

# On the existence of harmonic $\mathbf{Z}_2$ spinors

Aleksander Doan  
Stony Brook University

Thomas Walpuski  
Michigan State University

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## Abstract

We prove the existence of singular harmonic  $\mathbf{Z}_2$  spinors on 3-manifolds with  $b_1 > 1$ . The proof relies on a wall-crossing formula for solutions to the Seiberg–Witten equation with two spinors. The existence of singular harmonic  $\mathbf{Z}_2$  spinors and the shape of our wall-crossing formula shed new light on recent observations made by Joyce [Joy17] regarding Donaldson and Segal’s proposal for counting  $G_2$ -instantons [DS11].

**Keywords:** harmonic  $\mathbf{Z}_2$  spinors, Haydys correspondence, Seiberg–Witten equation, wall-crossing

**MSC2010:** 53C07; 57R57, 53C27

## 1 Introduction

The notion of harmonic  $\mathbf{Z}_2$  spinors was introduced by Taubes [Tau14] as an abstraction of various limiting objects appearing in compactifications of moduli spaces of flat  $\mathrm{PSL}_2(\mathbf{C})$ -connections over 3-manifolds [Tau13a] and solutions to the Kapustin–Witten equation [Tau13b], the Vafa–Witten equation [Tau17], and the Seiberg–Witten equation with multiple spinors [HW15; Tau16].

**Definition 1.1.** Let  $M$  be a Riemannian manifold and  $\mathbf{S}$  a Dirac bundle over  $M$ .<sup>1</sup> Denote by  $\not{D}: \Gamma(\mathbf{S}) \rightarrow \Gamma(\mathbf{S})$  the associated Dirac operator. A  $\mathbf{Z}_2$  **spinor** with values in  $\mathbf{S}$  is a triple  $(Z, \mathfrak{l}, \Psi)$  consisting of

1. a proper closed subset  $Z \subset M$ ,
2. a Euclidean line bundle  $\mathfrak{l} \rightarrow M \setminus Z$ , and
3. a section  $\Psi \in \Gamma(M \setminus Z, \mathbf{S} \otimes \mathfrak{l})$

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<sup>1</sup>A Dirac bundle is a bundle of Clifford modules together with a metric and a compatible connection; see, [LM89, Chapter II Definition 5.2].

such that  $|\Psi|$  extends to a continuous function on  $M$  with  $|\Psi|^{-1}(0) = Z$  and  $|\nabla\Psi|^2 \in L^2_{\text{loc}}(M)$ . We say that  $(Z, \mathfrak{l}, \Psi)$  is **singular** if  $\mathfrak{l}$  does not extend to a Euclidean line bundle on  $M$ . A  $\mathbf{Z}_2$  spinor  $(Z, \mathfrak{l}, \Psi)$  is called **harmonic** if

$$\not{D}\Psi = 0$$

holds on  $M \setminus Z$ .

*Remark 1.2.* Taubes [Tau14] proved that if  $M$  is three- or four-dimensional, then  $Z$  has Hausdorff codimension at least 2.

*Remark 1.3.* If  $\mathfrak{l}$  extends to a Euclidean line bundle on  $M$ , then  $\mathbf{S} \otimes \mathfrak{l}$  extends to a Dirac bundle on  $M$  and  $\Psi$  extends to a harmonic spinor with values in  $\mathbf{S} \otimes \mathfrak{l}$  vanishing precisely along  $Z$ .

The harmonic  $\mathbf{Z}_2$  spinors appearing as limits of flat  $\text{PSL}_2(\mathbf{C})$ -connections over 3-manifolds take values in  $\underline{\mathbf{R}} \oplus T^*M$ ; those appearing as limits of the Seiberg–Witten equation with two spinors in dimension three take values in  $\text{Re}(S \otimes E)$  where  $S$  is the spinor bundle of some spin structure  $\mathfrak{s}$  on the 3-manifold  $M$  and  $E$  is an rank two Hermitian bundle with trivial determinant and equipped with a compatible connection.

Henceforth, we specialize to the case  $\mathbf{S} = \text{Re}(S \otimes E)$ . The Dirac operator on  $\text{Re}(S \otimes E)$ , and thus also the notion of a harmonic  $\mathbf{Z}_2$  spinor, depends on the choice of a Riemannian metric on  $M$  and a connection on  $E$ .

**Definition 1.4.** Let  $\text{Met}(M)$  be the space of Riemannian metrics on  $M$  and  $\mathcal{A}(E)$  the space of  $\text{SU}(2)$  connections on  $E$ . The **space of parameters** is

$$\mathcal{P} := \text{Met}(M) \times \mathcal{A}(E)$$

equipped with the  $C^\infty$  topology. Given a spin structure  $\mathfrak{s}$  on  $M$  and  $\mathbf{p} \in \mathcal{P}$ , we denote by  $\not{D}_{\mathbf{p}}^{\mathfrak{s}}$  the corresponding Dirac operator on  $\Gamma(\text{Re}(S \otimes E))$ .

**Question 1.5.** For which parameters  $\mathbf{p} \in \mathcal{P}$  does there exist a singular harmonic  $\mathbf{Z}_2$  spinor with values in  $\text{Re}(S \otimes E)$ ?

The answer to this question for non-singular harmonic  $\mathbf{Z}_2$  spinors (that is: harmonic spinors) is well-understood. Let  $\mathcal{W}^{\mathfrak{s}}$  be the set of  $\mathbf{p} \in \mathcal{P}$  for which  $\dim \ker \not{D}_{\mathbf{p}}^{\mathfrak{s}} > 0$ . It is the closure of  $\mathcal{W}_1^{\mathfrak{s}}$ , the set of  $\mathbf{p}$  for which  $\dim \ker \not{D}_{\mathbf{p}}^{\mathfrak{s}} = 1$ . Moreover,  $\mathcal{W}_1^{\mathfrak{s}}$  is a cooriented, codimension one submanifold of  $\mathcal{P}$  and  $\mathcal{W}^{\mathfrak{s}} \setminus \mathcal{W}_1^{\mathfrak{s}}$  has codimension three. (See Proposition 9.3 and Proposition 9.9.) The intersection number of a path  $(\mathbf{p}_t)_{t \in [0,1]}$  with  $\mathcal{W}_1^{\mathfrak{s}}$  is given by the spectral flow of the path of operators  $(\not{D}_{\mathbf{p}_t}^{\mathfrak{s}})_{t \in [0,1]}$ . Therefore, along any path with non-zero spectral flow there

exists a parameter  $\mathbf{p}_\star$  such that  $\dim \ker \mathbb{D}_{\mathbf{p}_\star}^s > 0$ . Moreover, if the path is generic,<sup>2</sup> then the kernel is spanned by a nowhere vanishing spinor (because  $\dim M < \text{rank } \mathbf{S}$ ; see Definition 4.1 and Proposition 8.1)

By contrast, little is known about the existence of singular harmonic  $\mathbf{Z}_2$  spinors. The only examples known thus far have been obtained by means of complex geometry, on Riemannian 3-manifolds of the form  $M = S^1 \times \Sigma$  for a Riemann surface  $\Sigma$ ; see [Tau13a, Theorem 1.2] in the case  $\mathbf{S} = \underline{\mathbf{R}} \oplus T^*M$  and [Doa17b, Section 3] in the case  $\mathbf{S} = \text{Re}(S \otimes E)$ . We remedy this situation by proving—in a rather indirect way—that 3-manifolds abound with singular harmonic  $\mathbf{Z}_2$  spinors.

**Theorem 1.6.** *For every compact, connected, oriented 3-manifold  $M$  with  $b_1(M) > 1$  there exists  $\mathbf{p}_\star \in \mathcal{P}$  and a singular harmonic  $\mathbf{Z}_2$  spinor with respect to  $\mathbf{p}_\star$ . In fact, there is a closed subset  $\mathcal{W}_b \subset \mathcal{P}$  and a non-trivial cohomology class  $\omega \in H^1(\mathcal{P} \setminus \mathcal{W}_b, \mathbf{Z})$  with the property that if  $(\mathbf{p}_t)_{t \in S^1}$  is a generic loop in  $\mathcal{P} \setminus \mathcal{W}_b$  and*

$$\omega([\mathbf{p}_t]) \neq 0,$$

*then for some  $\mathbf{p}_\star$  in  $(\mathbf{p}_t)_{t \in S^1}$  there exists a singular harmonic  $\mathbf{Z}_2$  spinor.*

*Remark 1.7.* The definition of  $\mathcal{W}_b$  is given in Definition 9.1 and the precise meaning of generic loop is given in Definition 4.1 and Proposition 2.5.

*Remark 1.8.* Theorem 1.6 suggests that on 3-manifolds the appearance of singular harmonic  $\mathbf{Z}_2$  spinors is a codimension one phenomenon—as is the appearance of harmonic spinors. This is in consensus with the work of Takahashi [Tak15; Tak17], who proved that the linearized deformation theory of singular harmonic  $\mathbf{Z}_2$  spinors with  $Z = S^1$  is an index zero Fredholm problem (or index minus one, after scaling is taken into account).

*Remark 1.9.* The assumption that  $b_1(M) > 1$  has to do with reducible solutions to the Seiberg–Witten equation with two spinors. We expect it to be non-substantial; the wall-crossing caused by reducible solutions should be amenable to the methods used by Chen [Che97] and Lim [Lim00] in classical Seiberg–Witten theory.

The proof of Theorem 1.6 relies on the wall-crossing formula for  $n(\mathbf{p})$ , the signed count of solutions to the Seiberg–Witten equation with two spinors, which is stated as Theorem 4.9. The number  $n(\mathbf{p})$  is defined provided  $\mathbf{p}$  is generic and there are no singular harmonic  $\mathbf{Z}_2$  spinors with respect to  $\mathbf{p}$ . The wall-crossing formula can be stated as follows. Let  $\mathcal{W}_{1,\emptyset}^s$  be the set of  $\mathbf{p} \in \mathcal{P}$  for which  $\ker \mathbb{D}_{\mathbf{p}}^s = \mathbf{R}\langle \Psi \rangle$  with  $\Psi$  nowhere vanishing, and let  $\mathcal{W}_{1,\emptyset}$  be the union of all  $\mathcal{W}_{1,\emptyset}^s$  for all spin structures  $s$ . There is a closed subset  $\mathcal{W}_b \subset \mathcal{P}$ , as in Theorem 1.6, such that  $\mathcal{W}_{1,\emptyset}$  is an immersed,

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<sup>2</sup>Generic means from a residual subset of the space of objects in question. A subset of a topological space is residual if it contains a countable intersection of open and dense subsets. Baire’s theorem asserts that a residual subset of a complete metric space is dense.

closed, cooriented, codimension one submanifold of  $\mathcal{P} \setminus \mathcal{W}_b$ . If  $(\mathbf{p}_t)_{t \in [0,1]}$  is a generic path in  $\mathcal{P} \setminus \mathcal{W}_b$  and there are no singular harmonic  $\mathbf{Z}_2$  spinors along  $(\mathbf{p}_t)$ , then the difference

$$n(\mathbf{p}_1) - n(\mathbf{p}_0)$$

is equal to the intersection number of the path  $(\mathbf{p}_t)_{t \in [0,1]}$  with  $\mathcal{W}_{1,\emptyset}$ . In particular, if  $(\mathbf{p}_t)_{t \in S^1}$  is a generic loop whose intersection number with  $\mathcal{W}_{1,\emptyset}$  is non-zero, then there must be a singular harmonic  $\mathbf{Z}_2$  spinor for some  $\mathbf{p}_\star$  in  $(\mathbf{p}_t)$ .

*Remark 1.10.* Although the wall-crossing for  $n(\mathbf{p})$  does occur when the spectrum of  $\mathcal{D}_{\mathbf{p}}^s$  crosses zero, the contribution of a nowhere vanishing harmonic spinor to the wall-crossing formula is not given by the sign of the spectral crossing but instead by the degree of the spinor. Therefore, the wall-crossing formula is not given by the spectral flow. This should be contrasted with the wall-crossing phenomenon for the classical Seiberg–Witten equation caused by reducible solutions, as in Remark 1.9. Indeed, the wall-crossing described in this paper is a result of the non-compactness of the moduli spaces of solutions and as such it is a new phenomenon, with no counterpart in classical Seiberg–Witten theory.

The cohomology class  $\omega \in H^1(\mathcal{P} \setminus \mathcal{W}_b, \mathbf{Z})$  in Theorem 1.6 is defined by intersecting with  $\mathcal{W}_{1,\emptyset}$ . We prove that  $\omega$  is non-trivial, by exhibiting a loop  $(\mathbf{p}_t)_{t \in S^1}$  on which  $\omega$  evaluates as  $\pm 2$ . This is possible because  $\mathcal{W}_b$  behaves like a codimension two subset. In particular,  $\mathcal{P} \setminus \mathcal{W}_b$  is not simply-connected and we can take  $(\mathbf{p}_t)_{t \in S^1}$  to be a small loop linking  $\mathcal{W}_b$ .

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## 2 The Seiberg–Witten equation with two spinors

Fix a spin<sup>c</sup> structure  $w$  and denote its spinor bundle by  $W$ . Set

$$\mathcal{L} := \ker (d: \Omega^2(M, i\mathbf{R}) \rightarrow \Omega^3(M, i\mathbf{R})).$$

**Definition 2.1.** Let  $\eta \in \mathcal{L}$  and  $\mathbf{p} = (g, B) \in \mathcal{P}$ . The  $\eta$ -perturbed Seiberg–Witten equation with two spinors is the following differential equation for  $(\Psi, A) \in$

$\Gamma(\text{Hom}(E, W)) \times \mathcal{A}(\det W)$ :

$$(2.2) \quad \begin{aligned} \not{D}_A \Psi &= 0 \quad \text{and} \\ \frac{1}{2} F_A + \eta &= \mu(\Psi).^3 \end{aligned}$$

Here  $\not{D}_A = \not{D}_{A, \mathbf{p}}$  is the Dirac operator on  $\Gamma(\text{Hom}(E, W))$  induced by the spin connection on  $W$  obtained from  $A$  together with the Levi–Civita connection of  $g$  and the connection  $B$  on  $E$ ; and  $\mu(\Psi) = \mu_{\mathbf{p}}(\Psi) := \Psi \Psi^* - \frac{1}{2} |\Psi|^2 \text{id}_W$  is a section of  $\text{isu}(W)$  which is identified with an element of  $\Omega^2(M, i\mathbf{R})$  using the Clifford multiplication.

Let  $\mathcal{G}(\det W)$  be the gauge group of  $\det W$ . For  $(\mathbf{p}, \eta) \in \mathcal{P} \times \mathcal{L}$ , we denote by

$$\mathfrak{M}_{\mathbf{w}}(\mathbf{p}, \eta) := \left\{ [\Psi, A] \in \frac{\Gamma(\text{Hom}(E, W)) \times \mathcal{A}(\det W)}{\mathcal{G}(\det W)} : (\Psi, A) \text{ satisfies (2.2) with respect to } \mathbf{p} \text{ and } \eta \right\}$$

the **moduli space** of solutions to (2.2).

As discussed in [Doa17b, Section 2; DW17, Section 2], the infinitesimal deformation theory of (2.2) around a solution  $(\Psi, A)$ , is controlled by the linear operator

$$\begin{aligned} L_{\Psi, A} : \Gamma(\text{Hom}(E, W)) \oplus \Omega^1(M, i\mathbf{R}) \oplus \Omega^0(M, i\mathbf{R}) \\ \rightarrow \Gamma(\text{Hom}(E, W)) \oplus \Omega^1(M, i\mathbf{R}) \oplus \Omega^0(M, i\mathbf{R}) \end{aligned}$$

defined by

$$L_{\Psi, A} := \begin{pmatrix} -\not{D}_A & -\alpha_{\Psi} \\ -\alpha_{\Psi}^* & \mathfrak{d} \end{pmatrix}$$

with

$$(2.3) \quad \mathfrak{d} := \begin{pmatrix} * \mathfrak{d} & \mathfrak{d} \\ \mathfrak{d}^* & \end{pmatrix} \quad \text{and} \quad \alpha_{\Psi} := (\bar{\gamma}(\cdot) \Psi \quad \rho(\cdot) \Psi).^4$$

Here  $\bar{\gamma}$  is the Clifford multiplication by elements of  $T^*M \otimes i\mathbf{R}$  and  $\rho$  is the linearized action of the gauge group: pointwise multiplication by elements of  $i\mathbf{R}$ .

**Definition 2.4.** We say that a solution  $(\Psi, A)$  of (2.2) is **irreducible** if  $\Psi \neq 0$ , and **unobstructed** if  $L_{\Psi, A}$  is invertible.

If  $(\Psi, A)$  is irreducible and unobstructed, then it represents an isolated point in  $\mathfrak{M}_{\mathbf{w}}(\mathbf{p}, \eta)$ ; see [Doa17b, Lemma 2.15; DW17, Proposition 1.25].

<sup>3</sup>While (2.2) makes sense for any  $\eta \in \Omega^2(M, i\mathbf{R})$ , the existence of a solution implies that  $d\eta = 0$ ; see [Doa17b, Lemma 2.3; DW17, Proposition A.4].

<sup>4</sup>This linearization is obtained by writing a connection near  $A_0$  as  $A_0 + 2a$ .

