Tangent cones of Hermitian Yang–Mills connections with isolated singularities

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Abstract

We give a simple direct proof of uniqueness of tangent cones for singular projectively Hermitian Yang–Mills connections on reflexive sheaves at isolated singularities modelled on a sum of \( \mu \)-stable holomorphic bundles over \( \mathbb{P}^{n-1} \).

1 Introduction

A projectively Hermitian Yang–Mills (PHYM) connection \( A \) over a Kähler manifold \( X \) is a unitary connection \( A \) on a Hermitian vector bundle \((E, H)\) over \( X \) satisfying

\[
F^{0,2}_A = 0 \quad \text{and} \quad i \Lambda F_A - \frac{\text{tr}(i \Lambda F_A)}{\text{rk} E} \cdot \text{id}_E = 0.
\]

Since \( F^{0,2}_A = 0, \mathcal{E} := (E, \bar{\partial}_A) \) is a holomorphic vector bundle, and \( A \) is the Chern connection of \( H \). A Hermitian metric \( H \) on a holomorphic vector bundle is called PHYM if its Chern connection \( A_H \) is PHYM. The celebrated Donaldson–Uhlenbeck–Yau Theorem [Don85; Don87; UY86] asserts that a holomorphic vector bundle \( \mathcal{E} \) on a compact Kähler manifold admits a PHYM metric if and only if it is \( \mu \)-polystable; moreover, any two PHYM metrics are related by an automorphism of \( \mathcal{E} \) and by multiplication with a conformal factor. If \( H \) is a PHYM metric, then the connection \( A^o \) on \( PU(E, H) \), the principal \( PU(r) \)-bundle associated with \((E, H)\), induced by \( A_H \) is Hermitian Yang–Mills (HYM), that is, it satisfies \( F^{0,2}_{A^o} = 0 \) and \( i \Lambda F_{A^o} = 0 \); it depends only on the conformal class of \( H \). Conversely, any HYM connection \( A^o \) on \( PU(E, H) \) can be lifted to a PHYM connection \( A \); any two choices of lifts lead to isomorphic holomorphic vector bundles \( \mathcal{E} \) and conformal metrics \( H \).

An admissible PHYM connection is a PHYM connection \( A \) on a Hermitian vector bundle \((E, H)\) over \( X \setminus \text{sing}(A) \) with \( \text{sing}(A) \) a closed subset with locally finite \((2n-4)\)-dimensional Hausdorff measure and \( F_A \in L^2_{\text{loc}}(X) \).\(^1\) Bando [Ban91] proved that if \( A \) is an admissible PHYM connection,

\(^{1}\)It should be pointed out that our notion of admissible PHYM connection follows Bando and Siu [BS94] and not Tian [Tiao00]. The notion of admissible Yang–Mills connection introduced by Tian is stronger: it assumes that the Hermitian vector bundle extends to all of \( X \).
The main result of this article gives a description of the tangent cone to a stationary Yang–Mills connection—in particular, a unitary connection.

Proposition 1.4. Let \( (F, K) \) be a Hermitian vector bundle over \( \mathbb{P}^{n-1} \). If \( B \) is a unitary connection on \( \rho^*(F, K) \) such that \( A_* := \pi^*B \) is HYM with respect to \( \omega_0 := \frac{1}{2i} \partial \bar{\partial} |z|^2 \), then there is a \( k \in \mathbb{N} \) and, for each \( j \in \{1, \ldots, k\} \), there are \( \mu_j \in \mathbb{R} \), a Hermitian vector bundle \( (F_j, K_j) \) on \( \mathbb{P}^{n-1} \), and an irreducible unitary connection \( B_j \) on \( F_j \) satisfying

\[
F_{B_j}^{0,2} = 0 \quad \text{and} \quad i\Delta F_{B_j} = (2n-2)\pi \mu_j \cdot \text{id}_{F_j}
\]
such that

\[
F = \bigoplus_{j=1}^{k} F_j \quad \text{and} \quad B = \bigoplus_{j=1}^{k} \rho^* B_j + i \mu_j \text{id}_{\rho^* F_j} \cdot \theta.
\]

Here \( \theta \) denotes the standard contact structure\(^2\) on \( S^{2n-1} \). In particular,

\[
\mathcal{E} = (\sigma^* F, \tilde{\partial}_\Lambda) \cong \bigoplus_{j=1}^{k} \sigma^* \mathcal{F}_j
\]

with \( \mathcal{F}_j = (F_j, \tilde{\partial}_B) \) \( \mu \)-stable.

To conclude the introduction we discuss two concrete examples in which Theorem 1.2 can be applied.

Example 1.5 (Okonek, Schneider, and Spindler [OSS11, Example 1.1.13]). It follows from the Euler sequence that \( H^0(T_{P^3}(-1)) \cong \mathbb{C}^4 \). Denote by \( s_v \in H^0(T_{P^3}(-1)) \) the section corresponding to \( v \in \mathbb{C}^4 \). If \( v \neq 0 \), then the rank two sheaf \( \mathcal{E} = \mathcal{E}_v \) defined by

\[
0 \to \mathcal{O}_{P^3} \xrightarrow{s_v} T_{P^3}(-1) \to \mathcal{E}_v \to 0
\]

is reflexive and \( \text{sing} (\mathcal{E}) = \{ [v] \} \).

