On the existence of harmonic $\mathbf{Z}_2$ spinors

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Abstract

We prove the existence of singular harmonic $\mathbf{Z}_2$ spinors on 3–manifolds with $b_1 > 1$. The proof relies on a wall-crossing formula for solutions to the Seiberg–Witten equation with two spinors. The existence of singular harmonic $\mathbf{Z}_2$ spinors and the shape of our wall-crossing formula shed new light on recent observations made by Joyce [Joy17] regarding Donaldson and Segal's proposal for counting G_2–instantons [DS11].

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1 Introduction

The notion of harmonic $\mathbf{Z}_2$ spinors was introduced by Taubes [Tau14] as an abstraction of various limiting objects appearing in compactifications of moduli spaces of flat $\mathbf{PSL}_2(\mathbf{C})$–connections over 3–manifolds [Tau13a] and solutions to the Kapustin–Witten equation [Tau13b], the Vafa–Witten equation [Tau17], and the Seiberg–Witten equation with multiple spinors [HW15; Tau16].

Definition 1.1. Let M be a Riemannian manifold and $\mathbf{S}$ a Dirac bundle over M.[1] Denote by $\not{D}\colon \Gamma(\mathbf{S}) \to \Gamma(\mathbf{S})$ the associated Dirac operator. A $\mathbf{Z}_2$ **spinor** with values in $\mathbf{S}$ is a triple $(Z, \mathfrak{l}, \Psi)$ consisting of

1. a proper closed subset $Z \subset M$,

[1] A Dirac bundle is a bundle of Clifford modules together with a metric and a compatible connection; see, [LM89, Chapter II Definition 5.2].

2. a Euclidean line bundle $\mathfrak{l} \to M \backslash Z$, and

3. a section $\Psi \in \Gamma(M\backslash Z, \mathbf{S} \otimes \mathfrak{l})$

such that $|\Psi|$ extends to a continuous function on M with $|\Psi|^{-1}(0) = Z$ and $|\nabla\Psi|^2 \in L^2_{\mathrm{loc}}(M)$. We say that $(Z, \mathfrak{l}, \Psi)$ is **singular** if $\mathfrak{l}$ does not extend to a Euclidean line bundle on M. A $\mathbf{Z}_2$ spinor $(Z, \mathfrak{l}, \Psi)$ is called **harmonic** if

$$\not{D}\Psi = 0$$

holds on $M\backslash Z$.

Remark 1.2. Taubes [Tau14] proved that if M is three- or four-dimensional, then Z has Hausdorff codimension at least 2.

Remark 1.3. If $\mathfrak{l}$ extends to a Euclidean line bundle on M, then $\mathbf{S} \otimes \mathfrak{l}$ extends to a Dirac bundle on M and Ψ is extends to a harmonic spinor with values in $\mathbf{S} \otimes \mathfrak{l}$ vanishing precisely along Z.

The harmonic $\mathbf{Z}_2$ spinors appearing as limits of flat $\mathbf{PSL}_2(\mathbf{C})$–connections over 3–manifolds take values in $\underline{\mathbf{R}} \oplus T^*M$; those appearing as limits of the Seiberg–Witten equation with two spinors in dimension three take values in $\mathrm{Re}(S \otimes E)$ where S is the spinor bundle of some spin structure $\mathfrak{s}$ on the 3–manifold M and E is an rank two Hermitian bundle with trivial determinant and equipped with a compatible connection.[2]

Henceforth, we specialize to the case of M being 3–dimensional and $\mathbf{S} = \mathrm{Re}(S \otimes E)$. The Dirac operator on $\mathrm{Re}(S \otimes E)$, and thus also the notion of a harmonic $\mathbf{Z}_2$ spinor, depends on the choice of a Riemannian metric on M and a connection on E.

Definition 1.4. Let $\mathscr{M}et(M)$ be the space of Riemannian metrics on M and $\mathscr{A}(E)$ the space of SU(2) connections on E. The **space of parameters** is

$$\mathscr{P} := \mathscr{M}et(M) \times \mathscr{A}(E)$$

equipped with the C^∞ topology. Given a spin structure $\mathfrak{s}$ on M and $\mathbf{p} \in \mathscr{P}$, we denote by $\not{D}^{\mathfrak{s}}_{\mathbf{p}}$ the corresponding Dirac operator on $\Gamma(\mathrm{Re}(S \otimes E))$.

Question 1.5. For which parameters $\mathbf{p} \in \mathscr{P}$ does there exist a singular harmonic $\mathbf{Z}_2$ spinor with values in $\mathrm{Re}(S \otimes E)$?

[2] Both S and E are quaternionic vector bundles and therefore the complex vector bundle $S \otimes E$ carries a real structure; cf. Proposition A.1.

The answer to this question for non-singular harmonic $\mathbf{Z}_2$ spinors (that is: harmonic spinors) is well-understood. Let $\mathscr{W}^{\mathfrak{s}}$ be the set of $\mathbf{p} \in \mathscr{P}$ for which $\dim \ker \not{D}^{\mathfrak{s}}_{\mathbf{p}} > 0$. It is the closure of $\mathscr{W}^{\mathfrak{s}}_1$, the set of $\mathbf{p}$ for which $\dim \ker \not{D}^{\mathfrak{s}}_{\mathbf{p}} = 1$. Moreover, $\mathscr{W}^{\mathfrak{s}}_1$ is a cooriented, codimension one submanifold of $\mathscr{P}$ and $\mathscr{W}^{\mathfrak{s}} \backslash \mathscr{W}^{\mathfrak{s}}_1$ has codimension three. (See Proposition 9.3 and Proposition 9.9.) The intersection number of a path $(\mathbf{p}_t)_{t\in[0,1]}$ with $\mathscr{W}^{\mathfrak{s}}_1$ is given by the spectral flow of the path of operators $(\not{D}^{\mathfrak{s}}_{\mathbf{p}_t})_{t\in[0,1]}$. Therefore, along any path with non-zero spectral flow there exists a parameter $\mathbf{p}_\star$ such that $\dim \ker \not{D}^{\mathfrak{s}}_{\mathbf{p}_\star} > 0$. Moreover, if the path is generic,[3] then the kernel is spanned by a nowhere vanishing spinor (because $\dim M < \operatorname{rank} \mathbf{S}$; see Definition 4.1 and Proposition 8.1)

By contrast, little is known about the existence of singular harmonic $\mathbf{Z}_2$ spinors. The only examples known thus far have been obtained by means of complex geometry, on Riemannian 3–manifolds of the form $M = S^1 \times \Sigma$ for a Riemann surface Σ; see [Tau13a, Theorem 1.2] in the case $\mathbf{S} = \underline{\mathbf{R}} \oplus T^*M$ and [Doa17b, Section 3] in the case $\mathbf{S} = \operatorname{Re}(S \otimes E)$. We remedy this situation by proving—in a rather indirect way—that 3-manifolds abound with singular harmonic $\mathbf{Z}_2$ spinors.

Theorem 1.6. *For every compact, connected, oriented 3–manifold M with $b_1(M) > 1$ there exists $\mathbf{p}_\star \in \mathscr{P}$ and a singular harmonic $\mathbf{Z}_2$ spinor with respect to $\mathbf{p}_\star$. In fact, there is a closed subset $\mathscr{W}_b \subset \mathscr{P}$ and a non-trivial cohomology class $\omega \in H^1(\mathscr{P}\backslash\mathscr{W}_b, \mathbf{Z})$ with the property that if $(\mathbf{p}_t)_{t\in S^1}$ is a generic loop in $\mathscr{P}\backslash\mathscr{W}_b$ and*

$$\omega([\mathbf{p}_t]) \neq 0,$$

then for some $\mathbf{p}_\star$ in $(\mathbf{p}_t)_{t\in S^1}$ there exists a singular harmonic $\mathbf{Z}_2$ spinor.

Remark 1.7. The definition of $\mathscr{W}_b$ is given in Definition 9.1 and the precise meaning of generic loop is given in Definition 4.1 and Proposition 2.5.

Remark 1.8. Theorem 1.6 suggests that on 3–manifolds the appearance of singular harmonic $\mathbf{Z}_2$ spinors is a codimension one phenomenon—as is the appearance of harmonic spinors. This is in consensus with the work of Takahashi [Tak15; Tak17], who proved that the linearized deformation theory of singular harmonic $\mathbf{Z}_2$ spinors with $Z = S^1$ is an index zero Fredholm problem (or index minus one, after scaling is taken into account).

Remark 1.9. The assumption that $b_1(M) > 1$ has to do with reducible solutions to the Seiberg–Witten equation with two spinors. We expect it to be non-substantial;

[3] **Generic** means from a residual subset of the space of objects in question. A subset of a topological space is residual if it contains a countable intersection of open and dense subsets. Baire's theorem asserts that a residual subset of a complete metric space is dense.

the wall-crossing caused by reducible solutions should be amenable to the methods used by Chen [Che97] and Lim [Lim00] in classical Seiberg–Witten theory.

The proof of Theorem 1.6 relies on the wall-crossing formula for $n(\mathbf{p})$, the signed count of solutions to the Seiberg–Witten equation with two spinors, which is stated as Theorem 4.9. The number $n(\mathbf{p})$ is defined provided $\mathbf{p}$ is generic and there are no singular harmonic $\mathbf{Z}_2$ spinors with respect to $\mathbf{p}$. The wall-crossing formula can be stated as follows. Let $\mathscr{W}^{\mathfrak{s}}_{1,\varnothing}$ be the set of $\mathbf{p} \in \mathscr{P}$ for which $\ker \slashed{D}^{\mathfrak{s}}_{\mathbf{p}} = \mathbf{R}\langle\Psi\rangle$ with Ψ nowhere vanishing, and let $\mathscr{W}_{1,\varnothing}$ be the union of all $\mathscr{W}^{\mathfrak{s}}_{1,\varnothing}$ for all spin structures $\mathfrak{s}$. There is a closed subset $\mathscr{W}_b \subset \mathscr{P}$, as in Theorem 1.6, such that $\mathscr{W}_{1,\varnothing}$ is an immersed, closed, cooriented, codimension one submanifold of $\mathscr{P}\backslash\mathscr{W}_b$. If $(\mathbf{p}_t)_{t\in[0,1]}$ is a generic path in $\mathscr{P}\backslash\mathscr{W}_b$ and there are no singular harmonic $\mathbf{Z}_2$ spinors along $(\mathbf{p}_t)$, then the difference

$$n(\mathbf{p}_1) - n(\mathbf{p}_0)$$

is equal to the intersection number of the path $(\mathbf{p}_t)_{t\in[0,1]}$ with $\mathscr{W}_{1,\varnothing}$. In particular, if $(\mathbf{p}_t)_{t\in S^1}$ is a generic loop whose intersection number with $\mathscr{W}_{1,\varnothing}$ is non-zero, then there must be a singular harmonic $\mathbf{Z}_2$ spinor for some $\mathbf{p}_\star$ in $(\mathbf{p}_t)$.

Remark 1.10. Although the wall-crossing for $n(\mathbf{p})$ does occur when the spectrum of $\slashed{D}^{\mathfrak{s}}_{\mathbf{p}}$ crosses zero, the contribution of a nowhere vanishing harmonic spinor to the wall-crossing formula is not given by the sign of the spectral crossing but instead by the degree of the spinor. Therefore, the wall-crossing formula is not given by the spectral flow. This should be contrasted with the wall-crossing phenomenon for the classical Seiberg–Witten equation caused by reducible solutions, as in Remark 1.9. Indeed, the wall-crossing described in this paper is a result of the non-compactness of the moduli spaces of solutions and as such it is a new phenomenon, with no counterpart in classical Seiberg–Witten theory.

The cohomology class $\omega \in H^1(\mathscr{P}\backslash\mathscr{W}_b, \mathbf{Z})$ in Theorem 1.6 is defined by intersecting with $\mathscr{W}_{1,\varnothing}$. We prove that ω is non-trivial, by exhibiting a loop $(\mathbf{p}_t)_{t\in S^1}$ on which ω evaluates as ± 2. This is possible because $\mathscr{W}_b$ behaves like a codimension two subset. In particular, $\mathscr{P}\backslash\mathscr{W}_b$ is not simply-connected and we can take $(\mathbf{p}_t)_{t\in S^1}$ to be a small loop linking $\mathscr{W}_b$.

Remark 1.11. The discussion in the article is related to an observation made by Joyce [Joy17, Section 8.4] which points out potential issues with the Donaldson–Segal program for counting G_2–instantons. We discuss this in detail in Appendix B.

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2 The Seiberg–Witten equation with two spinors

Fix a spin^c structure $\mathfrak{w}$ and denote its spinor bundle by W. Set

$$\mathscr{Z} \coloneqq \ker\left(\mathrm{d}\colon\ \Omega^2(M, i\mathbf{R}) \to \Omega^3(M, i\mathbf{R})\right).$$

Definition 2.1. Let $\eta \in \mathscr{Z}$ and $\mathbf{p} = (g, B) \in \mathscr{P}$. The **$\eta$–perturbed Seiberg–Witten equation with two spinors** is the following differential equation for $(\Psi, A) \in \Gamma(\mathrm{Hom}(E, W)) \times \mathscr{A}(\det W)$:

$$\begin{aligned} \not{D}_A \Psi &= 0 \quad \text{and} \\ \frac{1}{2} F_A + \eta &= \mu(\Psi).^{4} \end{aligned} \tag{2.2}$$

Here $\not{D}_A = \not{D}_{A,\mathbf{p}}$ is the Dirac operator on $\Gamma(\mathrm{Hom}(E, W))$ induced by the spin connection on W obtained from A together with the Levi–Civita connection of g and the connection B on E; and $\mu(\Psi) = \mu_{\mathbf{p}}(\Psi) \coloneqq \Psi\Psi^* - \frac{1}{2}|\Psi|^2 \,\mathrm{id}_W$ is a section of $i\mathfrak{su}(W)$ which is identified with an element of $\Omega^2(M, i\mathbf{R})$ using the Clifford multiplication.

Let $\mathscr{G}(\det W)$ be the gauge group of $\det W$. For $(\mathbf{p}, \eta) \in \mathscr{P} \times \mathscr{Z}$, we denote by

$$\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta) \coloneqq \left\{ [\Psi, A] \in \frac{\Gamma(\mathrm{Hom}(E, W)) \times \mathscr{A}(\det W)}{\mathscr{G}(\det W)} : \begin{array}{l} (\Psi, A) \text{ satisfies (2.2)} \\ \text{with respect to } \mathbf{p} \text{ and } \eta \end{array} \right\}$$

the **moduli space** of solutions to (2.2).

As discussed in [Doa17b, Section 2; DW17, Section 2], the infinitesimal deformation theory of (2.2) around a solution (Ψ, A), is controlled by the linear operator

$$\begin{aligned} L_{\Psi,A}\colon\ &\Gamma(\mathrm{Hom}(E, W) \oplus \Omega^1(M, i\mathbf{R}) \oplus \Omega^0(M, i\mathbf{R}) \\ &\qquad\qquad \to \Gamma(\mathrm{Hom}(E, W)) \oplus \Omega^1(M, i\mathbf{R}) \oplus \Omega^0(M, i\mathbf{R}) \end{aligned}$$

[4] While (2.2) makes sense for any $\eta \in \Omega^2(M, i\mathbf{R})$, the existence of a solution implies that $d\eta = 0$; see [Doa17b, Lemma 2.3; DW17, Proposition A.4].

defined by

$$L_{\Psi,A} := \begin{pmatrix} -\not{D}_A & -\mathfrak{a}_\Psi \\ -\mathfrak{a}_\Psi^* & \mathfrak{d} \end{pmatrix}$$

with

$$\mathfrak{d} := \begin{pmatrix} *\mathrm{d} & \mathrm{d} \\ \mathrm{d}^* & \end{pmatrix} \quad \text{and} \quad \mathfrak{a}_\Psi := \begin{pmatrix} \bar{\gamma}(\cdot)\Psi & \rho(\cdot)\Psi \end{pmatrix}. \tag{2.3}$$

[5]

Here $\bar{\gamma}$ is the Clifford multiplication by elements of $T^*M \otimes i\mathbf{R}$ and ρ is the linearized action of the gauge group: pointwise multiplication by elements of $i\mathbf{R}$.

Definition 2.4. We say that a solution (Ψ, A) of (2.2) is **irreducible** if $\Psi \neq 0$, and **unobstructed** if $L_{\Psi,A}$ is invertible.

If (Ψ, A) is irreducible and unobstructed, then it represents an isolated point in $\mathfrak{M}_\mathfrak{w}(\mathbf{p}, \eta)$; see [Doa17b, Lemma 2.15; DW17, Proposition 1.25].

Proposition 2.5 ([Doa17b, Theorem 2.32]). *If $b_1(M) > 0$, then for every $(\mathbf{p}, \eta)$ from a residual subset of $\mathscr{P} \times \mathscr{Z}$ all solutions to (2.2) are irreducible and unobstructed.*

For every $(\mathbf{p}, \eta)$ as in Proposition 2.5, the moduli space $\mathfrak{M}_\mathfrak{w}(\mathbf{p}, \eta)$ is a zero-dimensional manifold. It can be oriented as explained in [Doa17b, Section 2.7]. Here is a brief outline of the **orientation procedure**: Let $\det L \to \Gamma(\mathrm{Hom}(E, W)) \times \mathscr{A}(\det W)$ be the determinant line bundle of the family of elliptic operators $L_{\Psi,A}$ as (Ψ, A) varies over all configurations. Over the locus $\{\Psi = 0\}$ we have $L_{0,A} = \not{D}_A \oplus \mathfrak{d}$. Since the operator $\not{D}_A$ is complex-linear and the kernel and cokernel of $\mathfrak{d}$ are canonically isomorphic to $H^1(M, i\mathbf{R}) \oplus H^0(M, i\mathbf{R})$, the determinant line bundle $\det L$ has a canonical orientation over $\{\Psi = 0\}$. Orientation transport along the path $s \mapsto L_{s\Psi,A}$ orients $\det L_{\Psi,A}$ for arbitrary Ψ. This procedure defines a $\mathscr{G}(\det W)$–equivariant orientation of $\det L$. If (Ψ, A) is an irreducible and unobstructed solution of (2.2), then the determinant of the tangent space to $\mathfrak{M}_\mathfrak{w}(\mathbf{p}, \eta)$ at $[\Psi, A]$ is canonically isomorphic to $\det L_{\Psi,A}$; hence, it is oriented by the above procedure.