**Proposition 2.5** ([Doa17b, Theorem 2.32]). *If  $b_1(M) > 0$ , then for every  $(\mathbf{p}, \eta)$  from a residual subset of  $\mathcal{P} \times \mathcal{X}$  all solutions to (2.2) are irreducible and unobstructed.*

For every  $(\mathbf{p}, \eta)$  as in Proposition 2.5, the moduli space  $\mathfrak{M}_w(\mathbf{p}, \eta)$  is a zero-dimensional manifold. It can be oriented as explained in [Doa17b, Section 2.7]. Here is a brief outline of the **orientation procedure**: Let  $\det L \rightarrow \Gamma(\text{Hom}(E, W)) \times \mathcal{A}(\det W)$  be the determinant line bundle of the family of elliptic operators  $L_{\Psi, A}$  as  $(\Psi, A)$  varies over all configurations. Over the locus  $\{\Psi = 0\}$  we have  $L_{0, A} = \mathbb{D}_A \oplus \mathfrak{d}$ . Since the operator  $\mathbb{D}_A$  is complex-linear and the kernel and cokernel of  $\mathfrak{d}$  are canonically isomorphic to  $H^1(M, i\mathbf{R}) \oplus H^0(M, i\mathbf{R})$ , the determinant line bundle  $\det L$  has a canonical orientation over  $\{\Psi = 0\}$ . Orientation transport along the path  $s \mapsto L_{s\Psi, A}$  orients  $\det L_{\Psi, A}$  for arbitrary  $\Psi$ . This procedure defines a  $\mathcal{G}(\det W)$ -equivariant orientation of  $\det L$ . If  $(\Psi, A)$  is an irreducible and unobstructed solution of (2.2), then the determinant of the tangent space to  $\mathfrak{M}_w(\mathbf{p}, \eta)$  at  $[\Psi, A]$  is canonically isomorphic to  $\det L_{\Psi, A}$ ; hence, it is oriented by the above procedure.

**Definition 2.6.** For  $[\Psi, A] \in \mathfrak{M}_w(\mathbf{p}, \eta)$ , we define

$$\text{sign}[\Psi, A] \in \{\pm 1\}$$

by comparing the orientation described above with the standard orientation of a point.

As we will see shortly, if  $(\mathbf{p}, \eta)$  is generic, and under the assumption that there are no singular harmonic  $\mathbf{Z}_2$  spinors with respect to  $\mathbf{p}$ , then  $\mathfrak{M}_w(\mathbf{p}, \eta)$  is a compact, oriented, zero-dimensional manifold. In this situation, we define

$$(2.7) \quad n_w(\mathbf{p}, \eta) := \sum_{[\Psi, A] \in \mathfrak{M}_w(\mathbf{p}, \eta)} \text{sign}[\Psi, A],$$

the signed count of solutions to the Seiberg–Witten equation with two spinors.

### 3 Compactness of the moduli space

In general,  $\mathfrak{M}_w(\mathbf{p}, \eta)$  might be non-compact; and even if it is compact for given  $(\mathbf{p}, \eta)$ , compactness might still fail as  $(\mathbf{p}, \eta)$  varies. This can only happen when, along a sequence of solutions to (2.2), the  $L^2$ -norm of the spinors goes to infinity. The following result describes in which sense one can still take a rescaled limit in this situation.

**Theorem 3.1** ([HW15, Theorem 1.5]). *Let  $(\mathbf{p}_i, \eta_i)$  be a sequence in  $\mathcal{P} \times \mathcal{X}$  such that  $(\mathbf{p}_i, \eta_i)$  converges to  $(\mathbf{p}, \eta)$  in  $C^\infty$ . Let  $(\Psi_i, A_i)$  be a sequence of solutions of (2.2) with respect to  $(\mathbf{p}_i, \eta_i)$ . If  $\limsup_{i \rightarrow \infty} \|\Psi_i\|_{L^2} = \infty$ , then after rescaling  $\tilde{\Psi}_i = \Psi_i / \|\Psi_i\|_{L^2}$  and passing to a subsequence the following hold:*

1. *The subset*

$$Z := \left\{ x \in M : \limsup_{i \rightarrow \infty} |\tilde{\Psi}_i(x)| = 0 \right\}$$

is closed and nowhere-dense. (In fact,  $Z$  has Hausdorff dimension at most one by [Tau14, Theorem 1.2].)

2. *There exist  $\Psi \in \Gamma(M \setminus Z, \text{Hom}(E, W))$  and a connection  $A$  on  $\det W|_{M \setminus Z}$  satisfying the limiting equation*

$$(3.2) \quad \mathbb{D}_A \Psi = 0 \quad \text{and} \quad \mu(\Psi) = 0$$

on  $M \setminus Z$  with respect to  $\mathfrak{p}$ . The pointwise norm  $|\Psi|$  extends to a Hölder continuous function on all of  $M$  and

$$Z = |\Psi|^{-1}(0).$$

Moreover,  $A$  is flat with monodromy in  $\mathbf{Z}_2$ .

3. *On  $M \setminus Z$ , up to gauge transformations,  $\tilde{\Psi}_i$  weakly converges to  $\Psi$  in  $W_{\text{loc}}^{2,2}$  and  $A_i$  weakly converges to  $A$  in  $W_{\text{loc}}^{1,2}$ . There is a constant  $\gamma > 0$  such that  $|\tilde{\Psi}_i|$  converges to  $|\Psi|$  in  $C^{0,\gamma}(M)$ .*

We expect that the convergence  $(\tilde{\Psi}_i, A_i) \rightarrow (\Psi, A)$  can be improved to  $C_{\text{loc}}^\infty$  on  $M \setminus Z$ ; cf. [Doa17a, Theorem 1.5]. In Section 5, we will show that this is indeed the case if  $Z$  is empty.

The following proposition will give use a concrete understanding of solutions to the limiting equation (3.2); it is a special case of the Haydys correspondence.

**Proposition 3.3** (cf. [HW15, Appendix A]). *If  $(\Psi, A) \in \Gamma(\text{Hom}(E, W)) \times \mathcal{A}(\det W)$  is solution of (3.2) and  $\Psi$  is nowhere vanishing, then:*

1.  *$\det W$  is trivial; in particular,  $w$  is induced by a spin structure,*
2. *after a gauge transformation we can assume that  $A$  is the product connection and there exists a unique spin structure  $\mathfrak{s}$  inducing  $w$  and such that  $\Psi$  takes values in  $\text{Re}(E \otimes S_{\mathfrak{s}}) \subset \Gamma(\text{Hom}(E, W))$ ; here  $S_{\mathfrak{s}}$  is the spinor bundle of  $\mathfrak{s}$ ,*
3.  *$\Psi$  lies in the kernel of  $\mathbb{D}_{\mathfrak{p}}^{\mathfrak{s}} : \Gamma(\text{Re}(E \otimes S_{\mathfrak{s}})) \rightarrow \Gamma(\text{Re}(E \otimes S_{\mathfrak{s}}))$ .*

Moreover, any nowhere vanishing element in the kernel  $\mathbb{D}_{\mathfrak{p}}^{\mathfrak{s}}$  for some spin structure  $\mathfrak{s}$  inducing  $w$  gives rise to a solution of (3.2).

*Remark 3.4.* The set of spin structures is a torsor over  $H^1(M, \mathbf{Z}_2)$  while the set of  $\text{spin}^c$  structures is a torsor over  $H^2(M, \mathbf{Z})$ . If  $\beta : H^1(M, \mathbf{Z}_2) \rightarrow H^2(M, \mathbf{Z})$  denotes the Bockstein homomorphism in the exact sequence

$$\dots \rightarrow H^1(M, \mathbf{Z}_2) \xrightarrow{\beta} H^2(M, \mathbf{Z}) \xrightarrow{2\times} H^2(M, \mathbf{Z}) \rightarrow \dots,$$

then the set of all spin structures  $\mathfrak{s}$  inducing the  $\text{spin}^c$  structure  $\mathfrak{w}$  is a torsor over  $\ker \beta$ . The set of all  $\text{spin}^c$  structures  $\mathfrak{w}$  with trivial determinant is a torsor over  $\ker 2\times$ , the 2-torsion subgroup of  $H^2(M, \mathbf{Z})$ .

*Proof of Proposition 3.3.* Fix a spin structure  $\mathfrak{s}_0$  and a Hermitian line bundle  $L$  which induce  $\mathfrak{w}$ ; in particular,  $W = S_{\mathfrak{s}_0} \otimes L$  and  $A$  induces a connection  $\sqrt{A}$  on  $L$ . By Proposition A.1,

$$\text{Hom}_{\mathbf{C}}(\mathbf{C}^2, \mathbf{H}) // \text{U}(1) = (\text{Re}(\mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^2) \setminus \{0\}) / \mathbf{Z}_2;$$

hence,  $\Psi$  gives rise to a section  $s \in \Gamma(\mathfrak{X})$  with

$$\mathfrak{X} = (\text{Re}(E \otimes S_{\mathfrak{s}_0}) \setminus \{0\}) / \mathbf{Z}_2$$

satisfying the Fueter equation (that is: local lifts of  $s$  to  $\text{Re}(\mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^2)$  satisfy the Dirac equation). The Haydys correspondence [DW17, Proposition 3.2] asserts that:

- any  $s \in \Gamma(\mathfrak{X})$  can be lifted; that is: there exists a Hermitian line bundle  $L$ ,  $\Psi \in \Gamma(\text{Hom}(E, S_{\mathfrak{s}_0} \otimes L)) = \Gamma(\text{Hom}(E, W))$ , and  $\sqrt{A} \in \mathcal{A}(L)$  satisfying (3.2), and
- $L$  is determined by  $s$  up to isomorphism and any two lifts of  $s$  are related by a unique gauge transformation in  $\mathcal{G}(L)$ .

It is easy to see that  $s$  can be lifted to a section  $\tilde{\Psi} \in \Gamma(\text{Re}(E \otimes S_{\mathfrak{s}_0}) \otimes \mathfrak{l})$  for some Euclidean line bundle  $\mathfrak{l}$  (whose Čech cocycle remembers the  $\mathbf{Z}_2$  ambiguity in choosing local lifts). Set  $\tilde{L} := \mathfrak{l} \otimes_{\mathbf{R}} \mathbf{C}$ . By Proposition A.1,

$$\text{Re}(E \otimes S_{\mathfrak{s}_0}) \otimes \mathfrak{l} \subset \mu^{-1}(0) \subset \text{Hom}(E, S_{\mathfrak{s}_0} \otimes \tilde{L}) = \text{Hom}(E, W);$$

and if  $\tilde{A} \in \mathcal{A}(\tilde{L})$  denotes the connection induced by the canonical connection on  $\mathfrak{l}$ , then  $\tilde{D}_{\tilde{A}} \tilde{\Psi} = 0$ .

It thus follows from the Haydys correspondence that  $L \cong \mathfrak{l} \otimes_{\mathbf{R}} \mathbf{C}$  and, after this identification has been made and a suitable gauge transformation has been applied,  $\Psi = \tilde{\Psi}$  and  $\sqrt{A} = \tilde{A}$ . This shows that  $\det W = L^2$  is trivial and  $\mathfrak{w}$  is induced by the spin structure  $\mathfrak{s}$  obtained by twisting  $\mathfrak{s}_0$  with  $\mathfrak{l}$ .  $\square$

**Definition 3.5.** Denote by  $\mathcal{P}^{\text{reg}} \subset \mathcal{P}$  the subset consisting of those  $\mathfrak{p}$  for which the Dirac operator

$$(3.6) \quad \tilde{D}_{\mathfrak{p}}^{\mathfrak{s}} : \Gamma(\text{Re}(E \otimes S_{\mathfrak{s}})) \rightarrow \Gamma(\text{Re}(E \otimes \tilde{S}_{\mathfrak{s}}))$$

is invertible for all spin structures  $\mathfrak{s}$ . Set

$$\mathcal{Q} := \mathcal{P} \times \mathcal{L},$$

and denote by  $\mathcal{Q}^{\text{reg}} \subset \mathcal{Q}$  the subset consisting of those  $(\mathfrak{p}, \eta)$  for which  $\mathfrak{p} \in \mathcal{P}^{\text{reg}}$  and every solution  $(\Psi, A)$  of (2.2) with respect to  $(\mathfrak{p}, \eta)$  is irreducible and unobstructed.



**Proposition 3.7** ([Doa17b, Theorem 2.32; Ang96, Theorem 1.5; Mai97, Theorem 1.2]). *If  $b_1(M) > 0$ , then  $\mathcal{Q}^{\text{reg}}$  is residual in  $\mathcal{Q}$ .*

**Proposition 3.8.** *If  $(\mathbf{p}, \eta) \in \mathcal{Q}^{\text{reg}}$  and there are no singular harmonic  $\mathbf{Z}_2$  spinors for  $\mathbf{p}$ , then  $\mathfrak{M}_w(\mathbf{p}, \eta)$  is compact. In particular,  $n_w(\mathbf{p}, \eta) \in \mathbf{Z}$  as in (2.7) is defined.*

*Proof.* By hypothesis we know that there are no singular harmonic  $\mathbf{Z}_2$  spinors and, by definition of  $\mathcal{P}^{\text{reg}}$  and Proposition 3.3, there are also no harmonic spinors. It thus follows from Theorem 3.1 that  $\mathfrak{M}_w(\mathbf{p}, \eta)$  is compact.  $\square$

## 4 Wall-crossing and the spectral flow

In the absence of singular harmonic  $\mathbf{Z}_2$  spinors, we can define  $n_w(\mathbf{p}, \eta)$  for every  $(\mathbf{p}, \eta) \in \mathcal{Q}^{\text{reg}}$ . However,  $\mathcal{Q}^{\text{reg}}$  is not path-connected and  $n_w(\mathbf{p}, \eta)$  does depend on the path-connected component of  $\mathcal{Q}^{\text{reg}}$  in which  $(\mathbf{p}, \eta)$  lies. We study the wall-crossing for  $n_w(\mathbf{p}, \eta)$  by analyzing the family of moduli spaces  $\mathfrak{M}_w(\mathbf{p}_t, \eta_t)$  along paths of the following kind.

**Definition 4.1.** Given  $\mathbf{p}_0, \mathbf{p}_1 \in \mathcal{P}^{\text{reg}}$ , denote by  $\mathcal{P}^{\text{reg}}(\mathbf{p}_0, \mathbf{p}_1)$  the space of smooth paths from  $\mathbf{p}_0$  to  $\mathbf{p}_1$  in  $\mathcal{P}$  such that for every spin structure  $s$ :

1. the path of Dirac operators  $(\mathcal{D}_{\mathbf{p}_t}^s)_{t \in [0,1]}$  has transverse spectral flow and
2. whenever the spectrum of  $\mathcal{D}_{\mathbf{p}_t}^s$  crosses zero,  $\ker \mathcal{D}_{\mathbf{p}_t}^s$  is spanned by a nowhere vanishing section  $\Psi \in \Gamma(\text{Re}(E \otimes S_s))$ .

Given  $(\mathbf{p}_0, \eta_0), (\mathbf{p}_1, \eta_1) \in \mathcal{Q}^{\text{reg}}$ , denote by  $\mathcal{Q}^{\text{reg}}((\mathbf{p}_0, \eta_0), (\mathbf{p}_1, \eta_1))$  the space of smooth paths  $(\mathbf{p}_t, \eta_t)_{t \in [0,1]}$  from  $(\mathbf{p}_0, \eta_0)$  to  $(\mathbf{p}_1, \eta_1)$  in  $\mathcal{Q}$  such that (1) and (2) hold and, moreover:

3. for every  $t_0 \in [0, 1]$ , every solution  $(\Psi, A)$  of (2.2) is irreducible and unobstructed with respect to  $(\mathbf{p}_t, \eta_t)_{t \in [0,1]}$ , that is: if  $L_{\Psi, A}$  is the linearized operator and  $\pi : \Gamma(\text{Hom}(E, W)) \oplus \Omega^1(M, i\mathbf{R}) \oplus \Omega^0(M, i\mathbf{R}) \rightarrow \text{coker } L_{\Psi, A}$  denotes the  $L^2$ -orthogonal projection, then

$$\pi \left( \left. \frac{d}{dt} \right|_{t=t_0} \left( \begin{array}{c} -\mathcal{D}_{\mathbf{p}_t, A} \Psi \\ *(F_A + \eta_t - \mu_{\mathbf{p}_t}(\Psi)) \\ 0 \end{array} \right) \right)$$

spans  $\text{coker } L_{\Psi, A}$ , and

4. for any  $\Psi$  as in (2) with  $\|\Psi\|_{L^2} = 1$ , denote by

$$\{(\Psi_\varepsilon = \Psi + \varepsilon^2 \psi + O(\varepsilon^4), A_\varepsilon; t(\varepsilon) = t_0 + O(\varepsilon^2)) : 0 \leq \varepsilon \ll 1\}$$

the family of solutions to

$$\begin{aligned} \mathbb{D}_{A_\varepsilon, \mathbf{p}_{t(\varepsilon)}} \Psi_\varepsilon &= 0, \\ \varepsilon^2 \left( \frac{1}{2} F_{A_\varepsilon} + \eta_{t(\varepsilon)} \right) &= \mu_{\mathbf{p}_{t(\varepsilon)}}(\Psi_\varepsilon), \quad \text{and} \\ \|\Psi_\varepsilon\|_{L^2} &= 1 \end{aligned}$$

obtained from [DW17, Theorem 1.38]; we require that

$$\delta = \delta(\Psi, \mathbf{p}_{t_0}, \eta_{t_0}) := \langle \mathbb{D}_{A_0, \mathbf{p}_{t_0}} \psi, \psi \rangle_{L^2} \neq 0.$$

Condition (1) is necessary since the wall-crossing formula will involve the spectral flow of  $\mathbb{D}_{\mathbf{p}_t}^s$ . Condition (2) ensures that the harmonic  $\mathbf{Z}_2$  spinors produced by Theorem 1.6 are indeed singular. Condition (3) is used to show that as  $t$  varies, the moduli spaces  $\mathfrak{M}_w(\mathbf{p}_t, \eta_t)$  form a one-dimensional cobordism. Finally, condition (4) ensures that we can use the local model from [DW17, Theorem 1.38] to study the wall-crossing phenomenon.

The following result shows that a generic path from  $(\mathbf{p}_0, \eta_0)$  to  $(\mathbf{p}_1, \eta_1)$  satisfies the conditions in Definition 4.1; its proof is postponed to Section 8.

