathbf{C},\mathbf{H}\}}} \mathcal{W}^{g,r}_{\mathrm{K},d+1}(\omega).$$

By Proposition 12.1, $\Pi(\mathcal{W}_{\geqslant 1}(\omega))$ is the set of those complex structures for which (1) fails. By definition, $\mathcal{J}_P(\omega)\backslash\mathcal{J}_{\geqslant 0}(\omega)$ and $\Pi(\mathcal{W}_{\not\subset}(\omega))$ are the sets of almost complex structures for which (2) and (3) fail, respectively. Consequently, the set $\mathcal{J}_P(\omega)\backslash\mathcal{J}_{\diamond,P}(\omega)$ of almost complex structures which are fail to be super-rigid satisfy

$$\mathcal{J}_P(\omega)\backslash\mathcal{J}_{\diamond,P}(\omega) \subset \mathcal{J}_P(\omega)\backslash\mathcal{J}_{\geqslant 0}(\omega) \cup \Pi(\mathcal{W}_{\not\subset}(\omega)) \cup \Pi(\mathcal{W}_{\geqslant 1}(\omega)).$$

Each of the sets in this union is meager:

1. By Proposition 8.3, $\mathcal{J}_{\geqslant 0}(\omega)$ is residual.
2. By Theorem 8.6, $\mathcal{W}_{\not\subset}(\omega)$ has codimension $2(n-2) \geqslant 2$. Hence, by Proposition 8.4 its image $\Pi(\mathcal{W}_{\not\subset}(\omega))$ under the index zero Fredholm map Π is meager.
3. By Proposition 13.4, $\mathcal{W}^{g,r}_{\mathrm{K},d+1}(\omega) \subset \mathcal{ML}^{g,r}_{\mathrm{K}}$ has codimension at least

$$k(d + (n-1)r) \geqslant 2r + 1;$$

   hence, its image under Π is meager.

This proves that $\mathcal{J}_\diamond(\omega)$ is residual. ◻

15 Super-rigidity along paths of almost complex structures

Even though super-rigidity holds for a generic almost complex structure, it may fail for some almost complex structures along a generic path. The purpose of this section is to prove Theorem 15.2, which describes this failure in detail.

Let M be a manifold and let $\boldsymbol{\omega} = (\omega_t)_{t\in[0,1]}$ be a path symplectic structures. Set

$$\mathcal{J}_P(\boldsymbol{\omega}) := \bigcup_{t\in[0,1]} \mathcal{J}_P(\omega_t) \quad \text{and} \quad \mathcal{M}_0(\boldsymbol{\omega}) := \bigcup_{t\in[0,1]} \mathcal{M}_0(\omega_t).$$

$\mathcal{J}_P(\boldsymbol{\omega})$ is a separable Banach manifold. The proof of Theorem 8.2 shows that $\mathcal{M}_0(\boldsymbol{\omega})$ is a separable Banach manifold and $\Pi\colon \mathcal{M}_0(\boldsymbol{\omega}) \to \mathcal{J}_P(\boldsymbol{\omega})$ is a Fredholm map of index zero. Similarly, in the notation of Section 14,

$$\mathcal{ML}^{g,r}_{\mathrm{K}}(\boldsymbol{\omega}) := \bigcup_{t\in[0,1]} \mathcal{ML}^{g,r}_{\mathrm{K}}(\omega_t)$$

is a separable Banach manifold and $\Pi\colon \mathscr{ML}_{\mathrm{K}}^{g,r} \to \mathscr{J}_P(\omega)$ is a Fredholm map of index $2r$.

