The Gopakumar–Vafa finiteness conjecture

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Abstract

The Gopakumar–Vafa conjecture predicts that the BPS invariants of a symplectic 6–manifold, defined in terms of the Gromov–Witten invariants, are integers and all but finitely many vanish in every homology class. The integrality part of this conjecture was proved by Ionel and Parker [IP18]. This article proves the finiteness part. The proof relies on a modification of Ionel and Parker's cluster formalism using results from geometric measure theory.

Contents

1 Introduction

Using ideas from M–theory Gopakumar and Vafa [GV98a; GV98b] predicted that there exist integer invariants $\mathrm{BPS}_{A,\mathrm{g}}(X,\omega)$ associated with a closed symplectic 6–manifold (X,ω); a Calabi–Yau class A, that is: $A \in \mathrm{H}_2(X,\mathbb{Z})$ with $c_1(A) := \langle c_1(X,\omega), A\rangle = 0$; and $\mathrm{g} \in \mathbb{N}_0$.

These invariants are interpreted in physics as the count of BPS states supported on J–holomorphic curves representing A and of genus g. Gopakumar and Vafa conjectured that their invariants are related to the Gromov–Witten invariants $\mathrm{GW}_{A,\mathrm{g}}(X,\omega)$ by the marvelous formula

$$\sum_{A\in\Gamma}\sum_{\mathrm{g}=0}^{\infty}\mathrm{GW}_{A,\mathrm{g}}(X,\omega)\cdot t^{2\mathrm{g}-2}q^A=\sum_{A\in\Gamma}\sum_{\mathrm{g}=0}^{\infty}\mathrm{BPS}_{A,\mathrm{g}}(X,\omega)\cdot\sum_{k=1}^{\infty}\frac{1}{k}(2\sin(kt/2))^{2\mathrm{g}-2}q^{kA} \tag{1.1}$$

with $\Gamma := \{A\in \mathrm{H}_2(X,\mathbb{Z}) : A\neq 0, c_1(A)=0\}$; see [GV98b, (3.2)]. This formula is to be understood as an equality of formal power series in variables q^A whose coefficients are Laurent series in t.

Gopakumar and Vafa did not give a direct mathematical definition of their invariants. Indeed, despite valiant efforts—especially by algebraic geometers [HST01; PT09; PT10; KL12; MT18]—mathematicians still do not know how to define them directly. Turning the problem on its head and regarding (1.1) as the *definition* of $\mathrm{BPS}_{A,\mathrm{g}}(X,\omega)$ led to the following conjecture.

Conjecture 1.2 (The Gopakumar–Vafa conjecture [GV98a; GV98b; BP01, Conjecture 1.2]). *Let (X,ω) be a closed symplectic 6–manifold. For every $A\in \mathrm{H}_2(X,\mathbb{Z})$ with $A\neq 0$ and $c_1(A)=0$ the numbers $\mathrm{BPS}_{A,\mathrm{g}}(X,\omega)$ defined by (1.1) satisfy:*

- **(integrality)** $\mathrm{BPS}_{A,\mathrm{g}}(X,\omega)\in\mathbb{Z}$ *for every* $\mathrm{g}\in\mathbb{N}_0$.
- **(finiteness)** *There is* $\mathrm{g}_A\in\mathbb{N}_0$ *such that* $\mathrm{BPS}_{A,\mathrm{g}}(X,\omega)=0$ *for every* $\mathrm{g}\geqslant \mathrm{g}_A$. •

Remark 1.3. The integrality part of Conjecture 1.2 was proved by Ionel and Parker [IP18]. •

Remark 1.4. There is an analogue of Conjecture 1.2 for Fano classes; that is: $A\in \mathrm{H}_2(X,\mathbb{Z})$ with $c_1(A)>0$; see Appendix A. This case is significantly easier because multiple covers can be avoided. Zinger [Zin11, Theorem 1.5] has proved integrality for Fano classes. Doan and Walpuski [DW19, Corollary 1.18] have proved finiteness for Fano classes and primitive Calabi–Yau classes. •

Remark 1.5. The finiteness part of Conjecture 1.2 implies that the coefficients of q^A in the Gromov–Witten series (1.1) are t^{-2} times analytic functions of t and rational functions of $u=-e^{it}$; cf. [PT09, Conjectures 3.2, 3.3, and 3.28]. •

Question 1.6. The finiteness part of Conjecture 1.2 predicts that for every $A\in\Gamma$ the **BPS Castelnuovo number**

$$\gamma_A^{\mathrm{BPS}}(X,\omega) := \inf\{\mathrm{g}\in\mathbb{N}_0 : \mathrm{BPS}_{A,\mathrm{g}}(X,\omega)=0\}\in\mathbb{N}_0\cup\{\infty\}$$

is finite. This is an invariant of (X,ω). It is interesting to ask: *are there effective bounds on $\gamma_A^{\mathrm{BPS}}(X,\omega)$ analogous to Castelnuovo's bound for the genus of an irreducible degree d curve in* $\mathbb{P}^n$ [Cas89; ACGH85, Chapter III Section 2]?

The purpose of this article is to prove the above conjecture.

Theorem 1.7. *Conjecture 1.2 holds.*

The strategy of the proof is similar to that in [IP18]. The key concept in [IP18] is that of a cluster: it controls every J–holomorphic curve which is close to a given curve, called the core of the cluster, and whose area and genus are below a certain threshold. Truncations in q and t of the Gromov–Witten series $\mathrm{GW}(X,\omega)$ appearing in (1.1) can be decomposed into contributions of clusters. For special clusters these contributions can be determined using the work of Pandharipande [Pan99], Bryan and Pandharipande [BP08], Lee [Lee09], and Zinger [Zin11]. Moreover, both integrality *and finiteness* can be verified for these contributions. The contributions of general clusters agree with those of special clusters up to contributions of higher order clusters. This allows Ionel and Parker to recursively prove integrality, but not finiteness because of the truncation in t.

The truncation in t is necessary in [IP18] because their cluster formalism relies on Gromov's compactness theorem for J–holomorphic maps. The crucial insight of this article is that a variation of the cluster formalism can instead be based on the compactness theorem for J–holomorphic cycles [DW21, Proposition 1.9] and Allard's regularity theorem [All72]; see Section 2. This upgraded cluster formalism is developed in Section 3.1, Section 3.2, and Section 3.3. Once it is in place, Ionel and Parker's argument proves both integrality and finiteness; see Section 3.4 and Section 3.5.

For completeness' sake Appendix A summarises the work of Zinger [Zin11, Theorem 1.5] and Doan and Walpuski [DW19, Corollary 1.18] on the analogue of Conjecture 1.2 for Fano classes. The theory developed in Section 2 allows for an alternative proof of [DW19, Theorem 1.1] as well as a partial strengthening of [DW21, Theorem 1.6]. This is discussed in Remark A.6 and Appendix B.

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2 The space of pseudo-holomorphic cycles

Throughout this section, assume the following.

Situation 2.1. Let X be a smooth manifold. A **Hermitian structure** on X is a pair (J, g) consisting of an almost complex structure J and a Riemannian metric g with respect to which J is orthogonal. Let $\mathcal{H}$ be a topological space of Hermitian structures (J, g) on X which are at least C^3_{loc}. Suppose that the topology on $\mathcal{H}$ is metrizable and at least as fine as the C^3_{loc} topology. •

Example 2.2. If (X, ω) is a symplectic manifold, then there are two natural choices for $\mathcal{H}$:

(1) $\mathcal{J}(\omega)$, the space of almost complex structures J which are compatible with ω; that is: $g \coloneqq \omega(\cdot, J\cdot)$ defines a Riemannian metric.

(2) $\mathcal{J}_\tau(\omega)$, the space of almost complex structures J which are tamed by ω; that is: $g \coloneqq \frac{1}{2}(\omega(\cdot, J\cdot) - \omega(J\cdot, \cdot))$ defines a Riemannian metric.

In either case, J is orthogonal with respect to g. •

Denote by $\overline{\mathcal{M}}$ the space of pairs $(J, g; [u])$ consisting of $(J, g) \in \mathcal{H}$ and an equivalence class $[u]$ of stable nodal J–holomorphic maps. Denote by $\mathcal{M}^{\mathrm{si}}$ and $\mathcal{M}^{\mathrm{emb}}$ the subsets of those $(J, g; [u])$ with u being simple and an embedding respectively. Denote by $\mathrm{pr}_{\mathcal{H}} \colon \overline{\mathcal{M}} \to \mathcal{H}$ the canonical projection map and by $\mathrm{g} \colon \overline{\mathcal{M}} \to \mathbb{N}_0$ and $E \colon \overline{\mathcal{M}} \to \mathbb{N}_0$ the maps which assign to a nodal pseudo-holomorphic map its genus and energy. Gromov's compactness theorem asserts that if X is compact, then the map

$$(\mathrm{pr}_{\mathcal{H}}, \mathrm{g}, E)\colon \overline{\mathcal{M}} \to \mathcal{H} \times \mathbb{N}_0 \times [0,\infty)$$

is proper with respect to the Gromov topology [Gro85, §1; Hum97] . *The genus component is crucial*; indeed: the map $(\mathrm{pr}_{\mathcal{H}}, E)\colon \overline{\mathcal{M}} \to \mathcal{H} \times [0,\infty)$ fails to be proper. A trivial reason for the failure properness are ghosts components; that is: components of the domain of a nodal map on which the map is constant. Evidently, there are ghosts components of arbitrary genus. A more interesting reason for the failure properness are multiple covers. If $u\colon (\Sigma, j) \to (X, J)$ is a pseudo-holomorphic map and $\pi\colon (\tilde{\Sigma}, \tilde{j}) \to (\Sigma, j)$ is a branched cover, then $u \circ \pi$ is pseudo-holomorphic and $E(u \circ \pi) = \deg(\pi) \cdot E(u)$. Furthermore, for every $d \geqslant 2$ and $\mathrm{g}_0 \in \mathbb{N}$ there is a branched cover with $\deg(\pi) = d$ and $\mathrm{g}(\tilde{\Sigma}) \geqslant \mathrm{g}_0$. In either case, the unboundedness of the genus is not reflected in the subsets $\operatorname{im} u$ parametrized by $[u]$.

These issues can be partially resolved by considering pseudo-holomorphic cycles instead of pseudo-holomorphic maps. The purpose of this section is to summarize the salient parts of the theory of pseudo-holomorphic cycles and add to it a few observations, which might appear to be minor but are crucial for the proof of the Gopakumar–Vafa finiteness conjecture.

2.1 Definitions and results

Definition 2.3. Denote by $\mathcal{K}$ the set of compact subsets of X. For $(J, g) \in \mathcal{H}$ denote by $d\colon X \times X \to [0,\infty)$ the metric induced by g. The **Hausdorff metric** $d_H\colon \mathcal{K} \times \mathcal{K} \to [0,\infty]$ is defined by

$$d_H(A,B) \coloneqq \max\left\{\sup_{x\in A}\inf_{y\in B} d(x,y), \sup_{x\in B}\inf_{y\in A} d(x,y)\right\}. \qquad \bullet$$

Theorem 2.4 (Blaschke [Bla56; BBI01, Theorem 7.3.8]). *If (X, d) is compact, then so is $(\mathcal{K}, d_H)$.*

Remark 2.5. The topology induced by d_H depends only the topology of X. •

Definition 2.6.

(1) Let $(J, g) \in \mathcal{H}$. An **irreducible J–holomorphic curve** is a subset $C \subset X$ which is the image of a simple J–holomorphic map $u\colon (\Sigma, j) \to X$ from a connected, closed Riemann surface. A **J–holomorphic cycle** C is a formal finite sum

$$C = \sum_{i=1}^{I} m_i C_i \qquad (2.7)$$

of distinct irreducible J–holomorphic curves $C_1, \ldots, C_I$ with coefficients $m_1, \ldots, m_I \in \mathbb{N}$.

(2) Let $(J, g) \in \mathcal{H}$. Let C be a J–holomorphic cycle. The **support of** C and the **current associated with** C are the closed subset $\operatorname{supp} C$ and the linear map $\delta_C\colon \Omega^2_c(X) \to \mathbb{R}$ defined by

$$\operatorname{supp} C \coloneqq \bigcup_{i=1}^{I} C_i \quad \text{and} \quad \delta_C(\alpha) \coloneqq \sum_{i=1}^{I} m_i \int_{\Sigma_i} u_i^* \alpha.$$

(3) Denote by $\mathcal{Z}$ the sets of pairs consisting of an almost Hermitian structure $(J, g) \in \mathcal{H}$ and a J–holomorphic cycle C in X. The **geometric topology** on $\mathcal{Z}$ is the coarsest topology with respect to which the maps

$$\mathrm{pr}_{\mathcal{H}}\colon \mathcal{Z} \to \mathcal{H}, \quad \mathrm{supp}\colon \mathcal{Z} \to \mathcal{K}, \quad \text{and} \quad \delta\colon \mathcal{Z} \to \mathrm{Hom}(\Omega^2_c(X), \mathbb{R})$$

are continuous. Here $\mathrm{Hom}(\Omega^2_c(X), \mathbb{R})$ is equipped with the weak–$*$ topology.

(4) Let $(J, g; C) \in \mathcal{Z}$. The **mass of** C with respect to g is

$$\mathbf{M}(C) = \mathbf{M}_g(C) := \sum_{i=1}^{I} m_i \,\mathrm{area}_g(C_i).$$

The **homology class** of C is

$$[C] := \sum_{i=1}^{I} m_i [C_i] \quad \text{with} \quad [C_i] := (u_i)_*[\Sigma_i] \in \mathrm{H}_2(X, \mathbb{Z}).$$ •

Remark 2.8. Every $(J, g) \in \mathcal{H}$ defines a **Hermitian form** 2–form $\sigma(\cdot,\cdot) := g(J\cdot,\cdot)$. It defines a semi-calibration. If C is J–holomorphic, then δ_C is semi-calibrated by σ. For further details see, e.g., [DW21, Definition 4.8, Remark 4.9, Remark 5.1].

If (X, ω) is a symplectic manifold and $\mathcal{H} = \mathcal{J}(\omega)$ as in Example 2.2 (1), then $\sigma = \omega$. Therefore, it is a calibration. If $\mathcal{H} = \mathcal{J}_\tau(\omega)$ as in Example 2.2 (2), then $\sigma = \frac{1}{2}(\omega + \omega(J\cdot, J,\cdot))$ and need not be a calibration; nevertheless: $\mathbf{M}(C) = \langle [\omega], [C] \rangle$; cf. [MS12, Lemma 2.2.1]. Therefore, $[C]$ determines $\mathbf{M}(C)$. •

Remark 2.9. The notion of geometric convergence for pseudo-holomorphic cycles was introduced by Taubes [Tau96b]. •

In light of the following, arguments regarding the geometric topology on $\mathcal{Z}$ can be carried out using sequences.

Proposition 2.10. *$\mathcal{Z}$ is metrizable.*

Proof. The Banach space $B := C^0\Omega^2(X)$ is separable. The weak–$*$ topology on the dual Banach space $B^* := \mathcal{L}(B, \mathbb{R})$ is metrizable. Since $\Omega^2_c(X)$ is dense in B, the map $B^* \hookrightarrow \mathrm{Hom}(\Omega^2_c(X), \mathbb{R})$ is an embedding with respect to the weak–$*$ topologies. This implies the assertion. ■

The Federer–Fleming Compactness Theorem for integral currents [FF60; Whi89] and the regularity theory for 2–dimensional semi-calibrated integral currents developed by De Lellis, Spadaro, and Spolaor [DSS17a; DSS17b] lead to the following compactness theorem for pseudo-holomorphic cycles.

Theorem 2.11 ([DW21, Proposition 1.9]). *The map*

$$(\mathrm{pr}_{\mathcal{H}}, \mathrm{supp}, \mathbf{M})\colon \mathcal{Z} \to \mathcal{H} \times \mathcal{K} \times [0, \infty)$$

is continuous and proper. ■

Remark 2.12. If $\mathcal{H} = \mathcal{J}(\omega)$ is as in Example 2.2 (1), then the proof of Theorem 2.11 can be based—instead of [DSS17a; DSS17b]—on the earlier work of Rivière and Tian [RT09] and, in dimension four, on the seminal work of Taubes [Tau96a]. •

Remark 2.13. By Remark 2.8 in either case of Example 2.2 the map $\mathbf{M}\colon \mathcal{Z} \to [0, \infty)$ in Theorem 2.11 can be replaced by $[\cdot]\colon \mathcal{Z} \to \mathrm{H}_2(X, \mathbb{Z})$. •

To understand the relation between Gromov's compactness theorem and Theorem 2.11 it is enlightening to introduce the following map.

Definition 2.14. Define the map $\mathfrak{z}\colon \overline{\mathcal{M}} \to \mathcal{Z}$ by

$$\mathfrak{z}(J,[u]) := (J,C) \quad \text{with} \quad C := \sum_{i=1}^{I} \deg \pi_i \cdot \operatorname{im} v_i.$$

Here $[u_1],\dots,[u_I]$ denote the irreducible components of $[u]$ and, for every $i = 1,\dots,I$, $u_i = v_i \circ \pi_i$ with v_i a simple J-holomorphic map. •

The map $\mathfrak{z}$ is continuous, but it fails to be proper. The failure of properness again is due to ghosts components and branched covers with the same degree but different numbers of ramification points.

The following definitions and results concern certain important subsets of the space of cycles.

Definition 2.15. (1) Let $(J,g) \in \mathcal{H}$. A **J–holomorphic curve** is a J–holomorphic cycle all of whose multiplicities m_i in (2.7) are equal to one. Set

$$\mathcal{Z}^{\mathrm{si}} := \{(J,g;C) \in \mathcal{Z} : C \text{ is a } J\text{–holomorphic curve}\}.$$

(2) Let $(J,g) \in \mathcal{H}$. A J–holomorphic curve C is **embedded** if its components C_i in (2.7) are disjoint and embedded. Set

$$\mathcal{Z}^{\mathrm{emb}} := \{(J,g;C) \in \mathcal{Z}^{\mathrm{si}} : C \text{ is embedded}\}.$$

(3) Set

$$\mathcal{C} := \{(J,g;C) \in \mathcal{Z} : \operatorname{supp} C \text{ is connected}\},$$
$$\mathcal{C}^{\mathrm{si}} := \mathcal{C} \cap \mathcal{Z}^{\mathrm{si}}, \quad \text{and} \quad \mathcal{C}^{\mathrm{emb}} := \mathcal{C} \cap \mathcal{Z}^{\mathrm{emb}}.$$ •

Remark 2.16. A moment's thought shows that $\mathcal{C} = \operatorname{im} \mathfrak{z}$. •

Since the subset of connected, compact subsets is closed in $\mathcal{K}$, $\mathcal{C}$ is closed in $\mathcal{Z}$.

Proposition 2.17. *$\mathcal{Z}^{\mathrm{si}}$ is open in $\mathcal{Z}$.*

Theorem 2.18. *$\mathcal{C}^{\mathrm{emb}}$ and $\mathcal{Z}^{\mathrm{emb}}$ are open in $\mathcal{Z}^{\mathrm{si}}$ (and, therefore, in $\mathcal{Z}$).*

The proof of Proposition 2.17 requires the monotonicity formula and is discussed in Section 2.2. Theorem 2.18 is proved in Section 2.3—using Allard's regularity theorem [All72] and an observation due to Gray [Gra65].

The following results compare the geometric topology on $\mathcal{Z}^{\mathrm{emb}}$ with the C^1 topology.

Definition 2.19. (1) Denote by $\mathcal{S}$ the set of C^2 submanifolds of X.