**Proposition 4.2.** *Given  $(\mathbf{p}_0, \eta_0), (\mathbf{p}_1, \eta_1) \in \mathcal{Q}^{\text{reg}}$ , the subspace  $\mathcal{Q}^{\text{reg}}((\mathbf{p}_0, \eta_0), (\mathbf{p}_1, \eta_1))$  is residual in the space of all smooth paths from  $(\mathbf{p}_0, \eta_0)$  to  $(\mathbf{p}_1, \eta_1)$  in  $\mathcal{Q}$ .*

The next three sections are occupied with studying the wall crossing along paths in  $\mathcal{Q}^{\text{reg}}((\mathbf{p}_0, \eta_0), (\mathbf{p}_1, \eta_1))$ . In order to state the wall-crossing formula, we need the following preparation.

**Proposition 4.3.** *Denote by  $s$  a spin structure inducing the spin<sup>c</sup> structure  $w$  and by  $A$  the product connection on  $\det W$ . If  $\Psi$  is a nowhere vanishing section of  $\text{Re}(E \otimes S_s)$ , then the following hold:*

1. Let  $\alpha_\Psi$  be the algebraic operator given by (2.3). The map

$$\tilde{\alpha}_\Psi := |\Psi|^{-1} \alpha_\Psi : (T^*M \oplus \mathbf{R}) \otimes i\mathbf{R} \rightarrow \text{Im}(E \otimes S_s)$$

is an isometry.

2. Denote by  $\mathbb{D}_{\text{Im}}$  the restriction of  $\mathbb{D}_{A, \mathbf{p}}$  to  $\text{Im}(E \otimes S_s) \subset \text{Hom}(E, W)$  and define the operator  $\mathfrak{d}_\Psi : \Omega^1(M, i\mathbf{R}) \oplus \Omega^0(M, i\mathbf{R}) \rightarrow \Omega^1(M, i\mathbf{R}) \oplus \Omega^0(M, i\mathbf{R})$  by

$$\mathfrak{d}_\Psi := \tilde{\alpha}_\Psi^* \circ \mathbb{D}_{\text{Im}} \circ \tilde{\alpha}_\Psi.$$

For each  $t \in [0, 1]$ ,

$$\mathfrak{d}_\Psi^t := (1-t)\mathfrak{d}_\Psi + t\mathfrak{d}$$

is a self-adjoint Fredholm operator.

*Proof.* The fact that  $\tilde{\alpha}_\Psi$  is an isometry is a consequence of

$$(4.4) \quad \alpha_\Psi^* \alpha_\Psi = |\Psi|^2$$

which in turn follows from

$$\begin{aligned} \langle \alpha_\Psi(a, \xi), \alpha_\Psi(b, \eta) \rangle &= \langle \bar{\gamma}(a)\Psi + \rho(\xi)\Psi, \bar{\gamma}(b)\Psi + \rho(\eta)\Psi \rangle \\ &= |\Psi|^2 (\langle a, b \rangle + \langle \xi, \eta \rangle). \end{aligned}$$

Since

$$(4.5) \quad \begin{aligned} \mathcal{D}_{\text{Im}} \alpha_\Psi(a, \xi) &= \mathcal{D}_{\text{Im}}(\bar{\gamma}(a)\Psi + \rho(\xi)\Psi) \\ &= \bar{\gamma}(*da)\Psi + \rho(d^*a)\Psi + \bar{\gamma}(d\xi)\Psi \\ &\quad - \bar{\gamma}(a)\mathcal{D}_{\text{Re}}\Psi + \rho(\xi)\mathcal{D}_{\text{Re}}\Psi - 2 \sum_{i=1}^3 \rho(a(e_i))\nabla_{e_i}\Psi \\ &= \alpha_\Psi \mathfrak{d}(a, \xi) - 2 \sum_{i=1}^3 \rho(a(e_i))\nabla_{e_i}\Psi, \end{aligned}$$

we have

$$\mathfrak{d}_\Psi = \mathfrak{d} + \mathfrak{e}_\Psi \quad \text{with} \quad \mathfrak{e}_\Psi = -2 \sum_{i=1}^3 \rho(a(e_i))\nabla_{e_i}\Psi - |\Psi|^{-3} \alpha_\Psi^* \gamma(d|\Psi|) \alpha_\Psi.$$

This implies that  $\mathfrak{d}_\Psi^t$  is a Fredholm operator. □

**Definition 4.6.** In the situation of Proposition 4.3, define

$$\sigma(\Psi, \mathbf{p}) := (-1)^{1+b_1(M)} \cdot (-1)^{\text{SF}((-d_\Psi^t)_{t \in [0,1]})}.$$

*Remark 4.7.* The operator  $\mathfrak{d}_\Psi$  only depends on  $\Psi$  up to multiplication by a constant in  $\mathbf{R}^*$ ; hence, the same holds for  $\sigma(\Psi, \mathbf{p})$ .

**Definition 4.8.** For a pair of nowhere vanishing sections  $\Psi, \Phi \in \Gamma(\text{Re}(E \otimes S_\mathfrak{s}))$  we define their **relative degree**  $\text{deg}(\Psi, \Phi) \in \mathbf{Z}$  as follows. Choose any trivializations of  $E$  and  $S_\mathfrak{s}$  compatible with the  $\text{SU}(2)$  structures. In the induced trivialization of  $\text{Re}(E \otimes S_\mathfrak{s})$  the sections  $\Psi/|\Psi|$  and  $\Phi/|\Phi|$  are represented by maps  $M \rightarrow S^3$ . Set

$$\text{deg}(\Psi, \Phi) := \text{deg}(\Psi/|\Psi|) - \text{deg}(\Phi/|\Phi|).$$

This number does not depend on the choice of the trivializations.

**Theorem 4.9.** Let  $(\mathbf{p}_t, \eta_t)_{t \in [0,1]} \in \mathcal{Q}^{\text{reg}}((\mathbf{p}_0, \eta_0), (\mathbf{p}_1, \eta_1))$ . For each spin structure  $s$  inducing the spin<sup>c</sup> structure  $w$ , denote

- by  $\{t_1^s, \dots, t_{N_s}^s\} \subset [0, 1]$  the finite set of times at which the spectrum of  $\mathcal{D}_{\mathbf{p}_t}^s$  crosses zero

and, for each  $i = 1, \dots, N_{\mathfrak{s}}$ , denote

- by  $\chi_i^{\mathfrak{s}} \in \{\pm 1\}$  the sign of the spectral crossing at  $t_i^{\mathfrak{s}}$  and
- by  $\Psi_i^{\mathfrak{s}}$  a nowhere vanishing spinor spanning  $\ker \mathcal{D}_{\mathbf{p}_t}^{\mathfrak{s}}$ .

If there are no singular harmonic  $\mathbb{Z}_2$  spinors with respect to  $\mathbf{p}_t$  for any  $t \in [0, 1]$ , then

$$(4.10) \quad n_w(\mathbf{p}_1, \eta_1) = n_w(\mathbf{p}_0, \eta_0) + \sum_{\mathfrak{s}} \sum_{i=1}^{N_{\mathfrak{s}}} \chi_i^{\mathfrak{s}} \cdot \sigma(\Psi_i^{\mathfrak{s}}, \mathbf{p}_{t_i^{\mathfrak{s}}})$$

or, equivalently,

$$(4.11) \quad n_w(\mathbf{p}_1, \eta_1) = n_w(\mathbf{p}_0, \eta_0) + \sum_{\mathfrak{s}} \chi_1^{\mathfrak{s}} \cdot \sigma(\Psi_1^{\mathfrak{s}}, \mathbf{p}_{t_1^{\mathfrak{s}}}) \cdot \sum_{i=1}^{N_{\mathfrak{s}}} (-1)^{i+1} \cdot (-1)^{\deg(\Psi_1^{\mathfrak{s}}, \Psi_i^{\mathfrak{s}})}.$$

Here the sums are over all spin structures  $\mathfrak{s}$  inducing  $w$ .

*Remark 4.12.* It follows from Theorem 4.9, that  $n_w(\mathbf{p}, \eta)$  does not depend on  $\eta$ .

The proof of the (4.10) proceeds by analyzing the 1-parameter family of moduli spaces

$$\mathfrak{B} := \bigcup_{t \in [0, 1]} \mathfrak{M}_w(\mathbf{p}_t, \eta_t).$$

By Definition 4.1(3),  $\mathfrak{B}$  is an oriented, one-dimensional manifold with oriented boundary

$$\partial \mathfrak{B} = \mathfrak{M}_w(\mathbf{p}_0, \eta_0) - \mathfrak{M}_w(\mathbf{p}_1, \eta_1).$$

If  $\mathfrak{B}$  were compact, then it would follow that  $n_w(\mathbf{p}_1, \eta_1) = n_w(\mathbf{p}_0, \eta_0)$ . However,  $\mathfrak{B}$  is non-compact.

## 5 Compactification of the cobordism

Set

$$\overline{\mathfrak{B}} := \left\{ (t, \varepsilon, [\Psi, A]) \in [0, 1] \times [0, \infty) \times \frac{\Gamma(\text{Hom}(E, W) \times \mathcal{A}(\det W))}{\mathcal{E}(\det W)} : (*) \right\}$$

with  $(*)$  meaning that:

- the differential equation

$$(5.1) \quad \begin{aligned} \mathcal{D}_{A, \mathbf{p}_t} \Psi &= 0, \\ \varepsilon^2 \left( \frac{1}{2} F_A + \eta_t \right) &= \mu_{\mathbf{p}_t}(\Psi), \quad \text{and} \\ \|\Psi\|_{L^2} &= 1 \end{aligned}$$

holds and

- if  $\varepsilon = 0$ , then  $\Psi$  is nowhere vanishing.

Equip  $\overline{\mathfrak{B}}$  with the  $C^\infty$ -topology. We have a natural embedding  $\mathfrak{B} \hookrightarrow \overline{\mathfrak{B}}$  given by  $(t, [\Psi, A]) \mapsto (t, \varepsilon, [\tilde{\Psi}, A])$  with  $\varepsilon := 1/\|\Psi\|_{L^2}$  and  $\tilde{\Psi} := \Psi/\|\Psi\|_{L^2}$ .

**Proposition 5.2.**  $\mathfrak{B}$  is dense in  $\overline{\mathfrak{B}}$ .

*Proof.* If  $(t_0, \varepsilon, [\Psi, A]) \in \overline{\mathfrak{B}} \setminus \mathfrak{B}$ , then  $\varepsilon = 0$ . It follows from Definition 4.1 and [DW17, Theorem 1.38], that there are is a family  $\{(\Psi_\varepsilon, A_\varepsilon; t(\varepsilon)) : 0 \leq \varepsilon \ll 1\}$  of solutions to

$$\begin{aligned} \mathcal{D}_{A_\varepsilon, \mathbf{p}_{t(\varepsilon)}} \Psi_\varepsilon &= 0 \quad \text{and} \\ \varepsilon^2 \left( \frac{1}{2} F_{A_\varepsilon} + \eta_{t(\varepsilon)} \right) &= \mu_{\mathbf{p}_{t(\varepsilon)}}(\Psi_\varepsilon) \end{aligned}$$

with  $(\Psi_\varepsilon, A_\varepsilon)$  converging to  $(\Psi, A)$  in  $C^\infty$  and  $t(\varepsilon)$  converging to  $t_0$  as  $\varepsilon$  tends to zero. Consequently,  $\mathfrak{B}$  is dense in  $\overline{\mathfrak{B}}$ .  $\square$

That  $\overline{\mathfrak{B}}$  is compact does not follows from Theorem 3.1; it does, however, follow from the next result, whose proof will occupy the remainder of this section.

**Proposition 5.3.** Let  $(\mathbf{p}_i, \eta_i)$  be a sequence in  $\mathcal{P} \times \mathcal{L}$  such that  $(\mathbf{p}_i, \eta_i)$  converges to  $(\mathbf{p}, \eta)$  in  $C^\infty$ . Let  $(\varepsilon_i, \Psi_i, A_i)$  be a sequence of solutions of

$$(5.4) \quad \begin{aligned} \mathcal{D}_{A_i, \mathbf{p}_i} \Psi_i &= 0, \\ \varepsilon_i^2 \left( \frac{1}{2} F_{A_i} + \eta_i \right) &= \mu_{\mathbf{p}_i}(\Psi_i), \quad \text{and} \\ \|\Psi_i\|_{L^2} &= 1 \end{aligned}$$

with  $\lim_{i \rightarrow \infty} \varepsilon_i = 0$ . If

$$Z := \left\{ x \in M : \limsup_{i \rightarrow \infty} |\Psi_i(x)| = 0 \right\},$$

is empty, then, after passing to a gauge transformed subsequence,  $(\Psi_i, A_i)$  converges in  $C^\infty$  to a solution  $(\Psi, A)$  of

$$\mathcal{D}_{A, \mathbf{p}} \Psi = 0, \quad \mu_{\mathbf{p}}(\Psi) = 0, \quad \text{and} \quad \|\Psi\|_{L^2} = 1.$$

**Proposition 5.5.**  $\overline{\mathfrak{B}}$  is compact.

*Proof.* We need to show that any sequence  $(t_i, \varepsilon_i, [\Psi_i, A_i])$  in  $\mathfrak{B}$  has a subsequence which converges in  $\overline{\mathfrak{B}}$ . If  $\liminf \varepsilon_i > 0$ , then this a consequence of standard elliptic estimates and Arzelà–Ascoli. It only needs to be pointed out that  $\varepsilon_i$  cannot tend to infinity, because otherwise there would be a reducible solution to (2.2) which is ruled out by Definition 4.1(3).

If  $\liminf_{i \rightarrow \infty} \varepsilon_i = 0$ , then [Theorem 3.1](#) asserts that a gauge transformed subsequence of  $(\Psi_i, A_i)$  converges weakly in  $W_{\text{loc}}^{2,2} \times W_{\text{loc}}^{1,2}$  outside  $Z$ . If  $Z$  is non-empty, then the limit represents a singular harmonic  $\mathbf{Z}_2$  spinors. However, by assumption there are no singular harmonic  $\mathbf{Z}_2$  spinors; hence,  $Z$  is empty and [Proposition 5.3](#) asserts that a gauge transformed subsequence of  $(\Psi_i, A_i)$  converges in  $C^\infty$ .  $\square$

The proof of [Proposition 5.3](#) relies on the following a priori estimate.

**Proposition 5.6.** *For each  $m_0 > 0$ , there is an  $\varepsilon_0 > 0$  and, for each,  $k \in \mathbf{N}$ , a non-decreasing function  $f_k: [0, \infty) \rightarrow [0, \infty)$  such that the following holds: if  $(\Psi, A, \varepsilon) \in \Gamma(M, \text{Hom}(E, W)) \times \mathcal{A}(\det W) \times (0, \varepsilon_0]$  satisfies (5.4) and*

$$\min|\Psi|^2 \geq m_0,$$

then

$$\|\Psi\|_{C_A^k} + \|F_A\|_{C^k} \leq f_k \left( \|F_A\|_{L^2} + \|\eta\|_{C^k} + \|F_B\|_{C^k} + \|R_g\|_{C^k} \right).$$

Here  $\|\Psi\|_{C_A^k} := \sum_{i=0}^k \|\nabla_A^i \Psi\|_{L^\infty}$  and  $R_g$  denotes the Riemann curvature tensor of the metric  $g$ .

The key ingredient for the proof of the above a priori estimate are the identity

$$(5.7) \quad \begin{aligned} (d^*d + dd^*)\mu(\Psi) + |\Phi|^2 F_A &= \sum_{i,j=1}^3 \frac{1}{2} \rho^* \left( ((F_{ij}^B + F_{ij}^W) \cdot \Phi) \Phi^* \right) e^{ij} \\ &\quad + \rho^* \left( (\nabla_j^A \Phi) (\nabla_i^A \Phi)^* \right) e^{ij}, \end{aligned}$$

whose proof can be found in [[DW17](#), [Proposition A.7](#)], and the following two propositions.

**Proposition 5.8.** *Let  $k \in \mathbf{N}_0, p \geq 2$ . There is a monotone function  $f_{k,p}: [0, \infty) \rightarrow [0, \infty)$  such that for every  $(\Psi, A) \in \Gamma(M, \text{Hom}(E, W)) \times \mathcal{A}(\det W)$  solving*

$$\not{D}_{A,p} \Psi = 0$$

we have

$$\|\Psi\|_{W_A^{k+2,p}} \lesssim f_{k,p} \left( \|F_A\|_{W^{k,p}} + \|\Psi\|_{L^\infty} + \|F_B\|_{C^{k+1}} + \|R_g\|_{C^{k+1}} \right).$$

Here  $\|\Psi\|_{W_A^{k,p}} := \sum_{i=0}^k \|\nabla_A^i \Psi\|_{L^p}$ .