Fix $J_0 \in \mathscr{J}_{\diamond,P}(\omega_0)$ and $J_1 \in \mathscr{J}_{\diamond,P}(\omega_1)$. Denote by $\boldsymbol{\mathscr{J}}_P = \boldsymbol{\mathscr{J}}_P(J_0, J_1; \boldsymbol{\omega})$ the separable Banach manifold of paths of almost complex structures $(J_t)_{t\in[0,1]}$ from J_0 to J_1 with $J_t \in \mathscr{J}_P(\omega_t)$. Define the evaluation map $\mathrm{ev}\colon \boldsymbol{\mathscr{J}}_P \times [0,1] \to \mathscr{J}_P(\boldsymbol{\omega})$ by

$$\mathrm{ev}\left((J_t)_{t\in[0,1]}, t_\star\right) := J_{t_\star}.$$

Denote by $\boldsymbol{\mathscr{M}}_0 = \boldsymbol{\mathscr{M}}_0(J_0, J_1; \boldsymbol{\omega})$ the space of triples consisting of a path of almost complex structures $(J_t)_{t\in[0,1]} \in \boldsymbol{\mathscr{J}}_P$, $t \in [0,1]$, and $[u_t, j_t]_{t\in[0,1]} \in \mathscr{M}_0(J_t) := \Pi^{-1}(J_t) \subset \mathscr{M}(\boldsymbol{\omega})$. In other words, $\boldsymbol{\mathscr{M}}_0$ is the fibered product

$$\boldsymbol{\mathscr{M}}_0 = (\boldsymbol{\mathscr{J}}_P \times [0,1]) \times_{\mathscr{J}_P(\boldsymbol{\omega})} \mathscr{M}(\boldsymbol{\omega}).$$

Similarly, define

$$\boldsymbol{\mathscr{ML}}_{\mathrm{K}}^{g,r} = (\boldsymbol{\mathscr{J}}_P \times [0,1]) \times_{\mathscr{J}_P(\boldsymbol{\omega})} \mathscr{ML}_{\mathrm{K}}^{g,r}(\boldsymbol{\omega}).$$

By slight abuse of notation, we also denote by

$$\Pi\colon \boldsymbol{\mathscr{M}}_0 \to \boldsymbol{\mathscr{J}}_P \quad \text{and} \quad \Pi\colon \boldsymbol{\mathscr{ML}}_{\mathrm{K}}^{g,r} \to \boldsymbol{\mathscr{J}}_P.$$

the canonical projection maps. The following result equips $\boldsymbol{\mathscr{M}}_0$ and $\boldsymbol{\mathscr{ML}}_{\mathrm{K}}^{g,r}$ with the structure of a separable Banach manifold and exhibits Π as a Fredholm map of index one and index $2r+1$, respectively.

Proposition 15.1. *Let X, Y, Z be Banach manifolds. Let $f\colon X \to Z$ be a smooth map and let $g\colon Y \to Z$ be a submersion.*

1. *The* ***fibered product*** *$X \times_Z Y$, defined by*

$$X \times_Z Y = g^*Y := \{(x,y) \in X \times Y : f(x) = g(y)\}$$

   is a smooth submanifold of $X \times Y$.

2. *If $f\colon X \to Z$ is a Fredholm map of index i, then the projection map*

$$\pi_Y\colon X \times_Z Y \to Y$$

   is a Fredholm map of index i as well.

3. *If $W \subset X$ is a submanifold of codimension d, then $W \times_Z Y \subset X \times_Z Y$ is a submanifold of codimension d as well.*

Proof. Denote the diagonal in $Y \times Y$ by Δ. By definition,

$$X \times_Z Y = (f \times g)^{-1}(\Delta).$$

Since g is a submersion, the map $f \times g$ is transverse to Δ. It thus follows from the Regular Value Theorem that $X \times_Z Y$ is smooth. This proves (1).

By definition, we have

$$T_{(x,y)}X \times_Z Y = \big\{(\hat{x}, \hat{y}) \in T_xX \oplus T_yY : \mathrm{d}_x f(\hat{x}) = \mathrm{d}_y g(\hat{y})\big\}.$$

Consequently,

$$\ker \mathrm{d}_{(x,y)}\pi_Y = \big\{(\hat{x}, 0) \in T_xX \oplus T_yY : \mathrm{d}_x f(\hat{x}) = 0\big\} \cong \ker \mathrm{d}_x f$$

and

$$\operatorname{im} \mathrm{d}_{(x,y)}\pi_Y = \big\{\hat{y} \in T_yY : \mathrm{d}_y g(\hat{y}) \in \operatorname{im} \mathrm{d}_x f\big\}.$$