(2) Let $S \in \mathcal{S}$. A **tubular neighborhood of** S consists of an open neighborhood U of the zero section in the normal bundle NS, an open neighborhood V of S in X, and a C^1 diffeomorphism $\jmath\colon U \to V$ which restricts to identity along the zero section.

(3) For $S \in \mathcal{S}$, a tubular neighborhood $\jmath\colon U \to V$ of S, and $\varepsilon > 0$ set

$$\mathcal{U}(S,\jmath,\varepsilon) := \{\jmath(\operatorname{graph}\xi) : \xi \in \Gamma(U) \text{ with } \|\xi\|_{C^1} < \varepsilon\}.$$

The C^1 **topology on** $\mathcal{S}$ is the coarsest topology with respect to which the subsets $\mathcal{U}(S,\jmath,\varepsilon)$ are open. •

Theorem 2.20. *The map* $(\mathrm{pr}_{\mathcal{H}}, \mathrm{supp})\colon\ \mathcal{Z}^{\mathrm{emb}} \to \mathcal{H}\times\mathcal{S}$ *is an embedding.*

The proof is presented in Section 2.5; it is based on an observation due to White [Whi05].

Proposition 2.21. *The map* $\mathfrak{z}\colon\ \mathcal{M}^{\mathrm{emb}} \to \mathcal{Z}$ *is an open embedding; its image is* $\mathcal{C}^{\mathrm{emb}}$*. In particular, the Gromov topology on* $\mathcal{M}^{\mathrm{emb}}$ *agrees with the geometric topology on* $\mathcal{C}^{\mathrm{emb}}$*.*

Proof. Evidently, the image of $\mathfrak{z}\colon\ \mathcal{M}^{\mathrm{emb}} \to \mathcal{Z}$ is $\mathcal{C}^{\mathrm{emb}}$. By Proposition 2.17 and Theorem 2.18, the latter is open in $\mathcal{Z}$. Since the Gromov topology on $\mathcal{M}^{\mathrm{emb}}$ agrees with the C^1 topology on the space of maps, the composition

$$\mathcal{M}^{\mathrm{emb}} \xrightarrow{\mathfrak{z}} \mathcal{Z}^{\mathrm{emb}} \xrightarrow{(\mathrm{pr}_{\mathcal{H}},\mathrm{supp})} \mathcal{H}\times\mathcal{S}$$

is an embedding; that is: a homeomorphism on its image. Therefore, by Theorem 2.20, $\mathfrak{z}$ is an embedding. ■

Remark 2.22. The reader should be warned that the map $\mathfrak{z}\colon\ \mathcal{M}^{\mathrm{si}} \to \mathcal{Z}^{\mathrm{si}}$ is a continuous injection but fails to be an embedding. To see this, consider a sequence $(u_n\colon\ (\Sigma, j) \to X)$ of simple J–holomorphic maps which Gromov converges to a nodal J–holomorphic map $u\colon\ \widehat{\Sigma} \to X$ with $\widehat{\Sigma} = \Sigma \vee S^2$ such that $u|_\Sigma$ is constant and $v \coloneqq u|_{S^2}$ is simple. The sequence of J–holomorphic curves $(\mathrm{im}\, u_n) \in (\mathcal{Z}^{\mathrm{si}})^{\mathbb{N}}$ geometrically converges to $\mathrm{im}\, v$; however, (u_n) does not converge to v. By Proposition 2.21, v cannot be an embedding. Indeed, this can also be proved by analysing the obstruction map in the Kuranishi model of a neighborhood of $[u] \in \overline{\mathcal{M}}$; cf. [Ion98; Zin09; DW19]. •

The following result compares the geometric topology on $\mathcal{Z}^{\mathrm{si}}$ with the topology induced by the Hausdorff metric.

Definition 2.23. For $A \in \mathrm{H}_2(X, \mathbb{Z})$ and $\Lambda > 0$ set

$$\mathcal{Z}^{\mathrm{si}}_{A,\Lambda} = \{(J, g; C) \in \mathcal{Z}^{\mathrm{si}} : [C] = A \text{ and } \mathbf{M}(C) \leqslant \Lambda\};$$

furthermore, in either situation of Example 2.2, abbreviate

$$\mathcal{Z}^{\mathrm{si}}_{A} \coloneqq \mathcal{Z}^{\mathrm{si}}_{A,\Lambda} \quad \text{with} \quad \Lambda = \langle [\omega], [C]\rangle. \qquad \bullet$$

Proposition 2.24. *If there exists no* $(J, g; C) \in \mathcal{Z}$ *with* $[C] = 0$ *but* $C \neq 0$*, then the map* $(\mathrm{pr}_{\mathcal{H}}, \mathrm{supp})\colon\ \mathcal{Z}^{\mathrm{si}}_{A,\Lambda} \to \mathcal{H}\times\mathcal{K}$ *is an embedding. In particular, the geometric topology on* $\mathcal{Z}^{\mathrm{si}}_{A,\Lambda}$ *agrees with the topology induced by the Hausdorff metric.*

Remark 2.25. The hypothesis of Proposition 2.24 holds in either situation of Example 2.2; cf. Remark 2.8. •

Remark 2.26. The hypothesis of Proposition 2.24 is necessary. Consider S^6 with the almost Hermitian structure (J, g) induced by the octonions. Choose a sequence of distinct geodesic J–holomorphic 2–spheres (S_n) converging to a J–holomorphic geodesic 2–sphere S. $(S_n \amalg S)$ converges to S with respect to the Hausdorff metric, but $(S_n + S)$ does not geometrically converge to S: it geometrically converges to $2S \notin \mathcal{Z}^{\mathrm{si}}$. This issue also occurs with irreducible J–holomorphic curves; cf. Hashimoto [Has04]. •

Remark 2.27. The reader should be warned that the map $(\mathrm{pr}_{\mathcal{H}}, \mathrm{supp})\colon\ \mathcal{Z}^{\mathrm{si}} \to \mathcal{H}\times\mathcal{K}$ is a continuous injection but fails to be an embedding. To see this, consider a sequence of pseudo-holomorphic curves (C_n) geometrically converging to a pseudo-holomorphic cycle mC with $m \geqslant 2$. The sequence $(\mathrm{supp}\, C_n)$ converges to $\mathrm{supp}\, C$, but (C_n) does not geometrically converge to C. •

Proof of Proposition 2.24. The map $(\mathrm{pr}_{\mathcal{H}}, \mathrm{supp})$ is continuous and injective [MS12, Proposition 2.4.4, Corollary 2.5.3, Theorem E.1.2]. To prove that it is an embedding, let $(J_n, g_n; C_n) \in (\mathcal{Z}^{\mathrm{si}}_{A,\Lambda})^{\mathbb{N}}$ be such that $(J_n, g_n; \operatorname{supp} C_n)$ converges to $(J, g; \operatorname{supp} C)$ with $(J, g; C) \in \mathcal{Z}^{\mathrm{si}}_{A,\Lambda}$. By Theorem 2.11, $(J_n, g_n; C_n)$ converges to $(J, g; C') \in \mathcal{Z}$. By continuity, $\operatorname{supp} C' = \operatorname{supp} C$. Therefore, if $C = \sum_{i=1}^{I} C_i$, then $C' = \sum_{i=1}^{I} m_i C_i$ with $m_1, \dots, m_I \in \mathbb{N}$. Since $[C'] = A = [C]$, and by the hypothesis, $m_1 = \dots = m_I = 1$; hence: $C' = C$. ■

Finally, here is a partial summary of the above results in the symplectic setting.

Definition 2.28. For $A \in \mathrm{H}_2(X, \mathbb{Z})$ and $\mathrm{g} \in \mathbb{N}_0$ set

$$\begin{aligned}
\mathcal{C}^{\mathrm{emb}}_A &:= \{(J, h; C) \in \mathcal{C}^{\mathrm{emb}} : [C] = A\}, \\
\mathcal{M}^{\mathrm{emb}}_A &:= \{(J, h; [u\colon \Sigma \to X]) \in \mathcal{M}^{\mathrm{emb}} : u_*[\Sigma] = A\}, \quad \text{and} \\
\mathcal{M}^{\mathrm{emb}}_{A,\mathrm{g}} &:= \{(J, h; [u\colon \Sigma \to X]) \in \mathcal{M}^{\mathrm{emb}} : u_*[\Sigma] = A, \mathrm{g}(\Sigma) = \mathrm{g}\}.
\end{aligned}$$

•

Corollary 2.29. *If (X, ω) is symplectic and $\mathcal{H} = \mathcal{J}_\tau$ as in Example 2.2 (2), then for every $A \in \mathrm{H}_2(X, \mathbb{Z})$ the map*

$$(\mathrm{pr}_{\mathcal{J}}, \mathrm{im})\colon \mathcal{M}^{\mathrm{emb}}_A \to \mathcal{C}^{\mathrm{emb}}_A$$

is a homeomorphism and $\mathcal{C}^{\mathrm{emb}}_A$ is open in $\mathcal{Z}$. In particular:

(1) *The Gromov topology on $\mathcal{M}^{\mathrm{emb}}_A$ agrees with the geometric topology as well as with the topology induced by the Hausdorff metric.*

(2) *If C is a irreducible, embedded J–holomorphic curve representing A and of genus* g, *then there is an open neighborhood of $(J, C) \in \mathcal{J}_\tau(\omega) \times \mathcal{K}$ which contains no other images of pseudo-holomorphic cycles except for those in the image of $\mathcal{M}^{\mathrm{emb}}_{A,\mathrm{g}}$.* ■

The remainder of this section contains the proofs of Proposition 2.17, Theorem 2.18, and Theorem 2.20. A reader who is solely interested in the applications of these results to symplectic geometry might proceed to the next section.

2.2 The monotonicity formula

The proofs of Proposition 2.17 and Theorem 2.20 require the following monotonicity formula. This result is standard and can be derived from [De 18, Theorem 2.1] and [Gra65, Proposition 5.3]. Variants of this result can be found in the literature on pseudo-holomorphic curves—e.g.: [Zin20, Proposition 3.12]. For the readers' convenience a proof is included below.

Lemma 2.30 (Monotonicity formula). *For every $\varepsilon \geqslant 0$ and $\delta = \frac{1}{2}\varepsilon$ the following holds. Let (J, g) be an almost Hermitian structure on X. Let $x \in X$ and $r_1, r_2 \in (0, \mathrm{inj}_g(x))$ with $r_1 \leqslant r_2$. Let $C \subset B_{r_2}(x)$ be a J–holomorphic submanifold. If*

$$\left\| r^{-2}(J - J_x) \right\|_{C^0(B_{r_2}(x))} \leqslant r_2^{-2}\delta \quad \text{and} \quad \left\| r^{-2}(g - g_x) \right\|_{C^0(B_{r_2}(x))} \leqslant r_2^{-2}\delta,$$

then

$$(1 + \varepsilon r_2^2)\frac{\mathrm{area}(C \cap B_{r_2}(x))}{r_2^2} - (1 - \varepsilon r_1^2)\frac{\mathrm{area}(C \cap B_{r_1}(x))}{r_1^2} \geqslant \int_{C \cap (B_{r_2}(x) \setminus B_{r_1}(x))} \frac{|\nabla r^\perp|^2}{r^2} \,\mathrm{vol}_C.$$

Here $r := d(\cdot, x)$ and $(\cdot)^\perp$ denotes the projection onto the orthogonal complement of T_yC.

Proof. The following argument is essentially due to Imagi [Ima15, §3]. The cognizant reader will realize that the proof immediately carries over to semi-calibrated cycles.

It suffices the prove the statement with $X = B_1(0) \subset \mathbf{C}^m$, $x = 0$, $r_2 = 1$, $r_1 = s$, $J_x = i$, $g_x = g_0$, $\exp_x^g = \mathrm{id}_{B_1(0)}$, and $\nabla r = \partial_r$. By hypothesis, C is semi-calibrated by $\sigma := g(J\cdot,\cdot)$.

Define $f_0\colon (0,1] \to [0,\infty)$ by

$$f_0(s) := s^{-2}\int_{B_s(0)\cap C} \sigma_0.$$

A moment's thought shows that

$$\sigma_0 = \frac{1}{2}\mathrm{d}(r^2\alpha) \quad \text{and} \quad i(\partial_r)\mathrm{d}r \wedge \sigma_0 = \frac{1}{2}r^2\mathrm{d}\alpha \quad \text{with} \quad \alpha := r^{-1}i(\partial_r)\sigma_0.$$

Therefore,

$$f_0(s) = \frac{1}{2}\int_{\partial B_s(0)\cap C} \alpha$$

and

$$f_0(1) - f_0(s) = \frac{1}{2}\int_{(B_1(0)\backslash B_s(0))\cap C} \mathrm{d}\alpha = \int_{(B_1(0)\backslash B_s(0))\cap C} r^{-2}i(\partial_r)\mathrm{d}r \wedge \sigma_0.$$

Since C is semi-calibrated by σ, if $v \perp T_yC$, then $i(v)\sigma|_C = 0$. Therefore, with (e_1, e_2) denoting a local orthonormal frame of C

$$(i(\partial_r)\mathrm{d}r \wedge \sigma)|_C = \langle \mathrm{d}r \wedge \sigma, \partial_r \wedge e_1 \wedge e_2\rangle \cdot \mathrm{vol}_C = \langle \mathrm{d}r \wedge \sigma, \partial_r^\perp \wedge e_1 \wedge e_2\rangle \cdot \mathrm{vol}_C = |\partial_r^\perp|^2 \cdot \mathrm{vol}_C$$

This proves the assertion with $\varepsilon = 0$.

The function $f\colon (0,1] \to [0,\infty)$ defined by

$$f(s) := s^{-2}\int_{B_s(0)\cap C} \sigma$$

satisfies

$$(1-\delta s^2)\cdot f(s) \leqslant f_0(s) \leqslant (1+\delta s^2)\cdot f;$$

moreover,

$$i(\partial_r)\mathrm{d}r \wedge \sigma_0|_C \geqslant i(\partial_r)\mathrm{d}r \wedge \sigma|_C - \delta r^2 \cdot \mathrm{vol}_C.$$

Therefore,

$$\begin{aligned}(1+\delta s^2)\cdot f(1) - (1-\delta s^2)\cdot f(s) &\geqslant f(1) - f(s)\\ &= \int_{(B_1(0)\backslash B_s(0))\cap C} r^{-2}i(\partial_r)\mathrm{d}r \wedge \sigma_0\\ &\geqslant \int_{(B_1(0)\backslash B_s(0))\cap C} \frac{|\partial_r^\perp|}{r^2}\cdot \mathrm{vol}_C - \delta s^2 \cdot f(1).\end{aligned}$$

This proves the assertion. ■

Corollary 2.31. *If C is an i–holomorphic 2–dimensional submanifold of $\mathbf{C}^n$ satisfying*

$$\mathrm{area}(C \cap B_r(x)) = \pi r^2$$

for every $r > 0$, then C is a complex line. ■

Proof of Proposition 2.17. Suppose $(J_n, g_n; C_n) \in \mathcal{Z}^{\mathbb{N}}$ geometrically converges to $(J, g; C) \in \mathcal{Z}^{\mathrm{si}}$. For every $n \in \mathbb{N}$ decompose C_n as

$$C_n = D_n + E_n \quad \text{with} \quad D_n := \sum_{i=1}^{I_n} C_{n,i} \quad \text{and} \quad E_n := \sum_{i=1}^{I_n} (m_{n,i} - 1) C_{n,i}.$$

By Theorem 2.11 every subsequence of $(J_n, g_n; D_n)$ has a subsequence which geometrically converges to a limit $(J, g; D)$. By construction $(\operatorname{supp} D_n)$ converges to $\operatorname{supp} C$; hence: $\operatorname{supp} D = \operatorname{supp} C$. A further moment's thought shows that $D = C$. Therefore, $(J_n, g_n; D_n)$ geometrically converges to $(J, g; C)$ and $\lim_{n\to\infty} \mathbf{M}(E_n) = 0$. The latter contradicts Lemma 2.30. ■

2.3 Allard's regularity theorem

Definition 2.32. Let g be Riemannian metric on X. Let $d \in \mathbb{N}_0$. Denote by $\mathscr{H}^d$ the d–dimensional Hausdorff measure.

(1) A Borel subset $S \subset X$ is **rectifiable of dimension** d if there is a countable set $\{S_i : i \in I\}$ of C^1 submanifolds with

$$\mathscr{H}^d(S \backslash \bigcup_{i \in I} S_i) = 0.$$

(2) An **integral varifold of dimension** d is a pair $V = (S, m)$ consisting of a rectifiable subset S of dimension d and a Borel function $m \colon S \to \mathbb{N}$.

Let $V = (S, m)$ be an integral varifold of dimension d.

(3) The **measure associated with** V and the **mass of** V are defined by

$$\mu_V := m \mathscr{H}^d|_S \quad \text{and} \quad \mathbf{M}(V) := \mu_V(X) = \int_S m \mathscr{H}^d.$$

(4) Let H_V be a Borel vector field over S. V has **mean curvature** H_V if for every compactly supported C^1 vector field v

$$\int \langle H_V, v\rangle \, \mu_V = -\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \int_{\mathrm{flow}_v^t(S)} m \circ \mathrm{flow}_v^{-t} \, \mathscr{H}^d.$$

Here flow_v^t denotes the flow of v.

(5) For $x \in X$ and $r > 0$ set

$$\theta_V(x, r) = \theta_V(x, r; g) := \frac{\mu_V(B_r(x))}{\omega_d r^d}.$$

Here $\omega_d := \mathrm{vol}(B_1^d(0))$. •

Theorem 2.33 (Allard [All72, §8]; see also [Sim83, Theorem 24.3; De 18, Theorem 3.2]). *Let $m, d \in \mathbb{N}_0$ with $d \leqslant m$ and $\alpha \in (0,1)$. There are $\varepsilon = \varepsilon(m, d, \alpha) > 0$ and $\gamma = \gamma(m, d, \alpha) > 0$ such that the following holds for every $r > 0$. If V is an integral varifold of dimension d in $(B_r^m(0), g_0)$ satisfying*

$$\theta_V(0, r) \leqslant 1 + \varepsilon \quad \text{and} \quad \|H_V\|_{L^\infty(B_r(0))} \leqslant \varepsilon / r,$$

then $V \cap B_{\gamma r}(0)$ is a $C^{1,\alpha}$ submanifold of $\mathbb{R}^m$. ■

Remark 2.34. This implies a corresponding result for Riemannian manifolds. Indeed, Nash proved that every Riemannian manifold (X, g) admits an isometric embedding $\iota\colon (X, g) \hookrightarrow (\mathbb{R}^m, g_0)$ with $m = m(\dim X)$. Moreover, if II_ι denotes the second fundamental form of this embedding, then

$$|H_{\iota(V)}| \leqslant |H_V| + |\mathrm{II}_\iota|. \quad \bullet$$

Remark 2.35. It is a nuisance that the dependence of ε on g is not explicit. It should be possible to prove Theorem 2.33 directly for $g = g_0 + O(r^2)$ on $B^m_r(0)$. By careful bookkeeping in the proof of Nash's (local) isometric embedding theorem, it should also be possible to obtain bounds on the second fundamental form II_ι depending on $g - g_0$ and its derivatives. Unfortunately, the authors failed to locate proofs of either result in the literature. •

Theorem 2.36 (Gray [Gra65, Proposition 5.3]). *Let (J, g) be an almost Hermitian structure on X. For every J–holomorphic cycle C*

$$|H_C| \leqslant |\nabla J|. \quad \blacksquare$$

Proposition 2.37. *Let $K \subset X$ be compact. Let (J_n, g_n) be sequence of almost Hermitian structures converging to an almost Kähler structure (J, g) in the C^2_{loc} topology. There are constants $r, \varepsilon > 0$ (depending on the above data) such that the following holds for every $n \in \mathbb{N}$. If C is J_n–holomorphic cycle with $\operatorname{supp} C \subset K$ and such that for every $x \in C$ there is an $s \in (0, r)$ with*

$$\theta_C(x, s; g_n) \leqslant 1 + \varepsilon,$$

then C is smooth.