*Proof.* Fix a smooth reference connection  $A_0$ . By Hodge theory, after applying a gauge transformation, we can write  $A = A_0 + a$  with  $d^*a = 0$  and

$$\|a\|_{W^{k+1,p}} \lesssim \|F_A\|_{W^{k,p}}$$

with the implicit constant depending on  $k, p$ , and  $\|R_g\|_{C^k}$ . It suffices to estimate  $\|\Psi\|_{W_{A_0}^{k+2,p}}$ , since

$$\begin{aligned} \|(\nabla_{A_0} + a)^2 \Phi\|_{L^p} &\lesssim \|\nabla_{A_0}^2 \Phi\|_{L^p} + \| |a| |\nabla_{A_0} \Phi| \|_{L^p} + \| |a|^2 |\Phi| \|_{L^p} + \| |\nabla a| |\Phi| \|_{L^p} \\ &\lesssim \|\nabla_{A_0}^2 \Phi\|_{L^p} + \|a\|_{L^{2p}} \|\nabla_{A_0} \Phi\|_{L^{2p}} + \|a\|_{L^{2p}}^2 \|\Phi\|_{L^\infty} + \|\nabla a\|_{L^p} \|\Phi\|_{L^\infty} \\ &\lesssim (1 + \|a\|_{W^{1,p}})^2 \cdot \|\Psi\|_{W_{A_0}^{2,p}}. \end{aligned}$$

by Hölder's inequality, Sobolev embedding, and Morrey's inequality; more generally,

$$\|\Psi\|_{W_A^{k+2,p}} \lesssim (1 + \|a\|_{W^{k+1,p}})^{k+2} \cdot \|\Psi\|_{W_{A_0}^{k+2,p}}.$$

The Dirac equation can be written as

$$\not{D}_{A_0,p} \Psi = -\bar{\gamma}(a)\Psi.$$

By standard elliptic estimates

$$(5.9) \quad \|\Psi\|_{W_{A_0}^{\ell+1,q}} \lesssim \|\bar{\gamma}(a)\Psi\|_{W_{A_0}^{\ell,q}} + \|\Psi\|_{L^\infty}$$

for any  $\ell \in \mathbb{N}_0$  and  $q \in (1, \infty)$  with the implicit constant depending on  $\ell, q, \|F_B\|_{C^\ell}$ , and  $\|R_g\|_{C^\ell}$ . By the Sobolev Multiplication Theorem,<sup>5</sup> we have

$$\|\bar{\gamma}(a)\Psi\|_{W_{A_0}^{k+1,p}} \lesssim \|a\|_{W^{k+1,p}} \|\Psi\|_{W_{A_0}^{k+1,q}}$$

provided  $(k+1)q > 3$ . Therefore, to bound  $\|\Psi\|_{W_{A_0}^{k+2,p}}$  in terms of

$$\spadesuit_k := \|F_A\|_{W^{k,p}} + \|\Psi\|_{L^\infty} + \|F_B\|_{C^k} + \|R_g\|_{C^k},$$

it suffices to show that for some such  $q$ ,  $\|\Psi\|_{W_{A_0}^{k+1,q}}$  can be bounded in terms of  $\spadesuit_k$ . If  $k \geq 1$  or  $k = 0$  and  $q \geq 3$ , such an estimate can be assumed by induction. Hence, we only need to prove that  $\|\Psi\|_{W_{A_0}^{1,q}}$  is bounded in terms of  $\spadesuit_k$  for some  $q > 3$ .

By (5.9),  $\|\Psi\|_{W_{A_0}^{1,2}}$  can be bounded in terms of  $\spadesuit_k$ . Consequently, we have a bound on  $\|\gamma(a)\Psi\|_{L^3(B_r)}$  and thus, using (5.9) again, a bound on  $\|\Psi\|_{W_{A_0}^{1,3}}$  in terms of  $\spadesuit_k$ . Since  $W^{1,3} \hookrightarrow L^p$  for any  $p \in [1, \infty)$ , it follows that  $\|\gamma(a)\Psi\|_{L^q} \leq \|a\|_{L^6} \|\Psi\|_{L^p}$  is bounded for any  $q$  with  $1/q = 1/6 + 1/p$ . This then implies the desired bound on  $\|\Psi\|_{W_{A_0}^{1,q}}$  for any  $q \in [1, 6)$ .  $\square$

<sup>5</sup>The Sobolev Multiplication Theorem asserts that  $\|fg\|_{W^{k,p}} \lesssim \|f\|_{W^{k_1,p_1}} \|g\|_{W^{k_2,p_2}}$  provided  $k - \frac{3}{p} < k_1 - \frac{3}{p_1} + k_2 - \frac{3}{p_2}$  and  $k \leq \min\{k_1, k_2\}$ .

**Proposition 5.10.** *Let  $k \in \mathbb{N}, p \in [2, \infty)$ . Let  $m \in C^\infty(M)$  with  $m_0 := \min m > 0$ . There are constants  $c > 0$  and  $\varepsilon_0 > 0$  depending on  $k, p, m_0, \|m\|_{C^k}$ , and  $\|R_g\|_{C^k}$  such that for all  $\varepsilon \in (0, \varepsilon_0)$*

$$\|\alpha\|_{W^{k,p}} \leq c \|(\varepsilon^2 \Delta + m)\alpha\|_{W^{k,p}}$$

for all  $\alpha \in \Omega^*(M)$ .

*Proof.* Denote by  $\mathfrak{R}$  the curvature operator appearing in the Weitzenböck formula  $\Delta\alpha = \nabla^* \nabla \alpha + \mathfrak{R}\alpha$ . If  $\varepsilon \ll 1$ , then by integration by parts we obtain

$$\begin{aligned} \int_M \langle (\varepsilon^2 \Delta + m)\alpha, \alpha \rangle |\alpha|^{p-2} &= \int_M \varepsilon^2 |\nabla \alpha|^2 |\alpha|^{p-2} + \frac{\varepsilon^2(p-2)}{4} |\nabla_A |\alpha|^2|^2 |\alpha|^{p-4} \\ &\quad + \varepsilon^2 \langle \mathfrak{R}\alpha, \alpha \rangle |\alpha|^{p-2} + m |\alpha|^p \\ &\geq \int_M \frac{m_0}{2} |\alpha|^p. \end{aligned}$$

By Young's inequality and with  $q = p/(p-1)$ , we have

$$\begin{aligned} \int_M \langle (\varepsilon^2 \Delta + m)\alpha, \alpha \rangle |\alpha|^{p-2} &\leq \int_M |(\varepsilon^2 \Delta + m)\alpha| |\alpha|^{p-1} \\ &\leq \int \frac{\delta^{-p}}{p} |(\varepsilon^2 \Delta + m)\alpha|^p + \frac{\delta^q}{q} |\alpha|^p \end{aligned}$$

for all  $\delta > 0$ . Choosing  $\delta$  sufficient small, yields

$$\int_M |\alpha|^p \lesssim \int_M |(\Delta + m)\alpha|^p;$$

that is, the desired estimate for  $k = 0$ .

If  $v_1, \dots, v_k$  are vector fields on  $M$ , then the commutator  $[\varepsilon^2 \Delta + m, \partial_{v_1} \cdots \partial_{v_k}]$  is a differential operator of order  $k-1$  with coefficients depending on only on the first  $k$  derivatives of  $m$  and  $R_g$ . Therefore, the estimates for  $k > 0$  follows from the one for  $k = 0$  using

$$(\varepsilon^2 \Delta + m)(\partial_{v_1} \cdots \partial_{v_k} \alpha) = \partial_{v_1} \cdots \partial_{v_k} (\varepsilon^2 \Delta + m)\alpha + [\varepsilon^2 \Delta + m, \partial_{v_1} \cdots \partial_{v_k}] \alpha$$

and induction. □

*Proof of Proposition 5.6.* It follows from [HW15, Proposition 2.1] that  $\|\Psi\|_{L^\infty}$  can be bounded in terms of  $\clubsuit_k := \|F_A\|_{L^2} + \|\eta\|_{C^k} + \|F_B\|_{C^{k+1}} + \|R_g\|_{C^{k+1}}$ . From Proposition 5.8 with  $k = 0$  and  $p = 2$  it follows that  $\|\Psi\|_{W_A^{2,2}}$  bounded in terms of  $\clubsuit_k$  as well. Since  $W_A^{2,2} \hookrightarrow W_A^{1,6}$ , it follows that the  $L^3$ -norm of the right-hand side of (5.7) can be bounded in terms of  $\clubsuit_k$ . Proposition 5.10 thus yields a bound on  $\|F_A\|_{L^3}$



in terms of  $\clubsuit_k$ . Proposition 5.8 with  $k = 0$  and  $p = 3$  bounds  $\|\Psi\|_{W_A^{2,3}}$  and, hence,  $\|\Psi\|_{W^{1,p}}$  for all  $p \in (1, \infty)$  in terms of  $\clubsuit_k$ . Consequently, Proposition 5.10 bounds  $\|F_A\|_{L^p}$  for any  $p$  in terms of  $\clubsuit_k$ . Another application of Proposition 5.8 shows that  $\|\Psi\|_{W_A^{2,p}}$  can be bounded in terms of  $\clubsuit_k$ . This yields bounds on the  $W^{1,p}$ -norm of right-hand side of (5.7) and  $\|\Psi\|_{C^1}$  in terms of  $\clubsuit_k$ . It follows from Proposition 5.10 that  $\|F_A\|_{W^{1,p}}$  is bounded in terms of  $\clubsuit_k$ . Iterating applications of Proposition 5.8 and Proposition 5.10 one shows that  $\|F_A\|_{W^{k+1,p}}$  and  $\|\Psi\|_{W^{k+3,p}}$  can be bounded in terms of  $\clubsuit_k$ . Applying Morrey's inequality,  $|f|_{C^0} \lesssim \|f\|_{W^{1,p}}$  for  $p > 3$ , completes the proof.  $\square$

*Proof of Proposition 5.3.* Since  $Z$  is empty, there exists  $m_0 > 0$  such that, after passing to a subsequence, we have  $\min|\Psi_i|^2 \geq m_0$  for all  $i$ . Moreover, it follows from [HW15, Definition 3.1 and Proposition 4.1] that  $\|F_{A_i}\|_{L^2}$  is uniformly bounded. Proposition 5.6 now yields uniform  $C^k$ -bounds for  $F_{A_i}$  and  $\Psi_i$  (using the  $A_i$ -dependent norm). After putting  $A_i$  in the Uhlenbeck gauge as in the proof of Proposition 5.8 we also obtain  $C^k$  bounds for  $A_i$  and the result follows from the Arzelà–Ascoli theorem.  $\square$

## 6 Orientation at infinity

Suppose that  $(t_0, 0, [\Psi, A]) \in \overline{\mathfrak{M}} \setminus \mathfrak{M}$  is a boundary point in  $\overline{\mathfrak{M}}$ . Set  $\mathbf{p} := \mathbf{p}_{t_0}$ . By Proposition 3.3, there exists a spin structure  $\mathfrak{s}$  inducing the  $\text{spin}^c$  structure  $\mathfrak{w}$  such that  $\Psi \in \Gamma(\text{Re}(E \otimes S_{\mathfrak{s}})) \subset \Gamma(\text{Hom}(E, W))$ ,  $\mathcal{D}_{\mathbf{p}}^{\mathfrak{s}}\Psi = 0$ , and  $A$  is trivial. By Definition 4.1(1), there exists a unique solution  $\{\Psi_t : |t - t_0| \ll 1\}$  to

$$\mathcal{D}_{\mathbf{p}_t}^{\mathfrak{s}}\Psi_t = \lambda(t)\Psi_t \quad \text{and} \quad \|\Psi_t\|_{L^2} = 1$$

with  $\Psi_{t_0} = \Psi$ . Moreover, it satisfies  $\dot{\lambda}(t_0) \neq 0$ . In this situation, [DW17, Section 6] shows that for any choice of  $r \in \mathbf{N}$  there exist  $\tau \ll 1$  and  $\varepsilon_0 \ll 1$ , a  $C^r$  map

$$\text{ob}: (t_0 - \tau, t_0 + \tau) \times [0, \varepsilon_0] \rightarrow \mathbf{R},$$

an open neighborhood  $V$  of  $(t_0, 0, [\Psi, A]) \in \overline{\mathfrak{M}}$ , and a homeomorphism

$$\mathfrak{x}: \text{ob}^{-1}(0) \rightarrow V$$

such that

1.  $\mathfrak{x}$  commutes with the projection to the  $t$ - and  $\varepsilon$ -coordinates,
2. the restriction  $\text{ob}|_{(t_0-\tau, t_0+\tau) \times (0, \varepsilon_0)}$  is smooth,

$$\begin{aligned} \text{ob}(t, \varepsilon) &= \dot{\lambda}(t_0) \cdot (t - t_0) - \delta\varepsilon^4 + O(t^2, \varepsilon^6), \quad \text{and} \\ \partial_{\varepsilon}\text{ob}(t, \varepsilon) &= -4\delta\varepsilon^3 + O(t^2, \varepsilon^5) \end{aligned}$$

with  $\delta$  as Definition 4.1(4), and

3. if  $\varepsilon \in (0, \varepsilon_0) \times (t_0 - \tau, t_0 + \tau)$  satisfies  $\text{ob}(\varepsilon, t) = 0$  and  $[\Psi_\varepsilon, A_\varepsilon] = \mathfrak{x}(\varepsilon, t)$ , then there is a short exact sequence

$$(6.1) \quad 0 \rightarrow \ker L_{\Psi_\varepsilon, A_\varepsilon} \rightarrow \mathbf{R}\langle \Psi \rangle \xrightarrow{\partial_\varepsilon \text{ob}} \mathbf{R}\langle \Psi \rangle \rightarrow \text{coker } L_{\Psi_\varepsilon, A_\varepsilon} \rightarrow 0.$$

It follows from the above that

$$\text{ob}^{-1}(0) = \{(\varepsilon, t_0 + (\delta/\dot{\lambda}(0))\varepsilon^4 + O(\varepsilon^6)) : \varepsilon \in [0, \varepsilon_0]\}.$$

Therefore,  $\overline{\mathfrak{M}}$  is a compact, oriented, one-dimensional manifold with boundary

$$\partial \overline{\mathfrak{M}} = \mathfrak{M}_w(\mathbf{p}_1, \eta_1) - \mathfrak{M}_w(\mathbf{p}_0, \eta_0) \cup (\overline{\mathfrak{M}} \setminus \mathfrak{M}).$$

If  $\delta/\dot{\lambda}(0) > 0$ , then as  $t$  passes through  $t_0$  a solution to (2.2) is created. If  $\delta/\dot{\lambda}(0) < 0$ , then as  $t$  passes through  $t_0$  a solution to (2.2) is annihilated. The orientation procedure described in Section 2 determines an orientation of  $\det(L_{\Psi, A})$  which in turn is isomorphic to  $\det(\partial_\varepsilon \text{ob}: \mathbf{R}\langle \Psi \rangle \rightarrow \mathbf{R}\langle \Psi \rangle) \cong \mathbf{R}$  by (6.1). This determines a sign

$$\sigma([\Psi, A], \mathbf{p}) \in \{\pm 1\}$$

depending on whether the orientation induced on  $\mathbf{R}$  in this way is the standard orientation or the opposite. The solution that is created/annihilated at  $t_0$  contributes with  $\sigma \cdot \text{sign}(\partial_\varepsilon \text{ob}) = -\sigma \cdot \text{sign}(\delta)$ . Consequently, for  $\tau \ll 1$ , we have the local wall-crossing formula

$$(6.2) \quad n_w(\mathbf{p}_{t+\tau}, \eta_{t+\tau}) = n_w(\mathbf{p}_{t-\tau}, \eta_{t-\tau}) + \text{sign}(\dot{\lambda}(0)) \cdot \sigma([\Psi, A], \mathbf{p}).$$

The following proposition explains how to compute  $\sigma([\Psi, A], \mathbf{p})$ .

**Proposition 6.3.** *In the above situation,*

$$(6.4) \quad \sigma([\Psi, A], \mathbf{p}) = \sigma(\Psi, \mathbf{p})$$

with  $\sigma(\Psi, \mathbf{p})$  as in Proposition 4.3.

*Remark 6.5.*  $\Psi$  is determined by  $[\Psi, A]$  up to a sign and, by Remark 4.7,  $\sigma(\Psi, \mathbf{p})$  only depends on  $\Psi$  up to multiplication by a constant in  $\mathbf{R}^*$ .

*Proof of Proposition 6.3.* We proceed in three steps.

**Step 1.** For  $s \gg 1$ , the inclusion

$$\Gamma(\text{Re}(E \otimes S_s)) \hookrightarrow \Gamma(\text{Hom}(E, W)) \oplus \Omega^1(M, i\mathbf{R}) \oplus \Omega^0(M, i\mathbf{R})$$

induces isomorphisms

$$(6.6) \quad \mathbf{R}\langle \Psi \rangle \cong \ker L_{s\Psi, A} \quad \text{and} \quad \mathbf{R}\langle \Psi \rangle \cong \text{coker } L_{s\Psi, A}.$$

Denote by  $\mathcal{D}_{\text{Re}}$  and  $\mathcal{D}_{\text{Im}}$  the restriction of  $\mathcal{D}_{A,\mathbf{p}}$  to  $\text{Re}(E \otimes S_s)$  and  $\text{Im}(E \otimes S_s)$  respectively. We can write  $L_{s\Psi,A}$  as

$$L_{s\Psi,A} = \begin{pmatrix} -\mathcal{D}_{\text{Re}} & \\ & \mathcal{D}_{s\Psi,A} \end{pmatrix} \quad \text{with} \quad \mathcal{D}_{s\Psi,A} = \begin{pmatrix} -\mathcal{D}_{\text{Im}} & -s\alpha_\Psi \\ -s\alpha_\Psi^* & \mathfrak{d} \end{pmatrix}.$$

By (4.5), we have

$$\mathcal{D}_{s\Psi,A}^* \mathcal{D}_{s\Psi,A} = \begin{pmatrix} \mathcal{D}_{\text{Im}}^* \mathcal{D}_{\text{Im}} + s^2 |\Psi|^2 & \\ & \mathfrak{d}^* \mathfrak{d} + s^2 |\Psi|^2 \end{pmatrix} + s \begin{pmatrix} & \tilde{\epsilon}_\Psi \\ \tilde{\epsilon}_\Psi^* & \end{pmatrix}$$

with

$$\tilde{\epsilon}_\Psi = -2 \sum_{i=1}^3 \rho(a(e_i)) \nabla_{e_i} \Psi;$$

see (4.5). Since  $\tilde{\epsilon}_\Psi$  is a bounded zeroth order operator, it follows that, for  $s \gg 1$ ,  $\mathcal{D}_{s\Psi,A}$  is invertible which implies the assertion.

**Step 2.** *With respect to the isomorphisms (6.6),  $\sigma([\Psi, A], \mathbf{p})$  is given by orientation transport from  $\det(L_{0,A})$  to  $\det(L_{s\Psi,A})$ .*

It follows from [DW17, Section 5] that, with respect to the isomorphisms in (6.6),  $\sigma([\Psi, A], \mathbf{p})$  can be computed as the orientation transport from  $\det(L_{0,A_\epsilon})$  via  $\det(L_{\Psi_\epsilon, A_\epsilon})$  to  $\det(L_{s\Psi,A})$  which in turn is the orientation transport from  $\det(L_{0,A})$  to  $\det(L_{s\Psi,A})$  because the orientation transport from  $\det(L_{0,A_\epsilon})$  to  $\det(L_{0,A})$  is trivial.