The latter gives rise to the following commutative diagram in which all rows and all columns are exact:

$$\begin{array}{ccccc}
\ker \mathrm{d}_y g & \xrightarrow{=} & \ker \mathrm{d}_y g & & \\
\big\downarrow & & \big\downarrow & & \\
\operatorname{im} \mathrm{d}_{(x,y)}\pi_Y & \hookrightarrow & T_yY & \twoheadrightarrow & \operatorname{coker} \mathrm{d}_{(x,y)}\pi_Y \\
\big\downarrow{\scriptstyle \mathrm{d}_y g} & & \big\downarrow{\scriptstyle \mathrm{d}_y g} & & \vdots \\
\operatorname{im} \mathrm{d}_x f & \hookrightarrow & T_{f(x)}Z & \twoheadrightarrow & \operatorname{coker} \mathrm{d}_x f.
\end{array}$$

A diagram chase constructs the dashed linear map making the bottom right square commutative. A further diagram chase shows that this linear maps is an isomorphism. (Alternatively, one can quotient $\operatorname{im} \mathrm{d}_{(x,y)}\pi_Y$ and T_yY by $\ker \mathrm{d}_y g$. The maps induced by $\mathrm{d}_y g$ then become isomorphisms and it follows that the cokernels are isomorphic.) This proves (2).

The argument used to prove (1) shows that $W \times_Z Y$ is a smooth submanifold of $X \times_Z Y$. To determine its codimension, consider the following diagram in which all rows and all columns are exact:

$$\begin{array}{ccccc}
T_{(x,y)}W \times_Z Y & \hookrightarrow & T_xW \oplus T_yY & \twoheadrightarrow & T_{f(x)}Z \\
\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle =} \\
T_{(x,y)}X \times_Z Y & \hookrightarrow & T_xX \oplus T_yY & \twoheadrightarrow & T_{f(x)}Z \\
\big\downarrow & & \big\downarrow & & \\
N_{(x,y)}W \times_Z Y & \dashrightarrow & N_xW. & &
\end{array}$$

A diagram chase constructs the dashed linear map making the bottom left square commutative. (Alternatively, replace $T_xW \oplus T_yY$ by the kernel of the map to $T_{f(x)}Z$ and do the same for $T_xX \oplus T_yY$.) A further diagram chase shows that this linear maps is an isomorphism. This proves (3). ☐

Theorem 15.2 (cf. [Wen16, Section 2.4]). *Let M be a manifold of dimension $\dim M = 2n \geqslant 6$. Given a path of symplectic structures $\boldsymbol{\omega} = (\omega_t)_{t\in[0,1]}$, $J_0 \in \mathscr{J}_{\diamond,P}(\omega_0)$, and $J_1 \in \mathscr{J}_{\diamond,P}(\omega_1)$, there is a residual subset $\boldsymbol{\mathscr{J}}_{\diamond,P} \subset \boldsymbol{\mathscr{J}}_P(J_0, J_1; \boldsymbol{\omega})$ such that for every $\mathbf{J} = (J_t)_{t\in[0,1]} \in \boldsymbol{\mathscr{J}}_{\diamond,P}$ the following hold:*

1. *The path* $\mathbf{J}$ *is a regular value of* $\Pi\colon \boldsymbol{\mathscr{M}}_0 \to \boldsymbol{\mathscr{J}}_P$. *In particular,*

$$\mathscr{M}_0(\mathbf{J}) := \Pi^{-1}(\mathbf{J})$$

is a 1*–dimensional manifold with boundary.*

2. *For every* $t \in [0,1]$ *every* J_t*–holomorphic map of index zero is embedded, and every two simple* J_t*–holomorphic maps of index zero either have disjoint images or are related by a reparametrization.*