Definition 2.6. For $[\Psi, A] \in \mathfrak{M}_\mathfrak{w}(\mathbf{p}, \eta)$, we define

$$\mathrm{sign}[\Psi, A] \in \{\pm 1\}$$

by comparing the orientation described above with the standard orientation of a point.

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[5] This linearization is obtained by writing a connection near A_0 as $A_0 + 2a$.

As we will see shortly, if $(\mathbf{p},\eta)$ is generic, and under the assumption that there are no singular harmonic $\mathbf{Z}_2$ spinors with respect to $\mathbf{p}$, then $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p},\eta)$ is a compact, oriented, zero-dimensional manifold. In this situation, we define

$$n_{\mathfrak{w}}(\mathbf{p},\eta) := \sum_{[\Psi,A]\in\mathfrak{M}_{\mathfrak{w}}(\mathbf{p},\eta)} \operatorname{sign}[\Psi,A], \tag{2.7}$$

the signed count of solutions to the Seiberg–Witten equation with two spinors.

3 Compactness of the moduli space

In general, $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p},\eta)$ might be non-compact; and even if it is compact for given $(\mathbf{p},\eta)$, compactness might still fail as $(\mathbf{p},\eta)$ varies. This can only happen when, along a sequence of solutions to (2.2), the L^2–norm of the spinors goes to infinity. The following result describes in which sense one can still take a rescaled limit in this situation.

Theorem 3.1 ([HW15, Theorem 1.5]). *Let $(\mathbf{p}_i,\eta_i)$ be a sequence in $\mathscr{P}\times\mathscr{Z}$ such that $(\mathbf{p}_i,\eta_i)$ converges to $(\mathbf{p},\eta)$ in C^∞. Let (Ψ_i,A_i) be a sequence of solutions of (2.2) with respect to $(\mathbf{p}_i,\eta_i)$. If $\limsup_{i\to\infty}\|\Psi_i\|_{L^2}=\infty$, then after rescaling $\tilde\Psi_i=\Psi_i/\|\Psi_i\|_{L^2}$ and passing to a subsequence the following hold:*

1. *The subset*
$$Z := \left\{x\in M : \limsup_{i\to\infty}|\tilde\Psi_i(x)| = 0\right\}$$
is closed and nowhere-dense. (In fact, Z has Hausdorff dimension at most one by [Tau14, Theorem 1.2].)

2. *There exist $\Psi\in\Gamma(M\backslash Z,\mathrm{Hom}(E,W))$ and a connection A on $\det W|_{M\backslash Z}$ satisfying the limiting equation*
$$\not{D}_A\Psi = 0 \quad \text{and} \quad \mu(\Psi)=0 \tag{3.2}$$
on $M\backslash Z$ with respect to $\mathbf{p}$. The pointwise norm $|\Psi|$ extends to a Hölder continuous function on all of M and
$$Z = |\Psi|^{-1}(0).$$
Moreover, A is flat with monodromy in $\mathbf{Z}_2$.

3. *On $M\backslash Z$, up to gauge transformations, $\tilde\Psi_i$ weakly converges to Ψ in $W^{2,2}_{\mathrm{loc}}$ and A_i weakly converges to A in $W^{1,2}_{\mathrm{loc}}$. There is a constant $\gamma>0$ such that $|\tilde\Psi_i|$ converges to $|\Psi|$ in $C^{0,\gamma}(M)$.*

We expect that the convergence $(\tilde{\Psi}_i, A_i) \to (\Psi, A)$ can be improved to C^∞_{loc} on $M\backslash Z$; cf. [Doa17a, Theorem 1.5]. In Section 5, we will show that this is indeed the case if Z is empty.

The following proposition will give use a concrete understanding of solutions to the limiting equation (3.2); it is a special case of the Haydys correspondence.

Proposition 3.3 (cf. [HW15, Appendix A]). *If $(\Psi, A) \in \Gamma(\mathrm{Hom}(E, W)) \times \mathscr{A}(\det W)$ is solution of (3.2) and Ψ is nowhere vanishing, then:*

1. $\det W$ *is trivial; in particular,* $\mathfrak{w}$ *is induced by a spin structure,*
2. *after a gauge transformation we can assume that A is the product connection and there exists a unique spin structure $\mathfrak{s}$ inducing $\mathfrak{w}$ and such that Ψ takes values in* $\mathrm{Re}(E \otimes S_{\mathfrak{s}}) \subset \Gamma(\mathrm{Hom}(E, W))$*; here $S_{\mathfrak{s}}$ is the spinor bundle of $\mathfrak{s}$,*
3. Ψ *lies in the kernel of* $\not{D}^{\mathfrak{s}}_{\mathbf{p}}\colon \Gamma(\mathrm{Re}(E \otimes S_{\mathfrak{s}})) \to \Gamma(\mathrm{Re}(E \otimes S_{\mathfrak{s}}))$.

Moreover, any nowhere vanishing element in the kernel $\not{D}^{\mathfrak{s}}_{\mathbf{p}}$ for some spin structure $\mathfrak{s}$ inducing $\mathfrak{w}$ gives rise to a solution of (3.2).

Remark 3.4. The set of spin structures is a torsor over $H^1(M, \mathbf{Z}_2)$ while the set of spin^c structures is a torsor over $H^2(M, Z)$. If $\beta\colon H^1(M, \mathbf{Z}_2) \to H^2(M, \mathbf{Z})$ denotes the Bockstein homomorphism in the exact sequence

$$\cdots \to H^1(M, \mathbf{Z}_2) \xrightarrow{\beta} H^2(M, \mathbf{Z}) \xrightarrow{2\times} H^2(M, \mathbf{Z}) \to \cdots,$$

then the set of a all spin structures $\mathfrak{s}$ inducing the spin^c structure $\mathfrak{w}$ is a torsor over $\ker \beta$. The set of all spin^c structures $\mathfrak{w}$ with trivial determinant is a torsor over $\ker 2\times$, the 2–torsion subgroup of $H^2(M, \mathbf{Z})$.

Proof of Proposition 3.3. Fix a spin structure $\mathfrak{s}_0$ and a Hermitian line bundle L which induce $\mathfrak{w}$; in particular, $W = S_{\mathfrak{s}_0} \otimes L$ and A induces a connection $\sqrt{A}$ on L. By Proposition A.1,

$$\mathrm{Hom}_{\mathbf{C}}(\mathbf{C}^2, \mathbf{H}) /\!\!/\!\!/ \mathrm{U}(1) = \big(\mathrm{Re}(\mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^2) \backslash \{0\}\big)/\mathbf{Z}_2;$$

hence, Ψ gives rise to a section $s \in \Gamma(\mathfrak{X})$ with

$$\mathfrak{X} = \big(\mathrm{Re}(E \otimes S_{\mathfrak{s}_0}) \backslash \{0\}\big)/\mathbf{Z}_2$$

satisfying the Fueter equation (that is: local lifts of s to $\mathrm{Re}(\mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^2)$ satisfy the Dirac equation). The Haydys correspondence [DW17, Proposition 3.2] asserts that:

- any $s \in \Gamma(\mathfrak{X})$ can be lifted; that is: there exists a Hermitian line bundle L, $\Psi \in \Gamma(\mathrm{Hom}(E, S_{\mathfrak{s}_0} \otimes L)) = \Gamma(\mathrm{Hom}(E, W))$, and $\sqrt{A} \in \mathscr{A}(L)$ satisfying (3.2), and
- L is determined by s up to isomorphism and any two lifts of s are related by a unique gauge transformation in $\mathscr{G}(L)$.

It is easy to see that s can be lifted to a section $\tilde{\Psi} \in \Gamma(\mathrm{Re}(E \otimes S_{\mathfrak{s}_0}) \otimes \mathfrak{l})$ for some Euclidean line bundle $\mathfrak{l}$ (whose Čech cocycle remembers the $\mathbf{Z}_2$ ambiguity in choosing local lifts). Set $\tilde{L} := \mathfrak{l} \otimes_{\mathbf{R}} \mathbf{C}$. By Proposition A.1,

$$\mathrm{Re}(E \otimes S_{\mathfrak{s}_0}) \otimes \mathfrak{l} \subset \mu^{-1}(0) \subset \mathrm{Hom}(E, S_{\mathfrak{s}_0} \otimes \tilde{L}) = \mathrm{Hom}(E, W);$$

and if $\tilde{A} \in \mathscr{A}(\tilde{L})$ denotes the connection induced by the canonical connection on $\mathfrak{l}$, then $\not{D}_{\tilde{A}}\tilde{\Psi} = 0$.

It thus follows from the Haydys correspondence that $L \cong \mathfrak{l} \otimes_{\mathbf{R}} \mathbf{C}$ and, after this identification has been made and a suitable gauge transformation has been applied, $\Psi = \tilde{\Psi}$ and $\sqrt{A} = \tilde{A}$. This shows that $\det W = L^2$ is trivial and $\mathfrak{w}$ is induced by the spin structure $\mathfrak{s}$ obtained by twisting $\mathfrak{s}_0$ with $\mathfrak{l}$. □

Definition 3.5. Denote by $\mathscr{P}^{\mathrm{reg}} \subset \mathscr{P}$ the subset consisting of those $\mathbf{p}$ for which the Dirac operator

$$\not{D}^{\mathfrak{s}}_{\mathbf{p}} \colon \Gamma(\mathrm{Re}(E \otimes S_{\mathfrak{s}})) \to \Gamma(\mathrm{Re}(E \otimes S_{\mathfrak{s}})) \tag{3.6}$$

is invertible for all spin structures $\mathfrak{s}$. Set

$$\mathscr{Q} := \mathscr{P} \times \mathscr{Z},$$

and denote by $\mathscr{Q}^{\mathrm{reg}} \subset \mathscr{Q}$ the subset consisting of those $(\mathbf{p}, \eta)$ for which $\mathbf{p} \in \mathscr{P}^{\mathrm{reg}}$ and every solution (Ψ, A) of (2.2) with respect to $(\mathbf{p}, \eta)$ is irreducible and unobstructed.

Proposition 3.7 ([Doa17b, Theorem 2.32; Ang96, Theorem 1.5; Mai97, Theorem 1.2]). *If $b_1(M) > 0$, then $\mathscr{Q}^{\mathrm{reg}}$ is residual in $\mathscr{Q}$.*

Proposition 3.8. *If $(\mathbf{p}, \eta) \in \mathscr{Q}^{\mathrm{reg}}$ and there are no singular harmonic $\mathbf{Z}_2$ spinors for $\mathbf{p}$, then $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta)$ is compact. In particular, $n_{\mathfrak{w}}(\mathbf{p}, \eta) \in \mathbf{Z}$ as in (2.7) is defined.*

Proof. By hypothesis we know that there are no singular harmonic $\mathbf{Z}_2$ spinors and, by definition of $\mathscr{P}^{\mathrm{reg}}$ and Proposition 3.3, there are also no harmonic spinors. It thus follows from Theorem 3.1 that $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta)$ is compact. □

4 Wall-crossing and the spectral flow

In the absence of singular harmonic $\mathbf{Z}_2$ spinors, we can define $n_{\mathfrak{w}}(\mathbf{p}, \eta)$ for every $(\mathbf{p}, \eta) \in \mathcal{Q}^{\mathrm{reg}}$. However, $\mathcal{Q}^{\mathrm{reg}}$ is not path-connected and $n_{\mathfrak{w}}(\mathbf{p}, \eta)$ does depend on the path-connected component of $\mathcal{Q}^{\mathrm{reg}}$ in which $(\mathbf{p}, \eta)$ lies. We study the wall-crossing for $n_{\mathfrak{w}}(\mathbf{p}, \eta)$ by analyzing the family of moduli spaces $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}_t, \eta_t)$ along paths of the following kind.

Definition 4.1. Given $\mathbf{p}_0, \mathbf{p}_1 \in \mathcal{P}^{\mathrm{reg}}$, denote by $\boldsymbol{\mathcal{P}}^{\mathrm{reg}}(\mathbf{p}_0, \mathbf{p}_1)$ the space of smooth paths from $\mathbf{p}_0$ to $\mathbf{p}_1$ in $\mathcal{P}$ such that for every spin structure $\mathfrak{s}$:

1. the path of Dirac operators $\left(\not{D}^{\mathfrak{s}}_{\mathbf{p}_t}\right)_{t\in[0,1]}$ has transverse spectral flow and

2. whenever the spectrum of $\not{D}^{\mathfrak{s}}_{\mathbf{p}_t}$ crosses zero, $\ker \not{D}^{\mathfrak{s}}_{\mathbf{p}_t}$ is spanned by a nowhere vanishing section $\Psi \in \Gamma(\mathrm{Re}(E \otimes S_{\mathfrak{s}}))$.

Given $(\mathbf{p}_0, \eta_0), (\mathbf{p}_1, \eta_1) \in \mathcal{Q}^{\mathrm{reg}}$, denote by $\boldsymbol{\mathcal{Q}}^{\mathrm{reg}}\big((\mathbf{p}_0, \eta_0), (\mathbf{p}_1, \eta_1)\big)$ the space of smooth paths $(\mathbf{p}_t, \eta_t)_{t\in[0,1]}$ from $(\mathbf{p}_0, \eta_0)$ to $(\mathbf{p}_1, \eta_1)$ in $\mathcal{Q}$ such that (1) and (2) hold and, moreover:

3. for every $t_0 \in [0, 1]$, every solution (Ψ, A) of (2.2) is irreducible and unobstructed with respect to $(\mathbf{p}_t, \eta_t)_{t\in[0,1]}$, that is: if $L_{\Psi,A}$ is the linearized operator and $\pi\colon\ \Gamma(\mathrm{Hom}(E, W)) \oplus \Omega^1(M, i\mathbf{R}) \oplus \Omega^0(M, i\mathbf{R}) \to \operatorname{coker} L_{\Psi,A}$ denotes the L^2–orthogonal projection, then

$$\pi\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0}\begin{pmatrix} -\not{D}_{\mathbf{p}_t,A}\Psi \\ *(F_A + \eta_t - \mu_{\mathbf{p}_t}(\Psi)) \\ 0 \end{pmatrix}\right)$$

   spans $\operatorname{coker} L_{\Psi,A}$, and

4. for any Ψ as in (2) with $\|\Psi\|_{L^2} = 1$, denote by

$$\left\{\left(\Psi_\varepsilon = \Psi + \varepsilon^2\psi + O(\varepsilon^4), A_\varepsilon; t(\varepsilon) = t_0 + O(\varepsilon^2)\right) : 0 \le \varepsilon \ll 1\right\}$$

   the family of solutions to

$$\begin{gathered} \not{D}_{A_\varepsilon,\mathbf{p}_{t(\varepsilon)}}\Psi_\varepsilon = 0, \\ \varepsilon^2\left(\frac{1}{2}F_{A_\varepsilon} + \eta_{t(\varepsilon)}\right) = \mu_{\mathbf{p}_{t(\varepsilon)}}(\Psi_\varepsilon), \quad \text{and} \\ \|\Psi_\varepsilon\|_{L^2} = 1 \end{gathered}$$

obtained from [DW17, Theorem 1.38]; we require that

$$\delta = \delta(\Psi, \mathbf{p}_{t_0}, \eta_{t_0}) := \langle \not{D}_{A_0, \mathbf{p}_{t_0}} \psi, \psi \rangle_{L^2} \neq 0.$$

Condition (1) is necessary since the wall-crossing formula will involve the spectral flow of $\not{D}^{\mathfrak{s}}_{\mathbf{p}_t}$. Condition (2) ensures that the harmonic $\mathbf{Z}_2$ spinors produced by Theorem 1.6 are indeed singular. Condition (3) is used to show that as t varies, the moduli spaces $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}_t, \eta_t)$ form a one-dimensional cobordism. Finally, condition (4) ensures that we can use the local model from [DW17, Theorem 1.38] to study the wall-crossing phenomenon.

The following result shows that a generic path from $(\mathbf{p}_0, \eta_0)$ to $(\mathbf{p}_1, \eta_1)$ satisfies the conditions in Definition 4.1; its proof is postponed to Section 8.

Proposition 4.2. *Given $(\mathbf{p}_0, \eta_0), (\mathbf{p}_1, \eta_1) \in \mathcal{Q}^{\mathrm{reg}}$, the subspace $\boldsymbol{\mathcal{Q}}^{\mathrm{reg}}\big((\mathbf{p}_0, \eta_0), (\mathbf{p}_1, \eta_1)\big)$ is residual in the space of all smooth paths from $(\mathbf{p}_0, \eta_0)$ to $(\mathbf{p}_1, \eta_1)$ in $\mathcal{Q}$.*

The next three sections are occupied with studying the wall crossing along paths in $\boldsymbol{\mathcal{Q}}^{\mathrm{reg}}\big((\mathbf{p}_0, \eta_0), (\mathbf{p}_1, \eta_1)\big)$. In order to state the wall-crossing formula, we need the following preparation.

Proposition 4.3. *Denote by $\mathfrak{s}$ a spin structure inducing the spin^c structure $\mathfrak{w}$ and by A the product connection on $\det W$. If Ψ is a nowhere vanishing section of $\mathrm{Re}(E \otimes S_{\mathfrak{s}})$, then the following hold:*

1. *Let $\mathfrak{a}_\Psi$ be the algebraic operator given by (2.3). The map*

$$\tilde{\mathfrak{a}}_\Psi := |\Psi|^{-1} \mathfrak{a}_\Psi \colon\ (T^*M \oplus \underline{\mathbf{R}}) \otimes i\mathbf{R} \longrightarrow \mathrm{Im}(E \otimes S_{\mathfrak{s}})$$

   is an isometry.