**Step 3.** *The identity (6.4) holds.*

Using the isometry  $\tilde{\alpha}_\Psi = |\Psi|^{-1} \alpha_\Psi : \Gamma(\text{Re}(E \otimes S_s)) \rightarrow \Omega^1(M, i\mathbf{R}) \oplus \Omega^0(M, i\mathbf{R})$  we can think of  $L_s$  as the operator

$$\tilde{L}_s : (\Omega^1(M, i\mathbf{R}) \oplus \Omega^0(M, i\mathbf{R}))^{\oplus 3} \rightarrow (\Omega^1(M, i\mathbf{R}) \oplus \Omega^0(M, i\mathbf{R}))^{\oplus 3}$$

defined by

$$\tilde{L}_s = \begin{pmatrix} -\mathfrak{d}_\Psi & & \\ & -\mathfrak{d}_\Psi & -s \\ & -s & \mathfrak{d} \end{pmatrix}.$$

By the preceding step,  $\sigma([\Psi, A], \mathbf{p})$  is the orientation transport from  $\tilde{L}_0$  to  $\tilde{L}_s$  for  $s \gg 1$ . This is the same as the product of the orientation transport from

$$\begin{pmatrix} -\mathfrak{d}_\Psi & \\ & \mathfrak{d} \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} -\mathfrak{d} & \\ & \mathfrak{d} \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} -\mathfrak{d} & -s \\ -s & \mathfrak{d} \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} -\mathfrak{d}_\Psi & -s \\ -s & \mathfrak{d} \end{pmatrix}.$$

The first orientation transport is exactly the orientation transport from  $-\mathfrak{d}_\Psi$  to  $-\mathfrak{d}$ . The second is given by a universal sign. This sign can be computed to be

$(-1)^{1+b_1(M)}$  by observing that if  $\mathcal{H} := \mathcal{H}^0 \oplus \mathcal{H}^1$  denotes the space of harmonic forms and  $\mathfrak{d}_0: \mathcal{H}^\perp \rightarrow \mathcal{H}^\perp$  is the restriction of  $\mathfrak{d}$ , then

$$\begin{pmatrix} -\mathfrak{d} & -s \\ -s & \mathfrak{d} \end{pmatrix}$$

can be diagonalized to

$$\begin{pmatrix} s \cdot \text{id}_{\mathcal{H}} & & & \\ & -s \cdot \text{id}_{\mathcal{H}} & & \\ & & +\sqrt{\mathfrak{d}_0^2 + s^2} & \\ & & & -\sqrt{\mathfrak{d}_0^2 + s^2} \end{pmatrix},^6$$

and the orientation transport along this family of operators is  $(-1)^{1+b_1(M)}$ . The final orientation transport is trivial since  $\mathfrak{d}_\Psi = \mathfrak{d} + e_\Psi$  and, for  $s \gg 1$  and  $t \in [0, 1]$ , the operator

$$\begin{pmatrix} -\mathfrak{d} - te_\Psi & -s \\ -s & \mathfrak{d} \end{pmatrix}^2 = \begin{pmatrix} (-\mathfrak{d} - te_\Psi)^2 + s^2 & \\ & \mathfrak{d}^2 + s^2 \end{pmatrix} + \begin{pmatrix} & ste_\Psi \\ ste_\Psi & \end{pmatrix}$$

is invertible. This completes the proof.  $\square$

## 7 Proof the wall-crossing formulae

This section will conclude the proof of Theorem 4.9. The local wall-crossing formula (6.2) and Proposition 6.3 directly imply the wall-crossing formula

$$(4.10) \quad n_w(\mathbf{p}_1, \eta_1) = n_w(\mathbf{p}_0, \eta_0) + \sum_s \sum_{i=1}^{N_s} \chi_i^s \cdot \sigma(\Psi_i^s, \mathbf{p}_{t_i^s}).$$

In order to prove (4.11), we need to relate  $\sigma(\Psi_0, \mathbf{p}_0)$  to  $\sigma(\Psi_1, \mathbf{p}_1)$  for two different nowhere vanishing  $\Psi_0$  and  $\Psi_1$  and two parameters  $\mathbf{p}_0$  and  $\mathbf{p}_1$ .

**Proposition 7.1.** *Let  $(\mathbf{p}_t)_{t \in [0,1]}$  be a path in  $\mathcal{P}$ . If  $\Psi_0$  and  $\Psi_1$  are two nowhere vanishing sections of  $\text{Re}(E \otimes S_s)$ , then*

$$(7.2) \quad \sigma(\Psi_1, \mathbf{p}_1) = \sigma(\Psi_0, \mathbf{p}_0) \cdot (-1)^{\text{SF}(-\mathcal{D}_{\mathbf{p}_t}^s)} \cdot (-1)^{\text{deg}(\Psi_0, \Psi_1)}.$$

Here  $\text{deg}(\Psi_0, \Psi_1)$  denotes the relative the degree of  $\Psi_0$  and  $\Psi_1$  as in Definition 4.8.

<sup>6</sup>The operator  $\mathfrak{d}_0^2 + s^2$  has strictly positive spectrum, so its square root is well-defined and invertible.

*Proof.* If  $\deg(\Psi_0, \Psi_1) = 0$ , then we can find a path  $(\Psi_t)_{t \in [0,1]}$  of nowhere vanishing sections from  $\Psi_0$  to  $\Psi_1$ . It is clear that

$$\sigma(\Psi_1, \mathbf{p}_1) = \sigma(\Psi_0, \mathbf{p}_0) \cdot (-1)^{\text{SF}(-\mathfrak{d}_{\Psi_t})}.$$

The spectral flow along the path of operators  $-\mathfrak{d}_{\Psi_t}$  is identical to that of the path of operators

$$-\tilde{\mathfrak{a}}_{\Psi_0} \circ \mathfrak{d}_{\Psi_t} \circ \tilde{\mathfrak{a}}_{\Psi_0}^{-1},$$

which is homotopic to the path of operators

$$-\tilde{\mathfrak{a}}_{\Psi_t} \circ \mathfrak{d}_{\Psi_t} \circ \tilde{\mathfrak{a}}_{\Psi_t}^{-1} = -\mathbb{D}_{\mathbf{p}_t}^{\mathfrak{s}}$$

followed by

$$-\tilde{\mathfrak{a}}_{\Psi_{1-t}} \tilde{\mathfrak{a}}_{\Psi_1}^{-1} \circ \mathbb{D}_{\mathbf{p}_1}^{\mathfrak{s}} \circ \tilde{\mathfrak{a}}_{\Psi_1} \tilde{\mathfrak{a}}_{\Psi_{1-t}}^{-1}.$$

Since  $\tilde{\mathfrak{a}}_{\Psi_1} \tilde{\mathfrak{a}}_{\Psi_{1-t}}^{-1}$  is a path of isometries, the second path of operators has trivial spectral flow. This proves (7.2) if  $\deg(\Psi_0, \Psi_1) = 0$ .

It remains to deal with the case  $\deg(\Psi_0, \Psi_1) \neq 0$ . By the above, we can assume that  $\mathbf{p}_t = \mathbf{p}$  for all  $t \in [0, 1]$ . Set  $V := (T^*M \oplus \underline{\mathbf{R}}) \otimes i\mathbf{R}$  and define an isometry  $f: V \rightarrow V$  by

$$f := \tilde{\mathfrak{a}}_{\Psi_0}^{-1} \tilde{\mathfrak{a}}_{\Psi_1}$$

By definition of  $f$ , we have

$$\mathfrak{d}_{\Psi_1} = f^{-1} \circ \mathfrak{d}_{\Psi_0} \circ f.$$

Thus, gluing the operators  $\partial_t - \mathfrak{d}_{\Psi_1}^t$  and  $\partial_t - \mathfrak{d}_{\Psi_0}^{1-t}$ , we obtain a first order elliptic operator  $\mathfrak{D}$  on the bundle  $\mathbf{V}$  over  $S^1 \times M$  defined as the mapping torus of  $f$ . By Atiyah–Patodi–Singer,

$$\sigma(\Psi_1, \mathbf{p}_1) \cdot \sigma(\Psi_0, \mathbf{p}_0) = (-1)^{\text{SF}(-\mathfrak{d}_{\Psi_1}^t)} \cdot (-1)^{\text{SF}(-\mathfrak{d}_{\Psi_0}^{1-t})} = (-1)^{\text{index } \mathfrak{D}};$$

hence, by the Atiyah–Singer Index Theorem,

$$\sigma(\Psi_1, \mathbf{p}_1) \cdot \sigma(\Psi_0, \mathbf{p}_0) = (-1)^{\deg(\Psi_0, \Psi_1)}.$$

This completes the proof of this proposition.  $\square$

**Proposition 7.3.** *For all  $i \in \{1, \dots, N_w\}$ , we have*

$$\chi_i^{\mathfrak{s}} \cdot (-1)^{\text{SF}(-\mathbb{D}_{\mathbf{p}_t}^{\mathfrak{s}} : t \in [t_i^{\mathfrak{s}}, t_i^{\mathfrak{s}}])} = (-1)^{i+1} \cdot \chi_1^{\mathfrak{s}}.$$

*Proof.* By induction it suffices to consider the case  $i = 2$ . The case  $i = 2$  can be verified directly case-by-case as follows:

$\chi_1$	$\chi_2$	SF	$\chi_1 \cdot \chi_2 \cdot (-1)^{\text{SF}}$
+1	+1	1	-1
+1	-1	0	-1
-1	+1	0	-1
-1	-1	-1	-1

□

Combining the two preceding propositions shows that (4.10) can equivalently be written as follows:

$$(4.11) \quad n_w(\mathbf{p}_1, \eta_1) = n_w(\mathbf{p}_0, \eta_0) + \sum_s \chi_1^s \cdot \sigma(\Psi_1^s, \mathbf{p}_{t_1^s}) \cdot \sum_{i=1}^{N_s} (-1)^{i+1} \cdot (-1)^{\deg(\Psi_1^s, \Psi_i^s)}.$$

This completes the proof of Theorem 4.9. □

## 8 Transversality for paths

The purpose of this section is to prove Proposition 4.2.

**Proposition 8.1.** *For any  $\mathbf{p}_0, \mathbf{p}_1 \in \mathcal{P}^{\text{reg}}$ , the subspace  $\mathcal{P}^{\text{reg}}(\mathbf{p}_0, \mathbf{p}_1)$  is residual in the space of all smooth paths from  $\mathbf{p}_0$  to  $\mathbf{p}_1$  in  $\mathcal{P}$ .*

*Proof.* The proof is an application of the Sard–Smale theorem. We will work with Sobolev spaces of sections and connections of class  $W^{k,p}$  such that  $(k-1)p > 3$ . The statement for  $C^\infty$  spaces follows then from a standard argument; see, for example, [Doa17b, Proof of Proposition 2.21].

Since there are only finitely many spin structures on  $M$ , it suffices to consider the conditions (1) and (2) in Definition 4.1 for a fixed spin structure  $s$ . Set

$$X := \mathcal{P}(\mathbf{p}_0, \mathbf{p}_1) \times [0, 1] \times \frac{\Gamma(\text{Re}(E \otimes S_s) \setminus \{0\})}{\mathbf{R}^*}$$

and  $V := \mathcal{P}(\mathbf{p}_0, \mathbf{p}_1) \times [0, 1] \times \frac{\Gamma(\text{Re}(E \otimes S_s) \setminus \{0\}) \times \Gamma(\text{Re}(E \otimes S_s))}{\mathbf{R}^*}.$

$V$  is a vector bundle over  $X$ . Define a section  $\mathbf{s} \in \Gamma(V)$  by

$$\sigma((\mathbf{p}_t)_{t \in [0,1]}, t_*, [\Psi]) := ((\mathbf{p}_t)_{t \in [0,1]}, t_*, [(\Psi, \mathcal{D}_{\mathbf{p}_{t_*}} \Psi)]).$$

We can identify a neighborhood of  $[\Psi] \in \Gamma(\text{Re}(E \otimes S_s) \setminus \{0\})/\mathbf{R}^*$  with the  $L^2$ -orthogonal complement  $\Psi^\perp \subset \Gamma(\text{Re}(E \otimes S_s))$ . This gives us a local trivialization of  $V$  in which  $\sigma$  can be identified with the map

$$\sigma((\mathbf{p}_t)_{t \in [0,1]}, t_*, \psi) = ((\mathbf{p}_t)_{t \in [0,1]}, t_*, \mathcal{D}_{\mathbf{p}_{t_*}}(\Psi + \psi)) \quad \text{for all } \psi \in \Psi^\perp.$$

In particular, for a fixed path  $(\mathbf{p}_t)_{t \in [0,1]}$ , the map  $\sigma((\mathbf{p}_t)_{t \in [0,1]}, \cdot)$  defines a Fredholm section of index zero, since  $\psi \mapsto \mathbb{D}_{\mathbf{p}_t}(\Psi + \psi)$  has index  $-1$  and  $\dim[0, 1] = 1$ .

Let  $\mathbf{x} := ((\mathbf{p}_t)_{t \in [0,1]}, t_*, [\Psi]) \in X$  and denote by  $(d\sigma)_{\mathbf{x}}$  the linearization of  $\sigma$  at  $\mathbf{x}$  (and computed in the above trivialization). We will prove that  $(d\sigma)_{\mathbf{x}}$  is surjective provided  $\sigma(\mathbf{x}) = 0$ . If  $\Phi \in V_{\mathbf{x}} = \Gamma(\text{Re}(E \otimes S_5))$  is orthogonal to the image of  $(d\sigma)_{\mathbf{x}}$ , then it follows that

$$(8.2) \quad \langle \bar{y}(b)\Psi, \Phi \rangle_{L^2} = 0 \quad \text{for all } b \in \Omega^1(M, \mathfrak{su}(E)).$$

Since  $\Psi$  is harmonic, its zero set must be nowhere dense. Clifford multiplication by  $T^*M \otimes \mathfrak{su}(E)$  on  $\text{Re}(E \otimes S_5)$  induces an isomorphism between  $T^*M \otimes \mathfrak{su}(E)$  and trace-free symmetric endomorphisms of  $\text{Re}(E \otimes S_5)$ . With this in mind it follows from (8.2) that  $\Psi = 0$ . This proves that  $d_{\mathbf{x}}\sigma$  is surjective.

It follows that  $\sigma^{-1}(0)$  is a smooth submanifold of  $X$  and the projection map  $\pi: \sigma^{-1}(0) \rightarrow \mathcal{P}(\mathbf{p}_0, \mathbf{p}_1)$  is a Fredholm map of index zero. The kernel of  $\pi$  at  $\mathbf{x}$  can be identified with the kernel of the linearization of  $\sigma$  in the directions of  $[0, 1]$  and  $\Gamma(\text{Re}(E \otimes S_5) \setminus \{0\})/\mathbb{R}^*$ . A moment's thought show that if this kernel is trivial, then  $\Psi$  must span  $\ker \mathbb{D}_{\mathbf{p}_{t_*}}$  and the  $t_*$  must be a regular crossing of the spectral flow of  $(\mathbb{D}_{\mathbf{p}_t})$ . By the Sard–Smale theorem, the subspace of regular values of  $\pi$  is residual; hence, set of those  $(\mathbf{p}_t)_{t \in [0,1]}$  in  $\mathcal{P}(\mathbf{p}_0, \mathbf{p}_1)$  for which the condition (1) in Definition 4.1 holds is residual.

To deal with condition (2) in Definition 4.1, we consider the vector bundle

$$W := \mathcal{P}(\mathbf{p}_0, \mathbf{p}_1) \times [0, 1] \times \frac{\Gamma(\text{Re}(E \otimes S_5) \setminus \{0\}) \times \Gamma(\text{Re}(E \otimes S_5)) \times \text{Re}(E \otimes S_5)}{\mathbb{R}^*}$$

over  $X \times M$  and define a section  $\tau \in \Gamma(W)$  by

$$\tau((\mathbf{p}_t)_{t \in [0,1]}, t_*, [\Psi], y) := ((\mathbf{p}_t)_{t \in [0,1]}, t_*, [(\Psi, \mathbb{D}_{\mathbf{p}_{t_*}} \Psi, \Psi(y))]).$$

For a fixed path  $(\mathbf{p}_t)_{t \in [0,1]}$ , the map  $\tau((\mathbf{p}_t)_{t \in [0,1]}, \cdot)$  defines a Fredholm section of index  $-1$ . Note that for  $\mathbf{x} := ((\mathbf{p}_t)_{t \in [0,1]}, t_*, [\Psi]) \in X$  and  $y \in M$  the condition  $\tau((\mathbf{p}_t)_{t \in [0,1]}, \mathbf{x}, y) = 0$  is equivalent to  $\mathbb{D}_{\mathbf{p}_{t_*}} \Psi = 0$  and  $\Psi(y) = 0$ . We prove that the linearization of  $\tau$  is surjective at any  $(\mathbf{x}, y)$  satisfying these equations. If  $(\Phi, \phi) \in W_{(\mathbf{x}, y)} = \Gamma(\text{Re}(E \otimes S_5)) \times \text{Re}(E \otimes S_5)_y$  is orthogonal to the image of  $(d\tau)_{(\mathbf{x}, y)}$ , then (8.2) holds and, moreover,

$$(8.3) \quad \langle \mathbb{D}_{\mathbf{p}_{t_*}}(\Psi + \psi), \Phi \rangle_{L^2} + \langle \psi(x), \phi \rangle = 0 \quad \text{for all } \psi \in \Psi^\perp$$

Since  $\mathbb{D}_{\mathbf{p}_{t_*}} \Psi = 0$  and  $\Psi(y) = 0$ , (8.3) holds in fact for all  $\psi \in \Gamma(\text{Re}(E \otimes S_5))$  and we conclude that  $\mathbb{D}_{\mathbf{p}_{t_*}} \Psi = 0$ . Plugging this back into (8.3) yields  $\langle \psi(x), \phi \rangle = 0$  for all  $\psi$ , which implies that  $\phi = 0$ . As before (8.2) implies that  $\Phi = 0$ .