\( \mathcal{E} \) is \( \mu \)-stable. To see this, because \( \mu (\mathcal{E}) = 1/2 \), it suffices to show that

\[
\text{Hom} (\mathcal{O}_{P^3}(k), \mathcal{E}) = H^0 (\mathcal{E}(-k)) = 0 \quad \text{for each } k \geq 1.
\]

However, by inspection of the Euler sequence, \( H^0 (\mathcal{E}(-k)) \cong H^0 (T_{P^3}(-k-1)) = 0 \). It follows that \( \mathcal{E} \) admits a PHYM metric \( H \) with \( F_H \in L^2 \) and a unique singular point at \( [v] \in P^3 \). To see that Theorem 1.2 applies, pick a standard affine neighborhood \( U \cong \mathbb{C}^3 \) in which \( [v] \) corresponds to 0.

In \( U \), the Euler sequence becomes

\[
0 \to \mathcal{O}_{\mathbb{C}^3} \overset{(1,z_1,z_2,z_3)}{\longrightarrow} \mathcal{O}_{\mathbb{C}^3}^{\oplus 4} \to T_{P^3}(-1)|_U \to 0,
\]

and \( s_v = [(1,0,0,0)] \); hence,

\[
0 \to \mathcal{O}_{\mathbb{C}^3} \overset{(z_1,z_2,z_3)}{\longrightarrow} \mathcal{O}_{\mathbb{C}^3}^{\oplus 3} \to \mathcal{E}_v|_U \to 0.
\]

On \( \mathbb{C}^3 \setminus \{0\} \), this is the pullback of the Euler sequence on \( P^2 \); therefore, \( \mathcal{E}_v|_U \cong i_* \sigma^* T_{P^2} \).

\(^2\)With respect to standard coordinates on \( \mathbb{C}^n \), the standard contact structure \( \theta \) on \( S^{2n-1} \) is such that \( \pi^* \theta = \sum_{j=1}^{n} (z_j dz_j - z_j d\bar{z}_j)/2|z|^2 \).
Example 1.6. For $t \in C$, define $f_t : \mathcal{O}_{P^3}(-2)^{\oplus 2} \to \mathcal{O}_{P^3}(-1)^{\oplus 5}$ by

$$f_t := \begin{pmatrix} z_0 & 0 \\ z_1 & z_0 \\ z_2 & z_1 \\ t \cdot z_3 & z_2 \\ 0 & z_3 \end{pmatrix},$$

and denote by $\mathcal{E}_t$ the cokernel of $f_t$, i.e.,

$$0 \to \mathcal{O}_{P^3}(-2)^{\oplus 2} \xrightarrow{f_t} \mathcal{O}_{P^3}(-1)^{\oplus 5} \to \mathcal{E}_t \to 0.$$  \hfill (1.7)

If $t \neq 0$, then $\mathcal{E}_t$ is locally free; $\mathcal{E}_0$ is reflexive with $\text{sing}(\mathcal{E}_0) = \{[0 : 0 : 0 : 1]\}$. The proof of this is analogous to that of the reflexivity of $\mathcal{E}_0$ from Example 1.5 given in [OSS11, Example 1.1.13].

For each $t$, $H^0(\mathcal{E}_t) = H^1(\mathcal{E}_t)^* = 0$; hence, $\mathcal{E}_t$ is $\mu$-stable according to the criterion of Okonek, Schneider, and Spindler [OSS11, Remark 1.2.6(b)]. The former vanishing is obvious since $H^0(\mathcal{O}_{P^3}(-1)) = H^1(\mathcal{O}_{P^3}(2)) = 0$. The latter follows by dualising (1.7), twisting by $\mathcal{E}_0^{-1}$ and observing that the induced map $H^0(f_0^*): H^0(\mathcal{O}_{P^3})^{\oplus 5} \to H^0(\mathcal{O}_{P^3}(1))^{\oplus 2}$, which is given by

$$\begin{pmatrix} z_0 & z_1 & z_2 & t \cdot z_3 & 0 \\ 0 & z_0 & z_1 & z_2 & z_3 \end{pmatrix},$$

is injective.

In a standard affine neighborhood $U \cong \mathbb{C}^3$ of $[0 : 0 : 0 : 1]$, we have $\mathcal{E}_0|_U \cong i_* \sigma^*(\mathcal{T}_{P^2} \oplus \mathcal{O}_{P^2}(1))$. To see this, note that the cokernel of the map $g : \mathcal{O}_{P^2}^{\oplus 4} \to \mathcal{O}_{P^2}(1)^{\oplus 4} \oplus \mathcal{O}_{P^2}$ defined by

$$g := \begin{pmatrix} z_0 & 0 \\ z_1 & z_0 \\ z_2 & z_1 \\ 0 & z_2 \\ 0 & 1 \end{pmatrix}$$

is $\mathcal{T}_{P^2} \oplus \mathcal{O}_{P^2}(1)$.