Proof. Choose an open neighborhood U of K such that

$$\varepsilon_n \coloneqq \sum_{m=n-1}^{n+1} \|g_m - g\|_{C^2(U)}$$

converges to zero. After passing to a subsequence, $\limsup_{n\to\infty} n^{-4}\varepsilon_n \leqslant 1$. Choose $\chi \in C^\infty(\mathbb{R}, [0,1])$ with $\chi|_{[-1/3,1/3]} = 1$, $\operatorname{supp}(\chi) \subset [-2/3, 2/3]$, and $\sum_{n\in\mathbb{Z}} \chi(\cdot + n) = 1$. Define a Riemannian metric G on $(0,1] \times X$ by

$$G \coloneqq \mathrm{d}t \otimes \mathrm{d}t + \sum_{n=1}^{\infty} \chi(1/t - n) g_n.$$

By construction, for $k, \ell \in \{0,1,2\}$ and $t \in [1/(n-1), 1/(n+1)]$

$$\sup_{x\in U} |\partial_t^k \nabla_x^\ell G|(t,x) \lesssim \|\chi\|_{C^k} n^{-2k} \varepsilon_n \leqslant \|\chi\|_{C^k}.$$

Therefore, G extends to a C^2 Riemannian metric on $[0,1] \times X$.

For every $n \in \mathbb{N}$ the map $\iota_n \coloneqq (1/n, \mathrm{id}_X)$ defines an isometric embedding $(U, g_n) \hookrightarrow ([0,1] \times U, G)$ with II_{ι_n} bounded independently of n. Choose an isometric embedding $\jmath\colon ([0,1] \times U, G) \hookrightarrow (\mathbb{R}^n, g_0)$. By Theorem 2.36,

$$|H_{\jmath\circ\iota_n(C)}| \leqslant \Lambda \quad \text{with} \quad \Lambda \coloneqq \sup_{n\in\mathbb{N}} \big(\|\nabla J_n\|_{L^\infty(K)} + \|\mathrm{II}_{\iota_n}\|_{L^\infty(K)}\big) + \|\mathrm{II}_\jmath\|_{L^\infty([0,1]\times K)} < \infty.$$

Therefore, the assertion follows from Theorem 2.33. ■

Proposition 2.38. *If $(J_n, g_n; C_n) \in \mathcal{Z}^{\mathbb{N}}$ geometrically converges to $(J, g; C) \in \mathcal{Z}$, then for every $r > 0$*

$$\limsup_{n\to\infty} \max_{x\in\operatorname{supp} C_n} \theta_{C_n}(x, r; g_n) \leqslant \max_{x\in\operatorname{supp} C} \theta_C(x, r; g).$$

Proof. If not, then for every $n \in \mathbb{N}$ there is a $x_n \in C_n$ with (x_n) converging to $x \in C$ and

$$\limsup_{n\to\infty} \theta_{C_n}(x_n, r; g_n) > \theta_C(x, r; g).$$

However, this is in contradiction to geometric convergence. ∎

Proof of Theorem 2.18. Suppose $(J_n, g_n; C_n) \in \mathcal{Z}^{\mathbb{N}}$ geometrically converges to $(J, g; C) \in \mathcal{Z}^{\mathrm{emb}}$. Let $r, \varepsilon > 0$ be as in Proposition 2.37. Choose $s \in (0, r)$ such that

$$\max_{x\in\operatorname{supp} C} \theta_{C_n}(x, s) \leqslant 1 + \frac{1}{2}\varepsilon.$$

By Proposition 2.38, for $n \gg 1$

$$\max_{x\in\operatorname{supp} C_n} \theta_{C_n}(x, s) \leqslant 1 + \varepsilon.$$

Therefore, by Proposition 2.37, C_n is embedded. This proves that $\mathcal{Z}^{\mathrm{emb}}$ is open in $\mathcal{Z}^{\mathrm{emb}}$.

Suppose that $\operatorname{supp} C$ is connected but $\operatorname{supp} C_n$ fails to be connected for $n \gg 1$. Decompose $C_n = D_n + E_n$ with $\operatorname{supp} D_n$ and $\operatorname{supp} E_n$ disjoint. After passing to a subsequence, (D_n) converges to D with $\operatorname{supp} D \subset \operatorname{supp} C$; hence: $\operatorname{supp} C = \operatorname{supp} D$. Similarly, (E_n) converges to E with $\operatorname{supp} E = \operatorname{supp} C$. This contradicts $C \in \mathcal{C}^{\mathrm{emb}}$. ∎

2.4 Convergence of submanifolds

The proof of Theorem 2.20 requires the following discussion of the convergence of submanifolds. This material is entirely standard and elementary. It is spelled out in detail for the readers' convenience. Throughout this subsection, set $m \coloneqq \dim X$, let $d \in \mathbb{N}_0$ with $d \leqslant m$, and $k \in 2 + \mathbb{N}_0$.

Notation 2.39.

(1) The **graph** of $f \in C^k(B_1^d(0), \mathbb{R}^{m-d})$ is defined by

$$\operatorname{graph} f \coloneqq \{(x, f(x) : x \in B_1^d(0)\} \subset B_1^d(0) \times \mathbb{R}^{m-d}.$$

(2) For $r > 0$ define $s_r \colon \mathbb{R}^m \to \mathbb{R}^m$ by

$$s_r(x) \coloneqq r \cdot x.$$

(3) Set

$$Q^d \coloneqq B_1^d(0) \times B_1^{m-d}(0).$$

(4) Let $x \in X$. A **frame** of (T_xX, g_x) is a linear isometry $\phi \colon (\mathbb{R}^m, g_0) \to (T_xX, g_x)$. The space of frames is denoted by

$$\operatorname{Fr}(T_xX, g_x).$$ •

Definition 2.40. Let g be a C^{k+1} Riemannian metric on X. Denote by $\mathcal{S}$ the set of closed C^k submanifolds of X of dimension d. Let $(S_n) \in \mathcal{S}^{\mathbb{N}}$ and $S \in \mathcal{S}$. (S_n) **weakly C^k_{loc} converges to** S if:

(1) For every compact $K \subset X$ the sequence $(S_n \cap K)$ converges to $S \cap K$ with respect to the Hausdorff metric.

(2) For every $x \in S$ there are $\phi \in \mathrm{Fr}(T_xX, g_x)$, $(f_n) \in C^k(B_1^d(0), \mathbb{R}^{m-d})^{\mathbb{N}}$, and $f \in C^k(B_1^d(0), \mathbb{R}^{m-d})$ such that:
   (a) $(\exp_x^g \circ \phi \circ s_r)^{-1}(S) \cap Q^d = \operatorname{graph} f$,
   (b) $(\exp_x^g \circ \phi \circ s_r)^{-1}(S_n) \cap Q^d = \operatorname{graph} f_n$ for $n \gg 1$,
   (c) $\limsup_{n\to\infty} \|f_n\|_{C^k} < \infty$, and
   (d) $\lim_{n\to\infty} \|f_n - f\|_{C^{k-1,\alpha}} = 0$ for every $\alpha \in (0,1)$. •

Definition 2.41. Let g be a C^{k+1} Riemannian metric on X. Let $S \subset X$. The C^k **regularity scale** is the map $r_S^k(\,\cdot\,; g)\colon S \to [0, \infty]$ defined by

$$r_S^k(x; g) \coloneqq \sup\{r_S^k(x, \phi; g) : \phi \in \mathrm{Fr}(T_xX, g_x)\}$$

with $r_S^k(x, \phi; g)$ denoting the supremum of those $r \in (0, \mathrm{inj}_g(x)/2]$ for which

$$(\exp_x^g \circ \phi \circ s_r)^{-1}(S) \cap Q^d = \operatorname{graph} f$$

with $f \in C^k(B_1^d(0), \mathbb{R}^{m-d})$ satisfying $\|f\|_{C^k} \leqslant 1$; if there is no such r (that is: if S fails to be a C^k submanifold in every neighborhood of x), then

$$r_S^k(x, \phi; g) \coloneqq 0. \qquad \bullet$$

Proposition 2.42. *Let g be a C^{k+1} Riemannian metric on X. Let (g_n) be a sequence of C^{k+1} Riemannian metrics on X converging to g in the C^{k+1}_{loc} topology. Let $(S_n) \in \mathcal{S}^{\mathbb{N}}$ and let $S \subset X$ be a closed subset. If for every compact $K \subset X$ the sequence $(S_n \cap K)$ converges to $S \cap K$ with respect to the Hausdorff metric and*

$$\liminf_{n\to\infty} \inf\{r_{S_n}^k(x, g_n) : x \in S_n \cap K\} > 0,$$

then S is a C^k submanifold, and (S_n) weakly C^k_{loc} converges to S.

The proof requires the following preparation.

Proposition 2.43. *For every $\varepsilon_0 \in (0,1)$ there is a constant $c = c(k, \varepsilon_0) > 0$ such that the following holds. Let $f \in C^k(B_1^d(0), \mathbb{R}^{m-d})$ and $\Phi \in C^k(Q, \mathbb{R}^m)$. If*

$$\|f\|_{C^k} \leqslant 1 \quad \textit{and} \quad \varepsilon \coloneqq \|\Phi - \mathrm{id}\|_{C^k} \leqslant \varepsilon_0,$$

then there is a $\tilde f \in C^k(B_{1-\varepsilon}^d(0), \mathbb{R}^{m-d})$ such that

$$\Phi(\operatorname{graph} f) \cap (B_{1-\varepsilon}^d(0) \times \mathbb{R}^{m-d}) = \operatorname{graph} \tilde f \quad \textit{and} \quad \|\tilde f - f\|_{C^k} \leqslant c\varepsilon.$$

Proof. Define $\xi \in C^k(B_1^d(0), \mathbb{R}^d)$ and $\phi \in C^k(B_1^d(0), \mathbb{R}^{m-d})$ by

$$(\xi(x), \phi(x)) \coloneqq \Phi(x, f(x)).$$

A moment's thought shows that

$$\|\xi - \mathrm{id}\|_{C^k} \leqslant \varepsilon \quad \text{and} \quad \|\phi - f\|_{C^k} \leqslant \varepsilon.$$

By the inverse function theorem, ξ is injective, $B_{1-\varepsilon}(0) \subset \operatorname{im} \xi$, and

$$\|\xi^{-1} - \mathrm{id}\|_{C^k} \leqslant c\varepsilon.$$

Define $\tilde{f}\colon \bar{B}^d_{1-\varepsilon}(0) \to \mathbb{R}^{m-d}$ by

$$\tilde{f} \coloneqq \phi \circ \xi^{-1}.$$

By construction,

$$\Phi(\operatorname{graph} f) \cap (B^d_{1-\varepsilon}(0) \times \mathbb{R}^{m-d}) = \operatorname{graph} \tilde{f}$$

and

$$\|\tilde{f} - f\|_{C^k} \leqslant \|\phi \circ \xi^{-1} - \phi\|_{C^k} + \|\phi - f\|_{C^k} \leqslant c\varepsilon. \qquad \blacksquare$$

Proof of Proposition 2.42. Let $x \in S$. For every $n \in \mathbb{N}$ choose $x_n \in S_n$ such that $x = \lim_{n\to\infty} x_n$. By hypothesis,

$$r \coloneqq \liminf_{n\to\infty} r_{S_n}(x_n; g_n) > 0.$$

By Definition 2.41, for every $n \in \mathbb{N}$ there are $r_n \in (0, \operatorname{inj}_{g_n}(x)/2]$, $\phi_n \in \operatorname{Fr}(T_{x_n}X, g_n)$, and $f_n \in C^k(B^d_1(0), \mathbb{R}^{m-d})$ with $\|f_n\|_{C^k} \leqslant 1$ such that

$$\tilde{S}_n \coloneqq \iota_n^{-1}(S_n) \cap Q^d = \operatorname{graph} f_n \quad \text{with} \quad \iota_n \coloneqq \exp^{g_n}_{x_n} \circ \phi_n \circ s_{r_n}$$

and

$$\liminf_{n\to\infty} r_n = r.$$

By the Arzelà–Ascoli theorem, after passing to a subsequence (without relabelling), (r_n) converges to r, (ϕ_n) converges to $\phi \in \operatorname{Fr}(T_xX, g)$, and (f_n) converges to $f \in C^k(B^d_1(0), \mathbb{R}^{m-d})$ with $\|f\|_{C^k} \leqslant 1$ in the $C^{k-1,\alpha}$ topology for every $\alpha \in (0,1)$. The sequence (ι_n) converges to

$$\iota \coloneqq \exp^g_x \circ \phi \circ s_r\colon B^d_2(0) \to X$$

in the C^k topology. Set

$$\tilde{S} \coloneqq \iota^{-1}(S) \cap Q^d$$

and for $\rho \in (0,1)$ set

$$Q^d_\rho \coloneqq B^d_\rho(0) \times B^{m-d}_\rho(0).$$

On the one hand, by hypothesis, $(\iota_n(\overline{Q^d_\rho}))$ converges to $\iota(\overline{Q^d_\rho})$ with respect to the Hausdorff metric, and, therefore $(\tilde{S}_n \cap \overline{Q^d_\rho})$ converges to $\tilde{S} \cap \bar{Q}^d_\rho$ with respect to the Hausdorff metric. On the other hand, evidently, $(\operatorname{graph} f_n)$ converges to $\operatorname{graph} f$ with respect to the Hausdorff metric. Therefore, since $\rho \in (0,1)$ was arbitrary,

$$\tilde{S} = \operatorname{graph} f.$$

In particular, $S \cap B_r(x)$ is a C^k submanifold.

Since $(\iota_n)_{n\in\mathbb{N}}$ converges to ι in the C^k topology, by Proposition 2.43, for every $n \gg 1$ there is an $\tilde{f}_n \in C^k(B^d_{1/2}(0), \mathbb{R}^{m-d})$ with $\|\tilde{f}_n\|_{C^k} \leqslant 2$ such that

$$\iota^{-1}(S_n) \cap Q^d_{1/2} = \operatorname{graph} \tilde{f}_n$$

and for every $\alpha \in (0,1)$

$$\lim_{n\to\infty} \|\tilde{f}_n - f\|_{C^{k-1,\alpha}} = 0.$$

Therefore, (S_n) weakly C^k_{loc} converges to S. $\blacksquare$

Proposition 2.44. *Let $(S_n) \in \mathcal{S}^{\mathbb{N}}$ and $S \in \mathcal{S}$. Suppose that S is compact. If (S_n) weakly C^k_{loc} converges to S, then for every $n \gg 1$ there is a $\xi_n \in \Gamma(NS)$ with $|\xi_n| < \mathrm{inj}_g$ such that*

$$S_n = \mathrm{graph}(\xi_n) := \{\exp^g_x \xi_n(x) : x \in S\}$$

and

$$\lim_{n\to\infty} \|\xi_n\|_{C^{k-1}} = 0.$$

Proof. Let $x \in S$ and $r \in (0, \mathrm{inj}_g(x)/4]$. Choose a frame $\phi \in \mathrm{Fr}(T_xX, g_x)$ with $\phi(T_xS) = \mathbb{R}^d \subset \mathbb{R}^m$. Define $\iota\colon B^m_2(0) \to X$ by

$$\iota := \exp^g_x \circ \phi \circ s_r$$

and define $\jmath\colon B^d_2(0) \times B^{m-d}_2(0) \to X$ by

$$\jmath \circ \phi^{-1} \circ s_r^{-1}(v, w) := \exp^g_{\exp^g_x(v)}(\tilde{w})$$

with $\tilde{w}$ denoting the parallel transport of w along the geodesic $t \mapsto \exp^g_x(tv)$. The map $\Phi := \jmath^{-1} \circ \iota\colon Q^d \to \mathbb{R}^m$ can be made arbitrarily C^{k-1}–close to id by choosing $r \ll 1$. Therefore, the assertion follows from Proposition 2.43. ∎

Corollary 2.45. *Let (S_n) be a sequence of C^2 submanifolds and let S be a C^2 submanifold. If (S_n) weakly C^2 converges to S, then it C^1 converges to S.* ∎

2.5 Convergence of embedded pseudo-holomorphic curves

Proposition 2.46. *Let $k \in 2+\mathbb{N}_0$ and $m \in \mathbb{N}$. For every $\Lambda > 0$ there are $\varepsilon = \varepsilon(m, k, \Lambda) > 0$ and $\delta = \delta(m, k, \Lambda) > 0$ such that the following holds. Let X be a smooth manifold of dimension $2m$, let (J, g) be a C^{k+1} almost Hermitian structure, let $x \in X$, let $r \in (0, \mathrm{inj}_g(x))$, and let $C \subset B_r(x)$ be a J–holomorphic submanifold. If*

$$\|(\exp^g \circ s_r)^*J - J_x\|_{C^{k+1}(B_1(0))} \leqslant \Lambda \quad \textit{and} \quad \|r^{-2}(\exp^g \circ s_r)^*g - g_x\|_{C^{k+1}(B_1(0))} \leqslant \Lambda,$$

and for every $y \in C$ and every $0 < s < d(y, \partial B_r(x))$

$$\theta_C(y, s; g) \leqslant 1 + \varepsilon,$$

then

$$r^k_C(y; g) \geqslant \delta \cdot d(y, \partial B_r(x)).$$

Proof. It suffices to prove the statement with $X = B_1(0) \subset \mathbf{C}^m$, $x = 0$, $r = 1$, $J_x = i$, and g satisfying $g_x = g_0$ and $\exp^g_x = \mathrm{id}$. Here g_0 and i are the standard Euclidean metric and complex structure on $\mathbf{C}^m$.

If the statement fails to hold, then for every $n \in \mathbb{N}$ there are a C^{k+1} almost Hermitian structure (g_n, J_n) on $B_1(0)$ and a J_n–holomorphic submanifold $C_n \subset B_1(0)$ such that

$$\|J_n - i\|_{C^{k+1}(B_1(0))} \leqslant \Lambda \quad \text{and} \quad \|g_n - g_0\|_{C^{k+1}(B_1(0))} \leqslant \Lambda$$

and for every $x \in C_n$ and every $0 < s < 1 - |x|$

$$\theta_C(x, s; g_n) \leqslant 1 + \varepsilon_n \quad \text{with} \quad \varepsilon_n := 1/n,$$

but the sequence $(\delta_n)_{n\in\mathbb{N}}$ defined by

$$\delta_n := \inf_{x\in B_1(0)} \frac{r^k_{C_n}(x;g_n)}{1-|x|}$$

converges to zero. Since C_n is a submanifold, $\delta_n > 0$.

For every $n \in \mathbb{N}$ choose $x_n \in B_1(0)$ such that

$$\frac{r^k_{C_n}(x_n;g_n)}{1-|x_n|} \leqslant 2\delta_n,$$

and rescale by declaring that

$$R_n := 1/r^k_{C_n}(x_n;g_n), \quad \tilde{J}_n := s^*_{1/R_n}J_n, \quad \tilde{g}_n := R_n^2 \cdot s^*_{1/R_n}g_n, \quad \text{and} \quad \tilde{C}_n := s^{-1}_{1/R_n}(C_n).$$

The following hold:

(1) For every $n \in \mathbb{N}$ the submanifold $\tilde{C}_n$ is $\tilde{J}_n$–holomorphic.