2. *Denote by $\not{D}_{\mathrm{Im}}$ the restriction of $\not{D}_{A,\mathbf{p}}$ to $\mathrm{Im}(E \otimes S_{\mathfrak{s}}) \subset \mathrm{Hom}(E, W)$ and define the operator $\mathfrak{d}_\Psi \colon\ \Omega^1(M, i\mathbf{R}) \oplus \Omega^0(M, i\mathbf{R}) \longrightarrow \Omega^1(M, i\mathbf{R}) \oplus \Omega^0(M, i\mathbf{R})$ by*

$$\mathfrak{d}_\Psi := \tilde{\mathfrak{a}}_\Psi^* \circ \not{D}_{\mathrm{Im}} \circ \tilde{\mathfrak{a}}_\Psi.$$

   For each $t \in [0, 1]$,

$$\mathfrak{d}_\Psi^t := (1 - t)\mathfrak{d}_\Psi + t\mathfrak{d}$$

   is a self-adjoint Fredholm operator.

Proof. The fact that $\tilde{\mathfrak{a}}_\Psi$ is an isometry is a consequence of

$$\mathfrak{a}_\Psi^*\mathfrak{a}_\Psi = |\Psi|^2 \tag{4.4}$$

which in turn follows from

$$\begin{aligned}\langle \mathfrak{a}_\Psi(a,\xi), \mathfrak{a}_\Psi(b,\eta)\rangle &= \langle \bar{\gamma}(a)\Psi + \rho(\xi)\Psi, \bar{\gamma}(b)\Psi + \rho(\eta)\Psi\rangle \\ &= |\Psi|^2(\langle a,b\rangle + \langle \xi,\eta\rangle).\end{aligned}$$

Since

$$\begin{aligned}\not{D}_{\mathrm{Im}}\mathfrak{a}_\Psi(a,\xi) &= \not{D}_{\mathrm{Im}}(\bar{\gamma}(a)\Psi + \rho(\xi)\Psi) \\ &= \bar{\gamma}(*\mathrm{d}a)\Psi + \rho(\mathrm{d}^*a)\Psi + \bar{\gamma}(\mathrm{d}\xi)\Psi \\ &\quad - \bar{\gamma}(a)\not{D}_{\mathrm{Re}}\Psi + \rho(\xi)\not{D}_{\mathrm{Re}}\Psi - 2\sum_{i=1}^3 \rho(a(e_i))\nabla_{e_i}\Psi \\ &= \mathfrak{a}_\Psi\mathfrak{d}(a,\xi) - 2\sum_{i=1}^3 \rho(a(e_i))\nabla_{e_i}\Psi,\end{aligned} \tag{4.5}$$

we have

$$\mathfrak{d}_\Psi = \mathfrak{d} + \mathfrak{e}_\Psi \quad \text{with} \quad \mathfrak{e}_\Psi = -2\sum_{i=1}^3 \rho(a(e_i))\nabla_{e_i}\Psi - |\Psi|^{-3}\mathfrak{a}_\Psi^*\gamma(\mathrm{d}|\Psi|)\mathfrak{a}_\Psi.$$

This implies that $\mathfrak{d}_\Psi^t$ is a Fredholm operator. □

Definition 4.6. In the situation of Proposition 4.3, define

$$\sigma(\Psi,\mathbf{p}) := (-1)^{1+b_1(M)} \cdot (-1)^{\mathrm{SF}\left((-\mathfrak{d}_\Psi^t)_{t\in[0,1]}\right)}.$$

Remark 4.7. The operator $\mathfrak{d}_\Psi$ only depends on Ψ up to multiplication by a constant in $\mathbf{R}^*$; hence, the same holds for $\sigma(\Psi,\mathbf{p})$.

Definition 4.8. For a pair of nowhere vanishing sections $\Psi,\Phi \in \Gamma(\mathrm{Re}(E\otimes S_{\mathfrak{s}}))$ we define their **relative degree** $\deg(\Psi,\Phi) \in \mathbf{Z}$ as follows. Choose any trivializations of E and $S_{\mathfrak{s}}$ compatible with the SU(2) structures. In the induced trivialization of $\mathrm{Re}(E\otimes S_{\mathfrak{s}})$ the sections $\Psi/|\Psi|$ and $\Phi/|\Phi|$ are represented by maps $M \to S^3$. Set

$$\deg(\Psi,\Phi) := \deg(\Psi/|\Psi|) - \deg(\Phi/|\Phi|).$$

This number does not depend on the choice of the trivializations.

Theorem 4.9. *Let $(\mathbf{p}_t, \eta_t)_{t\in[0,1]} \in \mathscr{Q}^{\mathrm{reg}}\big((\mathbf{p}_0, \eta_0), (\mathbf{p}_1, \eta_1)\big)$. For each spin structure $\mathfrak{s}$ inducing the spin^c structure $\mathfrak{w}$, denote*

- *by $\{t_1^{\mathfrak{s}}, \dots, t_{N_{\mathfrak{s}}}^{\mathfrak{s}}\} \subset [0,1]$ the finite set of times at which the spectrum of $\not{D}_{\mathbf{p}_t}^{\mathfrak{s}}$ crosses zero*

and, for each $i = 1, \dots, N_{\mathfrak{s}}$, denote

- *by $\chi_i^{\mathfrak{s}} \in \{\pm 1\}$ the sign of the spectral crossing at $t_i^{\mathfrak{s}}$ and*
- *by $\Psi_i^{\mathfrak{s}}$ a nowhere vanishing spinor spanning* $\ker \not{D}_{\mathbf{p}_t}^{\mathfrak{s}}$.

If there are no singular harmonic $\mathbf{Z}_2$ spinors with respect to $\mathbf{p}_t$ for any $t \in [0,1]$, then

$$n_{\mathfrak{w}}(\mathbf{p}_1, \eta_1) = n_{\mathfrak{w}}(\mathbf{p}_0, \eta_0) + \sum_{\mathfrak{s}} \sum_{i=1}^{N_{\mathfrak{s}}} \chi_i^{\mathfrak{s}} \cdot \sigma(\Psi_i^{\mathfrak{s}}, \mathbf{p}_{t_i^{\mathfrak{s}}}) \tag{4.10}$$

or, equivalently,

$$n_{\mathfrak{w}}(\mathbf{p}_1, \eta_1) = n_{\mathfrak{w}}(\mathbf{p}_0, \eta_0) + \sum_{\mathfrak{s}} \chi_1^{\mathfrak{s}} \cdot \sigma(\Psi_1^{\mathfrak{s}}, \mathbf{p}_{t_1^{\mathfrak{s}}}) \cdot \sum_{i=1}^{N_{\mathfrak{s}}} (-1)^{i+1} \cdot (-1)^{\deg(\Psi_1^{\mathfrak{s}}, \Psi_i^{\mathfrak{s}})}. \tag{4.11}$$

Here the sums are over all spin structures $\mathfrak{s}$ inducing $\mathfrak{w}$.

Remark 4.12. It follows from Theorem 4.9, that $n_{\mathfrak{w}}(\mathbf{p}, \eta)$ does not depend on η.

The proof of the (4.10) proceeds by analyzing the 1–parameter family of moduli spaces

$$\mathfrak{W} := \bigcup_{t\in[0,1]} \mathfrak{M}_{\mathfrak{w}}(\mathbf{p}_t, \eta_t).$$

By Definition 4.1(3), $\mathfrak{W}$ is an oriented, one-dimensional manifold with oriented boundary

$$\partial\mathfrak{W} = \mathfrak{M}_{\mathfrak{w}}(\mathbf{p}_0, \eta_0) - \mathfrak{M}_{\mathfrak{w}}(\mathbf{p}_1, \eta_1).$$

If $\mathfrak{W}$ were compact, then it would follow that $n_{\mathfrak{w}}(\mathbf{p}_1, \eta_1) = n_{\mathfrak{w}}(\mathbf{p}_0, \eta_0)$. However, $\mathfrak{W}$ is non-compact.

5 Compactification of the cobordism

Set

$$\overline{\mathfrak{W}} := \left\{(t, \varepsilon, [\Psi, A]) \in [0,1] \times [0,\infty) \times \frac{\Gamma(\mathrm{Hom}(E, W) \times \mathscr{A}(\det W)}{\mathscr{G}(\det W)} : (*)\right\}$$

with (∗) meaning that:

- the differential equation

$$\begin{aligned} \not{D}_{A,\mathbf{p}_t}\Psi &= 0, \\ \varepsilon^2\left(\frac{1}{2}F_A + \eta_t\right) &= \mu_{\mathbf{p}_t}(\Psi), \quad \text{and} \\ \|\Psi\|_{L^2} &= 1 \end{aligned} \tag{5.1}$$

  holds and

- if $\varepsilon = 0$, then Ψ is nowhere vanishing.

Equip $\overline{\mathfrak{W}}$ with the C^∞–topology. We have a natural embedding $\mathfrak{W} \hookrightarrow \overline{\mathfrak{W}}$ given by $(t, [\Psi, A]) \mapsto (t, \varepsilon, [\tilde{\Psi}, A])$ with $\varepsilon := 1/\|\Psi\|_{L^2}$ and $\tilde{\Psi} := \Psi/\|\Psi\|_{L^2}$.

Proposition 5.2. *$\mathfrak{W}$ is dense in $\overline{\mathfrak{W}}$.*

Proof. If $(t_0, \varepsilon, [\Psi, A]) \in \overline{\mathfrak{W}}\backslash\mathfrak{W}$, then $\varepsilon = 0$. It follows from Definition 4.1 and [DW17, Theorem 1.38], that there are is a family $\{(\Psi_\varepsilon, A_\varepsilon; t(\varepsilon)) : 0 \leq \varepsilon \ll 1\}$ of solutions to

$$\begin{aligned} \not{D}_{A_\varepsilon,\mathbf{p}_{t(\varepsilon)}}\Psi_\varepsilon &= 0 \quad \text{and} \\ \varepsilon^2\left(\frac{1}{2}F_{A_\varepsilon} + \eta_{t(\varepsilon)}\right) &= \mu_{\mathbf{p}_{t(\varepsilon)}}(\Psi_\varepsilon) \end{aligned}$$

with $(\Psi_\varepsilon, A_\varepsilon)$ converging to (Ψ, A) in C^∞ and $t(\varepsilon)$ converging to t_0 as ε tends to zero. Consequently, $\mathfrak{W}$ is dense in $\overline{\mathfrak{W}}$. □

That $\overline{\mathfrak{W}}$ is compact does not follows from Theorem 3.1; it does, however, follow from the next result, whose proof will occupy the remainder of this section.

Proposition 5.3. *Let $(\mathbf{p}_i, \eta_i)$ be a sequence in $\mathscr{P} \times \mathscr{Z}$ such that $(\mathbf{p}_i, \eta_i)$ converges to $(\mathbf{p}, \eta)$ in C^∞. Let $(\varepsilon_i, \Psi_i, A_i)$ be a sequence of solutions of*

$$\begin{aligned} \not{D}_{A_i,\mathbf{p}_i}\Psi_i &= 0, \\ \varepsilon^2\left(\frac{1}{2}F_{A_i} + \eta_i\right) &= \mu_{\mathbf{p}_i}(\Psi_i), \quad \textit{and} \\ \|\Psi_i\|_{L^2} &= 1 \end{aligned} \tag{5.4}$$

with $\lim_{i\to\infty} \varepsilon_i = 0$*. If*

$$Z := \left\{ x \in M : \limsup_{i\to\infty} |\Psi_i(x)| = 0 \right\},$$

is empty, then, after passing to a gauge transformed subsequence, (Ψ_i, A_i) *converges in* C^∞ *to a solution* (Ψ, A) *of*

$$\not{D}_{A,\mathfrak{p}}\Psi = 0, \quad \mu_{\mathfrak{p}}(\Psi) = 0, \quad \textit{and} \quad \|\Psi\|_{L^2} = 1.$$

Proposition 5.5. $\overline{\mathfrak{W}}$ *is compact.*

Proof. We need to show that any sequence $(t_i, \varepsilon_i, [\Psi_i, A_i])$ in $\mathfrak{W}$ has a subsequence which converges in $\overline{\mathfrak{W}}$. If $\liminf \varepsilon_i > 0$, then this a consequence of standard elliptic estimates and Arzelà–Ascoli. It only needs to be pointed out that ε_i cannot tend to infinity, because otherwise there would be a reducible solution to (2.2) which is ruled out by Definition 4.1(3).

If $\liminf_{i\to\infty} \varepsilon_i = 0$, then Theorem 3.1 asserts that a gauge transformed subsequence of (Ψ_i, A_i) converges weakly in $W^{2,2}_{\mathrm{loc}} \times W^{1,2}_{\mathrm{loc}}$ outside Z. If Z is non-empty, then the limit represents a singular harmonic $\mathbf{Z}_2$ spinors. However, by assumption there are no singular harmonic $\mathbf{Z}_2$ spinors; hence, Z is empty and Proposition 5.3 asserts that a gauge transformed subsequence of (Ψ_i, A_i) converges in C^∞. □

The proof of Proposition 5.3 relies on the following a priori estimate.

Proposition 5.6. *For each* $m_0 > 0$*, there is an* $\varepsilon_0 > 0$ *and, for each,* $k \in \mathbf{N}$*, a non-decreasing function* $f_k \colon [0,\infty) \to [0,\infty)$ *such that the following holds: if* $(\Psi, A, \varepsilon) \in \Gamma(M, \mathrm{Hom}(E, W)) \times \mathscr{A}(\det W) \times (0, \varepsilon_0]$ *satisfies* (5.4) *and*

$$\min|\Psi|^2 \geq m_0,$$

then

$$\|\Psi\|_{C^k_A} + \|F_A\|_{C^k} \leq f_k\left(\|F_A\|_{L^2} + \|\eta\|_{C^k} + \|F_B\|_{C^k} + \|R_g\|_{C^k}\right).$$

Here $\|\Psi\|_{C^k_A} := \sum_{i=0}^k \|\nabla_A^i \Psi\|_{L^\infty}$ *and* R_g *denotes the Riemann curvature tensor of the metric* g*.*

The key ingredient for the proof of the above a priori estimate are the identity

$$\begin{aligned}(\mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*)\mu(\Psi) + |\Phi|^2 F_A = \sum_{i,j=1}^3 \frac{1}{2}\rho^*\Big(((F^B_{ij} + F^{\mathfrak{w}}_{ij}) \cdot \Phi)\Phi^*\Big)e^{ij} \\ + \rho^*\Big((\nabla^A_j\Phi)(\nabla^A_i\Phi)^*\Big)e^{ij},\end{aligned} \tag{5.7}$$

whose proof can be found in [DW17, Proposition A.7], and the following two propositions.

Proposition 5.8. *Let $k \in \mathbf{N}_0, p \geq 2$. There is a monotone function $f_{k,p} \colon [0,\infty) \to [0,\infty)$ such that for every $(\Psi, A) \in \Gamma(M, \mathrm{Hom}(E,W)) \times \mathscr{A}(\det W)$ solving*

$$\not{D}_{A,\mathbf{p}}\Psi = 0$$

we have

$$\|\Psi\|_{W_A^{k+2,p}} \lesssim f_{k,p}\left(\|F_A\|_{W^{k,p}} + \|\Psi\|_{L^\infty} + \|F_B\|_{C^{k+1}} + \|R_g\|_{C^{k+1}}\right).$$

Here $\|\Psi\|_{W_A^{k,p}} \coloneqq \sum_{i=0}^k \|\nabla_A^i \Psi\|_{L^p}$.

Proof. Fix a smooth reference connection A_0. By Hodge theory, after applying a gauge transformation, we can write $A = A_0 + a$ with $\mathrm{d}^* a = 0$ and

$$\|a\|_{W^{k+1,p}} \lesssim \|F_A\|_{W^{k,p}}$$

with the implicit constant depending on k, p, and $\|R_g\|_{C^k}$. It suffices to estimate $\|\Psi\|_{W_{A_0}^{k+2,p}}$, since

$$\begin{aligned}
\|(\nabla_{A_0} + a)^2\Phi\|_{L^p} &\lesssim \|\nabla_{A_0}^2\Phi\|_{L^p} + \||a||\nabla_{A_0}\Phi|\|_{L^p} + \||a|^2|\Phi|\|_{L^p} + \||\nabla a||\Phi|\|_{L^p} \\
&\lesssim \|\nabla_{A_0}^2\Phi\|_{L^p} + \|a\|_{L^{2p}}\|\nabla_{A_0}\Phi\|_{L^{2p}} + \|a\|_{L^{2p}}^2\|\Phi\|_{L^\infty} + \|\nabla a\|_{L^p}\|\Phi\|_{L^\infty} \\
&\lesssim (1 + \|a\|_{W^{1,p}})^2 \cdot \|\Psi\|_{W_{A_0}^{2,p}}.
\end{aligned}$$

by Hölder's inequality, Sobolev embedding, and Morrey's inequality; more generally,

$$\|\Psi\|_{W_A^{k+2,p}} \lesssim \left(1 + \|a\|_{W^{k+1,p}}\right)^{k+2} \cdot \|\Psi\|_{W_{A_0}^{k+2,p}}.$$

The Dirac equation can be written as

$$\not{D}_{A_0,\mathbf{p}}\Psi = -\bar{\gamma}(a)\Psi.$$

By standard elliptic estimates

$$\|\Psi\|_{W_{A_0}^{\ell+1,q}} \lesssim \|\bar{\gamma}(a)\Psi\|_{W_{A_0}^{\ell,q}} + \|\Psi\|_{L^\infty} \tag{5.9}$$

for any $\ell \in \mathbf{N}_0$ and $q \in (1, \infty)$ with the implicit constant depending on ℓ, q, $\|F_B\|_{C^\ell}$, and $\|R_g\|_{C^\ell}$. By the Sobolev Multiplication Theorem,[6] we have

$$\|\bar{\gamma}(a)\Psi\|_{W^{k+1,p}_{A_0}} \lesssim \|a\|_{W^{k+1,p}}\|\Psi\|_{W^{k+1,q}_{A_0}}$$

provided $(k+1)q > 3$. Therefore, to bound $\|\Psi\|_{W^{k+2,p}_{A_0}}$ in terms of

$$\spadesuit_k := \|F_A\|_{W^{k,p}} + \|\Psi\|_{L^\infty} + \|F_B\|_{C^k} + \|R_g\|_{C^k},$$

it suffices to show that for some such q, $\|\Psi\|_{W^{k+1,q}_{A_0}}$ can be bounded in terms of $\spadesuit_k$. If $k \geq 1$ or $k = 0$ and $q \geq 3$, such an estimate can be assumed by induction. Hence, we only need to prove that $\|\Psi\|_{W^{1,q}_{A_0}}$ is bounded in terms of $\spadesuit_k$ for some $q > 3$.