It follows that  $\tau^{-1}(0)$  is smooth and the projection  $\rho: \tau^{-1}(0) \rightarrow \mathcal{P}(\mathbf{p}_0, \mathbf{p}_1)$  is a Fredholm map of index  $-1$ ; in particular, the preimage of a regular value must be

empty. It follows that the paths  $(\mathbf{p}_t)_{t \in [0,1]} \in \mathcal{P}(\mathbf{p}_0, \mathbf{p}_1)$  for which condition (2) in Definition 4.1 holds is residual.  $\square$

To address condition (4) in Definition 4.1 we compute  $\delta(\Psi, \mathbf{p}, \eta)$ .

**Proposition 8.4.** *Let  $(\mathbf{p}_t)_{t \in [0,1]} \in \mathcal{P}(\mathbf{p}_0, \mathbf{p}_1)$ , let  $(\eta_t)_{t \in [0,1]}$  be a path in  $\mathcal{L}$ , let  $t_0 \in (0, 1)$ , and let  $\Psi$  be a nowhere vanishing section of  $\text{Re}(E \otimes S_{\mathbb{S}})$  spanning  $\ker \mathbb{D}_{\mathbf{p}_{t_0}}^{\mathbb{S}}$  and satisfying  $\|\Psi\|_{L^2} = 1$ . There is a linear algebraic operator*

$$\mathfrak{f}_{\mathbf{p}_{t_0}, \Psi}: \Omega^2(M, i\mathbb{R}) \rightarrow \Omega^2(M, i\mathbb{R})$$

such that

$$(8.5) \quad \delta(\Psi, \mathbf{p}_{t_0}, \eta_{t_0}) = \int_M |\Psi|^{-2} \langle d * \eta_{t_0} + \mathfrak{f}_{\Psi, \mathbf{p}_{t_0}} \eta_{t_0}, \eta_{t_0} \rangle$$

with  $\delta(\Psi, \mathbf{p}_{t_0}, \eta_{t_0})$  is as in Definition 4.1(4).

*Proof.* If we denote by

$$\{(\Psi_\varepsilon = \Psi + \varepsilon^2 \psi + O(\varepsilon^4), A_\varepsilon; t(\varepsilon) = t_0 + O(\varepsilon^2)) : 0 \leq \varepsilon \ll 1\}$$

the family of solutions to

$$\begin{aligned} \mathbb{D}_{A_\varepsilon, \mathbf{p}_{t(\varepsilon)}} \Psi_\varepsilon &= 0, \\ \varepsilon^2 \left( \frac{1}{2} F_{A_\varepsilon} + \eta_{t(\varepsilon)} \right) &= \mu_{\mathbf{p}_{t(\varepsilon)}}(\Psi_\varepsilon), \quad \text{and} \\ \|\Psi_\varepsilon\|_{L^2} &= 1 \end{aligned}$$

obtained from [DW17, Theorem 1.38], then

$$\delta = \delta(\Psi, \mathbf{p}_{t_0}, \eta_{t_0}) = \langle \mathbb{D}_{A_0, \mathbf{p}_{t_0}} \psi, \psi \rangle_{L^2}.$$

The connection  $A = A_0$  corresponding to  $\Psi$  is flat; see Proposition 3.3. For the unperturbed equation we would have  $t(\varepsilon) = 0$ . Since we consider the perturbed equation, however,  $\eta$  enters into the computation of  $\delta$ . More precisely, by [DW17, Equations 6.1, 6.2] we have

$$\psi = -\mathbb{D}_{\text{Re} \gamma}^{-1} \Pi^* \nu - \nu \quad \text{with} \quad \nu := (\mathfrak{a}_\Psi^*)^{-1} * \eta.$$

By (4.4), we have

$$\nu = |\Psi|^{-2} \bar{\gamma}(*\eta)\Psi.$$



Denote by  $\pi_{\text{Re}}$  the projection onto  $\text{Re}(E \otimes S_s)$ . From [DW17, Proposition 3.8] we know that for any  $a \in \Omega^1(M, i\mathbf{R})$

$$-\gamma \Pi^* \bar{\gamma}(a) \Psi = \sum_{i=1}^3 \pi_{\text{Re}} \left( \rho(a(e_i)) \nabla_{e_i}^A \Psi \right) = 0.$$

It follows that

$$\psi = -v = |\Psi|^{-2} \bar{\gamma}(*\eta) \Psi.$$

Set

$$a := |\Psi|^{-2} * \eta.$$

Since

$$\mathbb{D}_A \bar{\gamma}(a) \Psi = \bar{\gamma}(*da) \Psi + \rho(d^*a) \Psi - 2 \sum_{i=1}^3 \rho(a_i) \nabla_i \Psi,$$

we have

$$\begin{aligned} \delta &= \int_M \langle \mathbb{D}_A \bar{\gamma}(a) \Psi, \bar{\gamma}(a) \Psi \rangle \\ &= \int_M \langle \bar{\gamma}(*da) \Psi, \bar{\gamma}(a) \Psi \rangle + \langle \rho(d^*a) \Psi, \bar{\gamma}(a) \Psi \rangle \\ &\quad - 2 \sum_{i=1}^3 \langle \rho(a_i) \nabla_i \Psi, \bar{\gamma}(a) \Psi \rangle. \end{aligned}$$

The first term in the integral is

$$|\Psi|^2 \langle *da, a \rangle = \langle d * (|\Psi|^{-2} \eta), \eta \rangle.$$

The second term vanishes. Therefore,

$$\delta = \int_M |\Psi|^{-2} \langle d * \eta, \eta \rangle + |\Psi|^{-2} \langle \mathfrak{f}_1 \eta, \mathfrak{f}_2 \eta \rangle$$

for linear operators  $\mathfrak{f}_1, \mathfrak{f}_2: \Omega^2(M, i\mathbf{R}) \rightarrow \Gamma(\text{Re}(E \otimes S_s))$  of order zero. Set  $\mathfrak{f}_{\Psi, \mathbf{p}} = \mathfrak{f}_2^* \mathfrak{f}_1$ .  $\square$

**Proposition 8.6.** For  $(\Psi, \mathbf{p}, \eta) \in \Gamma(\text{Re}(E \otimes S_s)) \times \mathcal{P} \times \mathcal{L}$  such that  $\Psi$  is nowhere vanishing, define  $\delta(\Psi, \mathbf{p}, \eta)$  by formula (8.5). For each  $(\Psi, \mathbf{p})$ , the set

$$\mathcal{L}^{\text{reg}}(\Psi, \mathbf{p}) = \{\eta \in \mathcal{L} : \delta(\Psi, \mathbf{p}, \eta) \neq 0\}$$

is open and dense in  $\mathcal{L}$ .

*Proof.* Replace all the spaces in question by their completions with respect to the  $W^{k,p}$  norm for any  $k$  and  $p$  satisfying  $(k-1)p > 3$ . We will prove the statement with respect to the Sobolev topology; the corresponding statement for  $C^\infty$  spaces follows then from the Sobolev embedding theorem and the fact that  $\delta$  is continuous with respect to any of these topologies.

By Proposition 8.4,

$$(\mathrm{d}\delta)_\eta[\hat{\eta}] = \int_M |\Psi|^{-2} \langle 2\mathrm{d} * \eta + \bar{f}_{\Psi,p} \eta, \hat{\eta} \rangle,$$

where

$$\bar{f}_{\Psi,p} \eta = (f_{\Psi,p} + f_{\Psi,p}^*) \eta - 2\mathrm{d}(\log|\Psi|) \wedge * \eta$$

is a linear algebraic operator. Thus, the derivative of  $\delta$  vanishes along the set  $\mathcal{X}_{\mathrm{crit}}(\Psi, \mathbf{p})$  of solutions  $\eta$  to the linear elliptic differential equation

$$\begin{aligned} \mathrm{d}\eta &= 0, \\ \mathrm{d} * \eta + * \bar{f}_{\Psi,p} \eta &= 0. \end{aligned}$$

$\mathcal{X}_{\mathrm{crit}}(\Psi, \mathbf{p})$  a closed, finite-dimensional subspace of  $\mathcal{X}$ . By the Implicit Function Theorem, away from  $\mathcal{X}_{\mathrm{crit}}(\Psi, \mathbf{p})$ , the zero set of  $\delta(\Psi, \mathbf{p}, \cdot)$  is a codimension one Banach submanifold of (the Sobolev completion of)  $\mathcal{X}$ . Hence, the set

$$\mathcal{X}^{\mathrm{reg}}(\Psi, \eta) \cap (\mathcal{X} \setminus \mathcal{X}_{\mathrm{crit}}(\Psi, \eta))$$

is dense. Since  $\delta$  is continuous,  $\mathcal{X}^{\mathrm{reg}}(\Psi, \eta)$  is open.  $\square$

*Proof of Proposition 4.2.* By [Doa17b, Theorem 2.34],  $\mathcal{Q}_3$ , the subspace of paths from  $(\mathbf{p}_0, \eta_0)$  to  $(\mathbf{p}_1, \eta_1)$  satisfying Definition 4.1(3), is residual.

Denote by  $\mathcal{Q}_{1,2}$  the space of paths from  $(\mathbf{p}_0, \eta_0)$  to  $(\mathbf{p}_1, \eta_1)$  satisfying conditions (1) and (2) in Definition 4.1. By Proposition 8.1,  $\mathcal{Q}_{1,2}$  is residual in the space of paths from  $(\mathbf{p}_0, \eta_0)$  to  $(\mathbf{p}_1, \eta_1)$ . Let  $\mathcal{Q}_{1,2,4} \subset \mathcal{Q}_{1,2}$  be the space of paths from  $(\mathbf{p}_0, \eta_0)$  to  $(\mathbf{p}_1, \eta_1)$  also satisfying Definition 4.1(4). Elementary arguments show that  $\mathcal{Q}_{1,2,4}$  is open in  $\mathcal{Q}_{1,2}$  and we will shortly prove that  $\mathcal{Q}_{1,2,4}$  is dense in  $\mathcal{Q}_{1,2}$ . A set which is open and dense in a residual set is itself residual. It follows that  $\mathcal{Q}_{1,2,4}$  is residual; hence, so is  $\mathcal{Q}^{\mathrm{reg}}((\mathbf{p}_0, \eta_0), (\mathbf{p}_1, \eta_1)) = \mathcal{Q}_{1,2,4} \cap \mathcal{Q}_3$ .

To prove that  $\mathcal{Q}_{1,2,4}$  is dense in  $\mathcal{Q}_{1,2}$ , suppose that  $(\mathbf{p}_t, \eta_t)_{t \in [0,1]} \in \mathcal{Q}_{1,2}$ . There are finitely many times  $0 < t_1 < \dots < t_n < 1$  for which the kernel of  $\mathcal{D}_{\mathbf{p}_{t_i}}$  is non-trivial. For  $i = 1, \dots, n$ , denote by  $\Psi_i$  a section spanning  $\ker \mathcal{D}_{\mathbf{p}_{t_i}}$ . By Proposition 8.6, for any  $\sigma > 0$ , there are closed forms  $\alpha_1, \dots, \alpha_n$  such that

$$\delta(\Psi_i, \mathbf{p}_{t_i}, \eta_{t_i} + \alpha_i) \neq 0 \quad \text{and} \quad \|\alpha_i\|_{L^\infty} \leq \sigma$$

for every  $i = 1, \dots, n$ . Let  $(\alpha_t)_{t \in [0,1]}$  be a path of closed forms such that  $\alpha_{t_i} = \alpha_i$  for  $i = 1, \dots, n$  and  $\|\alpha_t\|_{L^\infty} \leq \sigma$  for all  $t \in [0, 1]$ . The path  $(\mathbf{p}_t, \eta_t + \alpha_t)_{t \in [0,1]}$  satisfies conditions (1), and (2) in Definition 4.1 because these only depend on  $(\mathbf{p}_t)_{t \in [0,1]}$ . It also satisfies (4) by construction. We conclude that  $(\mathbf{p}_t, \eta_t + \alpha_t)_{t \in [0,1]} \in \mathcal{Q}_{1,2,4}$ . Since  $\sigma$  is arbitrary, it follows that  $\mathcal{Q}_{1,2,4}$  is dense in  $\mathcal{Q}_{1,2}$ .  $\square$

## 9 Proof of the existence of singular harmonic $Z_2$ spinors

In this section prove Theorem 1.6. We begin with defining the set  $\mathcal{W}_b$  appearing in its statement.

**Definition 9.1.** Given a spin structure  $\mathfrak{s}$ , set

$$\begin{aligned} \mathcal{W}^{\mathfrak{s}} &:= \left\{ \mathbf{p} \in \mathcal{P} : \dim \ker \mathcal{D}_{\mathbf{p}}^{\mathfrak{s}} > 0 \right\}, \\ \mathcal{W}_{1,\emptyset}^{\mathfrak{s}} &:= \left\{ \mathbf{p} \in \mathcal{P} : \ker \mathcal{D}_{\mathbf{p}}^{\mathfrak{s}} = \mathbf{R}\langle \Psi \rangle \text{ with } \Psi \text{ nowhere vanishing} \right\}, \\ \mathcal{W}_{1,\star}^{\mathfrak{s}} &:= \left\{ \mathbf{p} \in \mathcal{P} : \ker \mathcal{D}_{\mathbf{p}}^{\mathfrak{s}} = \mathbf{R}\langle \Psi \rangle \text{ and } \Psi \text{ has a single non-degenerate zero} \right\} \quad \text{and} \\ \mathcal{W}_b^{\mathfrak{s}} &:= \mathcal{W}^{\mathfrak{s}} \setminus \mathcal{W}_{1,\emptyset}^{\mathfrak{s}}. \end{aligned}$$

A zero  $x \in \Psi^{-1}(0)$  is non-degenerate if the linear map  $(\nabla \Psi)_x : T_x M \rightarrow \text{Re}(E \otimes S)_x$  has maximal rank (that is, rank three). Set

$$\mathcal{W} := \bigcup_{\mathfrak{s}} \mathcal{W}^{\mathfrak{s}}, \quad \mathcal{W}_b := \bigcup_{\mathfrak{s}} \mathcal{W}_b^{\mathfrak{s}}, \quad \text{and} \quad \mathcal{W}_{1,\emptyset} := \bigcup_{\mathfrak{s}} \mathcal{W}_{1,\emptyset}^{\mathfrak{s}} \setminus \mathcal{W}_b^{\mathfrak{s}}.$$

Here the union is taken over all spin structures  $\mathfrak{s}$ .

*Remark 9.2.* As we will see shortly,  $\mathcal{W}^{\mathfrak{s}}$  is a codimension one stratified space with smooth strata  $\mathcal{W}_1^{\mathfrak{s}}, \mathcal{W}_2^{\mathfrak{s}}, \dots$  corresponding to the dimension of the kernel of  $\mathcal{D}_{\mathbf{p}}^{\mathfrak{s}}$ . The top stratum  $\mathcal{W}_1^{\mathfrak{s}}$  contains an open and dense subset  $\mathcal{W}_{1,\emptyset}^{\mathfrak{s}}$  for which the harmonic spinor spanning the kernel satisfies  $|\Psi|^{-1} = \emptyset$ . The rest of  $\mathcal{W}_1^{\mathfrak{s}}$  stratified according to the size of the zero set of  $\Psi$ .

We think of  $\mathcal{W}_b^{\mathfrak{s}}$  as the set of particularly degenerate (“bad”) parameters. It is stratified and its top stratum is  $\mathcal{W}_{1,\star}^{\mathfrak{s}}$ , which has codimension two in  $\mathcal{P}$ .

**Proposition 9.3.**  $\mathcal{W}_{1,\emptyset}^{\mathfrak{s}}$  is a closed, codimension one submanifold of  $\mathcal{P} \setminus \mathcal{W}_b^{\mathfrak{s}}$ . It carries a coorientation such that the following holds. Let  $(\mathbf{p}_t)$  be a path in  $\mathcal{P} \setminus \mathcal{W}_b^{\mathfrak{s}}$  with  $\mathbf{p}_0, \mathbf{p}_1 \in \mathcal{P} \setminus \mathcal{W}^{\mathfrak{s}}$  which intersects  $\mathcal{W}_{1,\emptyset}^{\mathfrak{s}}$  transversely. Denote

- by  $\{t_1, \dots, t_N\} \subset [0, 1]$  the finite set of times at which the spectrum of  $\mathcal{D}_{\mathbf{p}_t}^{\mathfrak{s}}$  crosses zero, i.e.,  $\mathbf{p}_t \in \mathcal{W}_{1,\emptyset}^{\mathfrak{s}}$

and, for each  $i = 1, \dots, N$ , denote

- by  $\chi_i \in \{\pm 1\}$  the sign of the spectral crossing at  $t_i$  and
- by  $\Psi_i$  a nowhere vanishing spinor spanning  $\ker \mathcal{D}_{\mathbf{p}_{t_i}}$ .

The intersection number of  $(\mathbf{p}_t)$  with  $\mathcal{W}_{1,\emptyset}^s$  is

$$\sum_{i=1}^N \chi_i \cdot \sigma(\Psi_i, \mathbf{p}_{t_i}).$$

*Proof.* Let  $\mathbf{p}_0 \in \mathcal{W}_{1,\emptyset}^s$ . Let  $\Psi_0 \in \ker \mathcal{D}_{\mathbf{p}_0}^s$  be such that  $\|\Psi_0\|_{L^2} = 1$ . It follows from the Implicit Function Theorem that, for some open neighborhood  $U$  of  $\mathbf{p}_0 \in \mathcal{P}$ , there is a unique smooth map  $U \rightarrow \mathbf{R} \times \Gamma(\text{Re}(E \otimes S))$ ,

$$\mathbf{p} \mapsto (\lambda(\mathbf{p}), \Psi_{\mathbf{p}})$$

such that

$$\lambda(\mathbf{p}_0) = 0 \quad \text{and} \quad \Psi_{\mathbf{p}_0} = \Psi_0$$

as well as

$$\mathcal{D}_{\mathbf{p}}^s \Psi_{\mathbf{p}} = \lambda(\mathbf{p}) \Psi_{\mathbf{p}} \quad \text{and} \quad \|\Psi_0\|_{L^2} = 1.$$

A moment's thought shows that if  $U$  is sufficiently small, then

$$U \cap \mathcal{W}_{1,\emptyset}^s = \lambda^{-1}(0).$$

Since

$$(d_{\mathbf{p}_0} \lambda)(0, b) = \langle \gamma(b) \Psi, \Psi \rangle_{L^2}$$

and Clifford multiplication induces an isomorphism from  $T^*M \otimes \mathfrak{su}(E)$  to trace-free symmetric endomorphisms of  $\text{Re}(E \otimes S)$ ,  $\lambda$  is a submersion provided  $U$  is sufficiently small. Hence,  $\mathcal{W}_{1,\emptyset}^s$  is a codimension one submanifold. To see that  $\mathcal{W}_{1,\emptyset}^s$  is closed, observe that  $(\mathbf{p}_i)$  is a sequence on  $\mathcal{W}_{1,\emptyset}^s$  with  $\mathbf{p}_i \rightarrow \mathbf{p} \in \mathcal{P}$ , then  $\mathbf{p} \in \mathcal{W}^s$  and thus either in  $\mathcal{W}_{1,\emptyset}^s$  or  $\mathcal{W}_b^s$ .