Conventions and notation. Set $B \coloneqq B_r(0)$ and $\tilde{B} \coloneqq B_r(0) \setminus \{0\}$. We denote by $c > 0$ a generic constant, which depends only on $\mathcal{F}, \omega, s|_{B_1 \setminus B_{1/2}}, H_0$, and $\|F_H\|_{L^2(B_{P(0)})}$ (which will be introduced in the next section). Its value might change from one occurrence to the next. Should $c$ depend on further data we indicate this by a subscript. We write $x \lesssim y$ for $x \leq cy$. The expression $O(x)$ denotes a quantity $y$ with $|y| \lesssim x$. Since reflexive sheaves are locally free away from a closed subset of complex codimension three, without loss of generality, we will assume throughout that $n \geq 3$. 

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2 Reduction to the metric setting

In the situation of Theorem 1.2, the Hermitian metric $H$ on $E$ corresponds to a PHYM metric on $\sigma^*F$ via the isomorphism $(E, \bar{\partial}_A) \cong \sigma^*F$. By slight abuse of notation, we will denote this metric by $H$ as well.

Denote by $F_1, \ldots, F_k$ the $\mu$–stable summands of $F$. Denote by $K_j$ the PHYM metric on $F_j$ with

$$i\Lambda_{\omega_{FS}}F_{K_j} = \frac{2\pi}{(n-2)!\text{vol}(\mathbb{P}^{n-1})} \mu_j \cdot \text{id}_{F_j} = (2n-2)\pi \mu_j \cdot \text{id}_{F_j}$$

with $\omega_{FS}$ denoting the integral Fubini study form and for $\mu_j := \mu(F_j)$. The Kähler form $\omega_0$ associated with the standard Kähler metric on $\mathbb{C}^n$ can be written as

$$\omega_0 = \frac{1}{2i}\bar{\partial}\partial |z|^2 = \pi r^2 \sigma^* \omega_{FS} + r \text{d}r \wedge \pi^* \theta$$

with $\theta$ as in Proposition 1.4. Therefore, we have

$$i\Lambda_{\omega_0}F_{\sigma^*K_j} = (2n-2)\mu_j r^{-2} \cdot \text{id}_{\sigma^*F_j},$$

and $H_{\omega,j} := r^{2\mu_j} \cdot \sigma^*K_j$ satisfies

$$i\Lambda_{\omega_0}F_{H_{\omega,j}} = i\Lambda_{\omega_0}F_{\sigma^*K_j} + i\Lambda_{\omega_0}\bar{\partial}\partial \log r^{2\mu_j} \cdot \text{id}_{\sigma^*F_j}$$

$$= i\Lambda_{\omega_0}F_{\sigma^*K_j} + \frac{1}{2} \Delta \log r^{2\mu_j} \cdot \text{id}_{\sigma^*F_j} = 0.$$  

Denote by $A_{\omega,j}$ the Chern connection associated with $H_{\omega,j}$ and by $B_j$ the Chern connection associated with $K_j$. The isometry $r^{\mu_j} : (\sigma^*F_j, H_{\omega,j}) \to \sigma^*(F_j, K_j)$ transforms $A_{\omega,j}$ into

$$A_{\omega,j} := (r^{\mu_j})_*A_{\omega,j} = \sigma^*B_j + i\mu_j \text{id}_{\sigma^*F_j} \cdot \pi^* \theta.$$  

In particular,

$$A_* := \bigoplus_{j=1}^k A_{\omega,j}$$

is the pullback of a connection $B$ on $S^{2n-1}$. Moreover, $A_*$ is unitary with respect to

$$H_* := \bigoplus_{j=1}^k \sigma^*K_j.$$
Proposition 2.2. Assume the above situation. Set \( H_0 := \bigoplus_{j=1}^{k} H_{\omega, j} \) and fix \( R > 0 \). We have
\[
\left\| |z|^{2+\ell} \nabla_{H_0}^\ell F_{H_0} \right\|_{L^\infty(B_R)} < \infty \quad \text{for each } \ell \geq 0.
\] (2.3)

Proof. Using the isometry \( g := \bigoplus_{j=1}^{k} r^{\mu_j} \) both assertions can be translated to corresponding statements for \( A_\mu \). The first assertion then follows since \( A_\mu \) is the pullback of a connection \( B \) on \( S^{2n-1} \).

In the situation of Theorem 1.2, after a conformal change, which does not affect \( A_\omega \), we can assume that \( \det H = \det H_0 \). Setting
\[
s := \log(H_0^{-1}H) \in C^\omega(\tilde{B}_R, isu(\sigma^*F, H_0))^3
\]
and \( \Upsilon(s) := e^{ad_s} - 1 \), we have
\[
e_{s/2}^*H = H_0 \quad \text{and} \quad e_{s/2}^*A = A_\omega + a
\]
with \( a := \frac{1}{2} \Upsilon(-s/2)\bar{\partial}_{A_\omega} s - \frac{1}{2} \Upsilon(s/2)\bar{\partial}_{A_\omega} s \);

see, e.g., [JW18, Appendix A]. Moreover, with \( g := \bigoplus_{j=1}^{k} r^{\mu_j} \) we have
\[g_{s/2}^*A = A_\omega + gag^{-1}.
\]
Since
\[
|\nabla_{A_\omega}^k (gag^{-1})|_{H_0} = |\nabla_{H_0}^k a|_{H_0} \quad \text{for each } k \geq 0,
\]
Theorem 1.2 will be a consequence of Proposition 2.2 and the following result.