3. *Let* $g, r \in \mathbf{N}_0$, *and* $(J_t, t, [u, j; \underline{V}]) \in \mathscr{ML}_{\mathbf{K}}^{g,r}(\mathbf{J}) := \Pi^{-1}(\mathbf{J})$. *If* $\mathbf{K} = \mathbf{R}$, *then*

$$\dim\ker \mathfrak{d}_{\dot{u},\delta}^{N,\underline{V}} \leqslant 1 \quad \textit{for} \quad 0 < \delta \ll 1.$$

If $\mathbf{K} \in \{\mathbf{C}, \mathbf{H}\}$, *then*

$$\dim\ker \mathfrak{d}_{\dot{u},\delta}^{N,\underline{V}} = 0 \quad \textit{for} \quad 0 < \delta \ll 1.$$

4. *For every* $g, r \in \mathbf{N}_0$, *the subset*

$$\mathscr{W}_{\mathbf{R},1}^{g,r}(\mathbf{J}) := \left\{(t, [u, j; \underline{V}]) \in [0,1] \times \mathscr{ML}_{\mathbf{R}}^{g,r}(\mathbf{J}) : \dim\ker \mathfrak{d}_{\dot{u},\delta}^{N,\underline{V}} = 1 \textit{ for } \delta \ll 1\right\}$$

is a smooth submanifold of codimension $2r + 1$.

If $\dim M \geqslant 8$ *and* $r > 0$, *then* $\mathscr{W}_{\mathbf{R},1}^{g,r}(\mathbf{J})$ *is empty.*

If $(J_t; [u, j; \underline{V}]) \in \mathscr{W}_{\mathbf{R},1}^{g,r}(\mathbf{J})$, *then:*

   (a) $\dim(V/V^{\mu+x}) = 1$ *for every* $x \in Z$, *and*
   (b) *any irreducible local system on* $\Sigma\backslash Z$ *which is isomorphic to* $\underline{V}$ *but whose monodromy representation factors through the same quotient as that of* $\underline{V}$ *extends to* Σ.

Proof. Denote by $\boldsymbol{\mathscr{J}}^{\text{reg}}$ the set of regular values of Π.

Define $\mathscr{M}_0^{\amalg}(\boldsymbol{\omega})$ and $\Pi\colon \mathscr{M}_0^{\amalg}(\boldsymbol{\omega}) \to \mathscr{J}_P(\boldsymbol{\omega})$ in the obvious way. Set

$$\mathscr{W}_{\not\subset}(\boldsymbol{\omega}) = \bigcup_{t\in[0,1]} \mathscr{W}_{\not\subset}(\omega_t) \subset \mathscr{M}_0^{\amalg}(\boldsymbol{\omega}).$$

By Theorem 8.6, $\mathscr{W}_{\not\subset}(\boldsymbol{\omega}) \subset \mathscr{M}_0^{\amalg}(\boldsymbol{\omega})$ has codimension at least two. Therefore, $\mathrm{ev}^*\mathscr{W}_{\not\subset}(\omega) \subset \boldsymbol{\mathscr{M}}_0$ has codimension two in according to Proposition 15.1. Since $\Pi\colon \boldsymbol{\mathscr{M}}_0 \to \boldsymbol{\mathscr{J}}_P$ is a Fredholm map of index one, it follows from Proposition 8.4 that

$$\boldsymbol{\mathscr{J}}_{\not\subset} := \Pi(\mathrm{ev}^*\mathscr{W}_{\not\subset}(\boldsymbol{\omega}))$$

is meager.

Define $\mathscr{W}^{g,r}_{\mathbf{K},d}(\omega)$, and $\mathscr{W}^{g,r}_{\mathbf{R},1,\bullet}(\omega)$ in the obvious way. Set

$$\begin{aligned}\mathscr{J}^{g,r}_{\mathbf{K},r,d} &\coloneqq \Pi(\mathrm{ev}^*\mathscr{W}^{g,r}_{\mathbf{K},d}(\omega)) \subset \mathscr{J}_P, \quad \text{and} \\ \mathscr{J}^{g,r}_{\mathbf{R},1,\circ} &\coloneqq \Pi\left(\mathrm{ev}^*\left(\mathscr{W}^{g,r}_{\mathbf{R},1}(\omega)\backslash\mathscr{W}^{g,r}_{\mathbf{R},1,\bullet}(\omega)\right)\right) \subset \mathscr{J}_P.\end{aligned}$$