The above argument goes through with

$$\mathcal{W}_1^s = \{\mathbf{p} \in \mathcal{P} : \dim \ker \mathcal{D}_{\mathbf{p}}^s = 1\}$$

instead of  $\mathcal{W}_{1,\emptyset}^s$ . This induces a natural coorientation on  $\mathcal{W}_1^s$  by demanding that

$$d_{\mathbf{p}} \lambda: T_{\mathbf{p}} \mathcal{P} / T_{\mathbf{p}} \mathcal{W}_1^s \rightarrow \mathbf{R}$$

is orientation-preserving. If  $(\mathbf{p}_t)_{t \in [0,1]}$  is a path in  $\mathcal{P}$  such that  $\dim \ker \mathcal{D}_{\mathbf{p}_t}^s \leq 1$  and  $\dim \ker \mathcal{D}_{\mathbf{p}_t}^s = 0$  for  $t = 0, 1$ , then the intersection number of  $(\mathbf{p}_t)_{t \in [0,1]}$  with

$\mathcal{W}_1^s$  is precisely the spectral flow of  $\mathbb{D}_{\mathbf{p}_t}^s$ . The coorientation on  $\mathcal{W}_{1,\emptyset}^s$  is obtained by demanding that

$$\sigma(\Psi_{\mathbf{p}, \mathbf{p}}) \cdot d_{\mathbf{p}}\lambda : T_{\mathbf{p}}\mathcal{P} / T_{\mathbf{p}}\mathcal{W}_{1,\emptyset}^s \rightarrow \mathbf{R}$$

is orientation-preserving. This coorientation clearly has the asserted property.  $\square$

**Theorem 9.4.** *In the above situation, the following hold.*

1. *The cohomology class  $\omega \in H^1(\mathcal{P} \setminus \mathcal{W}_b, \mathbf{Z}) = \text{Hom}(\pi_1(\mathcal{P} \setminus \mathcal{W}_b), \mathbf{Z})$  defined by  $\mathcal{W}_{1,\emptyset}$  together with its natural coorientation from Proposition 9.3 is non-trivial.*
2. *If  $(\mathbf{p}_0, \eta_0) \in \mathcal{Q}^{\text{reg}}$  and  $(\mathbf{p}_t, \eta_t)$  is a loop in  $\mathcal{Q}^{\text{reg}}((\mathbf{p}_0, \eta_0), (\mathbf{p}_0, \eta_0))$ , then  $(\mathbf{p}_t)$  is a path in  $\mathcal{P} \setminus \mathcal{W}_b$  and if  $\omega([\mathbf{p}_t]) \neq 0$ , then there is exists a harmonic  $\mathbf{Z}_2$  spinor with respect to some  $\mathbf{p}_t$ .*

*Proof of Theorem 1.6.* Since the union of the projections of  $\mathcal{Q}^{\text{reg}}((\mathbf{p}_0, \eta_0), (\mathbf{p}_0, \eta_0))$  to  $\mathcal{P}(\mathbf{p}_0, \mathbf{p}_0)$  (as  $(\mathbf{p}_0, \eta_0)$  ranges over  $\mathcal{Q}^{\text{reg}}$ ) is a residual subset of the space of all loops in  $\mathcal{P}$ , Theorem 1.6 follows from Theorem 9.4.  $\square$

The idea of the proof that  $\omega \neq 0$  is to exhibit a loop  $(\mathbf{p}_t)$  in  $\mathcal{P} \setminus \mathcal{W}_b$  on which  $\omega$  evaluates non-trivially. More precisely, we will construct such a loop which intersects  $\mathcal{W}_{1,\emptyset}^s$  in two points as illustrated in Figure 1, which cannot be joined by a path in  $\mathcal{W}_{1,\emptyset}^s$ ; however, they are joined by a path in  $\mathcal{W}_1^s$  passing through  $\mathcal{W}_{1,\star}^s$  in a unique point.

Note that  $\mathcal{W}_{1,\star}^s$  is a subset of the bad set  $\mathcal{W}_b$  and we will prove that it is a codimension two submanifold of  $\mathcal{P}$ . While the coorientation on  $\mathcal{W}_1^s$  is preserved along this path, the one on  $\mathcal{W}_{1,\emptyset}^s$  is not. Consequently, the intersection number of the loop with  $\mathcal{W}_{1,\emptyset}^s$  is  $\pm 2$ . The above situation can be arranged so that  $(\mathbf{p}_t)$  does not intersect  $\mathcal{W}_{1,\emptyset}^s$  for any other spin structure  $\tilde{s}$ . It follows that

$$\omega([\mathbf{p}_t]) \pm 2 \neq 0.$$

If  $(\mathbf{p}_0, \eta_0) \in \mathcal{Q}^{\text{reg}}$  and  $(\mathbf{p}_t, \eta_t)$  is a loop in  $\mathcal{Q}^{\text{reg}}((\mathbf{p}_0, \eta_0), (\mathbf{p}_0, \eta_0))$  and  $\omega([\mathbf{p}_t]) \neq 0$ , then there is exists a singular harmonic  $\mathbf{Z}_2$  harmonic spinor with respect to some  $\mathbf{p}_t$  for otherwise

$$\omega([\mathbf{p}_t]) = 0$$

by Theorem 4.9.

*Remark 9.5.* This and the work of Takahashi [Tak15; Tak17] indicate the presence of a wall  $\mathcal{W}_{\mathbf{Z}_2} \subset \mathcal{P}$  caused by singular Fueter sections as depicted in Figure 1. In light of the above discussion it is a tantalizing question to ask:

Can the harmonic  $\mathbf{Z}_2$  spinors, whose abstract existence is guaranteed by Theorem 9.4, be constructed more directly by a gluing construction?

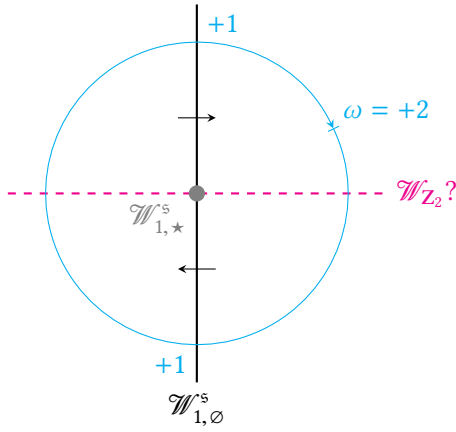


Figure 1: A loop linking  $\mathcal{W}_{1,\star}^s$  and pairing non-trivially with  $\omega$ .

We plan to investigate this problem in future work.

*Proof of Theorem 9.4.* To prove (2), note that if  $(\mathbf{p}_0, \eta_0) \in \mathcal{Q}^{\text{reg}}$  and  $(\mathbf{p}_t, \eta_t)$  is a loop in  $\mathcal{Q}^{\text{reg}}((\mathbf{p}_0, \eta_0), (\mathbf{p}_0, \eta_0))$  and  $\omega([\mathbf{p}_t]) \neq 0$ , then there exists a singular harmonic  $Z_2$  harmonic spinor with respect to some  $\mathbf{p}_t$  for otherwise

$$n(\mathbf{p}_0, \eta_0) = n(\mathbf{p}_0, \eta_0) + \omega([\mathbf{p}_t])$$

by Theorem 4.9. Here

$$n(\mathbf{p}, \eta) = \sum_{\mathfrak{w}} n_{\mathfrak{w}}(\mathbf{p}, \eta)$$

and we sum over all spin<sup>c</sup> structures  $\mathfrak{w}$  with trivial determinant.

In order to prove (1) we will produce a loop pairing non-trivially with  $\omega$ . The existence of such a loop is ensured by the following result provided we can exhibit a point  $\mathbf{p}_{\star} \in \mathcal{W}_{1,\star}^s$ .

**Proposition 9.6.** *Given  $\mathbf{p}_{\star} \in \mathcal{W}_{1,\star}^s$  and an open neighborhood  $U$  of  $\mathbf{p}_{\star} \in \mathcal{P}$ , there exists loop  $(\mathbf{p}_t)_{t \in S^1}$  in  $U \cap (\mathcal{P} \setminus \mathcal{W}_b^s)$  such that:*

1.  $\mathbf{p}_{1/4}, \mathbf{p}_{3/4} \in \mathcal{W}_{1,\emptyset}^s$ , and  $\mathbf{p}_t \in \mathcal{P} \setminus \mathcal{W}_1^s$  for all  $t \notin \{1/4, 3/4\}$ ,
2. if  $\Psi_{1/4}$  and  $\Psi_{3/4}$  denote spinors spanning  $\mathcal{D}_{\mathbf{p}_{1/4}}^s$  and  $\mathcal{D}_{\mathbf{p}_{3/4}}^s$ , then

$$\deg(\Psi_{1/4}, \Psi_{3/4}) = \pm 1;$$

and

3. the spectral crossings at  $\mathbf{p}_{1/4}$  and  $\mathbf{p}_{3/4}$  occur with opposite signs.

In particular, the intersection number of  $(\mathbf{p}_t)_{t \in [0,1]}$  with  $\mathcal{W}_{1,\emptyset}^s$  is  $\pm 2$ .

*Proof.* Let  $\Psi_\star \in \ker \mathcal{D}_{\mathbf{p}_\star}^s$  and  $x_\star \in M$  be such that  $\|\Psi_\star\|_{L^2} = 1$  and  $\Psi_\star(x_\star) = 0$ . Let  $\phi \in \Gamma(\text{Re}(E \otimes S_s))$  be such that

$$\text{im}((\nabla \Psi_\star)_{x_\star}) + \mathbf{R}\langle \phi_\star(x_\star) \rangle = \text{Re}(E \otimes S_s)_{x_\star} \quad \text{and} \quad |\phi_\star(x_\star)| = 1.$$

We can assume that  $U$  is sufficiently small for the Implicit Function Theorem to guarantee that there is a unique smooth map  $U \rightarrow \mathbf{R} \times \Gamma(\text{Re}(E \otimes S_s)) \times M \times \mathbf{R}$ ,

$$\mathbf{p} \mapsto (\lambda(\mathbf{p}), \Psi_{\mathbf{p}}, x_{\mathbf{p}}, \nu(\mathbf{p}))$$

such that

$$\lambda(\mathbf{p}_\star) = 0, \quad \Psi_{\mathbf{p}_\star} = \Psi_\star, \quad \text{and} \quad x_{\mathbf{p}_\star} = x_\star$$

as well as

$$(9.7) \quad \mathcal{D}_{\mathbf{p}}^s \Psi_{\mathbf{p}} = \lambda(\mathbf{p}) \Psi_{\mathbf{p}} \quad \Psi_{\mathbf{p}}(x_{\mathbf{p}}) = \nu(x_{\mathbf{p}}) \phi_\star(x_{\mathbf{p}}), \quad \text{and} \quad \|\Psi_0\|_{L^2} = 1.$$

As before

$$U \cap \mathcal{W}_1^s = \lambda^{-1}(0).$$

Set

$$\mathcal{N} := \nu^{-1}(0).$$

This is the set of those  $\mathbf{p} \in U$  for which the eigenspinor with smallest eigenvalue has a unique zero which is also non-degenerate.

From the proof of Proposition 9.3 we know that  $U \cap \mathcal{W}_1^s$  is a codimension one submanifold. We will now show that  $\mathcal{N}$  is a codimension one submanifold as well and that it intersects  $U \cap \mathcal{W}_1^s$  transversely in  $U \cap \mathcal{W}_{1,\star}^s$ , see Figure 2. Knowing this, the existence a loop  $(\mathbf{p})_{t \in S^1}$  with the desired properties follows easily because crossing  $\mathcal{N}$  changes the relative degree by  $\pm 1$ .

We will show that  $d_{\mathbf{p}_\star} \nu|_{T_{\mathbf{p}_\star} \mathcal{W}_1^s}$  is non-vanishing. This implies both that  $\mathcal{N}$  is a codimension one manifold and that it intersects  $U \cap \mathcal{W}_1^s$  transversely. For  $\hat{\mathbf{p}} = (0, b) \in \mathcal{T}_{\mathbf{p}_\star} \mathcal{P}$  to be determined, consider  $\mathbf{p}_\star + t\hat{\mathbf{p}}$  and set

$$\lambda_t = \lambda(\mathbf{p}_\star + t\hat{\mathbf{p}}), \quad \Psi_t = \Psi_{\mathbf{p}_\star + t\hat{\mathbf{p}}}, \quad x_t = x_{\mathbf{p}_\star + t\hat{\mathbf{p}}}, \quad \text{and} \quad \nu_t = \nu_{\mathbf{p}_\star + t\hat{\mathbf{p}}}.$$

as well as

$$\dot{\lambda} = \left. \frac{d}{dt} \right|_{t=0} \lambda_t, \quad \dot{\Psi} = \left. \frac{d}{dt} \right|_{t=0} \Psi_t, \quad \dot{x} = \left. \frac{d}{dt} \right|_{t=0} x_t, \quad \text{and} \quad \dot{\nu} = \left. \frac{d}{dt} \right|_{t=0} \nu_t.$$

Differentiating (9.7) we obtain

$$\bar{y}(b) \Psi_\star + \mathcal{D}_{\mathbf{p}_\star}^s \dot{\Psi} = \dot{\lambda} \Psi_\star, \quad \dot{\Psi}(x_\star) + (\nabla \Psi_\star)_{x_\star} \dot{x} = \dot{\nu} \phi(x_\star), \quad \text{and} \quad \langle \Psi_\star, \dot{\Psi} \rangle_{L^2} = 0.$$

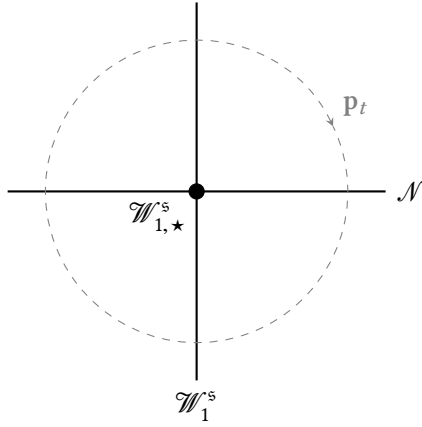


Figure 2:  $\mathcal{W}_1^s$  and  $\mathcal{N}$  intersecting in  $\mathcal{W}_{1, \star}^s$ .

From this it follows that

$$(9.8) \quad \begin{aligned} \dot{\lambda} &= \langle \gamma(b)\Psi_\star, \Psi_\star \rangle_{L^2}, \\ \mathcal{D}_{p_\star}^s \dot{\Psi} &= \gamma(b)\Psi_\star - \langle \gamma(b)\Psi_\star, \Psi_\star \rangle_{L^2} \Psi_\star, \quad \text{and} \\ \dot{v} &= \langle \dot{\Psi}(x_\star), \phi(x_\star) \rangle. \end{aligned}$$

Now, choose  $\dot{\Psi}$  such that:

1.  $\mathcal{D}_{p_\star}^s \dot{\Psi}$  vanishes in a neighborhood of  $x_\star$ .
2.  $\langle \dot{\Psi}(x_\star), \phi(x_\star) \rangle \neq 0$ , and
3.  $\langle \Psi_\star, \dot{\Psi} \rangle_{L^2} = 0$ .

Such a section can be defined by solving the Dirac equation in a neighborhood of  $x_\star$  subject to the constraint (2) and then extending to all of  $M$  so that (3) holds.

Clifford multiplication by  $T^*M \otimes \mathfrak{su}(E)$  on  $\text{Re}(E \otimes S_5)$  induces an isomorphism between  $T^*M \otimes \mathfrak{su}(E)$  and trace-free symmetric endomorphisms of  $\text{Re}(E \otimes S_5)$ . Since  $\mathcal{D}_{p_\star}^s \dot{\Psi}$  vanishes in a neighborhood of  $x_\star$  and  $\Psi_\star$  vanishes only at  $x_\star$ , one can find  $b \in \Omega^1(M, \mathfrak{su}(E))$  such that

$$\langle \gamma(b)\Psi_\star, \Psi_\star \rangle_{L^2} = 0 \quad \text{and} \quad \gamma(b)\Psi_\star = \mathcal{D}_{p_\star}^s \dot{\Psi}.$$

It follows from (9.8), that for this choice of  $b$  we have

$$\dot{\lambda} = 0 \quad \text{but} \quad \dot{v} \neq 0;$$

that is

$$\hat{p} = (0, b) \in T_{p_\star} \mathcal{W}_1^s \quad \text{and} \quad d_{p_\star} v(b) \neq 0.$$

This completes the proof. □



It remains to exhibit a point  $\mathbf{p}_\star \in \mathcal{W}_{1,\star}^{\mathfrak{s}}$  for some spin structure  $\mathfrak{s}$  but such that  $\mathbf{p}_\star \notin \mathcal{W}^{\tilde{\mathfrak{s}}}$  for any other spin structure  $\tilde{\mathfrak{s}}$ . This requires the following two propositions as preparation.