(2) Since (R_n) converges to ∞, $(\tilde{J}_n, \tilde{g}_n)$ converges to (i, g_0) in the C^{k+1}_{loc} topology.

(3) For every $n \in \mathbb{N}$ and $x \in B_{R_n}(0)$

$$r^k_{\tilde{C}_n}(x;\tilde{g}_n) = R_n \cdot r_{\tilde{C}_n}(s_{1/R_n}(x);g_n);$$

in particular:

$$r^k_{\tilde{C}_n}(\tilde{x}_n;g_n) = 1 \quad \text{with} \quad \tilde{x}_n := s^{-1}_{1/R_n}(x_n).$$

(4) For every $n \in \mathbb{N}$, $x \in B_{R_n}(0)$, and $0 < s < R_n - |x|$

$$\theta_{\tilde{C}_n}(x,s;\tilde{g}_n) \leqslant 1 + \varepsilon_n.$$

(5) The sequence $(\tilde{R}_n)$ defined by

$$\tilde{R}_n := \frac{1}{2}\cdot(R_n - |\tilde{x}_n|) = \frac{1}{2}\cdot\frac{1-|x_n|}{r^k_{C_n}(x_n;g_n)}$$

converges to ∞.

(6) For every $n \in \mathbb{N}$ and $x \in B_{R_n}(0)$

$$r^k_{\tilde{C}_n}(x;\tilde{g}_n) \geqslant \delta_n\cdot(R_n - |x|) \geqslant \frac{1}{2}\cdot\frac{R_n-|x|}{R_n-|\tilde{x}_n|} \geqslant \frac{1}{2}\cdot\left(1 - \frac{d(x,x_n)}{R_n-|\tilde{x}_n|}\right);$$

in particular:

$$\inf\left\{r^k_{\tilde{C}_n}(x;\tilde{g}_n) : x \in B_{\tilde{R}_n}(\tilde{x}_n)\right\} \geqslant \frac{1}{4}.$$

Translate by $-\tilde{x}_n$ in order to assume that $\tilde{x}_n = 0$. By Proposition 2.42, after passing to a subsequence (without relabeling), $(\tilde{C}_n)$ weakly C^k_{loc} converges to an i–holomorphic submanifold C. For every $x \in C$ and $s > 0$,

$$\theta_C(x,s;g_0) = 1.$$

Therefore, by Corollary 2.31, C is a complex line. Without loss of generality, $C = \mathbf{C}\times\{0\}$.

Since $(\tilde{C}_n)$ weakly C^k converges to C, by Proposition 2.43, there are $f_n \in C^k(B_4(0), \mathbf{C}^{m-1})$ for every $n \in \mathbb{N}$ such that $\limsup_{n\to\infty} \|f_n\|_{C^k} < \infty$, $\lim_{n\to\infty} \|f_n\|_{C^{k-1,\alpha}} = 0$ for every $\alpha \in (0,1)$, and for $n \gg 1$,

$$\tilde{C}_n \cap (B_4(0) \times \mathbf{C}^{m-1}) = \operatorname{graph} f_n.$$

The upcoming argument proves that, indeed, $\lim_{n\to\infty} \|f_n\|_{C^k(B_2(0))} = 0$. This contradicts $r^k_{\tilde{C}_n}(0; g_n) = 1$.

The map $F_n \in C^k(B_4(0), \mathbf{C}^m)$ defined by $F_n(z) \coloneqq (z, f_n(z))$ satisfies

$$(\tilde{J}_n \circ F_n) \cdot \mathrm{d}F_n - \mathrm{d}F_n \cdot j_n = 0 \tag{2.47}$$

with j_n denoting the C^{k-1} complex structure on $B_4(0)$ associated with $F_n^* \tilde{g}_n$. For every $\alpha \in (0,1)$

$$\lim_{n\to\infty} \|\tilde{J}_n \circ F_n - i\|_{C^{k-1,\alpha}} = 0 \quad \text{and} \quad \lim_{n\to\infty} \|j_n - j\|_{C^{k-2,\alpha}} = 0$$

with i and j denoting the standard complex structures on $\mathbf{C}^m$ and $\mathbf{C}$ respectively. With $\bar{\partial} F = i\mathrm{d}F - \mathrm{d}F j$ denoting the standard Cauchy–Riemann operator, (2.47) is rewritten as

$$\bar{\partial} F_n + (\tilde{J}_n \circ F_n - i) \cdot \mathrm{d}F_n - \mathrm{d}F_n \cdot (j_n - j) = 0.$$

Since $\bar{\partial} F_n = (0, \bar{\partial} f_n)$ and $\lim_{n\to\infty} \|\nabla f_n\|_{C^{k-1,\alpha}} = 0$, this implies a PDE of the form

$$\Delta f_n + \mathfrak{p}(\tilde{J}_n, f_n, \nabla f_n) \nabla^2 f_n + \mathfrak{q}(\tilde{J}_n, f_n, \nabla f_n) = 0$$

with

$$\lim_{n\to\infty} \|\mathfrak{p}(\tilde{J}_n, f_n, \nabla f_n)\|_{C^{k-2,\alpha}} = 0 \quad \text{and} \quad \lim_{n\to\infty} \|\mathfrak{q}(\tilde{J}_n, f_n, \nabla f_n)\|_{C^{k-2,\alpha}} = 0$$

for every $\alpha \in (0,1)$. Therefore, by interior Schauder estimates [GT01, Theorem 6.6],

$$\lim_{n\to\infty} \|f_n\|_{C^k(B_2(0))} \leqslant \lim_{n\to\infty} \|f_n\|_{C^{k,\alpha}(B_2(0))} = 0. \qquad \blacksquare$$

Proof of Theorem 2.20. The map $(\mathrm{pr}_{\mathfrak{H}}, \operatorname{supp})\colon \mathcal{Z}^{\mathrm{emb}} \to \mathcal{H} \times \mathcal{S}$ is injective. To see that it is continuous, suppose that $(C_n) \in (\mathcal{Z}^{\mathrm{emb}})^{\mathbb{N}}$ geometrically converges to $C \in \mathcal{Z}^{\mathrm{emb}}$. Let $\varepsilon > 0$ be as in Proposition 2.46. Since C is embedded, there is an $r \in (0, \mathrm{inj}_g)$ such that

$$\max_{x \in \operatorname{supp} C} \theta_C(x, r; g) \leqslant 1 + \varepsilon/2.$$

By Proposition 2.38 and Lemma 2.30, for $n \gg 1$ and $0 < s < r$

$$\max_{x \in \operatorname{supp} C_n} \theta_{C_n}(x, s; g_n) \leqslant 1 + \varepsilon.$$

Therefore, by Proposition 2.46,

$$\liminf_{n\to\infty} \inf_{x \in C_n} r^2_{C_n}(x; g) > 0.$$

Therefore, by Proposition 2.42 and Corollary 2.45, (C_n) C^1 converges to C. Evidently, if (C_n) C^1 converges to C, then it geometrically converges to C. ■

The same argument also proves the following.

Definition 2.48. Let $(J, g; C) \in \mathcal{Z}$. A point $x \in \operatorname{supp} C$ is **smooth** if

$$\limsup_{r \downarrow 0} \theta_C(x, r; g) = 1. \tag{2.49}$$

•

Proposition 2.50. *If $(J_n, g_n; C_n) \in \mathcal{Z}^{\mathbb{N}}$ geometrically converges to $(J, g; C) \in \mathcal{Z}$ and $x \in \operatorname{supp} C$ is smooth, then there is a neighborhood U of $x \in X$, such that, for every $n \gg 1$, $C_n \cap U$ is embedded and $(C_n \cap U)$ C^1 converges to $C \cap U$.* ■

3 The proof of the Gopakumar–Vafa conjecture

Throughout this section, assume the following.

Situation 3.1. Let (X, ω) be a closed symplectic 6–manifold. Denote by $\mathcal{J} := \mathcal{J}_\tau(\omega)$ the space of smooth almost complex structures J, which are tamed by ω, equipped with the C^∞ topology; cf. Example 2.2 (2). •

This section carries forward the notation from Section 2 with $\mathcal{H} = \mathcal{J}$. In particular, $\overline{\mathcal{M}}$ denotes the universal moduli space over $\mathcal{J}$ of stable nodal pseudo-holomorphic maps; moreover, $\mathcal{M}^{\mathrm{si}}$ and $\mathcal{M}^{\mathrm{emb}}$ denote the subspaces consisting of the equivalence classes of simple maps and of embeddings. For $A \in \mathrm{H}_2(M, \mathbb{Z})$ and $\mathrm{g} \in \mathbb{N}_0$ denote by $\overline{\mathcal{M}}_{A,\mathrm{g}}$ the subspace of nodal pseudo-holomorphic maps representing A and of genus g. For $J \in \mathcal{J}$ and $S \subset \mathcal{J}$ set

$$\overline{\mathcal{M}}(J) := \mathrm{pr}_{\mathcal{J}}^{-1}(J) \quad \text{and} \quad \overline{\mathcal{M}}(S) := \mathrm{pr}_{\mathcal{J}}^{-1}(S)$$

with $\mathrm{pr}_{\mathcal{J}} \colon \overline{\mathcal{M}} \to \mathcal{J}$ denoting the projection map. Analogous notation is used for the subspaces of $\overline{\mathcal{M}}$ introduced above.

The infinitesimal structure of the moduli space is controlled by the linearization of the Cauchy–Riemann operator.

Definition 3.2. Let $J \in \mathcal{J}$. Let $u \colon (\Sigma, j) \to (X, J)$ be a J–holomorphic map.

(1) Let $\mathcal{S}$ be an $\mathrm{Aut}(\Sigma, j)$–invariant slice of the Teichmüller space $\mathcal{T}(\Sigma)$ through j. The linearization of the Cauchy–Riemann operator defines a linear map

$$\mathfrak{d}_{u,J} \colon T_j\mathcal{S} \oplus \Gamma(u^*TX) \to \Omega^{0,1}(\Sigma, u^*TX).$$

If u is the inclusion of a J–holomorphic curve C, then $\mathfrak{d}_{C,J} := \mathfrak{d}_{u,J}$.

(2) The **index of** u is

$$\begin{aligned}\operatorname{index} u &:= \operatorname{index} \mathfrak{d}_{u,J} - \dim \mathrm{Aut}(\Sigma, j)\\ &= (\dim X - 6)(1-g) + 2c_1(A) = 2c_1(A)\end{aligned}$$

with $A := u_*[\Sigma] \in \mathrm{H}_2(X, \mathbb{Z})$ and $c_1(A) := \langle c_1(X, \omega), A\rangle$. If u is the inclusion of a J–holomorphic curve C, then the **index of** C is $\operatorname{index} u$.

(3) The map u is **unobstructed with respect to** J if $\operatorname{coker} \mathfrak{d}_{u,J} = 0$. If u is the inclusion of a J–holomorphic curve C, then C is **unobstructed with respect to** J if u is. •

3.1 Gromov–Witten invariants of symplectic 6–manifolds

For every $J \in \mathcal{J}$, $A \in \mathrm{H}_2(X, \mathbb{Z})$, and $\mathrm{g} \in \mathbb{N}_0$ the moduli space $\overline{\mathcal{M}}_{A,\mathrm{g}}(J)$ carries a **virtual fundamental class (VFC)**

$$[\overline{\mathcal{M}}_{A,\mathrm{g}}(J)]^{\mathrm{vir}} \in \check{\mathrm{H}}^{\mathrm{vdim}}(\overline{\mathcal{M}}_{A,\mathrm{g}}(J), \mathbb{Q})^\vee.$$

Here $\check{\mathrm{H}}^*(\cdot, \mathbb{Q})$ denotes Čech comology with rational coefficients, $(\cdot)^\vee := \mathrm{Hom}(\cdot, \mathbb{Q})$ denotes the dual vector space, and vdim is the **virtual dimension** of the moduli space

$$\mathrm{vdim} := (\dim X - 6)(1-g) + 2c_1(A) = 2c_1(A); \tag{3.3}$$

cf. Definition 3.2. The VFC is independent of J in the following sense. If $\mathbf{J} = (J_t)_{t\in[0,1]}$ is a path in $\mathcal{J}$, then

$$[\overline{\mathcal{M}}_{A,g}(J_0)]^{\mathrm{vir}} = [\overline{\mathcal{M}}_{A,g}(J_1)]^{\mathrm{vir}} \quad \text{in } \check{\mathrm{H}}^{\mathrm{vdim}}(\overline{\mathcal{M}}_{A,g}(\mathbf{J}), \mathbb{Q})^\vee \tag{3.4}$$

The reader can find the details of this in [Par16, Section 9.3].

If $A \in \mathrm{H}_2(X,\mathbb{Z})$ is a **Calabi–Yau class**; that is: $c_1(A) = 0$, then

$$\mathrm{vdim} = 0 \quad \text{and} \quad [\overline{\mathcal{M}}_{A,g}(J)]^{\mathrm{vir}} \in \check{\mathrm{H}}^0(\overline{\mathcal{M}}_{A,g}(J), \mathbb{Q})^\vee.$$

In that case, the **Gromov–Witten invariant** is obtained by pairing the VFC with $1 \in \check{\mathrm{H}}^0(\overline{\mathcal{M}}_{A,g}(J), \mathbb{Q})$:

$$\mathrm{GW}_{A,g} = \mathrm{GW}_{A,g}(X,\omega) := \int_{[\overline{\mathcal{M}}_{A,g}(J)]^{\mathrm{vir}}} 1 \in \mathbb{Q}$$

for $J \in \mathcal{J}$. Since $\mathcal{J}$ is path-connected and by (3.4), $\mathrm{GW}_{A,g}$ is independent of $\mathcal{J}$. It is convenient to package these into the **Gromov–Witten series**:

$$\mathrm{GW} = \mathrm{GW}(X,\omega) := \sum_{A\in\Gamma} \sum_{g=0}^{\infty} \mathrm{GW}_{A,g} \cdot t^{2g-2} q^A$$

with

$$\Gamma := \{A \in \mathrm{H}_2(X,\mathbb{Z}) : A \neq 0, c_1(A) = 0\}$$

denoting the set of non-zero Calabi–Yau classes.

For $A \in \Gamma$ and $g \in \mathbb{N}_0$ if

$$\overline{\mathcal{M}}_{A,g}(J) = \coprod_{i\in I} \mathcal{A}_i$$

is a finite decomposition into open and closed subsets, then the VFC decomposes accordingly

$$[\overline{\mathcal{M}}_{A,g}(J)]^{\mathrm{vir}} = \sum_{i\in I} [\mathcal{A}_i]^{\mathrm{vir}}; \tag{3.5}$$

see [Par16, Lemma 5.2.3]. Therefore,

$$\mathrm{GW}_{A,g} = \sum_{i\in I} \mathrm{GW}_{A,g}(\mathcal{A}_i) \quad \text{with} \quad \mathrm{GW}_{A,g}(\mathcal{A}_i) := \int_{[\mathcal{A}_i]^{\mathrm{vir}}} 1.$$

The number $\mathrm{GW}_{A,g}(\mathcal{A}_i)$ is the **Gromov–Witten contribution of** $\mathcal{A}_i$.

For the purpose of this article it is convenient to truncate the Gromov–Witten series GW according to an upper bound Λ on the mass, or energy, of pseudo-holomorphic maps. For $A \in \mathrm{H}_2(X,\mathbb{Z})$ set $\mathbf{M}(A) := \langle[\omega], A\rangle$. The **$\Lambda$–truncated Gromov–Witten series** is

$$\mathrm{GW}_\Lambda = \mathrm{GW}_\Lambda(X,\omega) := \sum_{A\in\Gamma_\Lambda} \sum_{g=0}^{\infty} \mathrm{GW}_{A,g} \cdot t^{2g-2} q^A$$

with

$$\Gamma_\Lambda := \{A \in \Gamma : \mathbf{M}(A) \leqslant \Lambda\} \tag{3.6}$$

denoting the set of non-zero Calabi–Yau classes of mass at most Λ. Denote by

$$\overline{\mathcal{M}}_\Lambda := \coprod_{A\in\Gamma_\Lambda} \coprod_{g=0}^{\infty} \overline{\mathcal{M}}_{A,g}$$

the universal moduli space of stable nodal pseudo-holomorphic maps of index zero and mass at most Λ; this is an open and closed subset of $\overline{\mathcal{M}}$. Moreover, the subspaces $\overline{\mathcal{M}}_\Lambda(J)$, $\mathcal{M}_\Lambda^{\mathrm{si}}$, $\mathcal{M}_\Lambda^{\mathrm{emb}}$, etc. are defined analogously. By the preceding discussion, if

$$\overline{\mathcal{M}}_\Lambda(J) = \coprod_{i \in I} \mathcal{A}_i$$

is a finite decomposition into open and closed subsets, then GW_Λ decomposes accordingly

$$\mathrm{GW}_\Lambda = \sum_{i \in I} \mathrm{GW}_\Lambda(\mathcal{A}_i).$$

The upcoming topological lemma describes a method for decomposing $\overline{\mathcal{M}}_\Lambda(J)$ into open and closed subsets. It is the foundation of the concept of Λ–cluster introduced in Section 3.2 which lies at the heart of the proof of Theorem 1.7.

Definition 3.7. Let $\Lambda > 0$, and let Γ_Λ be as in (3.6). Let $\mathcal{C}$ be the universal space of pseudo-holomorphic cycles with connected support over $\mathcal{J}$, as in Definition 2.15 (3).

(1) Set

$$\mathcal{C}_\Lambda := \{(J, C) \in \mathcal{C} : [C] \in \Gamma_\Lambda\}.$$

(2) For $J \in \mathcal{J}$ and $S \subset \mathcal{J}$ set

$$\mathcal{C}_\Lambda(J) := \mathcal{C}_\Lambda \cap \mathrm{pr}_{\mathcal{J}}^{-1}(J) \quad \text{and} \quad \mathcal{C}_\Lambda(S) := \mathcal{C}_\Lambda \cap \mathrm{pr}_{\mathcal{J}}^{-1}(S)$$

(3) Let $\mathcal{K}$ be the space of compact subsets of X, as in Definition 2.3, and let $\mathrm{supp}\colon \mathcal{C} \to \mathcal{K}$ be the map from Definition 2.6 (2). For $J \in \mathcal{J}$ and $S \subset \mathcal{J}$, and $\mathcal{U} \subset \mathcal{K}$ set

$$\mathcal{C}_\Lambda(J, \mathcal{U}) := \mathcal{C}_\Lambda(J) \cap \mathrm{supp}^{-1}(\mathcal{U}) \quad \text{and} \quad \mathcal{C}_\Lambda(S, \mathcal{U}) := \mathcal{C}_\Lambda(S) \cap \mathrm{supp}^{-1}(\mathcal{U}). \qquad \bullet$$

Lemma 3.8 (Open-Closed Contribution). *Let $\Lambda > 0$, $S \subset \mathcal{J}$, and $\mathcal{U} \subset \mathcal{K}$. If $\mathcal{C}_\Lambda(S, \mathcal{U})$ is open and closed in $\mathcal{C}_\Lambda(S)$, then the following hold:*

(1) *For every $J \in S$*

$$\overline{\mathcal{M}}_\Lambda(J; \mathcal{U}) := \mathfrak{z}^{-1}(\mathcal{C}_\Lambda(J, \mathcal{U}))$$

is open and closed in $\overline{\mathcal{M}}_\Lambda(J)$. Here $\mathfrak{z}\colon \overline{\mathcal{M}} \to \mathcal{C}$ is as in Definition 2.14. In particular, for every $J \in S$, $\mathcal{U}$ has a Gromov–Witten contribution

$$\mathrm{GW}_\Lambda(\mathcal{U}, J) := \mathrm{GW}_\Lambda(\overline{\mathcal{M}}_\Lambda(\mathcal{U}; J)).$$

(2) *The Gromov–Witten contribution $\mathrm{GW}_\Lambda(\cdot, \mathcal{U})$ of $\mathcal{U}$ is constant in paths in S, that is:*

$$\mathrm{GW}_\Lambda(\mathcal{U}, J_0) = \mathrm{GW}_\Lambda(\mathcal{U}, J_1)$$

for every path $\mathbf{J} = (J_t)_{t \in [0,1]}$ in S.