By (5.9), $\|\Psi\|_{W^{1,2}_{A_0}}$ can be bounded in terms of $\spadesuit_k$. Consequently, we have a bound on $\|\gamma(a)\Psi\|_{L^3(B_r)}$ and thus, using (5.9) again, a bound on $\|\Psi\|_{W^{1,3}_{A_0}}$ in terms of $\spadesuit_k$. Since $W^{1,3} \hookrightarrow L^p$ for any $p \in [1, \infty)$, it follows that $\|\gamma(a)\Psi\|_{L^q} \leq \|a\|_{L^6}\|\Psi\|_{L^p}$ is bounded for any q with $1/q = 1/6 + 1/p$. This then implies the desired bound on $\|\Psi\|_{W^{1,q}_{A_0}}$ for any $q \in [1, 6)$. □

Proposition 5.10. *Let $k \in \mathbf{N}, p \in [2, \infty)$. Let $m \in C^\infty(M)$ with $m_0 := \min m > 0$. There are constants $c > 0$ and $\varepsilon_0 > 0$ depending on k, p, m_0, $\|m\|_{C^k}$, and $\|R_g\|_{C^k}$ such that for all $\varepsilon \in (0, \varepsilon_0)$*

$$\|\alpha\|_{W^{k,p}} \leq c\|(\varepsilon^2\Delta + m)\alpha\|_{W^{k,p}}$$

for all $\alpha \in \Omega^*(M)$.

Proof. Denote by $\mathfrak{R}$ the curvature operator appearing in the Weitzenböck formula $\Delta\alpha = \nabla^*\nabla\alpha + \mathfrak{R}\alpha$. If $\varepsilon \ll 1$, then by integration by parts we obtain

$$\begin{aligned}\int_M \langle(\varepsilon^2\Delta + m)\alpha, \alpha\rangle|\alpha|^{p-2} &= \int_M \varepsilon^2|\nabla a|^2|a|^{p-2} + \frac{\varepsilon^2(p-2)}{4}\left|\nabla_A|a|^2\right|^2|a|^{p-4} \\ &\quad + \varepsilon^2\langle\mathfrak{R}\alpha, \alpha\rangle|\alpha|^{p-2} + m|\alpha|^p \\ &\geq \int_M \frac{m_0}{2}|\alpha|^p.\end{aligned}$$

[6] The Sobolev Multiplication Theorem asserts that $\|fg\|_{W^{k,p}} \lesssim \|f\|_{W^{k_1,p_1}}\|g\|_{W^{k_2,p_2}}$ provided $k - \frac{3}{p} < k_1 - \frac{3}{p_1} + k_2 - \frac{3}{p_2}$ and $k \leq \min\{k_1, k_2\}$.

By Young's inequality and with $q = p/(p-1)$, we have

$$\begin{aligned}\int_M \langle (\varepsilon^2\Delta + m)\alpha, \alpha\rangle |\alpha|^{p-2} &\leq \int_M |(\varepsilon^2\Delta + m)\alpha||\alpha|^{p-1} \\ &\leq \int \frac{\delta^{-p}}{p}|(\varepsilon^2\Delta + m)\alpha|^p + \frac{\delta^q}{q}|\alpha|^p\end{aligned}$$

for all $\delta > 0$. Choosing δ sufficient small, yields

$$\int_M |\alpha|^p \lesssim \int_M |(\Delta + m)\alpha|^p;$$

that is, the desired estimate for $k = 0$.

If $v_1, \ldots, v_k$ are vector fields on M, then the commutator $[\varepsilon^2\Delta + m, \partial_{v_1} \cdots \partial_{v_k}]$ is a differential operator of order $k-1$ with coefficients depending on only on the first k derivatives of m and R_g. Therefore, the estimates for $k > 0$ follows from the one for $k = 0$ using

$$(\varepsilon^2\Delta + m)(\partial_{v_1} \cdots \partial_{v_k}\alpha) = \partial_{v_1} \cdots \partial_{v_k}(\varepsilon^2\Delta + m)\alpha + [\varepsilon^2\Delta + m, \partial_{v_1} \cdots \partial_{v_k}]\alpha$$

and induction. □

Proof of Proposition 5.6. It follows from [HW15, Proposition 2.1] that $\|\Psi\|_{L^\infty}$ can be bounded in terms of $\clubsuit_k := \|F_A\|_{L^2} + \|\eta\|_{C^k} + \|F_B\|_{C^{k+1}} + \|R_g\|_{C^{k+1}}$. From Proposition 5.8 with $k = 0$ and $p = 2$ it follows that $\|\Psi\|_{W^{2,2}_A}$ bounded in terms of $\clubsuit_k$ as well. Since $W^{2,2}_A \hookrightarrow W^{1,6}_A$, it follows that the L^3–norm of the right-hand side of (5.7) can be bounded in terms of $\clubsuit_k$. Proposition 5.10 thus yields a bound on $\|F_A\|_{L^3}$ in terms of $\clubsuit_k$. Proposition 5.8 with $k = 0$ and $p = 3$ bounds $\|\Psi\|_{W^{2,3}_A}$ and, hence, $\|\Psi\|_{W^{1,p}_A}$ for all $p \in (1, \infty)$ in terms of $\clubsuit_k$. Consequently, Proposition 5.10 bounds $\|F_A\|_{L^p}$ for any p in terms of $\clubsuit_k$. Another application of Proposition 5.8 shows that $\|\Psi\|_{W^{2,p}_A}$ can be bounded in terms of $\clubsuit_k$. This yields bounds on the $W^{1,p}$–norm of right-hand side of (5.7) and $\||\Psi|\|_{C^1}$ in terms of $\clubsuit_k$. It follows from Proposition 5.10 that $\|F_A\|_{W^{1,p}}$ is bounded in terms of $\clubsuit_k$. Iterating applications of Proposition 5.8 and Proposition 5.10 one shows that $\|F_A\|_{W^{k+1,p}}$ and $\|\Psi\|_{W^{k+3,p}}$ can be bounded in terms of $\clubsuit_k$. Applying Morrey's inequality, $|f|_{C^0} \lesssim \|f\|_{W^{1,p}}$ for $p > 3$, completes the proof. □

Proof of Proposition 5.3. Since Z is empty, there exists $m_0 > 0$ such that, after passing to a subsequence, we have $\min|\Psi_i|^2 \geq m_0$ for all i. Moreover, it follows from [HW15, Definition 3.1 and Proposition 4.1] that $\|F_{A_i}\|_{L^2}$ is uniformly bounded. Proposition 5.6

now yields uniform C^k–bounds for F_{A_i} and and Ψ_i (using the A_i–dependent norm). After putting A_i in the Uhlenbeck gauge as in the proof of Proposition 5.8 we also obtain C^k bounds for A_i and the result follows from the Arzelà–Ascoli theorem. □

6 Orientation at infinity

Suppose that $(t_0, 0, [\Psi, A]) \in \overline{\mathfrak{W}}\backslash\mathfrak{W}$ is a boundary point in $\overline{\mathfrak{W}}$. Set $\mathbf{p} := \mathbf{p}_{t_0}$. By Proposition 3.3, there exists a spin structure $\mathfrak{s}$ inducing the spin^c structure $\mathfrak{w}$ such that $\Psi \in \Gamma(\mathrm{Re}(E \otimes S_{\mathfrak{s}})) \subset \Gamma(\mathrm{Hom}(E, W))$, $\not{D}^{\mathfrak{s}}_{\mathbf{p}}\Psi = 0$, and A is trivial. By Definition 4.1(1), there exists a unique solution $\{\Psi_t : |t - t_0| \ll 1\}$ to

$$\not{D}^{\mathfrak{s}}_{\mathbf{p}_t}\Psi_t = \lambda(t)\Psi_t \quad \text{and} \quad \|\Psi_t\|_{L^2} = 1$$

with $\Psi_{t_0} = \Psi$. Moreover, it satisfies $\dot{\lambda}(t_0) \neq 0$. In this situation, [DW17, Section 6] shows that for any choice of $r \in \mathbf{N}$ there exist $\tau \ll 1$ and $\varepsilon_0 \ll 1$, a C^r map

$$\mathrm{ob}\colon (t_0 - \tau, t_0 + \tau) \times [0, \varepsilon_0) \to \mathbf{R},$$

an open neighborhood V of $(t_0, 0, [\Psi, A]) \in \overline{\mathfrak{W}}$, and a homeomorphism

$$\mathfrak{x}\colon \mathrm{ob}^{-1}(0) \to V$$

such that

1. $\mathfrak{x}$ commutes with the projection to the t– and ε–coordinates,
2. the restriction $\mathrm{ob}|_{(t_0-\tau, t_0+\tau)\times(0,\varepsilon_0)}$ is smooth,

$$\begin{aligned}\mathrm{ob}(t, \varepsilon) &= \dot{\lambda}(t_0)\cdot(t - t_0) - \delta\varepsilon^4 + O(t^2, \varepsilon^6), \quad \text{and}\\ \partial_\varepsilon \mathrm{ob}(t, \varepsilon) &= -4\delta\varepsilon^3 + O(t^2, \varepsilon^5)\end{aligned}$$

   with δ as Definition 4.1(4), and
3. if $\varepsilon \in (0, \varepsilon_0] \times (t_0 - \tau, t_0 + \tau)$ satisfies $\mathrm{ob}(\varepsilon, t) = 0$ and $[\Psi_\varepsilon, A_\varepsilon] = \mathfrak{x}(\varepsilon, t)$, then there is a short exact sequence

$$0 \to \ker L_{\Psi_\varepsilon, A_\varepsilon} \to \mathbf{R}\langle\Psi\rangle \xrightarrow{\partial_\varepsilon \mathrm{ob}} \mathbf{R}\langle\Psi\rangle \to \operatorname{coker} L_{\Psi_\varepsilon, A_\varepsilon} \to 0. \tag{6.1}$$

It follows from the above that

$$\mathrm{ob}^{-1}(0) = \left\{(\varepsilon, t_0 + (\delta/\dot{\lambda}(0))\varepsilon^4 + O(\varepsilon^6)) : \varepsilon \in [0, \varepsilon_0)\right\}.$$

Therefore, $\overline{\mathfrak{W}}$ is a compact, oriented, one–dimensional manifold with boundary

$$\partial\overline{\mathfrak{W}} = \mathfrak{M}_{\mathfrak{w}}(\mathbf{p}_1, \eta_1) - \mathfrak{M}_{\mathfrak{w}}(\mathbf{p}_0, \eta_0) \cup \left(\overline{\mathfrak{W}}\backslash\mathfrak{W}\right).$$

If $\delta/\dot{\lambda}(0) > 0$, then as t passes through t_0 a solution to (2.2) is created. If $\delta/\dot{\lambda}(0) < 0$, then as t passes through t_0 a solution to (2.2) is annihilated. The orientation procedure described in Section 2 determines an orientation of $\det(L_{\Psi,A})$ which in turn is isomorphic to $\det(\partial_\varepsilon \mathrm{ob}\colon \mathbf{R}\langle\Psi\rangle \to \mathbf{R}\langle\Psi\rangle) \cong \mathbf{R}$ by (6.1). This determines a sign

$$\sigma([\Psi, A], \mathbf{p}) \in \{\pm 1\}$$

depending on whether the orientation induced on $\mathbf{R}$ in this way is the standard orientation or the opposite. The solution that is created/annihilated at t_0 contributes with $\sigma \cdot \mathrm{sign}(\partial_\varepsilon \mathrm{ob}) = -\sigma \cdot \mathrm{sign}(\delta)$. Consequently, for $\tau \ll 1$, we have the local wall-crossing formula

$$n_{\mathfrak{w}}(\mathbf{p}_{t+\tau}, \eta_{t+\tau}) = n_{\mathfrak{w}}(\mathbf{p}_{t-\tau}, \eta_{t-\tau}) + \mathrm{sign}(\dot{\lambda}(0)) \cdot \sigma([\Psi, A], \mathbf{p}). \tag{6.2}$$

The following proposition explains how to compute $\sigma([\Psi, A], \mathbf{p})$.

Proposition 6.3. *In the above situation,*

$$\sigma([\Psi, A], \mathbf{p}) = \sigma(\Psi, \mathbf{p}) \tag{6.4}$$

with $\sigma(\Psi, \mathbf{p})$ as in Proposition 4.3.

Remark 6.5. Ψ is determined by $[\Psi, A]$ up to a sign and, by Remark 4.7, $\sigma(\Psi, \mathbf{p})$ only depends on Ψ up to multiplication by a constant in $\mathbf{R}^*$.

Proof of Proposition 6.3. We proceed in three steps.

Step 1. *For $s \gg 1$, the inclusion*

$$\Gamma(\mathrm{Re}(E \otimes S_{\mathfrak{s}})) \hookrightarrow \Gamma(\mathrm{Hom}(E, W)) \oplus \Omega^1(M, i\mathbf{R}) \oplus \Omega^0(M, i\mathbf{R})$$

induces isomorphisms

$$\mathbf{R}\langle\Psi\rangle \cong \ker L_{s\Psi,A} \quad \textit{and} \quad \mathbf{R}\langle\Psi\rangle \cong \operatorname{coker} L_{s\Psi,A}. \tag{6.6}$$

Denote by $\not{D}_{\mathrm{Re}}$ and $\not{D}_{\mathrm{Im}}$ the restriction of $\not{D}_{A,\mathbf{p}}$ to $\mathrm{Re}(E \otimes S_{\mathfrak{s}})$ and $\mathrm{Im}(E \otimes S_{\mathfrak{s}})$ respectively. We can write $L_{s\Psi,A}$ as

$$L_{s\Psi,A} = \begin{pmatrix} -\not{D}_{\mathrm{Re}} & \\ & \mathfrak{D}_{s\Psi,A} \end{pmatrix} \quad \text{with} \quad \mathfrak{D}_{s\Psi,A} = \begin{pmatrix} -\not{D}_{\mathrm{Im}} & -s\mathfrak{a}_\Psi \\ -s\mathfrak{a}_\Psi^* & \mathfrak{d} \end{pmatrix}.$$

By (4.5), we have

$$\mathfrak{D}_{s\Psi,A}^*\mathfrak{D}_{s\Psi,A} = \begin{pmatrix} \not{D}_{\mathrm{Im}}^*\not{D}_{\mathrm{Im}} + s^2|\Psi|^2 & \\ & \mathfrak{d}^*\mathfrak{d} + s^2|\Psi|^2 \end{pmatrix} + s \begin{pmatrix} & \tilde{\mathfrak{e}}_\Psi \\ \tilde{\mathfrak{e}}_\Psi^* & \end{pmatrix}$$

with

$$\tilde{\mathfrak{e}}_\Psi = -2\sum_{i=1}^{3} \rho(a(e_i))\nabla_{e_i}\Psi;$$

see (4.5). Since $\tilde{\mathfrak{e}}_\Psi$ is a bounded zeroth order operator, it follows that, for $s \gg 1$, $\mathfrak{D}_{s\Psi,A}$ is invertible which implies the assertion.

Step 2. *With respect to the isomorphisms* (6.6), $\sigma([\Psi, A], \mathbf{p})$ *is given by orientation transport from* $\det(L_{0,A})$ *to* $\det(L_{s\Psi,A})$.

It follows from [DW17, Section 5] that, with respect to the isomorphisms in (6.6), $\sigma([\Psi, A], \mathbf{p})$ can be computed as the orientation transport from $\det(L_{0,A_\varepsilon})$ via $\det(L_{\Psi_\varepsilon,A_\varepsilon})$ to $\det(L_{s\Psi,A})$ which in turn is the orientation transport from $\det(L_{0,A})$ to $\det(L_{s\Psi,A})$ because the orientation transport from $\det(L_{0,A_\varepsilon})$ to $\det(L_{0,A})$ is trivial.

Step 3. *The identity* (6.4) *holds.*

Using the isometry $\tilde{\mathfrak{a}}_\Psi = |\Psi|^{-1}\mathfrak{a}_\Psi \colon \Gamma(\mathrm{Re}(E \otimes S_{\tilde{\mathfrak{s}}})) \to \Omega^1(M, i\mathbf{R}) \oplus \Omega^0(M, i\mathbf{R})$ we can think of L_s as the operator

$$\tilde{L}_s \colon (\Omega^1(M, i\mathbf{R}) \oplus \Omega^0(M, i\mathbf{R}))^{\oplus 3} \to (\Omega^1(M, i\mathbf{R}) \oplus \Omega^0(M, i\mathbf{R}))^{\oplus 3}$$

defined by

$$\tilde{L}_s = \begin{pmatrix} -\mathfrak{d}_\Psi & & \\ & -\mathfrak{d}_\Psi & -s \\ & -s & \mathfrak{d} \end{pmatrix}.$$

By the preceding step, $\sigma([\Psi, A], \mathbf{p})$ is the orientation transport from $\tilde{L}_0$ to $\tilde{L}_s$ for $s \gg 1$. This is the same as the product of the orientation transport from

$$\begin{pmatrix} -\mathfrak{d}_\Psi & \\ & \mathfrak{d} \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} -\mathfrak{d} & \\ & \mathfrak{d} \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} -\mathfrak{d} & -s \\ -s & \mathfrak{d} \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} -\mathfrak{d}_\Psi & -s \\ -s & \mathfrak{d} \end{pmatrix}.$$

The first orientation transport is exactly the orientation transport from $-\mathfrak{d}_\Psi$ to $-\mathfrak{d}$. The second is a given by a universal sign. This sign can be computed to be $(-1)^{1+b_1(M)}$ by observing that if $\mathscr{H} := \mathscr{H}^0 \oplus \mathscr{H}^1$ denotes the space of harmonic forms and $\mathfrak{d}_0 \colon \mathscr{H}^\perp \to \mathscr{H}^\perp$ is the restriction of $\mathfrak{d}$, then

$$\begin{pmatrix} -\mathfrak{d} & -s \\ -s & \mathfrak{d} \end{pmatrix}$$

can be diagonalized to

$$\begin{pmatrix} s \cdot \mathrm{id}_{\mathscr{H}} & & & \\ & -s \cdot \mathrm{id}_{\mathscr{H}} & & \\ & & +\sqrt{\mathfrak{d}_0^2 + s^2} & \\ & & & -\sqrt{\mathfrak{d}_0^2 + s^2} \end{pmatrix},[7]$$

and the orientation transport along this family of operators is $(-1)^{1+b_1(M)}$. The final orientation transport is trivial since $\mathfrak{d}_\Psi = \mathfrak{d} + \mathfrak{e}_\Psi$ and, for $s \gg 1$ and $t \in [0, 1]$, the operator

$$\begin{pmatrix} -\mathfrak{d} - t\mathfrak{e}_\Psi & -s \\ -s & \mathfrak{d} \end{pmatrix}^2 = \begin{pmatrix} (-\mathfrak{d} - t\mathfrak{e}_\Psi)^2 + s^2 & \\ & \mathfrak{d}^2 + s^2 \end{pmatrix} + \begin{pmatrix} & st\mathfrak{e}_\psi \\ st\mathfrak{e}_\psi & \end{pmatrix}$$

is invertible. This completes the proof. □

7 Proof the wall-crossing formulae

This section will conclude the proof of Theorem 4.9. The local wall-crossing formula (6.2) and Proposition 6.3 directly imply the wall-crossing formula

$$n_{\mathfrak{w}}(\mathbf{p}_1, \eta_1) = n_{\mathfrak{w}}(\mathbf{p}_0, \eta_0) + \sum_{\mathfrak{s}} \sum_{i=1}^{N_{\mathfrak{s}}} \chi_i^{\mathfrak{s}} \cdot \sigma(\Psi_i^{\mathfrak{s}}, \mathbf{p}_{t_i^{\mathfrak{s}}}). \tag{4.10}$$

In order to prove (4.11), we need to relate $\sigma(\Psi_0, \mathbf{p}_0)$ to $\sigma(\Psi_1, \mathbf{p}_1)$ for two different nowhere vanishing Ψ_0 and Ψ_1 and two parameters $\mathbf{p}_0$ and $\mathbf{p}_1$.