Since $\Pi\colon \mathscr{ML}^{g,r}_{\mathbf{K}} \to \mathscr{J}_P$ is a Fredholm map of index $2r+1$ and by Proposition 13.4, $\mathscr{J}^{g,r}_{\mathbf{K},1,\circ}$ is meager; moreover, if $d \geqslant 2$, or $d = 1$ and $\mathbf{K} \neq \mathbf{R}$, or $d = 1$, $r \geqslant 1$, and $n \geqslant 4$, then $\mathscr{J}^{g}_{\mathbf{K},r,d}$ is meager.

For $n = 3$, define $\mathscr{J}_{\diamond,P} \subset \mathscr{J}^{\mathrm{reg}}$ by

$$\mathscr{J}^{\mathrm{reg}}\backslash\mathscr{J}_{\diamond,P} \coloneqq \mathscr{J}_{\not\subset} \cup \bigcup_{r,d,g\in\mathbf{N}_0} \mathscr{J}^{g,r}_{\mathbf{R},d+2} \cup \bigcup_{\substack{r,d,g\in\mathbf{N}_0\\ \mathbf{K}\in\{\mathbf{C},\mathbf{H}\}}} \mathscr{J}^{g,r}_{\mathbf{K},d+1} \cup \bigcup_{r,g\in\mathbf{N}_0} \mathscr{J}^{g,r}_{\mathbf{R},1,\circ}.$$

For $n \geqslant 4$, define $\mathscr{J}_{\diamond} \subset \mathscr{J}^{\mathrm{reg}}$ by

$$\mathscr{J}^{\mathrm{reg}}\backslash\mathscr{J}_{\diamond,P} \coloneqq \mathscr{J}_{\not\subset} \cup \bigcup_{r,d,g\in\mathbf{N}_0} \mathscr{J}^{g,r}_{\mathbf{R},d+2} \cup \bigcup_{\substack{r,d,g\in\mathbf{N}_0\\ \mathbf{K}\in\{\mathbf{C},\mathbf{H}\}}} \mathscr{J}^{g,r}_{\mathbf{K},d+1} \cup \bigcup_{\substack{r,g\in\mathbf{N}_0\\ \mathbf{K}\in\{\mathbf{R},\mathbf{C},\mathbf{H}\}}} \mathscr{J}^{g,r+1}_{\mathbf{K},1} \cup \bigcup_{g\in\mathbf{N}_0} \mathscr{J}^{g,0}_{\mathbf{R},1,\circ}.$$

By construction $\mathscr{J}_{\diamond,P}$ is a residual subset of $\mathscr{J}_P$ and every $(J_t)_{t\in[0,1]} \in \mathscr{J}_{\diamond,P}$ satisfies (1), (2), (3), and (4). ◻

A The normal Cauchy–Riemann operator

The normal Cauchy–Riemann operator for embedded J–holomorphic maps can be traced back to the work of Gromov [Gro85, 2.1.B]. It was observed by Ivashkovich and Shevchishin [IS99, Section 1.3], and Wendl [Wen10, Section 3] that normal Cauchy–Riemann operator can be defined even for non-embedded J–holomorphic maps, and that it plays an important role in understanding the deformation theory of J–holomorphic curves. In this section we will briefly explain the construction of Tu and Nu, and discuss the proof of Proposition 7.9.