**Proposition 9.9.** *The subset*

$$\mathcal{W}_k^{\mathfrak{s}} := \{\mathbf{p} \in \mathcal{P} : \dim \ker \mathcal{D}_{\mathbf{p}}^{\mathfrak{s}} = k\} \subset \mathcal{P}$$

*is a smooth submanifold of codimension  $k(k+1)/2$ . Moreover,  $\mathcal{W}_k^{\mathfrak{s}} \cap \overline{\mathcal{W}_1^{\tilde{\mathfrak{s}}}} = \mathcal{W}_k^{\mathfrak{s}}$ .*

*Proof.* Let  $\mathbf{p}_0 \in \mathcal{P}$  such that  $\dim \ker \mathcal{D}_{\mathbf{p}_0}^{\mathfrak{s}} = k$ . Choose an  $L^2$ -orthonormal basis  $\{\Psi_i\}$  of  $\ker \mathcal{D}_{\mathbf{p}_0}^{\mathfrak{s}}$ . For a sufficiently small neighborhood  $U$  of  $\mathbf{p}_0$ , by the Implicit Function Theorem, there exists a unique smooth map  $U \rightarrow \Gamma(\operatorname{Re}(E \otimes S_{\mathfrak{s}}))^{\oplus k} \times S^2\mathbf{R}^k$

$$\mathbf{p} \mapsto (\Psi_{1,\mathbf{p}}, \dots, \Psi_{k,\mathbf{p}}, \Lambda(\mathbf{p}) = (\lambda_{ij}(\mathbf{p})))$$

such that

$$\Psi_{i,\mathbf{p}_0} = \Psi_i \quad \text{and} \quad \Lambda(\mathbf{p}) = 0$$

as well as

$$\mathcal{D}_{\mathbf{p}}^{\mathfrak{s}} \Psi_{i,\mathbf{p}} = \sum_{j=1}^k \lambda_{ij} \Psi_{j,\mathbf{p}} \quad \text{and} \quad \langle \Psi_{i,\mathbf{p}}, \Psi_{j,\mathbf{p}} \rangle_{L^2} = \delta_{ij}.$$

(It follows from the fact that  $\mathcal{D}_{\mathbf{p}}^{\mathfrak{s}}$  is symmetric, that  $\lambda_{ij} = \lambda_{ji}$ .)

We have

$$U \cap \mathcal{W}_k^{\mathfrak{s}} = \Lambda^{-1}(0) \quad \text{and} \quad U \cap \mathcal{W}_1^{\mathfrak{s}} = \Lambda^{-1}(S_{n-1})$$

with  $S_1 \subset S^2\mathbf{R}^n$  denoting the set of symmetric matrices with rank  $n-1$ . Using

$$(\mathbf{d}_{\mathbf{p}_0} \lambda_{ij})(0, b) = \langle \gamma(b) \Psi_i, \Psi_j \rangle_{L^2}$$

and the fact that Clifford multiplication induces an isomorphism from  $T^*M \otimes \mathfrak{su}(E)$  to trace-free symmetric endomorphisms of  $\operatorname{Re}(E \otimes S_{\mathfrak{s}})$ , one can show that  $\mathbf{d}\Lambda$  is a submersion  $U \rightarrow S^2\mathbf{R}^k$ . This implies that  $\mathcal{W}_k^{\mathfrak{s}}$  is a submanifold of codimension  $\dim S^2\mathbf{R}^k = k(k+1)/2$ . That  $\mathcal{W}_k^{\mathfrak{s}} \cap \overline{\mathcal{W}_1^{\tilde{\mathfrak{s}}}} = \mathcal{W}_k^{\mathfrak{s}}$  follows from the fact that  $0 \in \overline{S_{n-1}}$ .  $\square$

**Proposition 9.10.** *If  $\mathfrak{s}_1, \mathfrak{s}_2$  are two distinct spin structures, then  $\mathcal{W}_1^{\mathfrak{s}_1}$  and  $\mathcal{W}_1^{\mathfrak{s}_2}$  intersect transversely.*

*Proof.* Let  $\mathbf{p} \in \mathcal{W}_1^{\mathfrak{s}_1} \cap \mathcal{W}_1^{\mathfrak{s}_2}$ . Denote by  $\Psi_1$  and  $\Psi_2$  spinors spanning  $\ker \mathcal{D}_{\mathbf{p}}^{\mathfrak{s}_1}$  and  $\ker \mathcal{D}_{\mathbf{p}}^{\mathfrak{s}_2}$  respectively. The spin structures  $\mathfrak{s}_1$  and  $\mathfrak{s}_2$  differ by twisting by a  $\mathbf{Z}_2$ -bundle  $\mathbb{I}$ . This bundle corresponds to a double cover  $\pi: \tilde{M} \rightarrow M$  and upon pulling back to the cover the spin structures  $\mathfrak{s}_1$  and  $\mathfrak{s}_2$  both correspond to the same spin structure  $\tilde{\mathfrak{s}}$ .

The natural involution on  $\tilde{M}$  acts as  $-1$  on  $\tilde{\Psi}_1 = \pi^*\Psi_1$  and as  $+1$  on  $\tilde{\Psi}_2 = \pi^*\Psi_2$ . In particular,  $\tilde{\Psi}_1$  and  $\tilde{\Psi}_2$  are distinct.

Let  $\lambda^{s_1}$ ,  $\lambda^{s_2}$ , and  $\lambda^s$  be as in Proposition 9.3. A moment's thought shows that

$$d_{\Psi_i}\lambda^{s_i}(0, b) = \frac{1}{2}\langle \bar{\gamma}(\pi^*b)\tilde{\Psi}_i, \tilde{\Psi}_i \rangle_{L^2}.$$

With this in mind it is not difficult to find a  $b \in \Omega^1(M, \mathfrak{su}(E))$  such that  $d_{\Psi_i}\lambda^{s_1}(0, b) = 0$  but  $d_{\Psi_i}\lambda^{s_2}(0, b) \neq 0$ . Hence,  $\mathcal{W}_1^{s_1}$  and  $\mathcal{W}_1^{s_2}$  intersect transversely.  $\square$

Finally, we are in a position to construct  $\mathbf{p}_\star \in \mathcal{W}_{1,\star}^s$ . Fix a spin structure  $s$  as well as  $\mathbf{p}_0 = (g_0, B_0)$  and  $x_\star \in M$  such that  $g_0$  and  $B_0$  are flat on a small ball around a point  $x_\star \in M$ . Choose local coordinates  $(y_1, y_2, y_3)$  around  $x_\star$  and a local trivialization of  $\text{Re}(E \otimes S_s)$  in which  $g_0$  is given by the identity matrix and  $B_0$  is the trivial connection. Let  $\Psi \in \Gamma(\text{Re} \otimes S_s)$  be any section which is nowhere vanishing away from  $x_\star$  and around  $x_\star$  agrees with map  $\mathbf{R}^3 \rightarrow \mathbf{H}$  given by

$$(y_1, y_2, y_3) \mapsto 2iy_1 - jy_2 - ky_3.$$

In particular,  $\Psi$  has a single non-degenerate zero at  $x_\star$  and satisfies  $\mathcal{D}_{\mathbf{p}_0}^s \Psi = 0$  in a neighborhood of  $x_\star$ . Using the same argument as in the proof of Proposition 9.6, we find  $b \in \Omega^1(\mathfrak{su}(E))$  vanishing in a neighborhood of  $x_\star$  and such that for  $\mathbf{p}_\star = (g_0, B_0 + b)$  we have

$$0 = \mathcal{D}_{\mathbf{p}_\star}^s \Psi = \mathcal{D}_{\mathbf{p}_0}^s \Psi + \bar{\gamma}(b)\Psi.$$

This shows that  $\Psi$  is harmonic with respect to  $\mathbf{p}_\star$ . If  $\dim \ker \mathcal{D}_{\mathbf{p}_\star} > 1$ , then Proposition 9.9 and the argument from Proposition 9.6 can be used to slightly perturb  $\mathbf{p}_\star$  to arrange that  $\dim \ker \mathcal{D}_{\mathbf{p}_\star} = 1$  and any spinor spanning  $\mathcal{D}_{\mathbf{p}_\star}$  has a non-degenerate zero (close to  $x_\star$ ). Similarly, Proposition 9.9 and Proposition 9.10 can be used to ensure that there are no non-trivial harmonic spinors with respect to  $\mathbf{p}_\star$  for any other spin structure  $\tilde{s}$ .  $\square$

## A Computation of the hyperkähler quotient

Set

$$S := \text{Hom}_{\mathbf{C}}(\mathbf{C}^2, \mathbf{H})$$

with  $\mathbf{H}$  considered as a complex vector space whose complex structure is given by right-multiplication with  $i$ .  $S$  is a quaternionic Hermitian vector space: its  $\mathbf{H}$ -module structure arises by left-multiplication. The action of  $\text{U}(1)$  on  $S$  given by  $\rho(e^{i\theta})\Psi = e^{i\theta}\Psi$  is a quaternionic representation with associated moment map

$$\mu(\Psi) = \Psi\Psi^* - \frac{1}{2}|\Psi|^2 \text{id}_{\mathbf{H}}.$$

The standard complex volume form  $\Omega = e^1 \wedge e^2 \in \Lambda^2(\mathbb{C}^2)^*$  and the standard Hermitian metric on  $\mathbb{C}^2$ , define a complex anti-linear map  $J: \mathbb{C}^2 \rightarrow \mathbb{C}^2$  by

$$-\langle v, Jw \rangle = \Omega(v, w).$$

This makes  $\mathbb{C}^2$  into a  $\mathbf{H}$ -module.

**Proposition A.1.** *We have*

$$S^{\text{reg}} // \text{U}(1) = (\text{Re}(\mathbf{H} \otimes_{\mathbb{C}} \mathbb{C}^2) \setminus \{0\}) / \mathbb{Z}_2.$$

Here the real structure on  $\mathbf{H} \otimes_{\mathbb{C}} \mathbb{C}^2$  is given by  $\overline{q \otimes v} := jq \otimes Jv$ .

*Proof.* The complex volume form  $\Omega$  defines a complex linear isomorphism  $(\mathbb{C}^2)^* \cong \mathbb{C}^2$ ; hence, we can identify  $S \cong \mathbf{H} \otimes_{\mathbb{C}} \mathbb{C}^2$ . We will further identify  $\mathbb{C}^2$  with  $\mathbf{H}$  via  $(z, w) \mapsto z + wj$ . With respect to this identification the complex structure is given by left-multiplication with  $i$  and  $J$  becomes left-multiplication by  $j$ . If we denote by  $H_+$  ( $H_-$ ) the quaternions equipped with their right (left)  $\mathbf{H}$ -module structure, then we can identify  $S$  with

$$\mathbf{H}_+ \otimes_{\mathbb{C}} \mathbf{H}_-.$$

In this identification the action of  $\text{U}(1)$  is given by

$$\rho(e^{i\theta})q_+ \otimes q_- = q_+ e^{i\theta} \otimes q_- = q_+ \otimes e^{i\theta} q_-,$$

and the moment map becomes

$$\mu(q_1 \otimes 1 + q_2 \otimes j) = -i \otimes \frac{1}{2}(q_1 i \bar{q}_1 + q_2 i \bar{q}_2) \in i\mathbf{R} \otimes \text{Im } \mathbf{H}.$$

If  $\mu(q_1 \otimes 1 + q_2 \otimes j) = 0$ , then

$$(A.2) \quad q_1 i \bar{q}_1 = -q_2 i \bar{q}_2.$$

This implies that  $|q_1| = |q_2|$ . Unless  $q_1$  and  $q_2$  both vanish, there is a unique  $p \in \mathbf{H}$  satisfying

$$|p| = 1 \quad \text{and} \quad q_1 = q_2 p.$$

From (A.2), it follows that

$$pi = -ip;$$

hence,  $p = je^{i\phi}$  for some  $\phi \in \mathbf{R}$ . It follows that, for any  $\theta \in \mathbf{R}$ ,

$$q_1 e^{i\theta} = q_2 e^{i\theta} \cdot j e^{i(\phi+2\theta)}.$$

Since

$$\overline{q_1 \otimes 1 + q_2 \otimes j} = -q_2 j \otimes 1 + q_1 j \otimes j,$$

the real part of  $\mathbf{H}_+ \otimes_{\mathbb{C}} \mathbf{H}_-$  consists of those  $q_1 \otimes 1 + q_2 \otimes j$  with

$$q_1 = -q_2 j.$$

Consequently, the  $\text{U}(1)$ -orbit of each non-zero  $\mathbf{q} = q_1 \otimes 1 + q_2 \otimes j$  intersects  $\text{Re}(\mathbf{H}_+ \otimes_{\mathbb{C}} \mathbf{H}_-)$  twice: in  $\pm \rho(e^{i(-\phi/2+\pi/2)})\mathbf{q}$ .  $\square$

## B Relation to gauge theory on $G_2$ -manifolds

The discussion in the article is related to an observation made by Joyce [Joy17, Section 8.4] which points out potential issues with the Donaldson–Segal program for counting  $G_2$ -instantons. Donaldson and Segal [DS11, Section 6] proposed to compensate the jump in the number of  $G_2$ -instantons, which arises from bubbling along 3-dimensional associative submanifolds and is caused by the appearance of (nowhere vanishing) harmonic spinors of  $\text{Re}(E \otimes S)$ , with a counter-term consisting of a weighted count of associative submanifolds.

Joyce [Joy17, Example 8.5] poses the following scenario. Consider a one-parameter family of  $G_2$ -manifolds  $\{(Y, \phi_t) : t \in [0, 1]\}$  together with an  $\text{SU}(2)$ -bundle  $E$  where:

- there is a smooth family of irreducible connections  $(A_t)_{t \in [0, 1]} \in \mathcal{A}(E)^{[0, 1]}$  such that  $A_t$  is an unobstructed  $G_2$ -instantons with respect to  $\phi_t$  for each  $t \in [0, 1]$ ,
- there are no relevant associatives in  $(Y, \phi_t)$  for  $t \in [0, 1/3) \cup (2/3, 1]$ , and
- there are is an obstructed associative  $P_{1/3}$  in  $(Y, \phi_{1/3})$ , which splits into two unobstructed associatives  $P_t^\pm$  in  $(Y, \phi_t)$  for  $t \in (1/3, 2/3)$ , and which then annihilate each other in an obstructed associative  $P_{2/3}$  in  $(Y, \phi_{2/3})$ .

According to [Wal17, Theorem 1.2] a regular crossing of the spectral flow of family of Dirac operators  $\mathcal{D}_t^\pm : \Gamma(\text{Re}(E|_{P_t^\pm} \otimes \mathcal{S}_{P_t^\pm})) \rightarrow \Gamma(\text{Re}(E|_{P_t^\pm} \otimes \mathcal{S}_{P_t^\pm}))$  causes a jump in the signed count of  $G_2$ -instantons; however, the sign of this jump has not been analyzed.<sup>7</sup> Donaldson and Segal [DS11, Section 6] and Joyce [Joy17, Section 8.4] suggest that this is the only source of jumping phenomena. The difference in the spectral flows of Dirac operators  $\mathcal{D}_t^\pm$  on is a topological invariant, say  $k \in \mathbf{Z}$ , which may be non-zero. [Joy17, Section 8.4] thus concludes that passing from  $t < 1/3$  to  $t > 2/3$  the signed number of  $G_2$ -instantons should change by  $k \cdot |H_1(M, \mathbf{Z}_2)|$ ; and, since there are no associatives for  $t \in [0, 1/3) \cup (2/3, 1]$ , no counter-term involving a weighted count of associatives could compensate this jump.

It is proposed in [HW15] that the weight associated with each associative 3-manifold should be the signed count of solutions to the Seiberg–Witten equation with two spinors. The loop of associatives can equivalently be seen as a path of parameters  $(\mathbf{p}_t)_{t \in [0, 1]}$  on a fixed 3-manifold  $P$ , with  $\mathbf{p}_1$  gauge equivalent to  $\mathbf{p}_0$ . Therefore, one can ask how  $n(\mathbf{p}_t)$  varies in this scenario. Suppose that  $b_1(P_{1/3}) > 1$ . Assuming there are no harmonic  $\mathbf{Z}_2$  spinors along the path  $(\mathbf{p}_t)_{t \in [0, 1]}$ , a jump in  $n(\mathbf{p}_t)$  would occur precisely when the spectrum of one of the Dirac operators  $\mathcal{D}_{\mathbf{p}_t}^s$

<sup>7</sup>To be more precise, the jump occurs in the signed count of  $G_2$ -instantons on a bundle  $E'$ , which is related to  $E$  by  $c_2(E') = c_2(E) + \text{PD}[P]$  with  $[P] = [P_{1/3}] = [P_t^\pm] = [P_{2/3}]$ .

crosses zero. If the wall-crossing formula for  $n(\mathbf{p}_t)$  were given by the sum of the spectral flows of  $(\mathcal{D}_{\mathbf{p}_t}^s)_{t \in [0,1]}$ , then we would have arrived at a contradiction just like in Joyce’s argument:

$$0 \neq k \cdot |H_1(M, \mathbf{Z}_2)| = n(\mathbf{p}_1) - n(\mathbf{p}_0) = 0$$

since  $\mathbf{p}_1$  and  $\mathbf{p}_0$  are gauge equivalent. However, the conclusion of our work is that:

1. The wall-crossing for  $n(\mathbf{p}_t)$  caused by harmonic spinors is not given by the spectral flow.
2. There exist singular harmonic  $\mathbf{Z}_2$  spinors which cause additional wall-crossing.

It is possible that the same happens for the signed count of  $G_2$ -instantons. To evaluate the viability of the proposal in [HW15] it is important to answer the following questions.

**Question B.1.** What is the sign of the jump in the number of  $G_2$ -instantons caused by a harmonic spinor?

**Question B.2.** Do singular harmonic  $\mathbf{Z}_2$  spinors cause a jump in the number of (possibly singular)  $G_2$ -instantons?

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