Proof. Since $\mathfrak{z}$ is continuous, $\mathfrak{z}^{-1}(\mathcal{C}_\Lambda(S, \mathcal{U})) \subset \overline{\mathcal{M}}_\Lambda(S)$ is open and closed. The same holds for $\{J\}$ and $\mathbf{J}$ instead of S. This implies (1) and, together with (3.4) and (3.5), also (2). ∎

3.2 Cluster formalism

This section relies on the results from Section 2. The reader might find it helpful to review the definitions and results from Section 2.1. In particular, the following facts will be used:

(1) For every $\Lambda > 0$, $\mathcal{C}_\Lambda$ is open and closed in $\mathcal{C}$ and the projection map $\mathrm{pr}_{\mathcal{J}} \colon \mathcal{C}_\Lambda \to \mathcal{J}$ is proper and, therefore, also closed with respect to the geometric topology; see Theorem 2.11.

(2) The map $\mathfrak{z} \colon \mathcal{M}^{\mathrm{emb}} \to \mathcal{C}^{\mathrm{emb}}$ is a homeomorphism with respect to the Gromov topology and the geometric topology respectively; see Proposition 2.21.

(3) For every $A \in \mathrm{H}_2(X, \mathbb{Z})$ the map $(\mathrm{pr}_{\mathcal{J}}, \mathrm{supp}) \colon \mathcal{C}_A^{\mathrm{si}} \to \mathcal{J} \times \mathcal{K}$ is an embedding with respect to the geometric topology and the topology induced by the Hausdorff metric respectively; see Proposition 2.24.

Definition 3.9. Let $\Lambda > 0$. A Λ**–cluster** is a triple $\mathcal{O} = (\mathcal{U}, J, C)$ consisting of an open subset $\mathcal{U} \subset \mathcal{K}$, an almost complex structure $J \in \mathcal{J}$, and an irreducible, embedded J–holomorphic curve C, the **core** of $\mathcal{O}$, such that:

(1) There is no J–holomorphic curve C' with $\mathbf{M}(C') \leqslant \Lambda$ and $\mathrm{supp}\, C' \in \partial\mathcal{U} := \overline{\mathcal{U}} \backslash \mathcal{U}$.

(2) There is a Calabi–Yau class $A \in \Gamma$ such that for every J–holomorphic curve C' with $\mathbf{M}(C') \leqslant \Lambda$ and $\mathrm{supp}\, C' \in \overline{\mathcal{U}}$ there is a $k \in \mathbb{N}$ with $[C'] = kA$.
(In particular, every such C' is of index zero.)

(3) C is the unique J–holomorphic curve with $\mathrm{supp}\, C \in \mathcal{U}$ and $[C] = A$. •

Proposition 3.10 (Cluster Contribution). *Let $J \in \mathcal{J}$ and $\Lambda > 0$. If an open set $\mathcal{U} \subset \mathcal{K}$ satisfies Definition 3.9 (1), then there is a connected open neighborhood $\mathcal{V}$ of J in $\mathcal{J}$ such that the subset $\mathcal{C}_\Lambda(\mathcal{V}, \mathcal{U})$ is open and closed in $\mathcal{C}_\Lambda(\mathcal{V})$. In particular, by Lemma 3.8, for every $J' \in \mathcal{V}$, $\mathcal{U}$ has a Gromov–Witten contribution $\mathrm{GW}_\Lambda(\mathcal{U}, J')$ satisfying*

$$\mathrm{GW}_\Lambda(\mathcal{U}, J') = \mathrm{GW}_\Lambda(\mathcal{U}, J).$$

Notation 3.11. Let $\Lambda > 0$. The Gromov–Witten contribution of a Λ–cluster $\mathcal{O} = (\mathcal{U}, J, C)$ is

$$\mathrm{GW}_\Lambda(\mathcal{O}) := \mathrm{GW}_\Lambda(\mathcal{U}, J). \quad \bullet$$

Proof of Proposition 3.10. Since $\partial\mathcal{U}$ is closed in $\mathcal{K}$, $\mathcal{C}_\Lambda(\mathcal{J}, \partial\mathcal{U})$ is closed in $\mathcal{C}_\Lambda$ and thus in $\mathcal{C}$. Since $\mathrm{pr}_{\mathcal{J}} \colon \mathcal{C} \to \mathcal{J}$ is proper by Theorem 2.11, and therefore closed, the set

$$\mathcal{V} := \mathcal{J} \backslash \mathrm{pr}_{\mathcal{J}}(\mathcal{C}_\Lambda(\mathcal{J}, \partial\mathcal{U}))$$

is open; moreover, $J \in \mathcal{V}$ because it satisfies Definition 3.9 (1). By construction, $\mathcal{C}_\Lambda(\mathcal{V}, \partial\mathcal{U}) = \varnothing$. Therefore,

$$\mathcal{C}_\Lambda(\mathcal{V}, \mathcal{U}) = \mathcal{C}_\Lambda(\mathcal{V}, \overline{\mathcal{U}}) \tag{3.12}$$

is open and closed. Finally, replace $\mathcal{V}$ with its connected component containing J. ∎

The goal of the cluster formalism is to decompose the space of J–holomorphic cycles of mass at most Λ into finitely many Λ–clusters with the aim of analysing the Gromov–Witten contribution of each cluster. This can be done provided J belongs to the following class of generic almost complex structures.

Definition 3.13. Denote by $\mathcal{J}_{\mathrm{isol}}$ the subset of those $J \in \mathcal{J}$ for which:

(1) Every simple J–holomorphic map has non-negative index.

(2) Every simple J–holomorphic map of index zero is an embedding.

(3) Every pair of distinct simple J–holomorphic maps of index zero have disjoint images or are related by a reparametrization.

(4) The moduli space of simple J–holomorphic maps of index zero is discrete with respect to the Gromov topology.

Denote by $\mathcal{J}_*$ the subset of those $J \in \mathcal{J}$ satisfying (1), (2), (3), and—instead of (4)—the stronger condition:

(4^+) Every simple J–holomorphic map of index zero is unobstructed; cf. Definition 3.2. •

Proposition 3.14 ([IP18, Lemma 1.2]). *The subset $\mathcal{J}_*$ is comeager in $\mathcal{J}$.*

Recall that a subset of a topological space is **comeager** if it contains a countable intersection of open dense subsets. By the Baire category theorem, a comeager subset of a complete metric space is dense. This applies, in particular, to $\mathcal{J}$.

The significance of $\mathcal{J}_{\mathrm{isol}}$ stems from the following results and the fact that it is path-connected—while the complement of $\mathcal{J}_*$ in $\mathcal{J}$ is of codimension one and, therefore, $\mathcal{J}_*$ is not path-connected; cf. [IP18, Corollary 6.6].

Notation 3.15. For $A, B \in \mathrm{H}_2(X, \mathbb{Z})$ write

$$B|A$$

if there is a $k \in \mathbb{N}$ with $A = kB$. •

Lemma 3.16 (Clustering Behaviour). *For every $J \in \mathcal{J}_{\mathrm{isol}}$ and $A \in \Gamma$ the following hold:*

(1) $\mathcal{C}^{\mathrm{si}}_A(J) = \mathcal{C}^{\mathrm{emb}}_A(J)$.

(2) $\mathcal{C}^{\mathrm{emb}}_A(J)$ *is countable.*

(3) $\operatorname{supp} \mathcal{C}^{\mathrm{emb}}_A(J)$ *is discrete.*

(4) $\operatorname{supp} \mathcal{C}_A(J) = \bigcup_{B|A} \operatorname{supp} \mathcal{C}^{\mathrm{emb}}_B(J)$*; in particular, the latter is compact.*

Proof. Let $(J, C) \in \mathcal{C}$ with $C = \sum_{i=1}^{I} m_i C_i$ and $[C] = A \in \Gamma$, that is, $[C]$ is a Calabi–Yau class. By Definition 3.13 (1) and (3.3), $[C_1], \ldots, [C_I]$ must also be Calabi–Yau classes. By Definition 3.13 (2) and (3), $I = 1$; that is: $C = m_1 C_1$ and C_1 is embedded. This implies (1) and (4).

By Definition 3.13 (4), $\mathcal{M}^{\mathrm{emb}}_A(J) = \mathcal{M}^{\mathrm{si}}_A(J)$ is discrete and, therefore, countable. This implies (2). By Proposition 2.21 and Proposition 2.24, the map $\mathrm{im}\colon \mathcal{M}^{\mathrm{emb}}_A(J) \to \operatorname{supp} \mathcal{C}^{\mathrm{emb}}_A(J)$ is a homeomorphism. This implies (3). ■

Proposition 3.17 (Cluster Existence). *Let $J \in \mathcal{J}_{\mathrm{isol}}$ and $\Lambda > 0$. Let C be an irreducible, embedded J–holomorphic curve of index zero with $\mathbf{M}(C) \leqslant \Lambda$. There is an $\varepsilon_0 > 0$ such that the subset*

$$\{\varepsilon \in (0, \varepsilon_0] : \mathcal{O} = (B_\varepsilon(C), J, C) \text{ is a } \Lambda\text{–cluster}\}$$

is open and dense in $(0, \varepsilon_0]$. Here $B_\varepsilon(C)$ denotes the ball of radius ε centered at C in $\mathcal{K}$.

Proof. By Lemma 3.16 (3), there is an $\varepsilon_0 > 0$ with $\mathcal{C}_A(J, B_{\varepsilon_0}(C)) = \{(J, C)\}$. After possibly decreasing ε_0, C is a deformation retract of $\{x \in X : d(x, C) \leqslant \varepsilon_0\}$. Therefore if C' is a J–holomorphic curve with $d_H(C, \operatorname{supp} C') \leqslant \varepsilon_0$, then $[C'] = k[C]$ with $k \in \mathbb{N}$.

By Lemma 3.16 (2) and Theorem 2.11,

$$\Delta := \{d_H(C, \operatorname{supp} C') : (J, C') \in \mathcal{C}_\Lambda^{\mathrm{emb}}(J)\}$$

is countable and compact. Consequently, $(0, \varepsilon_0]\backslash\Delta$ is open and dense. ∎

Proposition 3.18 (Cluster Decomposition). *Let $J \in \mathcal{J}_{\mathrm{isol}}$ and $\Lambda > 0$. Let $\mathcal{U} \subset \mathcal{K}$ be such that $\mathcal{C}_\Lambda(J, \mathcal{U})$ is open and closed in $\mathcal{C}_\Lambda(J)$. There is a finite set $\{\mathcal{O}_i = (\mathcal{U}_i, J, C_i) : i \in I\}$ of Λ–clusters such that*

$$\mathcal{C}_\Lambda(J, \mathcal{U}) = \coprod_{i \in I} \mathcal{C}_\Lambda(J, \mathcal{U}_i);$$

in particular,

$$\mathrm{GW}_\Lambda(\mathcal{O}) = \sum_{i \in I} \mathrm{GW}_\Lambda(\mathcal{O}_i).$$

Proof. For every $d \in \mathbb{N}$ set

$$\mathcal{N}_d := \bigcup \operatorname{supp} \mathcal{C}_A^{\mathrm{emb}}(J) \cap \mathcal{U} \subseteq \operatorname{supp} \mathcal{C}_\Lambda(J, \mathcal{U}).$$

with the union taken over those $A \in \Gamma_\Lambda$ with divisibility at most d. Since $\mathcal{C}_\Lambda(J)$ is compact by Theorem 2.11, only finitely many $A \in \Gamma_\Lambda$ are represented by J–holomorphic curves. Therefore, these unions are finite and the sequence $\mathcal{N}_1 \subseteq \mathcal{N}_2 \subseteq \ldots$ eventually becomes constant.

By Lemma 3.16 (3) and (4), $\mathcal{N}_1$ is discrete and compact; hence: finite. Enumerate $\mathcal{N}_1$ as $\{C_1, \ldots, C_{n_1}\}$. For every $i \in \{1, \ldots, n_1\}$ choose $\varepsilon_i > 0$—by means of Proposition 3.17—such that for $\mathcal{U}_i := B_{\varepsilon_i}(C_i) \subset \mathcal{U}$ the triple $\mathcal{O}_i := (\mathcal{U}_i, J, C_i)$ is a Λ–cluster, and $\mathcal{U}_i \cap \mathcal{U}_j = \varnothing$ for $i \neq j \in \{1, \ldots, n_1\}$. Set

$$\hat{\mathcal{N}}_2 := \mathcal{N}_2 \backslash \coprod_{i=1}^{n_1} B_{\varepsilon_i}(C_i).$$

By Lemma 3.16 (3) and (4), $\hat{\mathcal{N}}_2$ is discrete and compact; hence: finite. Enumerate $\hat{\mathcal{N}}_2$ as $\{C_{n_1+1}, \ldots, C_{n_2}\}$. For $i \in \{n_1 + 1, \ldots, n_2\}$ choose $\varepsilon_i > 0$ such that for $\mathcal{U}_i := B_{\varepsilon_i}(C_i)$ the triple $\mathcal{O}_i := (\mathcal{U}_i, J, C_i)$ is a Λ–cluster, and $\mathcal{U}_i \cap \mathcal{U}_j = \varnothing$ for $i \neq j \in \{1, \ldots, n_2\}$. Continuing in this fashion constructs the desired decomposition. ∎

Proposition 3.19 (Cluster Refinement). *Let $J \in \mathcal{J}_{\mathrm{isol}}$ and $\Lambda > 0$. The following hold:*

1. *If $\mathcal{O}_0 = (\mathcal{U}_0, J, C)$ and $\mathcal{O}_0 = (\mathcal{U}_1, J, C)$ are Λ–clusters with identical cores, then there exists a $\mathcal{U} \subset \mathcal{U}_0 \cap \mathcal{U}_1$ such that $\mathcal{O} = (\mathcal{U}, J, C)$ is a Λ–cluster.*
2. *If $\mathcal{O}_+ = (\mathcal{U}_+, J, C)$ and $\mathcal{O}_- = (\mathcal{U}_-, J, C)$ are Λ–clusters with identical cores and $\mathcal{U}_- \subset \mathcal{U}_+$, then there is a finite set $\{\mathcal{O}_i = (\mathcal{U}_i, J, C_i) : i \in I\}$ of Λ–clusters such that*

$$\mathcal{C}_\Lambda(J, \mathcal{U}_+) = \mathcal{C}_\Lambda(J, \mathcal{U}_-) \amalg \coprod_{i \in I} \mathcal{C}_\Lambda(J, \mathcal{U}_i),$$

and, for every $i \in I$, $[C_i] = d_i[C]$ with $d_i \geqslant 2$; in particular,

$$\mathrm{GW}_\Lambda(\mathcal{O}_+) = \mathrm{GW}_\Lambda(\mathcal{O}_-) + \sum_{i \in I} \mathrm{GW}_\Lambda(\mathcal{O}_i).$$

Proof. Since $\mathcal{U}_0, \mathcal{U}_1$ are open neighborhoods of C, Proposition 3.17 implies (1).

To prove (2), observe the following. Since $\mathcal{C}_\Lambda(J, \mathcal{U}_+)\backslash\mathcal{C}_\Lambda(J, \mathcal{U}_-)$ is open and closed in $\mathcal{C}_\Lambda(J)$, Proposition 3.18 constructs $\{\mathcal{O}_i = (\mathcal{U}_i, J, C_i) : i \in I\}$. Since $\mathcal{O}_+$ is a Λ–cluster, the cores C_i must satisfy $[C_i] = d_i[C]$ with $d_i \geqslant 2$. ∎

Proposition 3.20 (Cluster Stability). *Let* $\Lambda > 0$ *and* $J \in \mathcal{J}$. *Let* $\mathcal{O} = (\mathcal{U}, J, C)$ *be a* Λ*–cluster. Set* $A \coloneqq [C]$ *and* $\mathrm{g} \coloneqq \mathrm{g}(C)$. *For every* $0 < \varepsilon \ll 1$ *there is an open neighborhood* $\mathcal{V}$ *of* J *in* $\mathcal{J}$ *such that for every* $J' \in \mathcal{V}$ *Definition 3.9 (1) and (2) hold; that is:*

(1) *There is no* J'*–holomorphic curve* C' *with* $\mathbf{M}(C') \leqslant \Lambda$ *and* $\operatorname{supp} C' \in \partial\mathcal{U}$.

(2) *For every* J'*–holomorphic curve* C' *with* $\mathbf{M}(C') \leqslant \Lambda$ *and* $\operatorname{supp} C' \in \overline{\mathcal{U}}$ *there is a* $k \in \mathbb{N}$ *with* $[C'] = kA$.

Moreover:

(3) *The map* $\mathfrak{z}\colon \mathcal{M}^{\mathrm{emb}}_{A,\mathrm{g}}(\mathcal{V}, B_\varepsilon(C)) \to \mathcal{C}_A(\mathcal{V}, \mathcal{U})$ *is a homeomorphism; in particular:* $\mathcal{C}_A(\mathcal{V}, \mathcal{U}) = \mathcal{C}^{\mathrm{emb}}_A(\mathcal{V}, \mathcal{U})$.

Proof. The subset

$$\Delta \coloneqq \{(J', C') \in \mathcal{C}_\Lambda : \operatorname{supp} C' \in \partial\mathcal{U} \text{ or } (\operatorname{supp} C' \in \overline{\mathcal{U}} \text{ and } [C'] \notin \mathbb{N} \cdot [C])\}$$

is closed because the map $(\operatorname{supp}, [\cdot])\colon \mathcal{C}_\Lambda \to \mathcal{K} \times \mathrm{H}_2(X, \mathbb{Z})$ is continuous. Since $\mathrm{pr}_{\mathcal{J}}\colon \mathcal{C}_\Lambda \to \mathcal{J}$ is closed by Theorem 2.11, $\mathrm{pr}_{\mathcal{J}}(\Delta)$ is closed. Set

$$\mathcal{V} \coloneqq \mathcal{J} \backslash \mathrm{pr}_{\mathcal{J}}(\Delta).$$

By construction, $J \in \mathcal{V}$, $\mathcal{V}$ is open, and $\Delta \cap \mathrm{pr}_{\mathcal{J}}^{-1}(\mathcal{V}) = \varnothing$. This proves (1) and (2).