Proposition 7.1. *Let $(\mathbf{p}_t)_{t \in [0,1]}$ be a path in $\mathscr{P}$. If Ψ_0 and Ψ_1 are two nowhere vanishing sections of $\mathrm{Re}(E \otimes S_{\mathfrak{s}})$, then*

$$\sigma(\Psi_1, \mathbf{p}_1) = \sigma(\Psi_0, \mathbf{p}_0) \cdot (-1)^{\mathrm{SF}(-\not{D}^{\mathfrak{s}}_{\mathbf{p}_t})} \cdot (-1)^{\deg(\Psi_0, \Psi_1)}. \tag{7.2}$$

Here $\deg(\Psi_0, \Psi_1)$ *denotes the relative the degree of* Ψ_0 *and* Ψ_1 *as in Definition 4.8.*

[7] The operator $\mathfrak{d}_0^2 + s^2$ has strictly positive spectrum, so its square root is well-defined and invertible.

Proof. If $\deg(\Psi_0, \Psi_1) = 0$, then we can find a path $(\Psi_t)_{t\in[0,1]}$ of nowhere vanishing sections from Ψ_0 to Ψ_1. It is clear that

$$\sigma(\Psi_1, \mathbf{p}_1) = \sigma(\Psi_0, \mathbf{p}_0) \cdot (-1)^{\mathrm{SF}(-\mathfrak{d}_{\Psi_t})}.$$

The spectral flow along the path of operators $-\mathfrak{d}_{\Psi_t}$ is identical to that of the path of operators

$$-\tilde{\mathfrak{a}}_{\Psi_0} \circ \mathfrak{d}_{\Psi_t} \circ \tilde{\mathfrak{a}}_{\Psi_0}^{-1},$$

which is homotopic to the path of operators

$$-\tilde{\mathfrak{a}}_{\Psi_t} \circ \mathfrak{d}_{\Psi_t} \circ \tilde{\mathfrak{a}}_{\Psi_t}^{-1} = -\not{D}^{\mathfrak{s}}_{\mathbf{p}_t}$$

followed by

$$-\tilde{\mathfrak{a}}_{\Psi_{1-t}}\tilde{\mathfrak{a}}_{\Psi_1}^{-1} \circ \not{D}^{\mathfrak{s}}_{\mathbf{p}_1} \circ \tilde{\mathfrak{a}}_{\Psi_1}\tilde{\mathfrak{a}}_{\Psi_{1-t}}^{-1}.$$

Since $\tilde{\mathfrak{a}}_{\Psi_1}\tilde{\mathfrak{a}}_{\Psi_{1-t}}^{-1}$ is a path of isometries, the second path of operators has trivial spectral flow. This proves (7.2) if $\deg(\Psi_0, \Psi_1) = 0$.

It remains to deal with the case $\deg(\Psi_0, \Psi_1) \neq 0$. By the above, we can assume that $\mathbf{p}_t = \mathbf{p}$ for all $t \in [0, 1]$. Set $V := (T^*M \oplus \underline{\mathbf{R}}) \otimes i\mathbf{R}$ and define an isometry $f\colon V \to V$ by

$$f := \tilde{a}_{\Psi_0}^{-1}\tilde{a}_{\Psi_1}$$

By definition of f, we have

$$\mathfrak{d}_{\Psi_1} = f^{-1} \circ \mathfrak{d}_{\Psi_0} \circ f.$$

Thus, gluing the operators $\partial_t - \mathfrak{d}^t_{\Psi_1}$ and $\partial_t - \mathfrak{d}^{1-t}_{\Psi_0}$, we obtain a first order elliptic operator $\mathfrak{D}$ on the bundle $\mathbf{V}$ over $S^1 \times M$ defined as the mapping torus of f. By Atiyah–Patodi–Singer,

$$\sigma(\Psi_1, \mathbf{p}_1) \cdot \sigma(\Psi_0, \mathbf{p}_0) = (-1)^{\mathrm{SF}(-\mathfrak{d}^t_{\Psi_1})} \cdot (-1)^{\mathrm{SF}(-\mathfrak{d}^{1-t}_{\Psi_0})} = (-1)^{\operatorname{index}\mathfrak{D}};$$

hence, by the Atiyah–Singer Index Theorem,

$$\sigma(\Psi_1, \mathbf{p}_1) \cdot \sigma(\Psi_0, \mathbf{p}_0) = (-1)^{\deg(\Psi_0, \Psi_1)}.$$

This completes the proof of this proposition. □

Proposition 7.3. *For all $i \in \{1, \dots, N_{\mathfrak{w}}\}$, we have*

$$\chi^{\mathfrak{s}}_i \cdot (-1)^{\mathrm{SF}\left(-\not{D}^{\mathfrak{s}}_{\mathbf{p}_t} : t\in\left[t^{\mathfrak{s}}_1, t^{\mathfrak{s}}_i\right]\right)} = (-1)^{i+1} \cdot \chi^{\mathfrak{s}}_1.$$

Proof. By induction it suffices to consider the case $i = 2$. The case $i = 2$ can be verified directly case-by-case as follows:

χ_1	χ_2	SF	$\chi_1 \cdot \chi_2 \cdot (-1)^{\mathrm{SF}}$
$+1$	$+1$	1	-1
$+1$	-1	0	-1
-1	$+1$	0	-1
-1	-1	-1	-1

□

Combining the two preceding propositions shows that (4.10) can equivalently be written as follows:

$$n_{\mathfrak{w}}(\mathbf{p}_1, \eta_1) = n_{\mathfrak{w}}(\mathbf{p}_0, \eta_0) + \sum_{\mathfrak{s}} \chi_1^{\mathfrak{s}} \cdot \sigma(\Psi_1^{\mathfrak{s}}, \mathbf{p}_{t_1^{\mathfrak{s}}}) \cdot \sum_{i=1}^{N_{\mathfrak{s}}} (-1)^{i+1} \cdot (-1)^{\deg(\Psi_1^{\mathfrak{s}}, \Psi_i^{\mathfrak{s}})}. \tag{4.11}$$

This completes the proof of Theorem 4.9. □

8 Transversality for paths

The purpose of this section is to prove Proposition 4.2.

Proposition 8.1. *For any $\mathbf{p}_0, \mathbf{p}_1 \in \mathscr{P}^{\mathrm{reg}}$, the subspace $\boldsymbol{\mathscr{P}}^{\mathrm{reg}}(\mathbf{p}_0, \mathbf{p}_1)$ is residual in the space of all smooth paths from $\mathbf{p}_0$ to $\mathbf{p}_1$ in $\mathscr{P}$.*

Proof. The proof is an application of the Sard–Smale theorem. We will work with Sobolev spaces of sections and connections of class $W^{k,p}$ such that $(k-1)p > 3$. The statement for C^∞ spaces follows then from a standard argument; see, for example, [Doa17b, Proof of Proposition 2.21].

Since there are only finitely many spin structures on M, it suffices to consider the conditions (1) and (2) in Definition 4.1 for a fixed spin structure $\mathfrak{s}$. Set

$$\begin{aligned} X &\coloneqq \boldsymbol{\mathscr{P}}(\mathbf{p}_0, \mathbf{p}_1) \times [0,1] \times \frac{\Gamma(\mathrm{Re}(E \otimes S_{\mathfrak{s}}) \backslash \{0\})}{\mathbf{R}^*} \\ \text{and} \quad V &\coloneqq \boldsymbol{\mathscr{P}}(\mathbf{p}_0, \mathbf{p}_1) \times [0,1] \times \frac{\Gamma(\mathrm{Re}(E \otimes S_{\mathfrak{s}}) \backslash \{0\}) \times \Gamma(\mathrm{Re}(E \otimes S_{\mathfrak{s}}))}{\mathbf{R}^*}. \end{aligned}$$

V is a vector bundle over X. Define a section $\mathbf{s} \in \Gamma(V)$ by

$$\boldsymbol{\sigma}\big((\mathbf{p}_t)_{t\in[0,1]}, t_*, [\Psi]\big) \coloneqq \big((\mathbf{p}_t)_{t\in[0,1]}, t_*, [(\Psi, \not{D}_{\mathbf{p}_{t_*}} \Psi)]\big).$$

We can identify a neighborhood of $[\Psi] \in \Gamma(\mathrm{Re}(E \otimes S_{\mathfrak{s}})\backslash\{0\})/\mathbf{R}^*$ with the L^2–orthogonal complement $\Psi^\perp \subset \Gamma(\mathrm{Re}(E \otimes S_{\mathfrak{s}})$. This gives us a local trivialization of V in which $\boldsymbol{\sigma}$ can be identified with the map

$$\boldsymbol{\sigma}\big((\mathbf{p}_t)_{t\in[0,1]}, t_*, \psi\big) = \big((\mathbf{p}_t)_{t\in[0,1]}, t_*, \not{D}_{\mathbf{p}_{t_*}}(\Psi + \psi)\big) \quad \text{for all } \psi \in \Psi^\perp.$$

In particular, for a fixed path $(\mathbf{p}_t)_{t\in[0,1]}$, the map $\boldsymbol{\sigma}((\mathbf{p}_t)_{t\in[0,1]}, \cdot)$ defines a Fredholm section of index zero, since $\psi \mapsto \not{D}_{\mathbf{p}_t}(\Psi + \psi)$ has index -1 and $\dim[0,1] = 1$.

Let $\mathbf{x} := \big((\mathbf{p}_t)_{t\in[0,1]}, t_*, [\Psi]\big) \in X$ and denote by $(\mathrm{d}\boldsymbol{\sigma})_{\mathbf{x}}$ the linearization of $\boldsymbol{\sigma}$ at $\mathbf{x}$ (and computed in the above trivialization). We will prove that $(\mathrm{d}\boldsymbol{\sigma})_{\mathbf{x}}$ is surjective provided $\boldsymbol{\sigma}(\mathbf{x}) = 0$. If $\Phi \in V_{\mathbf{x}} = \Gamma(\mathrm{Re}(E \otimes S_{\mathfrak{s}}))$ is orthogonal to the image of $(\mathrm{d}\boldsymbol{\sigma})_{\mathbf{x}}$, then it follows that

$$\langle \bar{\gamma}(b)\Psi, \Phi\rangle_{L^2} = 0 \quad \text{for all } b \in \Omega^1(M, \mathfrak{su}(E)). \tag{8.2}$$

Since Ψ is harmonic, its zero set must be nowhere dense. Clifford multiplication by $T^*M \otimes \mathfrak{su}(E)$ on $\mathrm{Re}(E \otimes S_{\mathfrak{s}})$ induces a isomorphism between $T^*M \otimes \mathfrak{su}(E)$ and trace-free symmetric endomorphisms of $\mathrm{Re}(E \otimes S_{\mathfrak{s}})$. With this in mind it follows from (8.2) that $\Psi = 0$. This proves that $\mathrm{d}_{\mathbf{x}}\boldsymbol{\sigma}$ is surjective.

It follows that $\boldsymbol{\sigma}^{-1}(0)$ is a smooth submanifold of X and the projection map $\pi\colon \boldsymbol{\sigma}^{-1}(0) \to \mathscr{P}(\mathbf{p}_0, \mathbf{p}_1)$ is a Fredholm map of index zero. The kernel of π at $\mathbf{x}$ can be identified with the kernel of the linearization of $\boldsymbol{\sigma}$ in the directions of $[0,1]$ and $\Gamma(\mathrm{Re}(E \otimes S_{\mathfrak{s}})\backslash\{0\})/\mathbf{R}^*$. A moment's thought show that if this kernel is trivial, then Ψ must span $\ker \not{D}_{\mathbf{p}_{t_*}}$ and the t_* must be a regular crossing of the spectral flow of $(\not{D}_{\mathbf{p}_t})$. By the Sard–Smale theorem, the subspace of regular values of π is residual; hence, set of those $(\mathbf{p}_t)_{t\in[0,1]}$ in $\mathscr{P}(\mathbf{p}_0, \mathbf{p}_1)$ for which the condition (1) in Definition 4.1 holds is residual.

To deal with condition (2) in Definition 4.1, we consider the vector bundle

$$W := \mathscr{P}(\mathbf{p}_0, \mathbf{p}_1) \times [0,1] \times \frac{\Gamma(\mathrm{Re}(E \otimes S_{\mathfrak{s}})\backslash\{0\}) \times \Gamma(\mathrm{Re}(E \otimes S_{\mathfrak{s}})) \times \mathrm{Re}(E \otimes S_{\mathfrak{s}})}{\mathbf{R}^*}$$

over $X \times M$ and define a section $\boldsymbol{\tau} \in \Gamma(W)$ by

$$\boldsymbol{\tau}\big((\mathbf{p}_t)_{t\in[0,1]}, t_*, [\Psi], y\big) := \big((\mathbf{p}_t)_{t\in[0,1]}, t_*, [(\Psi, \not{D}_{\mathbf{p}_{t_*}}\Psi, \Psi(y))]\big).$$

For a fixed path $(\mathbf{p}_t)_{t\in[0,1]}$, the map $\boldsymbol{\tau}((\mathbf{p}_t)_{t\in[0,1]}, \cdot)$ defines a Fredholm section of index -1. Note that for $\mathbf{x} := \big((\mathbf{p}_t)_{t\in[0,1]}, t_*, [\Psi]\big) \in X$ and $y \in M$ the condition $\boldsymbol{\tau}((\mathbf{p}_t)_{t\in[0,1]}, \mathbf{x}, y) = 0$ is equivalent to $\not{D}_{\mathbf{p}_{t_*}}\Psi = 0$ and $\Psi(y) = 0$. We prove that the linearization of $\boldsymbol{\tau}$ is surjective at any $(\mathbf{x}, y)$ satisfying these equations. If $(\Phi, \phi) \in$

$W_{(\mathbf{x},y)} = \Gamma(\mathrm{Re}(E \otimes S_{\mathfrak{s}})) \times \mathrm{Re}(E \otimes S_{\mathfrak{s}})_y$ is orthogonal to the image of $(\mathrm{d}\tau)_{(\mathbf{x},y)}$, then (8.2) holds and, moreover,

$$\langle \not{D}_{\mathbf{p}_{t_*}}(\Psi + \psi), \Phi\rangle_{L^2} + \langle \psi(x), \phi\rangle = 0 \quad \text{for all } \psi \in \Psi^\perp \tag{8.3}$$

Since $\not{D}_{\mathbf{p}_{t_*}}\Psi = 0$ and $\Psi(y) = 0$, (8.3) holds in fact for all $\psi \in \Gamma(\mathrm{Re}(E \otimes S_{\mathfrak{s}}))$ and we conclude that $\not{D}_{\mathbf{p}_{t_*}}\Psi = 0$. Plugging this back into (8.3) yields $\langle \psi(x), \phi\rangle = 0$ for all ψ, which implies that $\phi = 0$. As before (8.2) implies that $\Phi = 0$.

It follows that $\boldsymbol{\tau}^{-1}(0)$ is smooth and the projection $\rho\colon \boldsymbol{\tau}^{-1}(0) \to \mathscr{P}(\mathbf{p}_0, \mathbf{p}_1)$ is a Fredholm map of index -1; in particular, the preimage of a regular value must be empty. It follows that the paths $(\mathbf{p}_t)_{t\in[0,1]} \in \mathscr{P}(\mathbf{p}_0, \mathbf{p}_1)$ for which condition (2) in Definition 4.1 holds is residual. □

To address condition (4) in Definition 4.1 we compute $\delta(\Psi, \mathbf{p}, \eta)$.