Let $u\colon (\Sigma, j) \to (M, J)$ be a non-constant J–holomorphic map. Denote by $\mathfrak{d}_{u,J}$ the real Cauchy–Riemann operator on u^*TM. Denote by $\bar{\partial}_{u,J}$ the complex linear part of $\mathfrak{d}_{u,J}$. This is a complex Cauchy–Riemann operator and gives u^*TM the structure of a holomorphic vector bundle

$$\mathscr{E} \coloneqq (u^*TM, \bar{\partial}_{u,J}).$$

Denote by $\mathscr{T}\Sigma$ the tangent bundle of Σ equipped with its natural holomorphic structure. The derivatives of u induce a holomorphic map $\mathrm{d}u\colon \mathscr{T}\Sigma \to \mathscr{E}$. The quotient of this map, thought of as a morphism of sheaves,

$$\mathscr{Q} \coloneqq \mathscr{E}/\mathscr{T}\Sigma$$

is a coherent sheaf on Σ. It is locally free outside the critical points of $\mathrm{d}u$. Denote by D the divisor of the critical points of $\mathrm{d}u$, counted with multiplicity. Near a critical point z_0 of order k we can write du as $(z - z_0)^k f(z)$ with $f(z_0) \neq 0$. Consequently, the torsion subsheaf of $\mathscr{Q}$ is $\mathscr{O}_D$, the structure sheaf of D. The quotient sheaf

$$\mathscr{N}u := \mathscr{Q}/\mathscr{O}_D$$

is torsion free, and so locally free because $\dim_{\mathbf{C}} \Sigma = 1$; that is, $\mathscr{N}u$ is a holomorphic vector bundle. Similarly, the sheaf

$$\mathscr{T}u := \ker(\mathscr{E} \to \mathscr{N}u)$$

is locally free. We call $\mathscr{N}u$ the **generalized normal bundle** of u and $\mathscr{T}u$ the **generalized tangent bundle** of u.

Proposition A.1. *We have*

$$\mathscr{T}u/\mathscr{T}\Sigma \cong \mathscr{O}_D \quad \textit{and} \quad \mathscr{T}u \cong \mathscr{T}\Sigma(D).$$

Proof. The following commutative diagram summarizes the construction of $\mathscr{T}u$ and $\mathscr{N}u$:

$$\begin{array}{ccccc}
 & & & & \mathscr{O}_D \\
 & & & & \downarrow \\
\mathscr{T}\Sigma & \hookrightarrow & \mathscr{E} & \twoheadrightarrow & \mathscr{Q} \\
\downarrow & & \downarrow = & & \downarrow \\
\mathscr{T}u & \hookrightarrow & \mathscr{E} & \twoheadrightarrow & \mathscr{N}u \\
\downarrow & & & & \\
\mathscr{T}u/\mathscr{T}\Sigma. & & & &
\end{array}$$

Since the columns and rows are exact sequences, it follows from the Snake Lemma that

$$\mathscr{O}_D \cong \mathscr{T}u/\mathscr{T}\Sigma.$$

This implies the assertion. □

Proof of Proposition 7.9. Let $\mathcal{S}$ be an $\mathrm{Aut}(\Sigma, j)$–invariant local slice around j of the Teichmüller space $\mathscr{T}(\Sigma)$. Recall that $\mathrm{d}_{u,j}\bar{\partial}_J \colon \Gamma(u^*TM) \oplus T_j\mathcal{S} \to \Omega^{0,1}(\Sigma, u^*TM)$ is the linearization of $\bar{\partial}_J$, defined in (7.2), restricted to $C^\infty(\Sigma, M) \times \mathcal{S}$. Denote by Tu the complex vector bundle underlying $\mathscr{T}u$ and by Nu the complex vector bundle underlying $\mathscr{N}u$. As was mentioned before Definition 7.8, $Tu \subset u^*TM$ is the unique complex subbundle of rank one containing d$u(T\Sigma)$. Using a Hermitian metric on u^*TM we obtain an isomorphism

$$u^*TM \cong Tu \oplus Nu.$$

With respect to this splitting $\mathfrak{d}_{J,u}$, the restriction of $\mathrm{d}_{u,j}\bar{\partial}_J$ to $\Gamma(u^*TM)$, can be written as

$$\mathfrak{d}_{J,u} = \begin{pmatrix} \mathfrak{d}^T_{u,J} & * \\ \dagger & \mathfrak{d}^N_{u,J} \end{pmatrix}$$

with $\mathfrak{d}_{u,J}^N$ denoting the normal Cauchy–Riemann operator introduced in Definition 7.8. Since

$$\bar{\partial}_{u,J} \circ \mathrm{d}u = \mathrm{d}u \circ \bar{\partial}_{T\Sigma} \quad \text{and} \quad \mathscr{T}u \cong \mathscr{T}\Sigma(D),$$

it follows that

$$\bar{\partial}_{u,J}^T = \bar{\partial}_{Tu} \quad \text{and} \quad \dagger = 0.$$