In light of Proposition 2.21, it suffices to show that $\mathfrak{z}\colon \mathcal{M}^{\mathrm{emb}}_{A,\mathrm{g}}(\mathcal{V}, B_\varepsilon(C)) \to \mathcal{C}_A(\mathcal{V}, \mathcal{U})$ is surjective to prove (3). Consider $(J_n, C_n) \in (\mathcal{C}_A(\mathcal{J}, \mathcal{U}))^{\mathbb{N}}$ with (J_n) converging to J. By Theorem 2.11, (C_n) geometrically converges to a J–holomorphic cycle C' with $(J, C') \in \mathcal{C}_A$ and $\operatorname{supp} C' \in \overline{\mathcal{U}}$. Since $\mathcal{O} = (\mathcal{U}, J, C)$ is a Λ–cluster, $C' = C$. Therefore, $\mathfrak{z}$ is surjective—possibly after shrinking $\mathcal{V}$. ■

Proposition 3.21 (Cluster Perturbation). *Let* $\Lambda > 0$ *and* $J \in \mathcal{J}$. *Let* $\mathcal{O} = (\mathcal{U}, J, C)$ *be a* Λ*–cluster. There is a connected open neighborhood* $\mathcal{V}$ *of* J *in* $\mathcal{J}$ *such that the subset* $\mathcal{C}_\Lambda(\mathcal{V}, \mathcal{U})$ *is open and closed in* $\mathcal{C}_\Lambda(\mathcal{V})$ *and the following hold:*

(1) *If* C *is unobstructed, then for every* $J' \in \mathcal{V}$ *there is a unique* J'*–holomorphic curve* C' *such that* $\mathcal{O}' = (\mathcal{U}, J', C')$ *is a* Λ*–cluster.*

(2) *For every* $J' \in \mathcal{J}_* \cap \mathcal{V}$ *(which is non-empty by Proposition 3.14) there is a finite set* $\{\mathcal{O}_i = (\mathcal{U}_i, J', C_i) : i \in I\}$ *of* Λ*–clusters such that*

$$\mathcal{C}_\Lambda(J', \mathcal{U}) = \coprod_{i \in I} \mathcal{C}_\Lambda(J', \mathcal{U}_i),$$

and, for every $i \in I$, $[C_i] = d_i[C]$ *with* $d_i \geqslant 1$ *and* C_i *is unobstructed with respect to* J'*; in particular,*

$$\mathrm{GW}_\Lambda(\mathcal{O}) = \sum_{i \in I} \mathrm{GW}_\Lambda(\mathcal{O}_i).$$

Proof. Let $\mathcal{V}$ be the connected component of J of the open subset constructed in Proposition 3.20. By Proposition 3.20 (1) and the argument in the proof of Proposition 3.10, $\mathcal{C}_\Lambda(\mathcal{V}, \mathcal{U})$ is open and closed in $\mathcal{C}_\Lambda(\mathcal{V})$.

By the deformation theory of pseudo-holomorphic maps, if C is unobstructed, then for $0 < \varepsilon \ll 1$ and after possibly shrinking $\mathcal{V}$ the map $\mathrm{pr}_{\mathcal{J}}\colon \mathcal{M}^{\mathrm{emb}}_{[C],\mathrm{g}}(\mathcal{V}, B_\varepsilon(C)) \to \mathcal{V}$ is a diffeomorphism. Therefore, by Proposition 3.20 (3), for every $J' \in \mathcal{V}$ there is a unique J'–holomorphic curve C' with $\operatorname{supp} C' \in \mathcal{U}$ and $[C'] = [C]$. This proves (1).

(2) is a consequence of Proposition 3.18. ■

In applications, it is convenient to choose the open sets $\mathcal{U}$ and $\mathcal{V}$ in Proposition 3.20 and Proposition 3.21 to be arbitrarily small.

Proposition 3.22. *Let $J \in \mathcal{J}_{\mathrm{isol}}$ and $\Lambda > 0$. Let C be an irreducible, embedded J–holomorphic curve of index zero with $\mathbf{M}(C) \leqslant \Lambda$. Set $A := [C]$ and $\mathrm{g} := \mathrm{g}(C)$. In this situation, there exists a basis of open neighborhoods of (J, C) in $\mathcal{C}_A$ consisting of subsets of the form $\mathcal{C}_A(\mathcal{V}, \mathcal{U})$ such that:*

(1) *$\mathcal{V}$ is open in $\mathcal{J}$ and $\mathcal{U}$ is open in $\mathcal{K}$ (we can take $\mathcal{U} = B_\varepsilon(C)$).*

(2) *$\mathcal{O} = (\mathcal{U}, J, C)$ is a Λ–cluster, and*

(3.23) $$\mathrm{GW}_\Lambda(\mathcal{U}, J') = \mathrm{GW}_\Lambda(\mathcal{O})$$

for every $J' \in \mathcal{V}$.

(3) *The maps*

(3.24) $$\mathcal{M}^{\mathrm{emb}}_{A,\mathrm{g}}(\mathcal{V}, \mathcal{U}) \to \mathcal{C}_A(\mathcal{V}, \mathcal{U}) \quad \text{and} \quad \mathcal{C}^{\mathrm{emb}}_A(\mathcal{V}, \mathcal{U}) \to \operatorname{supp} \mathcal{C}^{\mathrm{emb}}_A(\mathcal{V}, \mathcal{U})$$

are homeomorphisms.

Proof. Since C is embedded, by Corollary 2.29 the basis of the topology on $\mathcal{C}_A$ at (J, C) consists of subsets of the form

$$\{(J', C') \in \mathcal{C}_A : J' \in \mathcal{V} \text{ and } \operatorname{supp} C' \in B_\varepsilon(C)\}$$

with $\varepsilon > 0$ and $\mathcal{V}$ from a basis of open, connected neighborhoods of J in $\mathcal{J}$. Therefore, the corollary follows from Proposition 3.20 and Proposition 3.10. Note that Proposition 2.21 and Proposition 2.24 imply that $(\mathrm{pr}_{\mathcal{J}}, \mathrm{im})\colon \mathcal{M}^{\mathrm{emb}}_A(\mathcal{V}, \mathcal{U}) \to \mathcal{V} \times \mathcal{U}$ is an embedding whose image is $\operatorname{supp} \mathcal{C}^{\mathrm{emb}}_A(V, \mathcal{U})$ and whose domain is equal to $\mathcal{M}^{\mathrm{emb}}_{A,\mathrm{g}}(\mathcal{V}, \mathcal{U})$ by Proposition 3.20-(2). ■

The crucial result for the proof of Theorem 1.7 is the following.

Theorem 3.25 (Cluster Isotopy). *Let $\Lambda > 0$. Let $\mathcal{O}_0 = (\mathcal{U}_0, J_0, C)$ and $\mathcal{O}_1 = (\mathcal{U}_1, J_1, C)$ be Λ–clusters with identical cores. If $J_0, J_1 \in \mathcal{J}_{\mathrm{isol}}$ and C is unobstructed with respect to J_0 and J_1, then there is a finite set $\{\mathcal{O}_i = (\mathcal{U}_i, J_i, C_i) : i \in I\}$ of Λ–clusters such that*

$$\mathrm{GW}_\Lambda(\mathcal{O}_1) = \pm \mathrm{GW}_\Lambda(\mathcal{O}_0) + \sum_{i \in I} \pm \mathrm{GW}_\Lambda(\mathcal{O}_i);$$

moreover, for every $i \in I$, $J_i \in \mathcal{J}_{\mathrm{isol}}$, C_i is unobstructed with respect to J_i, and $[C_i] = d_i[C]$ with $d_i \geqslant 2$.

3.3 Proof of the cluster isotopy theorem

This section provides the proof of Theorem 3.25. The results of Section 3.2 are sufficient to carry out the argument from [IP18, Section 7] with only minor changes in notation. In this section, $\Lambda > 0$ is fixed and $A \in \mathrm{H}_2(X, \mathbb{Z})$ is a Calabi–Yau class satisfying $\mathbf{M}(A) \leqslant \Lambda$.

Notation 3.26. Following [IP18, Section 7], given two Λ–clusters $\mathcal{O}_0 = (\mathcal{U}_0, J_0, C_0)$ and $\mathcal{O}_1 = (\mathcal{U}_1, J_1, C_1)$ with $[C_0] = [C_1]$ write

(3.27) $$\mathrm{GW}_\Lambda(\mathcal{O}_0) \approx \mathrm{GW}_\Lambda(\mathcal{O}_1)$$

if there is a finite set $\{\mathcal{O}_i = (\mathcal{U}_i, J_i, C_i) : i \in I\}$ of Λ–clusters such that

$$\mathrm{GW}_\Lambda(\mathcal{O}_1) = \mathrm{GW}_\Lambda(\mathcal{O}_0) + \sum_{i \in I} \pm \mathrm{GW}_\Lambda(\mathcal{O}_i),$$

and, for every $i \in I$, $[C_i] = d_i[C_0]$ with $d_i \geqslant 2$. Similarly, we will write $\mathrm{GW}_\Lambda(\mathcal{O}_0) \approx -\mathrm{GW}_\Lambda(\mathcal{O}_1)$, $\mathrm{GW}_\Lambda(\mathcal{O}_0) \approx 0$, and so on, when the equality holds modulo finitely many contributions of Λ–clusters with cores representing homology classes $d[C_0]$ with $d \geqslant 2$. •

Remark 3.28. While the notation might suggest otherwise, it is worth pointing out that (3.27) is actually an equivalence relation of Λ–clusters rather than of power series, as the homology class of the core plays an important role in the definition of (3.27). •

Notation 3.29. Following [IP18, Section 5], we consider the following subspaces of $\mathcal{M}^{\mathrm{si}}$:

$$\mathcal{W} := \{(J, [u]) \in \mathcal{M}^{\mathrm{si}} : \dim\ker \mathfrak{d}_{J,u} > 0\} = \bigcup_{k \in \mathbb{N}} \mathcal{W}^k,$$
$$\text{with } \mathcal{W}^k := \{(J, [u]) \in \mathcal{M}^{\mathrm{si}} : \dim\ker \mathfrak{d}_{J,u} = k\},$$

with $\mathfrak{d}_{J,u}$ being the linearization of the Cauchy–Riemann operator; see Definition 3.2.

Denote by $\mathcal{A}$ the set of points in $\mathcal{W}^1$ where the projection map $\pi\colon \mathcal{W}^1 \to \mathcal{J}$ fails to be an immersion, cf. [IP18, Section 5.4]. •

By [IP18, Proposition 5.3], $\mathcal{W}^1 \cap \mathcal{M}^{\mathrm{si}}_\Lambda$ is a codimension one submanifold of $\mathcal{M}^{\mathrm{si}}_\Lambda$. By [IP18, Lemma 5.6], $\mathcal{A} \cap \mathcal{M}^{\mathrm{emb}}_\Lambda$ is a codimension one submanifold of $\mathcal{W}^1$. (Recall that, by definition, all pseudo-holomorphic maps in $\mathcal{M}^{\mathrm{si}}_\Lambda$ have index zero.)

Proposition 3.30 (Simple Isotopy). *Let $(J_t, [u_t])_{t \in [0,1]}$ be a path in $\mathcal{M}^{\mathrm{emb}}_{A,\mathrm{g}}$ disjoint from $\mathcal{W}$ and such that $J_t \in \mathcal{J}_{\mathrm{isol}}$ for all $t \in [0,1]$. Let C_i be the image of u_i for $i = 0, 1$. If $\mathcal{O}_i = (\mathcal{U}_i, J_i, C_i)$ is a Λ–cluster for $i = 0, 1$, then*

$$\mathrm{GW}_\Lambda(\mathcal{O}_0) \approx \mathrm{GW}_\Lambda(\mathcal{O}_1).$$

Proof. The proof is identical to that of [IP18, Lemma 7.2], except that we use a different definition of an Λ–cluster and invoke the results of Section 3.2 to control sequences of curves without an a priori genus bound.

Assume that $\mathcal{O}_i = (\mathcal{U}_i, J_i, C_i)$ is a Λ–cluster for $i = 0, 1$. Let $C_t = \operatorname{im} u_t$; this is a family of curves of genus g representing A; in particular, of index zero. Since the path $(J_t, [u_t])_{t \in [0,1]}$ is disjoint from $\mathcal{W}$, it follows from the standard deformation theory for pseudo-holomorphic maps that there is an open neighborhood $\mathcal{Q}$ of the path $(J_t, [u_t])_{t \in [0,1]}$ in $\mathcal{M}^{\mathrm{emb}}_{A,g}$ such that for every $t \in [0,1]$,

$$\mathcal{M}^{\mathrm{emb}}_{A,\mathrm{g}}(J_t) \cap \mathcal{Q} = \{(J_t, [u_t])\}; \tag{3.31}$$

see, for example, [IP18, Proposition 5.3]. Since $\mathbf{J}$ is contained in $\mathcal{J}_{\mathrm{isol}}$, for every $t \in [0,1]$ we can apply Proposition 3.22 to J_t and C_t to produce open subsets $\mathcal{U}_t \subset \mathcal{K}$ and $\mathcal{V}_t \subset \mathcal{J}$ with all the properties listed in Proposition 3.22. We may, moreover, choose them in such a way that

$$\mathcal{M}^{\mathrm{emb}}_{A,\mathrm{g}}(\mathcal{V}_t, \mathcal{U}_t) \subset \mathcal{Q}. \tag{3.32}$$

In particular, for every $t \in [0,1]$:

(1) The triple $(\mathcal{U}_t, J_t, C_t)$ is a Λ–cluster.

(2) The map

$$\mathcal{M}^{\text{emb}}_{A,g}(\mathcal{V}_t, \mathcal{U}_t) \to \mathcal{C}_A(\mathcal{V}_t, \mathcal{U}_t)$$

is a homeomorphism.

(3) Since $\mathbf{J} \subset \mathcal{J}_{\text{isol}}$, by (3.31), (3.32) and property (2) above, we have

$$\mathcal{C}_A(J_s, \mathcal{U}_t) = \mathcal{C}^{\text{emb}}_A(J_s, \mathcal{U}_t) = \{(J_s, C_s)\}$$

for every $s \in [0,1]$ such that $J_s \in \mathcal{V}_t$.

(4) Moreover, for every such $s \in [0,1]$,

$$\text{GW}_\Lambda(\mathcal{U}_t, J_s) = \text{GW}_\Lambda(\mathcal{U}_t, J_t).$$

Since $[0,1]$ is compact, there are

$$0 = t_0 < t_1 < \ldots < t_m = 1$$

and $\delta_{t_0}, \ldots, \delta_{t_m} > 0$ such that the intervals $I_i = \{s \in [0,1] : |s - t_i| < \delta_{t_i}\}$ cover $[0,1]$ and for $s \in I_i$ we have $J_s \in \mathcal{V}_{t_i}$. Let $s \in I_i \cap I_{i+1}$. It follows from the preceding discussion and Proposition 3.19 that

$$\text{GW}_\Lambda(\mathcal{U}_{t_i}, J_{t_i}) = \text{GW}_\Lambda(\mathcal{U}_{t_i}, J_s) \approx \text{GW}_\Lambda(\mathcal{U}_{t_{i+1}}, J_s) = \text{GW}_\Lambda(\mathcal{U}_{t_{i+1}}, J_{t_{i+1}}).$$

We conclude that $\text{GW}_\Lambda(\mathcal{O}_0) \approx \text{GW}_\Lambda(\mathcal{U}_{t_0}, J_{t_0}) \approx \cdots \approx \text{GW}_\Lambda(\mathcal{U}_{t_m}, J_{t_m}) \approx \text{GW}(\mathcal{O}_1)$. ■

Proposition 3.33 (Wall-crossing in $\mathcal{J}$). *Let $\mathbf{J} = (J_t)_{t\in[-1,1]}$ be a C^1 path in $\mathcal{J}$, contained in $\mathcal{J}_{\text{isol}}$. Suppose that $\pi\colon \mathcal{M}^{\text{emb}}_{A,g} \to \mathcal{J}$ is transverse to the path $\mathbf{J}$ at a point $p_0 = (J_0, [u_0]) \in \mathcal{W}^1\backslash\mathcal{A}$. Then:*

(1) *There exist $\delta > 0$, $\sigma = \pm$, and an open neighborhood $\mathcal{Q}$ of p_0 in $\mathcal{M}^{\text{emb}}_{A,g}$ such that for all $t \neq 0$,*

$$\mathcal{M}^{\text{emb}}_{A,g}(J_t) \cap \mathcal{Q} = \begin{cases} \{p_t^+, p_t^-\} & \textit{for } 0 < |t| < \delta,\ \operatorname{sign} t = \sigma, \\ \varnothing & \textit{for } 0 < |t| < \delta,\ \operatorname{sign} t = -\sigma, \end{cases} \tag{3.34}$$

where $p_t^\pm = (J_t, [u_t^\pm]) \in \mathcal{M}^{\text{emb}}_{A,g}\backslash\mathcal{W}$ and $\lim_{t\to 0^\sigma} p_t^\pm = p_0$.

(2) *Let t be such that $0 < |t| < \delta$ and $\operatorname{sign} t = \sigma$, and set $C_t^\pm = \operatorname{im} u_t^\pm$. If $\mathcal{O}_t^\pm = (\mathcal{U}_t^\pm, J_t, C_t^\pm)$ are Λ–clusters, then*

$$\text{GW}_\Lambda(\mathcal{O}_t^+) \approx -\text{GW}_\Lambda(\mathcal{O}_t^-).$$

(Note that such $\mathcal{U}_t^+$ and $\mathcal{U}_t^-$ exist by Proposition 3.17.)

Proof. The proof is identical to that of [IP18, Lemma 7.3]. Part (1) is a consequence of the standard local model for the birth-death bifurcation for simple pseudo-holomorphic maps; see, for example, [IP18, Theorem 6.2, Corollary 6.3].

It remains to prove part (2). Suppose without loss of generality that $\sigma = -$ and set

$$C_0 = \operatorname{im} u_0, \quad C_t^\pm = \operatorname{im} u_t^\pm.$$

Since $\mathbf{J}$ is contained in $\mathcal{J}_{\text{isol}}$, we can apply Proposition 3.22 to J_0 and C_0 to produce open neighborhoods $\mathcal{U} \subset \mathcal{K}$ and $\mathcal{V} \subset \mathcal{J}$ with all the properties listed in Proposition 3.22, and such that

$$\mathcal{M}^{\text{emb}}_{A,g}(\mathcal{V}, \mathcal{U}) \subset \mathcal{Q}. \tag{3.35}$$

where $\mathcal{Q}$ is the open neighborhood in (3.34). In particular,

(1) The triple $(\mathcal{U}_0, J_0, C_0)$ is a Λ–cluster.

(2) The map
$$\mathfrak{z} : \mathcal{M}^{\mathrm{emb}}_{A,\mathrm{g}}(\mathcal{V}, \mathcal{U}) \to \mathcal{C}_A(\mathcal{V}, \mathcal{U})$$
is a homeomorphism.

(3) Since $J_t \in \mathcal{J}_{\mathrm{isol}}$, then (3.34) and (3.35) imply
$$\mathcal{C}_A(J_t, \mathcal{U}) = \mathcal{C}^{\mathrm{emb}}_A(J_t, \mathcal{U}) = \begin{cases} \{C_t^+,\ C_t^-\} & \text{for } -\delta < t < 0, \\ \varnothing & \text{for } 0 < t < \delta. \end{cases}$$
for some $\delta > 0$ sufficiently small so that the path $(J_t)_{t\in[-\delta,\delta]}$ is contained in $\mathcal{V}$.