Proposition 8.4. *Let $(\mathbf{p}_t)_{t\in[0,1]} \in \mathscr{P}(\mathbf{p}_0, \mathbf{p}_1)$, let $(\eta_t)_{t\in[0,1]}$ be a path in $\mathscr{Z}$, let $t_0 \in (0,1)$, and let Ψ be a nowhere vanishing section of* $\mathrm{Re}(E \otimes S_{\mathfrak{s}})$ *spanning* $\ker \not{D}^{\mathfrak{s}}_{\mathbf{p}_{t_0}}$ *and satisfying $\|\Psi\|_{L^2} = 1$. There is a linear algebraic operator*

$$\mathfrak{f}_{\mathbf{p}_{t_0},\Psi}\colon\ \Omega^2(M, i\mathbf{R}) \to \Omega^2(M, i\mathbf{R})$$

such that

$$\delta(\Psi, \mathbf{p}_{t_0}, \eta_{t_0}) = \int_M |\Psi|^{-2}\langle \mathrm{d} * \eta_{t_0} + \mathfrak{f}_{\Psi,\mathbf{p}_{t_0}}\eta_{t_0}, \eta_{t_0}\rangle \tag{8.5}$$

with $\delta(\Psi, \mathbf{p}_{t_0}, \eta_{t_0})$ is as in Definition 4.1(4).

Proof. If we denote by

$$\left\{\left(\Psi_\varepsilon = \Psi + \varepsilon^2\psi + O(\varepsilon^4), A_\varepsilon; t(\varepsilon) = t_0 + O(\varepsilon^2)\right) : 0 \le \varepsilon \ll 1\right\}$$

the family of solutions to

$$\begin{gathered}\not{D}_{A_\varepsilon,\mathbf{p}_{t(\varepsilon)}}\Psi_\varepsilon = 0,\\ \varepsilon^2\left(\frac{1}{2}F_{A_\varepsilon} + \eta_{t(\varepsilon)}\right) = \mu_{\mathbf{p}_{t(\varepsilon)}}(\Psi_\varepsilon), \quad \text{and}\\ \|\Psi_\varepsilon\|_{L^2} = 1\end{gathered}$$

obtained from [DW17, Theorem 1.38], then

$$\delta = \delta(\Psi, \mathbf{p}_{t_0}, \eta_{t_0}) = \langle \not{D}_{A_0,\mathbf{p}_{t_0}}\psi, \psi\rangle_{L^2}.$$

The connection $A = A_0$ corresponding to Ψ is flat; see Proposition 3.3. For the unperturbed equation we would have $t(\varepsilon) = 0$. Since we consider the perturbed equation, however, η enters into the computation of δ. More precisely, by [DW17, Equations 6.1, 6.2] we have

$$\psi = -\not{D}_{\mathrm{Re}}^{-1}\gamma \mathrm{II}^* \nu - \nu \quad \text{with} \quad \nu := (\mathfrak{a}_\Psi^*)^{-1} * \eta.$$

By (4.4), we have

$$\nu = |\Psi|^{-2}\bar{\gamma}(*\eta)\Psi.$$

Denote by π_{Re} the projection onto $\mathrm{Re}(E \otimes S_{\mathfrak{s}})$. From [DW17, Proposition 3.8] we know that for any $a \in \Omega^1(M, i\mathbf{R})$

$$-\gamma \mathrm{II}^* \bar{\gamma}(a)\Psi = \sum_{i=1}^{3} \pi_{\mathrm{Re}}\left(\rho(a(e_i))\nabla_{e_i}^A \Psi\right) = 0.$$

It follows that

$$\psi = -\nu = |\Psi|^{-2}\bar{\gamma}(*\eta)\Psi.$$

Set

$$a := |\Psi|^{-2} * \eta.$$

Since

$$\not{D}_A \bar{\gamma}(a)\Psi = \bar{\gamma}(*\mathrm{d}a)\Psi + \rho(\mathrm{d}^* a)\Psi - 2\sum_{i=1}^{3} \rho(a_i)\nabla_i \Psi,$$

we have

$$\begin{aligned}
\delta &= \int_M \langle \not{D}_A \bar{\gamma}(a)\Psi, \bar{\gamma}(a)\Psi\rangle \\
&= \int_M \langle \bar{\gamma}(*\mathrm{d}a)\Psi, \bar{\gamma}(a)\Psi\rangle + \langle \rho(\mathrm{d}^* a)\Psi, \bar{\gamma}(a)\Psi\rangle \\
&\qquad - 2\sum_{i=1}^{3} \langle \rho(a_i)\nabla_i \Psi, \bar{\gamma}(a)\Psi\rangle.
\end{aligned}$$

The first term in the integral is

$$|\Psi|^2 \langle *\mathrm{d}a, a\rangle = \langle \mathrm{d} * \left(|\Psi|^{-2}\eta\right), \eta\rangle.$$

The second term vanishes. Therefore,

$$\delta = \int_M |\Psi|^{-2}\langle \mathrm{d} * \eta, \eta\rangle + |\Psi|^{-2}\langle \mathfrak{f}_1\eta, \mathfrak{f}_2\eta\rangle$$

for linear operators $\mathfrak{f}_1, \mathfrak{f}_2 \colon \Omega^2(M, i\mathbf{R}) \to \Gamma(\mathrm{Re}(E \otimes S_{\mathfrak{s}}))$ of order zero. Set $\mathfrak{f}_{\Psi,\mathbf{p}} = \mathfrak{f}_2^* \mathfrak{f}_1$. ∎

Proposition 8.6. *For $(\Psi, \mathbf{p}, \eta) \in \Gamma(\mathrm{Re}(E \otimes S_{\mathfrak{s}})) \times \mathscr{P} \times \mathscr{Z}$ such that Ψ is nowhere vanishing, define $\delta(\Psi, \mathbf{p}, \eta)$ by formula (8.5). For each $(\Psi, \mathbf{p})$, the set*

$$\mathscr{Z}^{\mathrm{reg}}(\Psi, \mathbf{p}) = \{\eta \in \mathscr{Z} : \delta(\Psi, \mathbf{p}, \eta) \neq 0\}$$

is open and dense in $\mathscr{Z}$.

Proof. Replace all the spaces in question by their completions with respect to the $W^{k,p}$ norm for any k and p satisfying $(k-1)p > 3$. We will prove the statement with respect to the Sobolev topology; the corresponding statement for C^∞ spaces follows then from the Sobolev embedding theorem and the fact that δ is continuous with respect to any of these topologies.

By Proposition 8.4,

$$(\mathrm{d}\delta)_\eta[\hat{\eta}] = \int_M |\Psi|^{-2} \langle 2\mathrm{d} * \eta + \bar{\mathfrak{f}}_{\Psi,\mathbf{p}}\eta, \hat{\eta}\rangle,$$

where

$$\bar{\mathfrak{f}}_{\Psi,\mathbf{p}}\eta = (\mathfrak{f}_{\Psi,\mathbf{p}} + \mathfrak{f}_{\Psi,\mathbf{p}}^*)\eta - 2\mathrm{d}(\log|\Psi|) \wedge *\eta$$

is a linear algebraic operator. Thus, the derivative of δ vanishes along the set $\mathscr{Z}_{\mathrm{crit}}(\Psi, \mathbf{p})$ of solutions η to the linear elliptic differential equation

$$\begin{aligned} \mathrm{d}\eta &= 0, \\ \mathrm{d} * \eta + *\bar{\mathfrak{f}}_{\Psi,\mathbf{p}}\eta &= 0. \end{aligned}$$

$\mathscr{Z}_{\mathrm{crit}}(\Psi, \mathbf{p})$ a closed, finite-dimensional subspace of $\mathscr{Z}$. By the Implicit Function Theorem, away from $\mathscr{Z}_{\mathrm{crit}}(\Psi, \mathbf{p})$, the zero set of $\delta(\Psi, \mathbf{p}, \cdot)$ is a codimension one Banach submanifold of (the Sobolev completion of) $\mathscr{Z}$. Hence, the set

$$\mathscr{Z}^{\mathrm{reg}}(\Psi, \eta) \cap (\mathscr{Z} \backslash \mathscr{Z}_{\mathrm{crit}}(\Psi, \eta))$$

is dense. Since δ is continuous, $\mathscr{Z}^{\mathrm{reg}}(\Psi, \eta)$ is open. ∎

Proof of Proposition 4.2. By [Doa17b, Theorem 2.34], $\mathcal{Q}_3$, the subspace of paths from $(\mathbf{p}_0, \eta_0)$ to $(\mathbf{p}_1, \eta_1)$ satisfying Definition 4.1(3), is residual.

Denote by $\mathcal{Q}_{1,2}$ the space of paths from $(\mathbf{p}_0, \eta_0)$ to $(\mathbf{p}_1, \eta_1)$ satisfying conditions (1) and (2) in Definition 4.1. By Proposition 8.1, $\mathcal{Q}_{1,2}$ is residual in the space of paths from $(\mathbf{p}_0, \eta_0)$ to $(\mathbf{p}_1, \eta_1)$. Let $\mathscr{Q}_{1,2,4} \subset \mathscr{Q}_{1,2}$ be the space of paths from $(\mathbf{p}_0, \eta_0)$ to $(\mathbf{p}_1, \eta_1)$ also satisfying Definition 4.1(4). Elementary arguments show that $\mathscr{Q}_{1,2,4}$ is open in $\mathscr{Q}_{1,2}$ and we will shortly prove that $\mathscr{Q}_{1,2,4}$ is dense in $\mathscr{Q}_{1,2}$. A set which is open and dense in a residual set is itself residual. It follows that $\mathscr{Q}_{1,2,4}$ is residual; hence, so is $\mathscr{Q}^{\mathrm{reg}}\big((\mathbf{p}_0, \eta_0), (\mathbf{p}_1, \eta_1)\big) = \mathscr{Q}_{1,2,4} \cap \mathscr{Q}_3$.

To prove that $\mathscr{Q}_{1,2,4}$ is dense in $\mathscr{Q}_{1,2}$, suppose that $(\mathbf{p}_t, \eta_t)_{t\in[0,1]} \in \mathscr{Q}_{1,2}$. There are finitely many times $0 < t_1 < \ldots < t_n < 1$ for which the kernel of $\not{D}_{\mathbf{p}_{t_i}}$ is non-trivial. For $i = 1, \ldots, n$, denote by Ψ_i a section spanning $\ker \not{D}_{\mathbf{p}_{t_i}}$. By Proposition 8.6, for any $\sigma > 0$, there are closed forms $\alpha_1, \ldots, \alpha_n$ such that

$$\delta(\Psi_i, \mathbf{p}_{t_i}, \eta_{t_i} + \alpha_i) \neq 0 \quad \text{and} \quad \|\alpha_i\|_{L^\infty} \leq \sigma$$

for every $i = 1, \ldots, n$. Let $(\alpha_t)_{t\in[0,1]}$ be a path of closed forms such that $\alpha_{t_i} = \alpha_i$ for $i = 1, \ldots, n$ and $\|\alpha_t\|_{L^\infty} \leq \sigma$ for all $t \in [0, 1]$. The path $(\mathbf{p}_t, \eta_t + \alpha_t)_{t\in[0,1]}$ satisfies conditions (1), and (2) in Definition 4.1 because these only depend on $(\mathbf{p}_t)_{t\in[0,1]}$. It also satisfies (4) by construction. We conclude that $(\mathbf{p}_t, \eta_t + \alpha_t)_{t\in[0,1]} \in \mathscr{Q}_{1,2,4}$. Since σ is arbitrary, it follows that $\mathscr{Q}_{1,2,4}$ is dense in $\mathscr{Q}_{1,2}$. □

9 Proof of the existence of singular harmonic Z_2 spinors

In this section prove Theorem 1.6. We begin with defining the set $\mathscr{W}_b$ appearing in its statement.

Definition 9.1. Given a spin structure $\mathfrak{s}$, set

$$\begin{aligned}
\mathscr{W}^{\mathfrak{s}} &:= \Big\{\mathbf{p} \in \mathscr{P} : \dim\ker \not{D}^{\mathfrak{s}}_{\mathbf{p}} > 0\Big\}, \\
\mathscr{W}^{\mathfrak{s}}_{1,\varnothing} &:= \Big\{\mathbf{p} \in \mathscr{P} : \ker \not{D}^{\mathfrak{s}}_{\mathbf{p}} = \mathbf{R}\langle\Psi\rangle \text{ with } \Psi \text{ nowhere vanishing}\Big\}, \\
\mathscr{W}^{\mathfrak{s}}_{1,\star} &:= \Big\{\mathbf{p} \in \mathscr{P} : \ker \not{D}^{\mathfrak{s}}_{\mathbf{p}} = \mathbf{R}\langle\Psi\rangle \text{ and } \Psi \text{ has a single non-degenerate zero}\Big\} \quad \text{and} \\
\mathscr{W}^{\mathfrak{s}}_b &:= \mathscr{W}^{\mathfrak{s}} \backslash \mathscr{W}^{\mathfrak{s}}_{1,\varnothing}.
\end{aligned}$$

A zero $x \in \Psi^{-1}(0)$ is non-degenerate if the linear map $(\nabla\Psi)_x \colon T_xM \to \mathrm{Re}(E \otimes S)_x$ has maximal rank (that is, rank three). Set

$$\mathscr{W} := \bigcup_{\mathfrak{s}} \mathscr{W}^{\mathfrak{s}}, \quad \mathscr{W}_b := \bigcup_{\mathfrak{s}} \mathscr{W}^{\mathfrak{s}}_b, \quad \text{and} \quad \mathscr{W}_{1,\varnothing} := \bigcup_{\mathfrak{s}} \mathscr{W}^{\mathfrak{s}}_{1,\varnothing} \backslash \mathscr{W}_b.$$

Here the union is taken over all spin structures $\mathfrak{s}$.

Remark 9.2. As we will see shortly, $\mathscr{W}^{\mathfrak{s}}$ is a codimension one stratified space with smooth strata $\mathscr{W}_1^{\mathfrak{s}}, \mathscr{W}_2^{\mathfrak{s}}, \ldots$ corresponding to the dimension of the kernel of $\not{D}_{\mathbf{p}}^{\mathfrak{s}}$. The top stratum $\mathscr{W}_1^{\mathfrak{s}}$ contains an open and dense subset $\mathscr{W}_{1,\varnothing}^{\mathfrak{s}}$ for which the harmonic spinor spanning the kernel satisfies $|\Psi|^{-1} = \varnothing$. The rest of $\mathscr{W}_1^{\mathfrak{s}}$ stratified according to the size of the zero set of Ψ.

We think of $\mathscr{W}_b^{\mathfrak{s}}$ as the set of particularly degenerate ("bad") parameters. It is stratified and its top stratum is $\mathscr{W}_{1,\star}^{\mathfrak{s}}$, which has codimension two in $\mathscr{P}$.

Proposition 9.3. *$\mathscr{W}_{1,\varnothing}^{\mathfrak{s}}$ is a closed, codimension one submanifold of $\mathscr{P}\backslash\mathscr{W}_b^{\mathfrak{s}}$. It carries a coorientation such that the following holds. Let $(\mathbf{p}_t)$ be a path in $\mathscr{P}\backslash\mathscr{W}_b^{\mathfrak{s}}$ with $\mathbf{p}_0, \mathbf{p}_1 \in \mathscr{P}\backslash\mathscr{W}^{\mathfrak{s}}$ which intersects $\mathscr{W}_{1,\varnothing}^{\mathfrak{s}}$ transversely. Denote*

- *by $\{t_1, \ldots, t_N\} \subset [0,1]$ the finite set of times at which the spectrum of $\not{D}_{\mathbf{p}_t}^{\mathfrak{s}}$ crosses zero, i.e., $\mathbf{p}_t \in \mathscr{W}_{1,\varnothing}^{\mathfrak{s}}$*

and, for each $i = 1, \ldots, N$, denote

- *by $\chi_i \in \{\pm 1\}$ the sign of the spectral crossing at t_i and*
- *by Ψ_i a nowhere vanishing spinor spanning* $\ker \not{D}_{\mathbf{p}_t}$.

The intersection number of $(\mathbf{p}_t)$ with $\mathscr{W}_{1,\varnothing}^{\mathfrak{s}}$ is

$$\sum_{i=1}^{N} \chi_i \cdot \sigma(\Psi_i, \mathbf{p}_{t_i}).$$

Proof. Let $\mathbf{p}_0 \in \mathscr{W}_{1,\varnothing}^{\mathfrak{s}}$. Let $\Psi_0 \in \ker \not{D}_{\mathbf{p}_0}^{\mathfrak{s}}$ be such that $\|\Psi_0\|_{L^2} = 1$. It follows from the Implicit Function Theorem that, for some open neighborhood U of $\mathbf{p}_0 \in \mathscr{P}$, there is a unique smooth map $U \to \mathbf{R} \times \Gamma(\mathrm{Re}(E \otimes S))$,

$$\mathbf{p} \mapsto (\lambda(\mathbf{p}), \Psi_{\mathbf{p}})$$

such that

$$\lambda(\mathbf{p}_0) = 0 \quad \text{and} \quad \Psi_{\mathbf{p}_0} = \Psi_0$$

as well as

$$\not{D}_{\mathbf{p}}^{\mathfrak{s}} \Psi_{\mathbf{p}} = \lambda(\mathbf{p}) \Psi_{\mathbf{p}} \quad \text{and} \quad \|\Psi_0\|_{L^2} = 1.$$

A moment's thought shows that if U is sufficiently small, then

$$U \cap \mathscr{W}_{1,\varnothing}^{\mathfrak{s}} = \lambda^{-1}(0).$$

Since

$$(\mathrm{d}_{\mathbf{p}_0}\lambda)(0,b) = \langle \gamma(b)\Psi, \Psi\rangle_{L^2}$$

and Clifford multiplication induces an isomorphism from $T^*M \otimes \mathfrak{su}(E)$ to trace-free symmetric endomorphisms of $\mathrm{Re}(E \otimes S)$, λ is a submersion provided U is sufficiently small. Hence, $\mathscr{W}_{1,\varnothing}^{\mathfrak{s}}$ is a codimension one submanifold. To see that $\mathscr{W}_{1,\varnothing}^{\mathfrak{s}}$ is closed, observe that $(\mathbf{p}_i)$ is a sequence on $\mathscr{W}_{1,\varnothing}^{\mathfrak{s}}$ with $\mathbf{p}_i \to \mathbf{p} \in \mathscr{P}$, then $\mathbf{p} \in \mathscr{W}^{\mathfrak{s}}$ and thus either in $\mathscr{W}_{1,\varnothing}^{\mathfrak{s}}$ or $\mathscr{W}_{b}^{\mathfrak{s}}$.