Theorem 2.4. Suppose \( \omega = \frac{1}{2} \bar{\partial} \partial |z|^2 + O(|z|^3) \) is a Kähler form on \( \tilde{B}_R \subset C^n \), \( \mathcal{E} \) is a holomorphic vector bundle over \( \tilde{B}_R \), and \( H_0 \) is a Hermitian metric on \( \mathcal{E} \) which is HYM with respect to \( \omega_0 \) and satisfies (2.3). If \( H \) is an admissible HYM metric on \( \mathcal{E} \) with \( \text{sing}(A_H) = \{0\} \) and \( \det H = \det H_0 \), then
\[
s := \log(H_0^{-1}H) \in C^\omega(\tilde{B}_R, isu(\sigma^*F, H_0))
\]

satisfies
\[
|s| \leq C_0 \quad \text{and} \quad |z|^k |\nabla_{H_0}^k s| \leq C_k |z|^\alpha \quad \text{for each } k \geq 1.
\]
The constants \( C_k, \alpha > 0 \) depend on \( \omega, H_0, s|_{\tilde{B}_R \setminus \tilde{B}_R/2} \), and \( \|F_H\|_{L^2(\tilde{B}_R)} \).

The next three sections of this paper are devoted to proving Theorem 2.4. Without loss of generality, we will assume that the radius \( R \) is one. We set \( B := B_1 \) and \( \tilde{B} := \tilde{B}_1 \).

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3If \( H, K \) are two Hermitian inner products on a complex vector space \( V \), then there is a unique endomorphism \( T \in \text{End}(V) \) which is self-adjoint with respect to \( H \) and \( K \), has positive spectrum, and satisfies \( H(Tv, w) = K(v, w) \). It is customary to denote \( T \) by \( H^{-1}K \), and thus \( \log(H^{-1}K) = \log(T) \).
3 A priori $C^0$ estimate

As a first step towards proving Theorem 2.4 we bound $|s|$, using an argument which is essentially contained in Bando and Siu [BS94, Theorem 2(a) and (b)].

**Proposition 3.1.** We have $|s| \in L^\infty(B)$ and $\|s\|_{L^\infty(B)} \leq c$.

**Proof.** The proof relies on the differential inequality

\[
\Delta \log tr H_0^{-1} H_1 \leq |K_{H_1} - K_{H_0}|
\]

for Hermitian metrics $H_0$ and $H_1$ with $\det H_0 = \det H_1$, and with

\[
K_H := i\Lambda H - \frac{\text{tr}(i\Lambda H)}{\text{rk} E} \cdot \text{id}_E;
\]

see [Siu87, p. 13] for a proof.

**Step 1.** We have $\log tr e^s \in W^{1,2}(B)$ and $\|\log tr e^s\|_{W^{1,2}(B)} \leq c$.

Choose $1 \leq i < j \leq n$ and define the projection $\pi : B \to \mathbb{C}^{n-2}$ by

\[
\pi(z) := (z_1, \ldots, \hat{z}_i, \ldots, \hat{z}_j, \ldots, z_n).
\]

For $\zeta \in \mathbb{C}^{n-2}$, denote by $\nabla_\zeta$ and $\Delta_\zeta$ the derivative and the Laplacian on the slice $\pi^{-1}(\zeta)$ respectively. Set $f_\zeta := \log tr e^{s|_{\pi^{-1}(\zeta)}}$. Applying (3.2) to $H|_{\pi^{-1}(\zeta)}$ and $H_0|_{\pi^{-1}(\zeta)}$ we obtain

\[
\Delta_\zeta f_{\zeta} \leq |F_H| + |F_{H_0}|.
\]

Fix $\chi \in C^\infty(\mathbb{C}^2; [0, 1])$ such that $\chi(\eta) = 1$ for $|\eta| \leq 1/2$ and $\chi(\eta) = 0$ for $|\eta| \geq 1/\sqrt{2}$. For $0 < |\zeta| \leq 1/\sqrt{2}$ and $\epsilon > 0$, we have

\[
\int_{\pi^{-1}(\zeta)} |\nabla_\zeta (\chi f_{\zeta})|^2 \leq \int_{\pi^{-1}(\zeta)} \chi^2 f_{\zeta}(|F_H| + |F_{H_0}|) + 1 \\
\leq \epsilon \int_{\pi^{-1}(\zeta)} |\chi f_{\zeta}|^2 + \epsilon^{-1} \int_{\pi^{-1}(\zeta)} |F_H|^2 + |F_{H_0}|^2 + 1.
\]