For $-\delta < t < 0$, let $\mathcal{U}_t^\pm$ be an open neighborhood of $C_t^\pm$ in $\mathcal{K}$ such that the triples $\mathcal{O}_t^\pm = (\mathcal{U}_t^\pm, J_t, C_t^\pm)$ are Λ–clusters and
$$\mathcal{U}_t^+ \cap \mathcal{U}_t^- = \varnothing, \quad \mathcal{U}_t^\pm \subset \mathcal{U}.$$
Such open neighborhoods exist by Proposition 3.17. By Proposition 3.10, $\mathcal{C}_\Lambda(J_t, \mathcal{U}_t^\pm)$ is open and closed in $\mathcal{C}_\Lambda(J_t)$. Since $J_t \in \mathcal{J}_{\mathrm{isol}}$, Proposition 3.18 implies that the set
$$\mathcal{C}_\Lambda(J_t) \backslash \left(\mathcal{C}_\Lambda(J_t, \mathcal{U}_t^+) \sqcup \mathcal{C}_\Lambda(J_t, \mathcal{U}_t^-)\right)$$
has a finite decomposition into Λ–clusters. The preceding discussion shows that the cores of the clusters appearing in this decomposition represent homology classes of the form dA for $d \geqslant 2$. Therefore, for $-\delta < t < 0$,
$$\mathrm{GW}_\Lambda(\mathcal{U}, J_t) \approx \mathrm{GW}_\Lambda(\mathcal{O}_t^+) + \mathrm{GW}_\Lambda(\mathcal{O}_t^-).$$
On the other hand, for $0 < t < \delta$, we similarly prove that
$$\mathrm{GW}_\Lambda(\mathcal{U}, J_t) \approx 0.$$
By Proposition 3.10, the contribution $\mathrm{GW}_\Lambda(\mathcal{U}, J_t)$ does not depend on $t \in (-\delta, \delta)$. We conclude that
$$\mathrm{GW}_\Lambda(\mathcal{O}_t^+) \approx -\mathrm{GW}_\Lambda(\mathcal{O}_t^-). \qquad \blacksquare$$

Definition 3.36. Given an embedded, oriented, closed surface $C \subset M$, denote by $\mathcal{J}_C \subset \mathcal{J}$ the subset consisting of all J for which C is J–holomorphic. •

Proposition 3.37 (Wall-crossing in $\mathcal{J}_C$). *Let $C \subset M$ be an embedded, oriented, connected, closed surface; denote by $\iota\colon C \to M$ the inclusion map. Let $\mathbf{J} = (J_t)_{t\in[-1,1]}$ be a C^1 path in $\mathcal{J}$, contained in $\mathcal{J}_C \cap \mathcal{J}_{\mathrm{isol}}$, and such that the path $(J_t, [\iota])_{t\in[-1,1]}$ in $\mathcal{M}^{\mathrm{emb}}_{A,\mathrm{g}}$ is transverse to $\mathcal{W}^1$ at the point $p_0 = (J_0, [\iota]) \in \mathcal{W}^1 \backslash \mathcal{A}$. Then there exist a $\delta > 0$ such that if $\mathcal{O}_\pm = (\mathcal{U}_\pm, J_{\pm\delta}, C)$ are Λ–clusters, then*
$$\mathrm{GW}_\Lambda(\mathcal{O}_+) \approx -\mathrm{GW}_\Lambda(\mathcal{O}_-).$$

Proof. The proof is identical to that of [IP18, Lemma 7.4]. Together with the path $\mathbf{J}$ we will consider its thickenings $\mathbb{J}$, which can be seen either as a 2–parameter family $\mathbb{J} = (J_{s,t})_{s,t}$ in $\mathcal{J}$, or as a 1–parameter family $\mathbb{J} = (\mathbf{J}_s)_s$ of paths $\mathbf{J}_s$ in $\mathcal{J}$ such that $\mathbf{J}_0 = \mathbf{J}$.

The results of [IP18, Section 6, in particular Corollary 6.3] provide an explicit Kuranishi model for the family of moduli spaces $\mathcal{M}^{\mathrm{emb}}_{A,\mathrm{g}}(\mathbf{J}_0)$ and $\mathcal{M}^{\mathrm{emb}}_{A,\mathrm{g}}(\mathbb{J})$ in a neighborhood of the point $p_0 = (J_{00}, [\iota_C])$ for a generic thickening $\mathbb{J}$. In this situation, and after reparametrizing the path $\mathbf{J}$, the following hold for a generic thickening $\mathbb{J}$:

(1) There is an open neighborhood $\mathcal{Q}$ of $p_0 = (J_{00}, [\iota_C])$ in $\mathcal{M}^{\text{emb}}_{A,\text{g}}$ such that $\mathcal{Q} \cap \mathcal{M}^{\text{emb}}_{A,\text{g}}(\mathbb{J})$ is diffeomorphic to a neighborhood of the point $(0, 0, 0)$ in the surface

$$S = \{(s, t, x) \in [-1, 1]^3 : s = x(x \pm t)\}. \tag{3.38}$$

(Without loss of generality, assume that the above sign is negative).

(2) Under this diffeomorphism, the projection $\mathcal{M}^{\text{emb}}_{A,\text{g}}(\mathbb{J}) \to \mathcal{J}$ agrees with

$$(s, t, x) \mapsto (s, t),$$

with the path $\mathbf{J} = (J_{0,t})_{t\in[-1,1]}$ corresponding to $\{s = 0, t \in [-1, 1]\}$.

(3) Under this diffeomorphism, the set

$$\mathcal{W} \cap \mathcal{M}^{\text{emb}}_{A,\text{g}}(\mathbb{J}) = (\mathcal{W}^1 \backslash \mathcal{A}) \cap \mathcal{M}^{\text{emb}}_{A,\text{g}}(\mathbb{J})$$

is identified with the curve

$$S \cap \{(s, t, x) : 2x - t = 0\}$$

and its tangent space is identified with $TS \cap \ker(2\mathrm{d}x - \mathrm{d}t)$.

In the proof, we will need two additional properties of the generic thickening $\mathbb{J}$.

(4) There exists a countable set $\Delta \subset [-1, 1]$ such that $\mathbf{J}_s$ is a path in $\mathcal{J}_{\text{isol}}$ for all $s \in [-1, 1]\backslash\Delta$.

This can be achieved as in [IP18, Lemma 6.5], by perturbing $\mathbb{J}$ as a 1–parameter family of paths $\mathbb{J} = (\mathbf{J}_s)_s$ with $\mathbf{J}_0 = \mathbf{J}$ fixed. First, as in the proof of [IP18, Lemmas 6.4 and 6.5] the Sard-Smale Theorem implies that the restriction $(\mathbf{J}_s)_{s\neq 0}$ of a generic thickening $\mathbb{J}$ is transverse to the projection $\pi : \mathcal{M}^{\text{si}}_{\Lambda} \to \mathcal{J}$ as well as to the restriction of π to all the strata of

- $\mathcal{M}^{\text{si}}_{\Lambda} \backslash \mathcal{M}^{\text{emb}}$,
- $\mathcal{W}^k$ for $k \geqslant 1$,
- $\mathcal{A}$.

Therefore, for a generic thickening $\mathbb{J}$ and generic $s \neq 0$, $\mathcal{M}^{\text{si}}_{\Lambda}(\mathbf{J}_s)$ is a 1-dimensional manifold, transverse to $\mathcal{W}^1$ and disjoint from $\mathcal{M}^{\text{si}}_{\Lambda} \backslash \mathcal{M}^{\text{emb}}$, $\mathcal{W}^k$ for $k \geqslant 2$ and $\mathcal{A}$, which all have codimension at least 2. The local Kuranishi models for $\mathcal{M}^{\text{si}}_{\Lambda}(\mathbf{J}_s)$ then imply that the path $\mathbf{J}_s$ is contained in $\mathcal{J}_{\text{isol}}$.

In addition, by Proposition 3.17 and Proposition 3.22, we can guarantee the following.

(5) There are open neighborhoods $\mathcal{U}$ of C in $\mathcal{K}$ and $\mathcal{V}$ of J_{00} in $\mathcal{J}$ with all the properties listed in Proposition 3.22 and such that

$$\mathcal{M}^{\text{emb}}_{A,\text{g}}(\mathcal{V}, \mathcal{U}) \subset \mathcal{Q}, \tag{3.39}$$

where $\mathcal{Q}$ is the open neighborhood of $p_0 = (J_{00}, [\iota_C])$ from property (1). In particular,

$$\mathfrak{z} : \mathcal{M}^{\text{emb}}_{A,\text{g}}(\mathcal{V}, \mathcal{U}) \to \mathcal{C}_A(\mathcal{V}, \mathcal{U})$$

is a homeomorphism. Therefore, a neighborhood of the point (J_{00}, C) in $\mathcal{C}_A(\mathbb{J}, \mathcal{U})$ is homeomorphic to a neighborhood of $(0, 0, 0)$ in the surface S from (3.38). Without loss of generality we will assume that the entire family $\mathbb{J}$ is contained in $\mathcal{V}$.

Since $\mathbf{J} \subset \mathcal{J}_{\text{isol}}$, it follows from (5) and (3.38) that $\mathcal{C}_A(\mathbf{J}, \mathcal{U})$ is homeomorphic to a neighborhood of the point $(0,0)$ in

$$\{(t,x) \in [-1,1]^2 : 0 = x(x-t)\}.$$

The curve $\{x = 0\}$ corresponds to the path $(J_t, C)_{t\in[-1,1]}$, while the curve $\{x = t\}$ corresponds to another 1–parameter family C'_t of irreducible, embedded J_t–holomorphic curves representing the class A. Note that $C'_t \neq C$ for $t \neq 0$, and $C'_0 = C$. It follows from (5) and the Kuranishi model (3.38) that for sufficiently small $\delta > 0$ and $s \neq 0$,

$$\mathcal{C}_A(J_{s,\pm\delta}, \mathcal{U}) = \{p_{s,\pm},\ p'_{s,\pm}\}$$

consists of two points, which, as $s \longrightarrow 0$, converge to

$$p_\pm = (J_{0,\pm\delta}, C) \quad \text{and} \quad p'_\pm = (J_{0,\pm\delta}, C'_{\pm\delta}).$$

Since the path $\mathbf{J} = \mathbf{J}_0$ is contained in $\mathcal{J}_{\text{isol}}$ and $C'_t \neq C$ for all $t \neq 0$, by Proposition 3.17 there are open neighborhoods $\mathcal{U}_\pm$ of C and $\mathcal{U}'_\pm$ of $C'_{\pm\delta}$ in $\mathcal{K}$ such that the triples

$$\mathcal{O}_\pm = (\mathcal{U}_\pm, J_{0,\pm\delta}, C) \quad \text{and} \quad \mathcal{O}'_\pm = (\mathcal{U}'_\pm, J_{0,\pm\delta}, C'_{\pm\delta})$$

are Λ–clusters and

$$\mathcal{U}_\pm \subset \mathcal{U}, \quad \mathcal{U}'_\pm \subset \mathcal{U}, \quad \text{and} \quad \mathcal{U}_\pm \cap \mathcal{U}'_\pm = \varnothing.$$

Combining the local description of the cycle space $\mathcal{C}^{\text{emb}}_A(\mathbb{J})$, given by properties (1) and (5), with Proposition 3.19 and Proposition 3.20, we obtain

$$\text{(3.40)}\quad \text{GW}_\Lambda(\mathcal{O}_-)+\text{GW}_\Lambda(\mathcal{O}'_-) \approx \text{GW}_\Lambda(\mathcal{U}, J_{0,-\delta}) = \text{GW}_\Lambda(\mathcal{U}, J_{0,\delta}) \approx \text{GW}_\Lambda(\mathcal{O}_+)+\text{GW}_\Lambda(\mathcal{O}'_+).$$

We will show that

$$\text{(3.41)}\qquad \text{GW}_\Lambda(\mathcal{O}_-) \approx \text{GW}_\Lambda(\mathcal{O}'_+) \approx -\text{GW}_\Lambda(\mathcal{O}'_-),$$

which, in conjunction with (3.40), will complete the proof.

We prove (3.41) by considering the restrictions of the local Kuranishi model from property (1) over the paths $\mathbf{J}_s$. It follows from (3.38) that for $s \neq 0$, $\mathcal{M}^{\text{emb}}_A(\mathbf{J}_s, \mathcal{U})$ is a 1–dimensional manifold such that:

i. For $s \neq 0$, $\mathcal{M}^{\text{emb}}_A(\mathbf{J}_s, \mathcal{U})$ has two connected components, one corresponding to $x > 0$ and the other to $x < 0$ in the description provided by property (1).

ii. For $s > 0$, the component with $x > 0$ is a path in $\mathcal{M}^{\text{emb}}_A$ disjoint from the wall $\mathcal{W}$, and the projection to $\mathcal{J}$ is a injective when restricted to this path. For s small, this path intersects the fiber over $J_{s,-\delta}$ in $p_{s,-}$ and the fiber over $J_{s,-\delta}$ in $p'_{s,+}$.

iii. For $s < 0$, the component with $x > 0$ is a path in $\mathcal{M}^{\text{emb}}_A$ which intersects the wall transversally in precisely one point q_s with $t = x/2 = \sqrt{-s}$, and $q_s \in \mathcal{W}^1 \backslash \mathcal{A}$. For $s < 0$ small, the intersection of this path with the fiber over $J_{s,\delta}$ consists of the points $p_{s,+}$ and $p'_{s,+}$.

For small $s \notin \Delta$, $\mathbf{J}_s$ is a path in $\mathcal{J}_{\text{isol}}$, as was the path $\mathbf{J}$. Let $\mathcal{U}_\pm$ and $\mathcal{U}'_\pm$ be four open sets in $\mathcal{K}$ appearing in the definition of the Λ–clusters $\mathcal{O}_\pm$ and $\mathcal{O}'_\pm$ above. It follows from the preceding discussion that for s sufficiently small,

$$\mathcal{C}_A(J_{s,\pm\delta}, \mathcal{U}_\pm) = p_{s,\pm}, \quad \text{and} \quad \mathcal{C}_A(J_{s,\pm\delta}, \mathcal{U}'_\pm) = p'_{s,\pm}.$$

Therefore, by part (1) of Proposition 3.20, the triples

$$\mathcal{P}_{s,\pm} = (\mathcal{U}_\pm, J_{s,\pm\delta}, p_{s,\pm}) \quad \text{and} \quad \mathcal{P}'_{s,\pm} = (\mathcal{U}'_\pm, J_{s,\pm\delta}, p'_{s,\pm})$$

are Λ–clusters. (We engage here in a slight abuse of notation by identifying the points $p_{s,\pm}$ and $p'_{s,\pm}$ in $\mathcal{M}^{\mathrm{emb}}_\Lambda$ with the corresponding pseudo-holomorphic curves in $\mathcal{C}^{\mathrm{emb}}$.) Moreover,

$$\mathrm{GW}_\Lambda(\mathcal{O}_\pm) = \mathrm{GW}_\Lambda(\mathcal{P}_{s,\pm}) \quad \text{and} \quad \mathrm{GW}_\Lambda(\mathcal{O}'_\pm) = \mathrm{GW}_\Lambda(\mathcal{P}'_{s,\pm}).$$

On the other hand, for $s > 0$ small and $s \notin \Delta$, we get a path of type (ii.) which misses the wall $\mathcal{W}$, thus Proposition 3.30 implies that

$$\mathrm{GW}_\Lambda(\mathcal{P}_{s,+}) \approx \mathrm{GW}_\Lambda(\mathcal{P}'_{s,+}).$$

For $s < 0$ small and $s \notin \Delta$, we get a path of type (iii.) which intersects the wall $\mathcal{W}$ transversally at precisely one point q_s in $\mathcal{W}^1 \backslash \mathcal{A}$, thus Proposition 3.33 implies

$$\mathrm{GW}_\Lambda(\mathcal{P}_{s,-}) \approx -\mathrm{GW}_\Lambda(\mathcal{P}'_{s,+}).$$

Combining the last four displayed equations implies

$$\mathrm{GW}_\Lambda(\mathcal{O}_-) \approx \mathrm{GW}_\Lambda(\mathcal{O}'_+) \approx -\mathrm{GW}_\Lambda(\mathcal{O}'_-),$$

and therefore completes the proof. ∎

Proof of Theorem 3.25. Set $A := [C] \in \Gamma$ and $\mathrm{g} := \mathrm{g}(C)$. Denote by $\iota\colon C \hookrightarrow X$ the inclusion map. Since $\mathcal{J}_C$ is path connected it follows from [IP18, proof of Lemma 6.7] that there exists a path $\mathbf{J} = (J_t)_{t\in[0,1]}$ in $\mathcal{J}_C$ connecting J_0 and J_1 with the following properties.

- The path $(J_t, [\iota])_{t\in[0,1]}$ in $\mathcal{M}^{\mathrm{emb}}_{A,\mathrm{g}}$ intersects $\mathcal{W}^1 \backslash \mathcal{A}$ transversely at finitely many points and is otherwise disjoint from $\mathcal{W}$.
- Away from the points of intersection with $\mathcal{W}^1 \backslash \mathcal{A}$, for all $d \geq 1$, the subset $\mathcal{M}^{\mathrm{si}}_\Lambda(\mathbf{J})$ of the moduli space of index 0 simple maps is a 1-dimensional manifold, consisting of embeddings, and intersecting the wall transversely at points in $\mathcal{W}^1 \backslash \mathcal{A}$.

In particular, as in [IP18, proof of Lemma 6.7], the local Kuranishi models for $\mathcal{M}^{\mathrm{si}}_\Lambda(\mathbf{J})$ imply that the path $\mathbf{J}$ is contained in $\mathcal{J}_{\mathrm{isol}}$. Therefore the theorem follows by Proposition 3.21 (2), combined with Proposition 3.30 and Proposition 3.37 after dividing the path $(J_t, [\iota])_{t\in[0,1]}$ into finitely many paths, each either disjoint from $\mathcal{W}$ or intersecting $\mathcal{W}^1 \backslash \mathcal{A}$ transversally at one point. ∎

3.4 Contributions of super-rigid curves

Definition 3.42. Let $J \in \mathcal{J}$. Let C be an irreducible, embedded J–holomorphic curve. Set $j := J|_{TC}$.

(1) The operator $\mathfrak{d}_{C,J}$ restricts to the **normal Cauchy–Riemann operator**

$$\mathfrak{d}^N_{C,J}\colon \Gamma(NC) \to \Omega^{0,1}(C, NC).$$

(2) If $\pi\colon (\tilde{C}, \tilde{j}) \to (C, j)$ is a nodal j–holomorphic map, then $\mathfrak{d}^N_{C,J}$ induces

$$\pi^*\mathfrak{d}^N_{C,J}\colon \Gamma(\pi^*NC) \to \Omega^{0,1}(\tilde{C}, \pi^*NC)$$

by pulling back; cf. [Zin11, §2.2; DW18, Definition 1.2.1].

(3) C is **super-rigid with respect to** J if $\ker \pi^*\mathfrak{d}_{C,J}^N = 0$ for every j–holomorphic map π. •

Assume the situation of Definition 3.42. Denote by $\overline{\mathcal{H}}_{d,\mathrm{g}}(C)$ the moduli space of stable degree d genus g nodal J–holomorphic maps to C. The space $\overline{\mathcal{H}}_{d,\mathrm{g}}(C)$ is an orbispace and parametrizes the family of Fredholm operators

$$\underline{\mathfrak{d}}_{C,J}^N = \left(\pi^*\mathfrak{d}_{C,J}^N\right)_{[\pi]\in\overline{\mathcal{H}}_{d,\mathrm{g}}}.$$

If C is super-rigid, then following hold:

(1) The cokernels of $\pi^*\mathfrak{d}_{C,J}^N$ form an orbibundle $\operatorname{coker}\underline{\mathfrak{d}}_{C,J}^N$ over $\overline{\mathcal{H}}_{d,\mathrm{g}}(C)$.
(2) By [DW21, Theorem 1.6], $\mathcal{C}_\Lambda(J,\{C\})$ is open and closed in $\mathcal{C}_\Lambda(J)$ for every $\Lambda \geqslant \mathbf{M}(C)$. In particular, $\overline{\mathcal{M}}(J,\{C\})$ is open and closed in $\overline{\mathcal{M}}(J)$.
(3) $\overline{\mathcal{M}}(J,\{C\})$ agrees with $\coprod_{d=1}^\infty \coprod_{\mathrm{g}=0}^\infty \overline{\mathcal{H}}_{d,\mathrm{g}}(C)$.
(4) According to Zinger [Zin11, Theorem 1.2], the Gromov–Witten contribution of C is

$$\mathrm{GW}(C,J) := \sum_{\mathrm{g}=0}^\infty \sum_{d=1}^\infty \int_{\left[\overline{\mathcal{H}}_{d,\mathrm{g}}(C)\right]^{\mathrm{vir}}} e(\operatorname{coker}\underline{\mathfrak{d}}_{C,J}^N)\cdot t^{2\mathrm{g}-2}q^{d[C]}. \tag{3.43}$$

Here $e(\cdot)$ denotes the Euler class.