The above argument goes through with

$$\mathscr{W}_1^{\mathfrak{s}} = \{\mathbf{p} \in \mathscr{P} : \dim\ker \not{D}_{\mathbf{p}}^{\mathfrak{s}} = 1\}$$

instead of $\mathscr{W}_{1,\varnothing}^{\mathfrak{s}}$. This induces a natural coorientation on $\mathscr{W}_1^{\mathfrak{s}}$ by demanding that

$$\mathrm{d}_{\mathbf{p}}\lambda \colon\ T_{\mathbf{p}}\mathscr{P}/T_{\mathbf{p}}\mathscr{W}_1^{\mathfrak{s}} \to \mathbf{R}$$

is orientation-preserving. If $(\mathbf{p}_t)_{t\in[0,1]}$ is a path in $\mathscr{P}$ such that $\dim\ker \not{D}_{\mathbf{p}_t}^{\mathfrak{s}} \le 1$ and $\dim\ker \not{D}_{\mathbf{p}_t}^{\mathfrak{s}} = 0$ for $t = 0, 1$, then the intersection number of $(\mathbf{p}_t)_{t\in[0,1]}$ with $\mathscr{W}_1^{\mathfrak{s}}$ is precisely the spectral flow of $\not{D}_{\mathbf{p}_t}^{\mathfrak{s}}$. The coorientation on $\mathscr{W}_{1,\varnothing}^{\mathfrak{s}}$ is obtained by demanding that

$$\sigma(\Psi_{\mathbf{p}}, \mathbf{p}) \cdot \mathrm{d}_{\mathbf{p}}\lambda \colon\ T_{\mathbf{p}}\mathscr{P}/T_{\mathbf{p}}\mathscr{W}_{1,\varnothing}^{\mathfrak{s}} \to \mathbf{R}$$

is orientation-preserving. This coorientation clearly has the asserted property. □

Theorem 9.4. *In the above situation, the following hold.*

1. *The cohomology class* $\omega \in H^1(\mathscr{P}\backslash\mathscr{W}_b, \mathbf{Z}) = \mathrm{Hom}(\pi_1(\mathscr{P}\backslash\mathscr{W}_b), \mathbf{Z})$ *defined by* $\mathscr{W}_{1,\varnothing}$ *together with its natural coorientation from Proposition 9.3 is non-trivial.*

2. *If* $(\mathbf{p}_0, \eta_0) \in \mathscr{Q}^{\mathrm{reg}}$ *and* $(\mathbf{p}_t, \eta_t)$ *is a loop in* $\boldsymbol{\mathscr{Q}}^{\mathrm{reg}}\big((\mathbf{p}_0,\eta_0),(\mathbf{p}_0,\eta_0)\big)$, *then* $(\mathbf{p}_t)$ *is a path in* $\mathscr{P}\backslash\mathscr{W}_b$ *and if* $\omega([\mathbf{p}_t]) \neq 0$, *then there is exists a harmonic* $\mathbf{Z}_2$ *spinor with respect to some* $\mathbf{p}_t$.

Proof of Theorem 1.6. Since the union of the projections of $\boldsymbol{\mathscr{Q}}^{\mathrm{reg}}\big((\mathbf{p}_0,\eta_0),(\mathbf{p}_0,\eta_0)\big)$ to $\boldsymbol{\mathscr{P}}(\mathbf{p}_0,\mathbf{p}_0)$ (as $(\mathbf{p}_0,\eta_0)$ ranges over $\mathscr{Q}^{\mathrm{reg}}$) is a residual subset of the space of all loops in $\mathscr{P}$, Theorem 1.6 follows from Theorem 9.4. □

The idea of the proof that $\omega \neq 0$ is to exhibit a loop $(\mathbf{p}_t)$ in $\mathscr{P}\backslash\mathscr{W}_b$ on which ω evaluates non-trivially. More precisely, we will construct such a loop which intersects $\mathscr{W}_{1,\varnothing}^{\mathfrak{s}}$ in two points as illustrated in Figure 1, which cannot be joined by a path in $\mathscr{W}_{1,\varnothing}^{\mathfrak{s}}$; however, they are joined by a path in $\mathscr{W}_1^{\mathfrak{s}}$ passing through $\mathscr{W}_{1,\star}^{\mathfrak{s}}$ in a unique point.

Note that $\mathscr{W}^{\mathfrak{s}}_{1,\star}$ is a subset of the bad set $\mathscr{W}_b$ and we will prove that it is a codimension two submanifold of $\mathscr{P}$. While the coorientation on $\mathscr{W}^{\mathfrak{s}}_{1}$ is preserved along this path, the one on $\mathscr{W}^{\mathfrak{s}}_{1,\varnothing}$ is not. Consequently, the intersection number of the loop with $\mathscr{W}^{\mathfrak{s}}_{1,\varnothing}$ is ± 2. The above situation can be arranged so that $(\mathbf{p}_t)$ does not intersect $\mathscr{W}^{\tilde{\mathfrak{s}}}_{1,\varnothing}$ for any other spin structure $\tilde{\mathfrak{s}}$. It follows that

$$\omega([\mathbf{p}_t]) \pm 2 \neq 0.$$

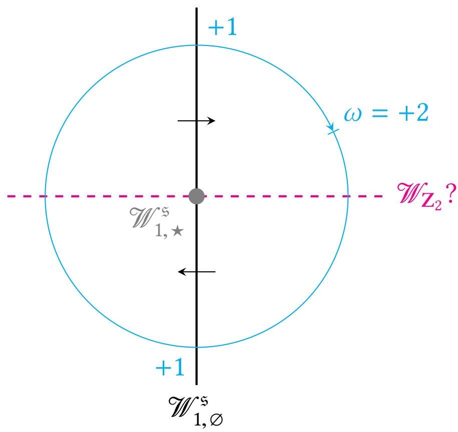

Figure 1: A loop linking $\mathscr{W}^{\mathfrak{s}}_{1,\star}$ and pairing non-trivially with ω.

If $(\mathbf{p}_0, \eta_0) \in \mathscr{Q}^{\mathrm{reg}}$ and $(\mathbf{p}_t, \eta_t)$ is a loop in $\boldsymbol{\mathscr{Q}}^{\mathrm{reg}}\big((\mathbf{p}_0, \eta_0), (\mathbf{p}_0, \eta_0)\big)$ and $\omega([\mathbf{p}_t]) \neq 0$, then there is exists a singular harmonic $\mathbf{Z}_2$ harmonic spinor with respect to some $\mathbf{p}_t$ for otherwise

$$\omega([\mathbf{p}_t]) = 0$$

by Theorem 4.9.

Remark 9.5. This and the work of Takahashi [Tak15; Tak17] indicate the presence of a wall $\mathscr{W}_{\mathbf{Z}_2} \subset \mathscr{P}$ caused by singular Fueter sections as depicted in Figure 1. In light of the above discussion it is a tantalizing question to ask:

> Can the harmonic $\mathbf{Z}_2$ spinors, whose abstract existence is guaranteed by Theorem 9.4, be constructed more directly by a gluing construction?

We plan to investigate this problem in future work.

Proof of Theorem 9.4. To prove (2), note that if $(\mathbf{p}_0, \eta_0) \in \mathcal{Q}^{\mathrm{reg}}$ and $(\mathbf{p}_t, \eta_t)$ is a loop in $\boldsymbol{\mathcal{Q}}^{\mathrm{reg}}\big((\mathbf{p}_0, \eta_0), (\mathbf{p}_0, \eta_0)\big)$ and $\omega([\mathbf{p}_t]) \neq 0$, then there is exists a singular harmonic $\mathbf{Z}_2$ harmonic spinor with respect to some $\mathbf{p}_t$ for otherwise

$$n(\mathbf{p}_0, \eta_0) = n(\mathbf{p}_0, \eta_0) + \omega([\mathbf{p}_t])$$

by Theorem 4.9. Here

$$n(\mathbf{p}, \eta) = \sum_{\mathfrak{w}} n_{\mathfrak{w}}(\mathbf{p}, \eta)$$

and we sum over all spin^c structures $\mathfrak{w}$ with trivial determinant.

In order to prove (1) we will produce a loop pairing non-trivially with ω. The existence of such a loop is ensured by the following result provided we can exhibit a point $\mathbf{p}_\star \in \mathscr{W}^{\mathfrak{s}}_{1,\star}$.

Proposition 9.6. *Given $\mathbf{p}_\star \in \mathscr{W}^{\mathfrak{s}}_{1,\star}$ and an open neighborhood U of $\mathbf{p}_\star \in \mathscr{P}$, there exists a loop $(\mathbf{p}_t)_{t\in S^1}$ in $U \cap (\mathscr{P}\backslash\mathscr{W}^{\mathfrak{s}}_b)$ such that:*

1. *$\mathbf{p}_{1/4}, \mathbf{p}_{3/4} \in \mathscr{W}^{\mathfrak{s}}_{1,\varnothing}$, and $\mathbf{p}_t \in \mathscr{P}\backslash\mathscr{W}^{\mathfrak{s}}_1$ for all $t \notin \{1/4, 3/4\}$,*
2. *if $\Psi_{1/4}$ and $\Psi_{3/4}$ denote spinors spanning $\not{D}^{\mathfrak{s}}_{\mathbf{p}_{1/4}}$ and $\not{D}^{\mathfrak{s}}_{\mathbf{p}_{3/4}}$, then*

   $$\deg(\Psi_{1/4}, \Psi_{3/4}) = \pm 1;$$

   and
3. *the spectral crossings at $\mathbf{p}_{1/4}$ and $\mathbf{p}_{3/4}$ occur with opposite signs.*

In particular, the intersection number of $(\mathbf{p}_t)_{t\in[0,1]}$ with $\mathscr{W}^{\mathfrak{s}}_{1,\varnothing}$ is ± 2.

Proof. Let $\Psi_\star \in \ker \not{D}^{\mathfrak{s}}_{\mathbf{p}_\star}$ and $x_\star \in M$ be such that $\|\Psi_\star\|_{L^2} = 1$ and $\Psi_\star(x_\star) = 0$. Let $\phi \in \Gamma(\mathrm{Re}(E \otimes S_{\mathfrak{s}}))$ be such that

$$\mathrm{im}((\nabla\Psi_\star)_{x_\star}) + \mathbf{R}\langle\phi_\star(x_\star)\rangle = \mathrm{Re}(E \otimes S_{\mathfrak{s}})_{x_\star} \quad \text{and} \quad |\phi_\star(x_\star)| = 1.$$

We can assume that U is sufficiently small for the Implicit Function Theorem to guarantee that there is a unique smooth map $U \longrightarrow \mathbf{R} \times \Gamma(\mathrm{Re}(E \otimes S_{\mathfrak{s}})) \times M \times \mathbf{R}$,

$$\mathbf{p} \mapsto (\lambda(\mathbf{p}), \Psi_{\mathbf{p}}, x_{\mathbf{p}}, \nu(\mathbf{p}))$$

such that

$$\lambda(\mathbf{p}_\star) = 0, \quad \Psi_{\mathbf{p}_\star} = \Psi_\star, \quad \text{and} \quad x_{\mathbf{p}_\star} = x_\star$$

as well as

$$\not{D}^{\mathfrak{s}}_{\mathbf{p}}\Psi_{\mathbf{p}} = \lambda(\mathbf{p})\Psi_{\mathbf{p}} \quad \Psi_{\mathbf{p}}(x_{\mathbf{p}}) = \nu(x_{\mathbf{p}})\phi_{\star}(x_{\mathbf{p}}), \quad \text{and} \quad \|\Psi_0\|_{L^2} = 1. \tag{9.7}$$

As before

$$U \cap \mathscr{W}^{\mathfrak{s}}_1 = \lambda^{-1}(0).$$

Set

$$\mathscr{N} := \nu^{-1}(0).$$

This is the set of those $\mathbf{p} \in U$ for which the eigenspinor with smallest eigenvalue has a unique zero which is also non-degenerate.

From the proof of Proposition 9.3 we know that $U \cap \mathscr{W}^{\mathfrak{s}}_1$ is a codimension one submanifold. We will now show that $\mathscr{N}$ is a codimension one submanifold as well and that it intersects $U \cap \mathscr{W}^{\mathfrak{s}}_1$ transversely in $U \cap \mathscr{W}^{\mathfrak{s}}_{1,\star}$, see Figure 2. Knowing this, the existence a loop $(\mathbf{p})_{t \in S^1}$ with the desired properties follows easily because crossing $\mathscr{N}$ changes the relative degree by ± 1.

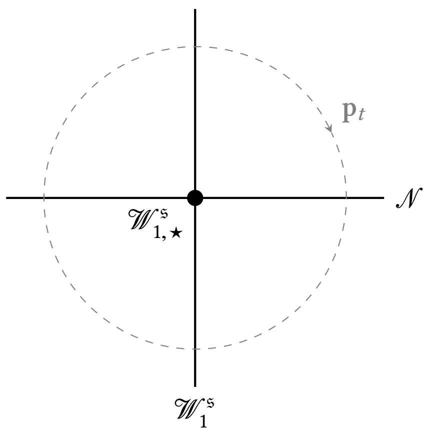

Figure 2: $\mathscr{W}^{\mathfrak{s}}_1$ and $\mathscr{N}$ intersecting in $\mathscr{W}^{\mathfrak{s}}_{1,\star}$.

We will show that $\mathrm{d}_{\mathbf{p}_\star}\nu|_{T_{\mathbf{p}_\star}\mathscr{W}^{\mathfrak{s}}_1}$ is non-vanishing. This implies both that $\mathscr{N}$ is a codimension one manifold and that it intersects $U \cap \mathscr{W}^{\mathfrak{s}_1}$ transversely. For $\hat{\mathbf{p}} = (0, b) \in \mathscr{T}_{\mathbf{p}_\star}\mathscr{P}$ to be determined, consider $\mathbf{p}_\star + t\hat{\mathbf{p}}$ and set

$$\lambda_t = \lambda(\mathbf{p}_\star + t\hat{\mathbf{p}}), \quad \Psi_t = \Psi_{\mathbf{p}_\star + t\hat{\mathbf{p}}}, \quad x_t = x_{\mathbf{p}\star + t\hat{\mathbf{p}}}, \quad \text{and} \quad \nu_t = \nu_{\mathbf{p}_\star + t\hat{\mathbf{p}}}.$$

as well as

$$\dot{\lambda} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \lambda_t, \quad \dot{\Psi} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Psi_t, \quad \dot{x} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} x_t, \quad \text{and} \quad \dot{\nu} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \nu_t.$$

Differentiating (9.7) we obtain

$$\bar{\gamma}(b)\Psi_\star + \not{D}^{\mathfrak{s}}_{\mathbf{p}_\star}\dot{\Psi} = \dot{\lambda}\Psi_\star, \quad \dot{\Psi}(x_\star) + (\nabla\Psi_\star)_{x_\star}\dot{x} = \dot{\nu}\phi(x_\star), \quad \text{and} \quad \langle\Psi_\star, \dot{\Psi}\rangle_{L^2} = 0.$$

From this it follows that

$$\begin{aligned} \dot{\lambda} &= \langle\gamma(b)\Psi_\star, \Psi_\star\rangle_{L^2}, \\ \not{D}^{\mathfrak{s}}_{\mathbf{p}_\star}\dot{\Psi} &= \gamma(b)\Psi_\star - \langle\gamma(b)\Psi_\star, \Psi_\star\rangle_{L^2}\Psi_\star, \quad \text{and} \\ \dot{\nu} &= \langle\dot{\Psi}(x_\star), \phi(x_\star)\rangle. \end{aligned} \tag{9.8}$$

Now, choose $\dot{\Psi}$ such that:

1. $\not{D}^{\mathfrak{s}}_{\mathbf{p}_\star}\dot{\Psi}$ vanishes in a neighborhood of $x_\star$.
2. $\langle\dot{\Psi}(x_\star), \phi(x_\star)\rangle \neq 0$, and
3. $\langle\Psi_\star, \dot{\Psi}\rangle_{L^2} = 0$.

Such a section can be defined by solving the Dirac equation in a neighborhood of $x_\star$ subject to the constraint (2) and then extending to all of M so that (3) holds.

Clifford multiplication by $T^*M \otimes \mathfrak{su}(E)$ on $\mathrm{Re}(E \otimes S_{\mathfrak{s}})$ induces a isomorphism between $T^*M \otimes \mathfrak{su}(E)$ and trace-free symmetric endomorphisms of $\mathrm{Re}(E \otimes S_{\mathfrak{s}})$. Since $\not{D}^{\mathfrak{s}}_{\mathbf{p}_\star}\dot{\Psi}$ vanishes in a neighborhood of $x_\star$ and $\Psi_\star$ vanishes only at $x_\star$, one can find $b \in \Omega^1(M, \mathfrak{su}(E))$ such that

$$\langle\gamma(b)\Psi_\star, \Psi_\star\rangle_{L^2} = 0 \quad \text{and} \quad \gamma(b)\Psi_\star = \not{D}^{\mathfrak{s}}_{\mathbf{p}_\star}\dot{\Psi}.$$

It follows from (9.8), that for this choice of b we have

$$\dot{\lambda} = 0 \quad \text{but} \quad \dot{\nu} \neq 0;$$

that is

$$\hat{\mathbf{p}} = (0, b) \in T_{\mathbf{p}_\star}\mathscr{W}^{\mathfrak{s}}_1 \quad \text{and} \quad \mathrm{d}_{\mathbf{p}_\star}\nu(b) \neq 0.$$

This completes the proof. □

It remains to exhibit a point $\mathbf{p}_\star \in \mathscr{W}^{\mathfrak{s}}_{1,\star}$ for some spin structure $\mathfrak{s}$ but such that $\mathbf{p}_\star \notin \mathscr{W}^{\tilde{\mathfrak{s}}}$ for any other spin structure $\tilde{\mathfrak{s}}$. This requires the following two propositions as preparation.

Proposition 9.9. *Let $k \in \{2, 3, \dots\}$. The subset*

$$\mathscr{W}_k^{\mathfrak{s}} := \{\mathbf{p} \in \mathscr{P} : \dim\ker \not{D}_{\mathbf{p}}^{\mathfrak{s}} = k\} \subset \mathscr{P}$$

is contained in a submanifold of codimension three. Moreover, $\mathscr{W}_k^{\mathfrak{s}} \cap \overline{\mathscr{W}_1^{\mathfrak{s}}} = \mathscr{W}_k^{\mathfrak{s}}$.