Denote by $\iota\colon T_j\mathcal{S} \to \Omega^{0,1}(\Sigma, u^*TM)$ the restriction of $\mathrm{d}_{u,j}\bar{\partial}_J$ to $T_j\mathcal{S}$. The tangent space to the Teichmüller space $\mathscr{T}(\Sigma)$ at $[j]$ can be identified with $\operatorname{coker} \bar{\partial}_{T\Sigma} \cong \ker \bar{\partial}_{T\Sigma}^*$. With respect to this identification, ι is the restriction of $\mathrm{d}u\colon T\Sigma \to u^*TM$ to $\ker \bar{\partial}_{T\Sigma}^*$. Consequently, we can write $\mathrm{d}_{u,j}\bar{\partial}_J\colon \Gamma(Tu) \oplus T_j\mathcal{S} \oplus \Gamma(Nu) \to \Gamma(Tu) \oplus \Gamma(Nu)$ as

$$\mathrm{d}_{u,j}\bar{\partial}_J = \begin{pmatrix} \bar{\partial}_{Tu} & \iota & * \\ 0 & 0 & \mathfrak{d}_{u,J}^N \end{pmatrix}.$$

The short exact sequence

$$0 \to \mathscr{T}\Sigma \to \mathscr{T}u \to \mathscr{O}_D \to 0$$

induces the following long exact sequence in cohomology

$$0 \to H^0(\mathscr{T}\Sigma) \to H^0(\mathscr{T}u) \to H^0(\mathscr{O}_D) \to H^1(\mathscr{T}\Sigma) \to H^1(\mathscr{T}u) \to 0.$$

It follows that

$$\operatorname{index} \bar{\partial}_{Tu} = 2\chi(\mathscr{T}u) = 2\chi(\mathscr{T}\Sigma) + 2h^0(\mathscr{O}_D) = \operatorname{index} \bar{\partial}_{T\Sigma} + 2Z(\mathrm{d}u),$$

that $\ker \bar{\partial}_{T\Sigma} \to \ker \bar{\partial}_{Tu}$ is injective, and that $\operatorname{coker} \bar{\partial}_{T\Sigma} \to \operatorname{coker} \bar{\partial}_{Tu}$ is surjective. The latter implies that $\bar{\partial}_{Tu} \oplus \iota$ is surjective. Therefore, there are an exact sequence

$$0 \to \ker \bar{\partial}_{Tu} \oplus \iota \to \ker \mathrm{d}_{u,j}\bar{\partial}_J \to \ker \mathfrak{d}_{u,J}^N \to 0,$$

and an isomorphism

$$\operatorname{coker} \mathrm{d}_{u,j}\bar{\partial}_J \cong \operatorname{coker} \mathfrak{d}_{u,J}^N.$$

The kernel of $\bar{\partial}_{Tu} \oplus \iota$ contains $\mathfrak{aut}(\Sigma, j) = \ker \bar{\partial}_{T\Sigma}$ and

$$\begin{aligned} \dim \ker \bar{\partial}_{Tu} \oplus \iota &= \operatorname{index} \bar{\partial}_{Tu} \oplus \iota \\ &= \operatorname{index} \bar{\partial}_{Tu} + \dim T_j\mathcal{S} \\ &= \operatorname{index} \bar{\partial}_{T\Sigma} + \dim T_j\mathcal{S} + 2Z(\mathrm{d}u) \\ &= \dim \mathfrak{aut}(\Sigma, j) + 2Z(\mathrm{d}u). \end{aligned}$$

This completes the proof of Proposition 7.9. □

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