Using the Dirichlet–Poincaré inequality and rearranging, we obtain

\[
\int_{\pi^{-1}(\zeta)} |\chi f_{\zeta}|^2 + |\nabla_\zeta (\chi f_{\zeta})|^2 \leq \int_{\pi^{-1}(\zeta)} |F_H|^2 + |F_{H_0}|^2 + 1.
\]

Integrating over $0 < |\zeta| \leq 1/\sqrt{2}$ yields

\[
\int_B |\log tr e^s|^2 + |\nabla' \log tr e^s|^2 \leq \int_B |F_H|^2 + |F_{H_0}|^2 + 1
\]

with $\nabla'$ denoting the derivative along the fibres of $\pi$. Using (2.3) and $n \geq 3$, $F_{H_0} \in L^2(B)$. Since the choice of $i, j$ defining $\pi$ was arbitrary, the asserted inequality follows.
Step 2. The differential inequality
\[ \Delta \log tr e^s \lesssim |K_{H_r}| \]
holds on \( B \) in the sense of distributions.

Fix a smooth function \( \chi : [0,\infty) \to [0,1] \) which vanishes on \([0,1]\) and is equal to one on \([2,\infty)\). Set \( \chi_\varepsilon := \chi(|\cdot|/\varepsilon) \). By (3.2), for \( \phi \in C_0^\infty(B) \), we have
\[
\int_B \Delta \phi \cdot \log tr e^s = \lim_{\varepsilon \to 0} \int_B \chi_\varepsilon \cdot \Delta \phi \cdot \log tr e^s \\
\leq \int_B \phi \cdot |K_{H_r}| + \lim_{\varepsilon \to 0} \int_B \phi \cdot (\Delta \chi_\varepsilon \cdot \log tr e^s - 2 \langle \nabla \chi_\varepsilon, \nabla \log tr e^s \rangle).
\]
Since \( n \geq 3 \), we have \( \|\chi_\varepsilon\|_{W^{2,2}(B)} \leq \varepsilon^2 \). Because \( \log tr e^s \in W^{1,2}(B) \) this shows that the limit vanishes.

Step 3. We have \( \log tr e^s \in L^\infty(B) \) and \( \|\log tr e^s\|_{L^\infty(B)} \leq c \).

Since \( tr s = 0 \), we have \( |s| \leq \text{rk}(\mathcal{F}) \cdot \log tr e^s \); in particular, \( \log tr e^s \) is non-negative. By hypothesis \( K_{H_r} = 0 \). Since \( H_0 \) is HYM with respect to \( \omega_0 \) and \( |F_{H_r}| \leq |z|^{-2} \) by hypothesis (2.3), we have \( |K_{H_r}| \leq c \). The asserted inequality thus follows from Step 2 via Moser iteration; see [GT01, Theorem 8.1].

4 A priori Morrey estimates

The following decay estimate is the crucial ingredient of the proof of Theorem 2.4.

Proposition 4.1. There is a constant \( \alpha > 0 \), such that for \( r \in [0,1] \) we have
\[
\int_{B_r} |\nabla H_r s| \leq r^{2n-2+2\alpha}.
\]

The proof of this proposition relies on a Neumann–Poincaré type inequality, which we describe in what follows. Denote by \( \nabla_{T,r} \) the connection on \( isu(E, H_r)|_{\partial B_r} \) induced by \( \nabla_{H_r} \). The linear operator \( \nabla_{T,r} : \Gamma(\partial B_r, isu(E, H_r)) \to \Omega^1(\partial B_r, isu(E, H_r)) \) has a finite dimensional kernel. Since \( \nabla_{H_r} \) is conical, we can identify
\[
\ker \nabla_{T,r} = \ker \nabla_{T,1} =: K.
\]
Moreover, we can regard \( K \) as a subset of constant sections: \( K \subset \Gamma(\hat{B}_r, isu(E, H_0)) \). Denote by \( \pi_r : \Gamma(\partial B_r, isu(E, H_0)) \to K \) the \( L^2 \)-orthogonal projection onto \( K \) and define \( \Pi_r : \Gamma(\hat{B}_r, isu(E, H_0)) \to K \) by
\[
\Pi_r s := \frac{1}{r} \int_r^{2r} \pi_t(s|_{\partial B_t}) \, dt.
\]

\(^4K\) can be determined explicitly from the from the decomposition of \( \mathcal{F} \) into \( \mu \)-stable summands, but we will not need a precise description of \( K \).
Proposition 4.2. We have

\[ \int_{B_{2r}\setminus B_r} |s - \Pi_r s|^2 \lesssim r^2 \int_{B_{2r}\setminus B_r} |\nabla_H s|^2. \]

Proof. The asserted estimate is scale-invariant; hence, we may assume \( r = 1/2 \). To prove the estimate in this case it suffices to prove the cylindrical estimate

\[ \int_{1/2}^1 \int_{\partial B} |s(t, \hat{x}) - \Pi s(t, \cdot)|^2 \, d\hat{x} \, dt \lesssim \int_{1/2}^1 \int_{\partial B} |\partial_t s(t, \hat{x})|^2 + |\nabla_T s(t, \hat{x})|^2 \, d\hat{x} \, dt \]

with \( s \) denoting a section over \([1/2, 1] \times \partial B\), \( \pi := \pi_1 \), \( \Pi s := 2 \int_{1/2}^1 \pi s(t, \cdot) \, dt \), and \( \nabla_T := \nabla_{T,1} \).