Proof. Let $\mathbf{p}_0 \in \mathscr{P}$ such that $\dim\ker \not{D}_{\mathbf{p}_0}^{\mathfrak{s}} = k$. Choose an L^2–orthonormal basis $\{\Psi_i\}$ of $\ker \not{D}_{\mathbf{p}_0}^{\mathfrak{s}}$. For a sufficiently small neighborhood U of $\mathbf{p}_0$, by the Implicit Function Theorem, there exists a unique smooth map $U \to \Gamma(\mathrm{Re}(E \otimes S_{\mathfrak{s}}))^{\oplus k} \times S^2\mathbf{R}^k$

$$\mathbf{p} \mapsto \big(\Psi_{1,\mathbf{p}}, \dots, \Psi_{k,\mathbf{p}}, \Lambda(\mathbf{p}) = (\lambda_{ij}(\mathbf{p}))\big)$$

such that

$$\Psi_{i,\mathbf{p}_0} = \Psi_i \quad \text{and} \quad \Lambda(\mathbf{p}) = 0$$

as well as

$$\not{D}_{\mathbf{p}}^{\mathfrak{s}}\Psi_{i,\mathbf{p}} = \sum_{j=1}^{k} \lambda_{ij}\Psi_{j,\mathbf{p}} \quad \text{and} \quad \langle \Psi_{i,\mathbf{p}}, \Psi_{j,\mathbf{p}}\rangle_{L^2} = \delta_{ij}.$$

(It follows from the fact that $\not{D}_{\mathbf{p}}^{\mathfrak{s}}$ is symmetric, that $\lambda_{ij} = \lambda_{ji}$.) We have

$$U \cap \mathscr{W}_k^{\mathfrak{s}} = \Lambda^{-1}(0)$$

We will show that $\mathrm{d}\Lambda\colon T_{\mathbf{p}_0}U \to S^2\mathbf{R}^k$ has rank at least three. This will imply that $\mathscr{W}_k^{\mathfrak{s}}$ has codimension at least three.

Suppose $\Psi_2 = f\Psi_1$ for some function $f \in C^\infty(M)$. (Here we dropped the subscript $\mathbf{p}_0$.) It follows that

$$0 = \not{D}\Psi_2 = \gamma(\nabla f)\Psi_1$$

This in turn implies that f is constant because Ψ_1 is non-vanishing on an dense open subset of M. However, this is non-sense because $\langle \Psi_i, \Psi_j\rangle_{L^2} = \delta_{ij}$. It follows that there is an $x \in M$ such that $\Psi_1(x)$ and $\Psi_2(x)$ are linearly independent. Clifford multiplication induces an isomorphism from $T^*M \otimes \mathfrak{su}(E)$ to trace-free symmetric endomorphisms of $\mathrm{Re}(E \otimes S_{\mathfrak{s}})$. Therefore, given any $(\mu_{ij}) \in S^2\mathbf{R}^2$, we can find $\hat{\mathbf{p}} = (0, b) \in T_{\mathbf{p}_0}U$ such that

$$\langle \gamma(b)\Psi_i, \Psi_j\rangle_{L^2} = \mu_{ij} \quad \text{for} \quad i, j \in \{1, 2\}.$$

Since

$$\mathrm{d}_{\mathbf{p}_0}\Lambda(\hat{\mathbf{p}}) = \big(\langle \gamma(b)\Psi_i, \Psi_j\rangle_{L^2}\big) \in S^2\mathbf{R}^k,$$

it follows that $\mathrm{d}_{\mathbf{p}_0}\Lambda$ has rank at least three.

It follows from the above that, for any $\mathbf{p}_0 \in \mathscr{W}_k^{\mathfrak{s}}$, there exists an arbitarily close $\mathbf{p} \in \mathscr{P}$ with $0 < \dim\ker \not{D}_{\mathbf{p}}^{\mathfrak{s}} < k$. From this it follows by induction that $\mathscr{W}_k^{\mathfrak{s}} \cap \overline{\mathscr{W}_1^{\mathfrak{s}}} = \mathscr{W}_k^{\mathfrak{s}}$. □

Proposition 9.10. *If $\mathfrak{s}_1, \mathfrak{s}_2$ are two distinct spin structures, then $\mathscr{W}_1^{\mathfrak{s}_1}$ and $\mathscr{W}_1^{\mathfrak{s}_2}$ intersect transversely.*

Proof. Let $\mathbf{p} \in \mathscr{W}_1^{\mathfrak{s}_1} \cap \mathscr{W}_1^{\mathfrak{s}_2}$. Denote by Ψ_1 and Ψ_2 spinors spanning $\ker \not{D}_{\mathbf{p}}^{\mathfrak{s}_1}$ and $\ker \not{D}_{\mathbf{p}}^{\mathfrak{s}_2}$ respectively. The spin structures $\mathfrak{s}_1$ and $\mathfrak{s}_2$ differ by twisting by a $\mathbf{Z}_2$–bundle $\mathfrak{l}$. This bundle corresponds to a double cover $\pi\colon \tilde{M} \to M$ and upon pulling back to the cover the spin structures $\mathfrak{s}_1$ and $\mathfrak{s}_2$ both correspond to the same spin structure $\tilde{\mathfrak{s}}$. The natural involution on $\tilde{M}$ acts as -1 on $\tilde{\Psi}_1 = \pi^*\Psi_1$ and as $+1$ on $\tilde{\Psi}_2 = \pi^*\Psi_2$. After renormalization, we can assume that $\langle \tilde{\Psi}_i, \tilde{\Psi}_2\rangle_{L^2} = \delta_{ij}$. In particular, there is an $x \in M$ such that, $\tilde{\Psi}_1(x)$ and $\tilde{\Psi}_2(x)$ are linearly independent (by the argument from Proposition 9.9).

Let $\lambda^{\mathfrak{s}_1}$, $\lambda^{\mathfrak{s}_2}$, and $\lambda^{\mathfrak{s}}$ be as in Proposition 9.3. A moment's thought shows that

$$\mathrm{d}_{\Psi_i}\lambda^{\mathfrak{s}_i}(0,b) = \frac{1}{2}\langle \bar{\gamma}(\pi^* b)\tilde{\Psi}_i, \tilde{\Psi}_i\rangle_{L^2}.$$

With this in mind it is not difficult to find a $b \in \Omega^1(M, \mathfrak{su}(E))$ such that $\mathrm{d}_{\Psi_i}\lambda^{\mathfrak{s}_1}(0,b) = 0$ but $\mathrm{d}_{\Psi_i}\lambda^{\mathfrak{s}_2}(0,b) \neq 0$. Hence, $\mathscr{W}_1^{\mathfrak{s}_1}$ and $\mathscr{W}_1^{\mathfrak{s}_2}$ intersect transversely. □

Finally, we are in a position to construct $\mathbf{p}_\star \in \mathscr{W}_{1,\star}^{\mathfrak{s}}$. Fix a spin structure $\mathfrak{s}$ as well as $\mathbf{p}_0 = (g_0, B_0)$ and $x_\star \in M$ such that g_0 and B_0 are flat on a small ball around a point $x_\star \in M$. Choose local coordinates (y_1, y_2, y_3) around $x_\star$ and a local trivialization of $\mathrm{Re}(E \otimes S_{\mathfrak{s}})$ in which g_0 is given by the identity matrix and B_0 is the trivial connection. Let $\Psi \in \Gamma(\mathrm{Re} \otimes S_{\mathfrak{s}})$ be any section which is nowhere vanishing away from $x_\star$ and around $x_\star$ agrees with map $\mathbf{R}^3 \to \mathbf{H}$ given by

$$(y_1, y_2, y_3) \mapsto 2iy_1 - jy_2 - ky_3.$$

In particular, Ψ has a single non-degenerate zero at $x_\star$ and satisfies $\not{D}_{\mathbf{p}}^{\mathfrak{s}} = 0$ in a neighborhood of $x_\star$. Using the same argument as in the proof of Proposition 9.6, we find $b \in \Omega^1(\mathfrak{su}(E))$ vanishing in a neighborhood of $x_\star$ and such that for $\mathbf{p}_\star = (g_0, B_0 + b)$ we have

$$0 = \not{D}_{\mathbf{p}_\star}^{\mathfrak{s}}\Psi = \not{D}_{\mathbf{p}_0}^{\mathfrak{s}}\Psi + \bar{\gamma}(b)\Psi.$$

This shows that Ψ is harmonic with respect to $\mathbf{p}_\star$. If $\dim\ker \not{D}_{\mathbf{p}_\star} > 1$, then Proposition 9.9 and the argument from Proposition 9.6 can be used to slightly perturb $\mathbf{p}_\star$ to arrange that $\dim\ker \not{D}_{\mathbf{p}_\star} = 1$ and any spinor spanning $\not{D}_{\mathbf{p}_\star}$ has a non-degenerate zero (close to $x_\star$). Similarly, Proposition 9.9 and Proposition 9.10 can be used to ensure that there are no non-trivial harmonic spinors with respect to $\mathbf{p}_\star$ for any other spin structure $\tilde{\mathfrak{s}}$. □

A Computation of the hyperkähler quotient

Set

$$S \coloneqq \mathrm{Hom}_{\mathbf{C}}(\mathbf{C}^2, \mathbf{H})$$

with $\mathbf{H}$ considered as a complex vector space whose complex structure is given by right-multiplication with i. S is a quaternionic Hermitian vector space: its $\mathbf{H}$–module structure arises by left-multiplication. The action of $\mathrm{U}(1)$ on S given by $\rho(e^{i\theta})\Psi = e^{i\theta}\Psi$ is a quaternionic representation with associated moment map

$$\mu(\Psi) = \Psi\Psi^* - \frac{1}{2}|\Psi|^2 \,\mathrm{id}_{\mathbf{H}}.$$

The standard complex volume form $\Omega = e^1 \wedge e^2 \in \Lambda^2(\mathbf{C}^2)^*$ and the standard Hermitian metric on $\mathbf{C}^2$, define a complex anti-linear map $J\colon \mathbf{C}^2 \to \mathbf{C}^2$ by

$$-\langle v, Jw\rangle = \Omega(v, w).$$

This makes $\mathbf{C}^2$ into a $\mathbf{H}$–module.

Proposition A.1. *We have*

$$S^{\mathrm{reg}} /\!\!/\!\!/ \mathrm{U}(1) = \big(\mathrm{Re}(\mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^2)\backslash\{0\}\big)/\mathbf{Z}_2.$$

Here the real structure on $\mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^2$ *is given by* $\overline{q \otimes v} \coloneqq qj \otimes Jv$.

Proof. The complex volume form Ω defines a complex linear isomorphism $(\mathbf{C}^2)^* \cong \mathbf{C}^2$; hence, we can identify $S \cong \mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^2$. We will further identify $\mathbf{C}^2$ with $\mathbf{H}$ via $(z, w) \mapsto z + wj$. With respect to this identification the complex structure is given by left-multiplication with i and J becomes left-multiplication by j. If we denote by H_+ (H_-) the quaternions equipped with their right (left) $\mathbf{H}$–module structure, then we can identify S with

$$\mathbf{H}_+ \otimes_{\mathbf{C}} \mathbf{H}_-.$$

In this identification the action of $\mathrm{U}(1)$ is given by

$$\rho(e^{i\theta})q_+ \otimes q_- = q_+e^{i\theta} \otimes q_- = q_+ \otimes e^{i\theta}q_-,$$

and the moment map becomes

$$\mu(q_1 \otimes 1 + q_2 \otimes j) = -i \otimes \frac{1}{2}(q_1 i \bar{q}_1 + q_2 i \bar{q}_2) \in i\mathbf{R} \otimes \operatorname{Im} \mathbf{H}.$$

If $\mu(q_1 \otimes 1 + q_2 \otimes j) = 0$, then

$$q_1 i \bar{q}_1 = -q_2 i \bar{q}_2. \tag{A.2}$$

This implies that $|q_1| = |q_2|$. Unless q_1 and q_2 both vanish, there is a unique $p \in \mathbf{H}$ satisfying

$$|p| = 1 \quad \text{and} \quad q_1 = q_2 p.$$

From (A.2), it follows that

$$pi = -ip;$$

hence, $p = je^{i\phi}$ for some $\phi \in \mathbf{R}$. It follows that, for any $\theta \in \mathbf{R}$,

$$q_1 e^{i\theta} = q_2 e^{i\theta} \cdot je^{i(\phi+2\theta)}.$$

Since

$$\overline{q_1 \otimes 1 + q_2 \otimes j} = -q_2 j \otimes 1 + q_1 j \otimes j,$$

the real part of $\mathbf{H}_+ \otimes_{\mathbf{C}} \mathbf{H}_-$ consists of those $q_1 \otimes 1 + q_2 \otimes j$ with

$$q_1 = -q_2 j.$$

Consequently, the U(1)–orbit of each non-zero $\mathbf{q} = q_1 \otimes 1 + q_2 \otimes j$ intersects $\operatorname{Re}(\mathbf{H}_+ \otimes_{\mathbf{C}} \mathbf{H}_-)$ twice: in $\pm\rho(e^{i(-\phi/2+\pi/2)})\mathbf{q}$. □

B Relation to gauge theory on G_2–manifolds

Donaldson and Thomas [DT98, Section 3] suggested that one might be able to construct G_2 analogues of the Casson invariant/instanton Floer homology associated to a natural functional whose critical point are G_2–instantons. One key difficulty with this proposal is that G_2–instantons can degenerate by bubbling along associative submanifolds. Donaldson and Segal [DS11, Section 6] explain that this bubbling could be caused by the appearance of (nowhere vanishing) harmonic spinors of $\operatorname{Re}(E \otimes S)$ over associative submanifolds. In particular, the signed count of G_2–instantons can jump along a one-parameter family. Donaldson and Segal propose to compensate this jump with a counter-term consisting of a weighted count of associative submanifolds.

Joyce [Joy17, Example 8.5] poses the following scenario. Consider a one–parameter family of G_2–manifolds $\{(Y, \phi_t) : t \in [0,1]\}$ together with an SU(2)–bundle E where:

- there is a smooth family of irreducible connections $(A_t)_{t\in[0,1]} \in \mathscr{A}(E)^{[0,1]}$ such that A_t is an unobstructed G_2–instanton with respect to ϕ_t for each $t \in [0,1]$,
- there are no relevant associatives in (Y, ϕ_t) for $t \in [0, 1/3) \cup (2/3, 1]$, and
- there is an obstructed associative $P_{1/3}$ in $(Y, \phi_{1/3})$, which splits into two unobstructed associatives $P_t^{\pm}$ in (Y, ϕ_t) for $t \in (1/3, 2/3)$, and which then annihilate each other in an obstructed associative $P_{2/3}$ in $(Y, \phi_{2/3})$.

According to [Wal12, Theorem 1.2] a regular crossing of the spectral flow of family of Dirac operators $\not{D}_t^{\pm} : \Gamma(\mathrm{Re}(E|_{P_t^{\pm}} \otimes \$_{P_t^{\pm}})) \to \Gamma(\mathrm{Re}(E|_{P_t^{\pm}} \otimes \$_{P_t^{\pm}}))$ causes a jump in the signed count of G_2–instantons; however, the sign of this jump has not been analyzed.[8] Donaldson and Segal [DS11, Section 6] and Joyce [Joy17, Section 8.4] suggest that this is the only source of jumping phenomena. The difference in the spectral flows of Dirac operators $\not{D}_t^{\pm}$ on is a topological invariant, say $k \in \mathbf{Z}$, which may be non-zero. [Joy17, Section 8.4] thus concludes that passing from $t < 1/3$ to $t > 2/3$ the signed number of G_2–instantons should change by $k \cdot |H_1(P_{1/3}, \mathbf{Z}_2)|$; and, since there are no associatives for $t \in [0, 1/3) \cup (2/3, 1]$, no counter-term involving a weighted count of associatives could compensate this jump.

It is proposed in [HW15] that the weight associated with each associative 3-manifold should be the signed count of solutions to the Seiberg–Witten equation with two spinors. The loop of associatives can equivalently be seen as a path of parameters $(\mathbf{p}_t)_{t\in[0,1]}$ on a fixed 3–manifold P, with $\mathbf{p}_1$ gauge equivalent to $\mathbf{p}_0$. Therefore, one can ask how $n(\mathbf{p}_t)$ varies in this scenario. Suppose that $b_1(P_{1/3}) > 1$. Assuming there are no harmonic $\mathbf{Z}_2$ spinors along the path $(\mathbf{p}_t)_{t\in[0,1]}$, a jump in $n(\mathbf{p}_t)$ would occur precisely when the spectrum of one of the Dirac operators $\not{D}_{\mathbf{p}_t}^{\mathfrak{s}}$ crosses zero. If the wall-crossing formula for $n(\mathbf{p}_t)$ were given by the sum of the spectral flows of $(\not{D}_{\mathbf{p}_t}^{\mathfrak{s}})_{t\in[0,1]}$, then we would have arrive a contradiction just like in Joyce's argument:

$$0 \neq k \cdot |H_1(P, \mathbf{Z}_2)| = n(\mathbf{p}_1) - n(\mathbf{p}_0) = 0$$

since $\mathbf{p}_1$ and $\mathbf{p}_0$ are gauge equivalent. However, the conclusion our work is that:

[8] To be more precise, the jump occurs in the signed count of G_2–instantons on a bundle E', which is related to E by $c_2(E') = c_2(E) + \mathrm{PD}[P]$ with $[P] = [P_{1/3}] = [P_t^{\pm}] = [P_{2/3}]$.

1. The wall-crossing for $n(\mathbf{p}_t)$ caused by harmonic spinors is not given by the spectral flow.

2. There exist singular harmonic $\mathbf{Z}_2$ spinors which cause additional wall-crossing.

It is possible that the same happens for the signed count of G_2–instantons. To evaluate the viability of the proposal in [HW15] it is important to answer the following questions.

Question B.1. What is the sign of the jump in the number of G_2–instantons caused by a harmonic spinor?

Question B.2. Do singular harmonic $\mathbf{Z}_2$ spinors cause a jump in the number of (possibly singular) G_2–instantons?

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