Corollary 3.44. *Let $J \in \mathcal{J}$. Let C be an irreducible, embedded J–holomorphic curve of index zero. Let $\Lambda \geqslant \mathbf{M}(C)$ If C is super-rigid, then there is an $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0,\varepsilon_0)$ the triple $\mathcal{O} = (B_\varepsilon(C), J, C)$ is a Λ–cluster and $\mathrm{GW}_\Lambda(\mathcal{O}) = \mathrm{GW}_\Lambda(C,J)$.* ■

Computing the contribution $\mathrm{GW}(C,J)$ in (3.43) is a formidable problem. Fortunately, it has been studied extensively by Bryan and Pandharipande [BP08].

Definition 3.45. For $h \in \mathbb{N}_0$ set

$$G_h(q,t) := \log\left(1 + \sum_{d=1}^\infty \sum_{\mu\vdash d} \prod_{\square\in\mu} (2\sin(h(\square)\cdot t/2))^{2h-2} q^d\right).$$

Here $\mu \vdash d$ indicates that the sum is taken over all partitions μ of d, $\square \in \mu$ indicates that $\square$ is a box in the Young diagram of μ, and $h(\square)$ denotes the hook length of $\square$. •

Proposition 3.46 ([Lee09, §2; IP18, Propositions 3.2 and 3.3; BP08, Corollary 7.3]). *Let $J \in \mathcal{J}$. Let C be an irreducible, embedded J–holomorphic curve of index zero. There is a $J_L \in \mathcal{J}_C$ with respect to which C is super-rigid and*

$$\mathrm{GW}(C,J_L) = G_h(q^{[C]},t) \quad \textit{with} \quad h := \mathrm{g}(C).$$ ■

The following combinatorial result verifies the Gopakumar–Vafa conjecture for G_h.

Proposition 3.47 ([IP18, Proposition 3.4]). *For every $h \in \mathbb{N}_0$ the coefficients $\mathrm{BPS}_{d,\mathrm{g}}(h)$ defined by*

$$G_h(q,t) = \sum_{d=1}^\infty \sum_{\mathrm{g}=0}^\infty \mathrm{BPS}_{d,\mathrm{g}}(h) \cdot \sum_{k=1}^\infty \frac{1}{k}(2\sin(kt/2))^{2\mathrm{g}-2} q^{kd}$$

satisfy:

(integrality) $\mathrm{BPS}_{d,g}(h) \in \mathbb{Z}$, *and*

(finiteness) $\mathrm{BPS}_{d,g}(h) = 0$ *for* $g \gg 1$. ∎

The following structure result for the contribution of a super-rigid J–holomorphic curve is a byproduct of the proof of Theorem 1.7 in Section 3.5.

Proposition 3.48 (Super-rigid Contributions). *Let $J \in \mathcal{J}$ and C be a irreducible, embedded J–holomorphic curve of index zero and genus* g. *If C is super-rigid with respect to J, then*

$$\mathrm{GW}(C,J) = \mathrm{sign}(C,J) \cdot G_g(q^{[C]},t) + \sum_{d=2}^{\infty}\sum_{h=g}^{\infty} e_{d,h}(C,J) \cdot G_h(q^{d[C]},t)$$

with

(integrality) $e_{d,h}(C,J) \in \mathbb{Z}$, *and*

(finiteness) $e_{d,h}(C,J) = 0$ *for* $g \gg 1$.

Remark 3.49. Wendl [Wen19, Theorem A] has recently proved that for a generic $J \in \mathcal{J}$ every J–holomorphic curve of index zero in a symplectic 6–manifold is super-rigid. Therefore, it is interesting to ask whether Proposition 3.48 can be proved directly. An obstacle to this appears to be the lack of understanding of the wall-crossing/bifurcation phenomena related to the failure of super-rigidity along a generic path $\mathbf{J} = (J_t)_{t\in[0,1]}$ in $\mathcal{J}$; cf. [Wen19, §2.4; DW18, §2.7]. •

3.5 Conclusion of the proof of the Gopakumar–Vafa conjecture

Theorem 1.7 is an immediate consequence of Proposition 3.47 and the following structure theorem.

Theorem 3.50. *There are unique coefficients $e_{A,g} = e_{A,g}(X,\omega)$ such that*

$$\mathrm{GW} = \sum_{A\in\Gamma}\sum_{g=0}^{\infty} e_{A,g} \cdot G_g(q^A,t); \tag{3.51}$$

moreover, they satisfy:

(integrality) $e_{A,g} \in \mathbb{Z}$, *and*

(finiteness) $e_{A,g} = 0$ *for* $g \gg 0$.

Remark 3.52. There is a version of the question raised in Question 1.6 with $\mathrm{BPS}_{A,g}$ replaced by $e_{A,g}$. •

The proof relies on the following result.

Notation 3.53. Consider a formal power series

$$S = \sum_{A\in\Gamma} c_A \cdot q^A.$$

For every $\Lambda > 0$ the **Λ–truncation** of S is the formal power series

$$S_\Lambda \coloneqq \sum_{A\in\Gamma_\Lambda} c_A \cdot q^A$$

with Γ_Λ as in (3.6). •

Proposition 3.54. *Let $\Lambda > 0$. Let $\mathcal{O} = (\mathcal{U}, J, C)$ be a Λ–cluster with $J \in \mathcal{J}_{\mathrm{isol}}$ and C unobstructed with respect to J. Set $d^* \coloneqq \lfloor \mathbf{M}(C)/\Lambda \rfloor$. There are unique coefficients $e_{d,\mathrm{g}}(\mathcal{O})$ such that*

$$\mathrm{GW}_\Lambda(\mathcal{O}) = \sum_{d=1}^{d^*} \sum_{\mathrm{g}=0}^{\infty} e_{d,\mathrm{g}}(\mathcal{O}) \cdot G_{\mathrm{g}}(q^{d[C]}, t)_\Lambda; \tag{3.55}$$

moreover, they satisfy:

(integrality) $e_{d,\mathrm{g}}(\mathcal{O}) \in \mathbb{Z}$, *and*

(finiteness) $e_{d,\mathrm{g}}(\mathcal{O}) = 0$ *for* $\mathrm{g} \gg 0$.

Proof. The uniqueness of the coefficients is a consequence of the fact that $G_{\mathrm{g}}(q,t) = t^{2\mathrm{g}-2}q +$ higher order terms.

Since the core C of $\mathcal{O} = (\mathcal{U}, J, C)$ is of index zero, by Corollary 3.44 and Proposition 3.46 there are $J' \in \mathcal{J}_C$ and an $\varepsilon > 0$ such that C is super-rigid with respect to J', $\mathcal{O}' \coloneqq (B_\varepsilon(C), J', C)$ is a Λ-cluster, and

$$\mathrm{GW}_\Lambda(\mathcal{O}') = G_{\mathrm{g}}(q^{[C]}, t)_\Lambda.$$

Since C is unobstructed with respect to J', by [IP18, Proof of Lemma 6.7] and Proposition 3.20, there is a $J'' \in \mathcal{J}_C \cap \mathcal{J}_*$ such that $\mathcal{O}'' \coloneqq (B_\varepsilon(C), J'', C)$ is a Λ-cluster, and

$$\mathrm{GW}_\Lambda(\mathcal{O}'') = \mathrm{GW}_\Lambda(\mathcal{O}').$$

By Theorem 3.25 with $\mathcal{O}_0 = \mathcal{O}$ and $\mathcal{O}_1 = \mathcal{O}''$, there are $e_{1,\mathrm{g}}(\mathcal{O}) \in \{\pm 1\}$ and a finite set $\{\mathcal{O}_i = (\mathcal{U}_i, J_i, C_i) : i \in I\}$ of Λ–clusters such that

$$\mathrm{GW}_\Lambda(\mathcal{O}) = e_{1,\mathrm{g}}(\mathcal{O}) \cdot G_{\mathrm{g}}(q^{[C]}, t)_\Lambda + \sum_{i \in I} \pm \mathrm{GW}_\Lambda(\mathcal{O}_i);$$

moreover, for every $i \in I$, $J_i \in \mathcal{J}_{\mathrm{isol}}$, C_i is unobstructed with respect to J_i, and $[C_i] = d_i[C]$ with $d_i \geqslant 2$.

This finishes the proof if $d^* = 1$. If $d^* \geqslant 2$, then $d_i^* \coloneqq \lfloor \mathbf{M}(C_i)/\Lambda \rfloor \leqslant d^* - 1$ and the assertion follows by induction. ■

Proof of Theorem 3.50. The uniqueness of the coefficients follows as in the proof of Proposition 3.54 because $G_{\mathrm{g}}(q^A, t) = t^{2\mathrm{g}-2}q^A +$ higher order terms.

Let $A \in \Gamma$ be a non-zero Calabi-Yau class. Set $\Lambda \coloneqq \langle [\omega], A \rangle$. Let $J \in \mathcal{J}_*$. By Proposition 3.18, there is a finite set $\{\mathcal{O}_i = (\mathcal{U}_i, J, C_i) : i \in I\}$ of Λ–clusters such that

$$\mathrm{GW}_\Lambda = \sum_{i \in I} \mathrm{GW}_\Lambda(\mathcal{O}_i).$$

By Proposition 3.54,

$$\mathrm{GW}_\Lambda = \sum_{i \in I} \mathrm{GW}_\Lambda(\mathcal{O}_i) = \sum_{i \in I} \sum_{d=1}^{\infty} \sum_{\mathrm{g}=0}^{\infty} e_{d,\mathrm{g}}(\mathcal{O}_i) \cdot G_{\mathrm{g}}(q^{d[C_i]}, t)_\Lambda.$$

Denote by I^* the subset of those $i \in I$ for which there is a $d_i \in \mathbb{N}$ with $A = d_i[C_i]$. By uniqueness of coefficients in (3.51),

$$e_{A,\mathrm{g}} = \sum_{i \in I^*} e_{d_i,\mathrm{g}}(\mathcal{O}).$$

By Proposition 3.54, these satisfy integrality and finiteness. ■

Proof of Proposition 3.48. Denote by $\iota\colon C \to M$ the inclusion. Since C is of index zero and is unobstructed with respect to J, the contribution of $[\iota] \in \overline{\mathcal{M}}_{A,g}$ to $\mathrm{GW}(C, J)$ is precisely $\mathrm{sign}(C, J) \cdot t^{2g-2}q^{[C]}$. The remaining contributions to $\mathrm{GW}(C, J)$ arise from $\overline{\mathcal{H}}_{d,h}(C)$ and vanish unless $h \geqslant g$. Therefore, the assertion is a consequence of Proposition 3.54. ■

A The Gopakumar–Vafa conjecture for Fano classes

There is an analogue of the Gopakumar–Vafa conjecture for **Fano classes**; that is: $A \in \mathrm{H}_2(X, \mathbb{Z})$ with $c_1(A) > 0$. (Gromov–Witten theory is trivial for $A \in \mathrm{H}_2(X, \mathbb{Z})$ with $c_1(A) < 0$.) Let $A \in \mathrm{H}_2(X, Z)$ be a Fano class, $g \in \mathbb{N}_0$, and $k \in \mathbb{N}_0$. Denote by $\overline{\mathcal{M}}_{A,g,k}$ the universal moduli space over $\mathcal{J}$ of stable nodal pseudo-holomorphic maps representing A, of genus g, and with k marked points. Evaluation at the marked points defines a map

$$\mathrm{ev}\colon \overline{\mathcal{M}}_{A,g,k} \to X^k.$$

As in Section 3.1, the fibers of $\overline{\mathcal{M}}_{A,g,k}$ carry a VFC of degree $2c_1(A) + 2k$ and these are consistent along paths in $\mathcal{J}$. If $\gamma_1, \ldots, \gamma_k \in \mathrm{H}^*(X, \mathbb{Z})$ satisfy

$$c_1(A) - \sum_{i=1}^{k}(\deg \gamma_i - 2) = 0, \tag{A.1}$$

then the **Gromov–Witten invariant** is defined by

$$\mathrm{GW}_{A,g}(\gamma_1, \ldots, \gamma_k) \coloneqq \int_{[\overline{\mathcal{M}}_{A,g,k}(J)]^{\mathrm{vir}}} \mathrm{ev}^*(\gamma_1 \times \cdots \times \gamma_k). \tag{A.2}$$

These can be packaged into a linear map

$$\mathrm{GW}_{A,g} = \mathrm{GW}_{A,g}(X, \omega)\colon\ \mathrm{Sym}^*\,\mathrm{H}^*(X, \mathbb{Z}) \to \mathbb{Q}. \tag{A.3}$$

Here $\mathrm{Sym}^*\,\mathrm{H}^*(X, \mathbb{Z})$ denotes the *graded* symmetric algebra on the graded abelian group $\mathrm{H}^*(X, \mathbb{Z})$. This map satisfies, in particular, the following axioms; cf. [MS12, §7.5]:

(grading) $\mathrm{GW}_{A,g}(\gamma_1 \cdots \gamma_k) = 0$ unless $\sum_{i=1}^{k} \deg \gamma_i = 2c_1(A) + 2k$.

(vanishing) For every $h \in \mathrm{H}^i(X, \mathbb{Z})$ with $i \in \{0, 1\}$

$$\mathrm{GW}_{A,g}(h \cdot \gamma) = 0.$$

(divisor) For every $h \in \mathrm{H}^2(X, \mathbb{Z})$

$$\mathrm{GW}_{A,g}(h \cdot \gamma) = \langle h, A\rangle \mathrm{GW}_{A,g}(\gamma).$$

The **Gopakumar–Vafa BPS invariant** $\mathrm{BPS}_{A,g} = \mathrm{BPS}_{A,g}(X, \omega)\colon\ \mathrm{Sym}^*\,\mathrm{H}^*(X, \mathbb{Z}) \to \mathbb{Q}$ is defined by

$$\sum_{g=0}^{\infty} \mathrm{GW}_{A,g}(\gamma) \cdot t^{2g-2} = \sum_{g=0}^{\infty} \mathrm{BPS}_{A,g}(\gamma) \cdot (2\sin(t/2))^{2g-2+2c_1(A)}. \tag{A.4}$$

Evidently, it satisfies the same axioms as $\mathrm{GW}_{A,g}$.

Theorem A.5 (Zinger [Zin11, Theorem 1.5] and Doan and Walpuski [DW19, Corollary 1.18]). *Let (X, ω) be a closed symplectic 6–manifold and let $A \in \mathrm{H}_2(X, \mathbb{Z})$ be a Fano class. The invariants $\mathrm{BPS}_{A,\mathrm{g}} = \mathrm{BPS}_{A,\mathrm{g}}(X, \omega)$ defined by (A.4) satisfy:*

(integrality) $\mathrm{BPS}_{A,\mathrm{g}}(\gamma) \in \mathbb{Z}$ *for every* $\gamma \in \mathrm{Sym}^* \mathrm{H}^*(X, \mathbb{Z})$.

(finiteness) *There exists* $\mathrm{g}_A \in \mathbb{N}_0$ *such that* $\mathrm{BPS}_{A,\mathrm{g}}(\gamma) = 0$ *for every* $\mathrm{g} \geqslant \mathrm{g}_A$ *and* $\gamma \in \mathrm{Sym}^* \mathrm{H}^*(X, \mathbb{Z})$.

Proof. The integrality statement was proved by Zinger [Zin11, Theorem 1.5].

By the vanishing and divisor axioms, it suffices to prove that there is a $\mathrm{g}_A \in \mathbb{N}_0$ such that $\mathrm{BPS}_{A,\mathrm{g}}(\gamma_1 \cdots \gamma_k) = 0$ whenever $\mathrm{g} \geqslant \mathrm{g}_A$ and $\deg \gamma_i \geqslant 3$. The latter implies $k \leqslant 2c_1(A)$. Since

$$\bigoplus_{k=0}^{2c_1(A)} \mathrm{Sym}^k \mathrm{H}^*(X, \mathbb{Z})$$

is a finitely generated abelian group, the finiteness statement follows from [DW19, Corollary 1.18]. ■

Remark A.6. The proof of [DW19, Corollary 1.18] relies on [DW19, Theorems 1.1]. To prove the latter, Doan and Walpuski carried out a somewhat delicate analysis of the Kuranishi model at nodal pseudo-holomorphic maps with ghost components—following ideas of Ionel [Ion98] and Zinger [Zin09, Theorem 1.2]. [DW19, Theorems 1.1], however, also is an immediate consequence of Proposition 2.50. •

B Castelnuovo's bound for primitive Calabi–Yau classes

Let (X, ω) be a closed symplectic 6–manifold. Denote by $\mathcal{J}$ the space of almost complex structures tamed by (or compatible with) ω; cf. Example 2.2.

Definition B.1. For $A \in \mathrm{H}_2(X, \mathbb{Z})$ and $J \in \mathcal{J}$ the **Castelnuovo number** $\gamma_A(X, J)$ is

$$\gamma_A(X, J) := \sup\{\mathrm{g}(C) : C \text{ is an irreducible } J\text{–holomorphic curve}\}. \quad \bullet$$

[DW21, Theorem 1.6] established that $\gamma_A(X, J) < \infty$ provided $J \in \mathcal{J}$ is k–rigid and A has divisibility at most k and $c_1(A) = 0$. The subset of these J is comeager [Eft16, Theorem 1.2; Wen19, Theorem A], but fails to be path-connected—even for $k = 1$. Therefore, [DW21, Theorem 1.6] does not establish Castelnuovo bounds in generic 1–parameter families. The results of Section 2, however, immediately yield such bounds for primitive Calabi–Yau classes $A \in \Gamma$.

Definition B.2. Denote by $\mathcal{J}_{\mathrm{emb}}$ the subset of those $J \in \mathcal{J}$ satisfying Definition 3.13 (1), (2), and (3). •

Theorem B.3 ([OZ09, Theorem 1.1; IP18, Proposition A.4]). *$\mathcal{J}_{\mathrm{emb}}$ has codimension two in $\mathcal{J}$; in particular: it is comeager and path-connected.*

Theorem B.4. *If $K \subset \mathcal{J}_{\mathrm{emb}}$ is compact, then for every primitive Calabi–Yau class $A \in \Gamma$*

$$\sup_{J \in K} \gamma_A(X, J) < \infty.$$

Proof of Theorem B.4. By Theorem 2.11, $\mathcal{C}_A(K)$ is compact.

Let $J \in K$ and $C = \sum_{i=1}^{I} m_i C_i$ with $(J, C) \in \mathcal{C}_A(K)$. As in the proof of Lemma 3.16, $I = 1$; that is: $C = m_1 C_1$ and C_1 is embedded. Since A is primitive, $m_1 = 1$. Therefore,

$$\mathcal{C}_A(K) = \mathcal{C}_A^{\mathrm{si}}(K) = \mathcal{C}_A^{\mathrm{emb}}(K).$$

By Proposition 2.21, the map $\mathrm{g}\colon \mathcal{C}_A^{\mathrm{emb}} \to \mathbb{N}_0$ assigning to (J, C) the genus of C is continuous. Since $\mathcal{C}_A^{\mathrm{emb}}(K)$ is compact, this implies the assertion. ■

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