We compute

\[ \int_{1/2}^1 \int_{\partial B} |s(t, \hat{x}) - \Pi s(t, \cdot)|^2 \, d\hat{x} \, dt \]

\[ = 4 \int_{1/2}^1 \int_{\partial B} \int_{1/2}^1 |s(t, \hat{x}) - \pi s(u, \cdot)| \, du \, d\hat{x} \, dt \]

\[ \lesssim \int_{1/2}^1 \int_{1/2}^1 \int_{\partial B} |s(t, \hat{x}) - \pi s(u, \cdot)|^2 \, d\hat{x} \, du \, dt \]

\[ \lesssim \int_{1/2}^1 \int_{1/2}^1 \int_{\partial B} |s(t, \hat{x}) - \pi s(t, \cdot)|^2 + |\pi s(t, \cdot) - \pi s(u, \cdot)|^2 \, d\hat{x} \, du \, dt. \]

The first summand can be bounded as follows

\[ \int_{1/2}^1 \int_{1/2}^1 \int_{\partial B} |s(t, \hat{x}) - \pi s(t, \cdot)|^2 \, d\hat{x} \, du \, dt \lesssim \int_{1/2}^1 \int_{1/2}^1 \int_{\partial B} |\nabla_T s(t, \hat{x})|^2 \, d\hat{x} \, du \, dt \]

\[ \lesssim \int_{1/2}^1 \int_{\partial B} |\nabla_T s(t, \hat{x})|^2 \, d\hat{x} \, dt. \]

The second summand can be controlled as in the usual proof of the Neumann–Poincaré inequality. We have

\[ |\pi s(t, \cdot) - \pi s(u, \cdot)| = \left| \int_0^1 \partial_v \pi s(t + v(t - u), \cdot) \, dv \right| \]

\[ \lesssim \left| \int_0^1 \pi(\partial_t s)(t + v(t - u), \cdot) \, dv \right| \]

\[ \lesssim \left( \int_0^1 \int_{\partial B} |(\partial_t s)(t + v(t - u), \hat{x})|^2 \, d\hat{x} \, dv \right)^{1/2}. \]
Plugging this into the second summand and symmetry considerations yield
\[
\int_{1/2}^{1} \int_{1/2}^{1} \int_{\partial B} |\pi s(t, \cdot) - \pi s(u, \cdot)|^2 \, d\hat{x} \, du \, dt \\
\leq \int_{1/2}^{1} \int_{1/2}^{1} \int_{0}^{1} |(\partial_t s)(t + v(t - u), \hat{x})|^2 \, d\hat{x} \, dv \, du \, dt \\
\leq \int_{1/2}^{1} \int_{\partial B} |\partial_t s(t, \hat{x})|^2 \, d\hat{x} \, dt.
\]
This finishes the proof. □

The proof of Proposition 4.1 also uses the following observation about \( \hat{s}_r := \log(e^{-\Pi_r s} e^s) \).

By construction, the section \( \hat{s}_r \) is self-adjoint with respect to \( H_\epsilon e^s \) as well as \( H_\epsilon e^{\Pi_r s} \), and
\[
H_\epsilon e^s = (H_\epsilon e^{\Pi_r s}) e^{\hat{s}_r}.
\]

**Proposition 4.3.** The section \( \hat{s}_r \) satisfies
\[
|\nabla H_\epsilon s| \leq |\nabla H_\epsilon \hat{s}_r|, \quad |\hat{s}_r| \leq |s - \Pi_r s|, \quad \text{and} \quad |\nabla H_\epsilon \hat{s}_r|^2 \leq 1 - \Delta |\hat{s}_r|^2.
\]

**Proof.** The first two inequalities follow by elementary considerations.

Since \( s \) is bounded in \( L^\infty(B) \), \( \Pi_r s \) is uniformly bounded and, consequently, so is \( \hat{s}_r \). By [JW18, Proposition A.9], we have
\[
\Delta |\hat{s}_r|^2 + 2|\nabla H_\epsilon e^{\Pi_r s} \hat{s}_r|^2 \leq |K_{H_\epsilon e^s}| + |K_{H_\epsilon e^{\Pi_r s}}|
\]
with
\[
u(\hat{s}_r) = \sqrt{1 - e^{-\text{ad}_{\hat{s}_r}} \text{ad}_{\hat{s}_r}} \in \text{End}(\mathfrak{g}(E)).
\]

\( H_\epsilon e^s \) is HYM; that is: \( K_{H_\epsilon e^s} = 0 \) Since \( \Pi_r s \) is constant with respect to \( \nabla H_\epsilon \), we have
\[
K_{H_\epsilon e^{\Pi_r s}} = i\Lambda(\epsilon^{\Pi_r s} \partial_{H_\epsilon} e^{-\Pi_r s}) = \text{Ad}(e^{\Pi_r s})K_{H_\epsilon},
\]
which is bounded. Moreover, \( \nabla H_\epsilon \) and \( \nabla H_\epsilon e^{\Pi_r s} \) differ by a bounded algebraic operator. Given this, the third inequality follows using
\[
\sqrt{\frac{1 - e^{-x}}{x}} \geq \frac{1}{\sqrt{1 + |x|}}.
\]

\( \|K_{H_\epsilon}\|_{L^\infty} \leq c \), which is a consequence of (2.3), and the fact that \( H_\epsilon \) is HYM with respect to \( \omega_0 \), and the bound on \( |s| \) established in Proposition 3.1. □
Proof of Proposition 4.1. Given the above discussion, the proof is very similar to that of [JW18, Proposition C.2]. Nevertheless, for the reader’s convenience we provide the necessary details.

Define \( g : [0, 1/2] \to [0, \infty) \) by

\[
g(r) := \int_{B_r} |z|^{2-2n} |\nabla H_s| \,dz.
\]

We will show that

\[
g(r) \leq cr^{2\alpha},
\]

which implies the asserted inequality.

**Step 1.** We have \( g \leq c \).

Fix a smooth function \( \chi : [0, \infty) \to [0, 1] \) which is equal to one on \([0, 1]\) and vanishes outside \([0, 2]\). Set \( \chi_r(\cdot) := \chi(|\cdot|/r) \). For \( r > \varepsilon > 0 \), using Proposition 4.3 and Proposition 3.1, and with \( G \) denoting Green’s function on \( B \) centered at 0, we have

\[
\int_{B_r \setminus B_{2r}} |z|^{2-2n} |\nabla H_s| \,dz \leq \int_{B_r \setminus B_{2r}} |z|^{2-2n} |\nabla H_\hat{s}_r| \,dz \\
\leq \int_{B_r \setminus B_{2r}} \chi_r(1 - \chi_{\varepsilon/2}) G(1 - \Delta |\hat{s}_r|^2) \\
\leq \int_{B_r \setminus B_{2r}} |z|^{-2n} |s - \Pi_r s|^2 + r^2 + \varepsilon^{-2n} \int_{B_r \setminus B_{2r}} |s - \Pi_r s|^2 \\
\leq c.
\]

**Step 2.** There are constants \( \gamma \in [0, 1) \) and \( A > 0 \) such that

\[
g(r) \leq \gamma g(2r) + Ar^2.
\]

Continuing the inequality from **Step 1** using Proposition 4.2, we have

\[
\int_{B_r \setminus B_{2r}} |z|^{2-2n} |\nabla H_s| \,dz \leq \int_{B_r \setminus B_{2r}} |z|^{2-2n} |\nabla H_s| \,dz + r^2 + \varepsilon^{-2n} \int_{B_r \setminus B_{2r}} |\nabla H_s| \,dz \\
\leq g(2r) - g(r) + r^2 + g(\varepsilon).
\]

By Lebesgue’s monotone convergence theorem, the last term vanishes as \( \varepsilon \) tends to zero; hence, the asserted inequality follows with \( \gamma = \frac{c}{c+1} \) and \( A = c \).

**Step 3.** We have \( g \leq cr^{2\alpha} \) for some \( \alpha \in (0, 1) \).

This follows from **Step 1** and **Step 2** and as in [JW18, Step 3 in the proof of Proposition C.2]. \( \square \)
5 Proof of Theorem 2.4

For \( r > 0 \), define \( m_r : C^n \rightarrow C^n \) by \( m_r(z) := rz \). Set
\[
s_r := m_r^*(s|_{B_{r/2} \setminus B_{r/4}}) \in C^\infty(B_1 \setminus B_{1/2}, \text{isim}(E, H_r)) \quad \text{and} \quad H_{0,r} := m_r^*H_0.
\]
The metric \( H_{0,r}e^{s_r} \) is HYM with respect to \( \omega_r := r^{-2}m_r^*\omega \) and \( \|F_{H_{0,r}}\|_{C^k(B_1 \setminus B_{1/2})} \leq c_k \).

Proposition 3.1, (2.3) and interior estimates for HYM metrics [JW18, Theorem C.1] imply that
\[
\|s_r\|_{C^k(B_1 \setminus B_{1/2})} \leq c_k.
\]
By Proposition 4.1, we have
\[
\|\nabla H_{0,r}s_r\|_{L^2(B_1 \setminus B_{1/2})} \leq r^\alpha.
\]
Schematically, \( K_{H_{0,r}}e^{s_r} = 0 \) can be written as
\[
\nabla^*_r H_{0,r} \nabla H_{0,r} s_r + B(\nabla H_{0,r} s \otimes \nabla H_{0,r} s_r) = C(K_{H_{0,r}}),
\]
where \( B \) and \( C \) are linear with coefficients depending on \( s \), but not on its derivatives; see, e.g., [JW18, Proposition A.1]. Since \( \|K_{H_{0,r}}\|_{C^k(B_1 \setminus B_{1/2})} \leq c_k r^2 \), as in [JW18, Step 3 in the proof of Proposition 5.1], standard interior estimates imply that
\[
\|\nabla H_{0,r}s_r\|_{L^2(B_1 \setminus B_{1/2})} \leq c_k r^\alpha
\]
and, hence, the asserted inequalities, for each \( k \geq 1 \). (The asserted inequality for \( k = 0 \) has already been proven in Proposition 3.1.) \( \square \)

6 Proof of Proposition 1.4

We will make use of the following general fact about connections over manifolds with free \( S^1 \)-actions.

**Proposition 6.1.** Let \( M \) be a manifold with a free \( S^1 \)-action. Denote the associated Killing field by \( \xi \in \text{Vect}(M) \) and let \( q : M \rightarrow M//S^1 \) be the canonical projection. Suppose \( \theta \in \Omega^1(M) \) is such that \( \theta(\xi) = 1 \) and \( \mathcal{L}_\xi \theta = 0 \). Let \( A \) be a unitary connection on a Hermitian vector bundle \((E, H)\) over \( M \). If \( i(\xi)F_A = 0 \), then there is a \( k \in \mathbb{N} \) and, for each \( j \in \{1, \ldots, k\} \), a Hermitian vector bundles \((F_j, K_j)\) over \( M//S^1 \) such that
\[
E = \bigoplus_{j=1}^k E_j \quad \text{and} \quad H = \bigoplus_{j=1}^k H_j
\]
with \( E_j := q^*F_j \) and \( H_j := q^*K_j \); moreover, the bundles \( E_j \) are parallel and, for each \( j \in \{1, \ldots, k\} \), there are a unitary connection \( B_j \) on \( F_j \) and \( \mu_j \in \mathbb{R} \) such that
\[
A = \bigoplus_{j=1}^k q^*B_j + i\mu_j \text{id}_{E_j} \cdot \theta.
\]
Proof. Denote by $\tilde{\xi} \in \text{Vect}(U(E))$ the $A$–horizontal lift of $\xi$. This vector field integrates to an $R$–action on $U(E)$. Thinking of $A$ as an $\mathfrak{u}(r)$–valued 1–form on $U(E)$ and $F_A$ as an $\mathfrak{u}(r)$–valued 2–form on $U(E)$, we have
\[
\mathcal{L}_\xi A = i(\tilde{\xi}) F_A = 0;
\]
hence, $A$ is invariant with respect to the $R$–action on $U(E)$.

The obstruction to the $R$–action on $U(E)$ inducing an $S^1$–action is the action of 1 $\in R$ and corresponds to a gauge transformation $g_A \in \mathcal{G}(U(E))$ fixing $A$. If this obstruction vanishes, i.e., $g_A = \text{id}_{U(E)}$, then $E \cong \bar{q}^* F$ with $F = E/S^1$ and there is a connection $A_0$ on $F$ such that $A = q^* A_0$.

If the obstruction does not vanish, we can decompose $E$ into pairwise orthogonal parallel subbundles $E_j$ such that $g_A$ acts on $E_j$ as multiplication with $e^{i\mu_j}$ for some $\mu_j \in R$. Set $A := A - \bigoplus_{j=1}^k i\mu_j \text{id}_{E_j} \cdot \theta$. This connection also satisfies $i(\tilde{\xi}) F_A = 0 \in \Omega^1(M, g_E)$ and the subbundles $E_j$ are also parallel with respect to $E_j$. Since $g_A = \text{id}_{E}$, the assertion follows.

In the situation of Proposition 1.4, with $\xi \in S^{2n-1}$ denoting the Killing field for the $S^1$–action we have $i(\tilde{\xi}) F_{A_0} = 0$; c.f., Tian [Tia80, discussion after Conjecture 2]. Therefore, we can write
\[
A_* = \bigoplus_{j=1}^k \sigma^* B_j + i\mu_j \text{id}_{E_j} \cdot \pi^* \theta.
\]
Since $d\theta = 2\pi \rho^* \omega_{FS}$, we have
\[
F_{A_*} = \bigoplus_{j=1}^k \sigma^* F_{B_j} + 2\pi i\mu_j \text{id}_{E_j} \cdot \sigma^* \omega_{FS}.
\]
Using (2.1), $A_*$ being HYM with respect to $\omega_0$ can be seen to be equivalent to
\[
F_{B_j}^{A_*} = 0 \quad \text{and} \quad i\Lambda F_{B_j} = (2n-2)\pi \mu_j \cdot \text{id}_{E_j}.
\]
The isomorphism $\mathcal{E} = (E, \bar{\partial}_{A_*}) \cong \bigoplus_{j=1}^k \rho^* \mathcal{F} / \mathcal{F}_j$ with $\mathcal{F}_j = (F_j, \bar{\partial}_{B_j})$ is given by $g^{-1}$ with $g := \bigoplus_{j=1}^k \rho^{\mu_j}$.

References


