

# Tangent cones of Hermitian Yang–Mills connections with isolated singularities

Adam Jacob

Henrique Sá Earp

Thomas Walpuski

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## Abstract

We give a simple direct proof of uniqueness of tangent cones for singular projectively Hermitian Yang–Mills connections on reflexive sheaves at isolated singularities modelled on a sum of  $\mu$ -stable holomorphic bundles over  $\mathbf{P}^{n-1}$ .

## 1 Introduction

A **projectively Hermitian Yang–Mills (PHYM)** connection  $A$  over a Kähler manifold  $X$  is a unitary connection  $A$  on a Hermitian vector bundle  $(E, H)$  over  $X$  satisfying

$$(1.1) \quad F_A^{0,2} = 0 \quad \text{and} \quad i\Lambda F_A - \frac{\text{tr}(i\Lambda F_A)}{\text{rk } E} \cdot \text{id}_E = 0.$$

Since  $F_A^{0,2} = 0$ ,  $\mathcal{E} := (E, \bar{\partial}_A)$  is a holomorphic vector bundle, and  $A$  is the Chern connection of  $H$ . A Hermitian metric  $H$  on a holomorphic vector bundle is called **PHYM** if its Chern connection  $A_H$  is PHYM. The celebrated Donaldson–Uhlenbeck–Yau Theorem [Don85; Don87; UY86] asserts that a holomorphic vector bundle  $\mathcal{E}$  on a compact Kähler manifold admits a PHYM metric if and only if it is  $\mu$ -polystable; moreover, any two PHYM metrics are related by an automorphism of  $\mathcal{E}$  and by multiplication with a conformal factor. If  $H$  is a PHYM metric, then the connection  $A^\circ$  on  $\text{PU}(E, H)$ , the principal  $\text{PU}(r)$ -bundle associated with  $(E, H)$ , induced by  $A_H$  is **Hermitian Yang–Mills (HYM)**, that is, it satisfies  $F_{A^\circ}^{0,2} = 0$  and  $i\Lambda F_{A^\circ} = 0$ ; it depends only on the conformal class of  $H$ . Conversely, any HYM connection  $A^\circ$  on  $\text{PU}(E, H)$  can be lifted to a PHYM connection  $A$ ; any two choices of lifts lead to isomorphic holomorphic vector bundles  $\mathcal{E}$  and conformal metrics  $H$ .

An **admissible PHYM** connection is a PHYM connection  $A$  on a Hermitian vector bundle  $(E, H)$  over  $X \setminus \text{sing}(A)$  with  $\text{sing}(A)$  a closed subset with locally finite  $(2n - 4)$ -dimensional Hausdorff measure and  $F_A \in L^2_{\text{loc}}(X)$ .<sup>1</sup> Bando [Ban91] proved that if  $A$  is an admissible PHYM connection,

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<sup>1</sup>It should be pointed out that our notion of admissible PHYM connection follows Bando and Siu [BS94] and not Tian [Tia00]. The notion of admissible Yang–Mills connection introduced by Tian is stronger: it assumes that the Hermitian vector bundle extends to all of  $X$ .

then  $(E, \bar{\partial}_A)$  extends to  $X$  as a reflexive sheaf  $\mathcal{E}$  with  $\text{sing}(\mathcal{E}) \subset \text{sing}(A)$ . Bando and Siu [BS94] proved that a reflexive sheaf on a compact Kähler manifold admits an admissible PHYM metric if and only if it is  $\mu$ -polystable.

The technique used by Bando and Siu does not yield any information on the behaviour of the admissible PHYM connection  $A_H$  near the singularities of the reflexive sheaf  $\mathcal{E}$ —not even at isolated singularities. The simplest example of a reflexive sheaf on  $\mathbf{C}^n$  with an isolated singularity at 0 is  $i_*\sigma^*\mathcal{F}$  with  $\mathcal{F}$  a holomorphic vector bundle over  $\mathbf{P}^{n-1}$ ; cf. Hartshorne [Har80, Example 1.9.1]. Here we use the obvious maps summarised in the following diagram:

$$\begin{array}{ccccc} \mathbf{C}^n & \xleftarrow{i} & \mathbf{C}^n \setminus \{0\} & \xrightarrow{\pi} & S^{2n-1} & \xrightarrow{\rho} & \mathbf{P}^{n-1}. \\ & & & & \searrow \sigma & & \nearrow \end{array}$$

The main result of this article gives a description of PHYM connections near singularities modelled on  $i_*\sigma^*\mathcal{F}$  with  $\mathcal{F}$  a sum of  $\mu$ -stable holomorphic vector bundles.

**Theorem 1.2.** *Let  $\omega = \frac{1}{2i}\bar{\partial}\partial|z|^2 + O(|z|^2)$  be a Kähler form on  $\bar{B}_R(0) \subset \mathbf{C}^n$ . Let  $A$  be an admissible PHYM connection on a Hermitian vector bundle  $(E, H)$  over  $B_R(0) \setminus \{0\}$  with  $\text{sing}(A) = \{0\}$  and  $(E, \bar{\partial}_A) \cong \sigma^*\mathcal{F}$  for some holomorphic vector bundle  $\mathcal{F}$  over  $\mathbf{P}^{n-1}$ . Denote by  $F$  the complex vector bundle underlying  $\mathcal{F}$ .*

*If  $\mathcal{F}$  is a sum of  $\mu$ -stable holomorphic vector bundles, then there exist a Hermitian metric  $K$  on  $F$ , a connection  $A_*$  on  $\sigma^*(F, K)$  which is the pullback of a connection on  $\rho^*(F, K)$ , and an isometry  $(E, H) \cong \sigma^*(F, K)$  such that with respect to this isometry we have*

$$|z|^{k+1} |\nabla_{A_*}^k (A^\circ - A_*^\circ)| \leq C_k |z|^\alpha \quad \text{for each } k \geq 0.$$

*The constants  $C_k, \alpha > 0$  depend on  $\omega, \mathcal{F}, A|_{B_R(0) \setminus B_{R/2}(0)}$ , and  $\|F_A\|_{L^2(B_R(0))}$ .*

**Remark 1.3.** Using a gauge theoretic Łojasiewicz–Simon gradient inequality, Yang [Yano3, Theorem 1] proved that the tangent cone to a stationary Yang–Mills connection—in particular, a  $\text{PU}(r)$  HYM connection—with an isolated singularity at  $x$  is unique provided

$$|F_A| \lesssim d(x, \cdot)^{-2}.$$

In our situation, such a curvature bound can be obtained from Theorem 1.2. Our proof of this result, however, proceeds more directly—without making use of Yang’s theorem.

The hypothesis that  $\mathcal{F}$  be a sum of  $\mu$ -stable holomorphic vector bundles is optimal. This is a consequence of the following observation, which will be proved in Section 6.

**Proposition 1.4.** *Let  $(F, K)$  be a Hermitian vector bundle over  $\mathbf{P}^{n-1}$ . If  $B$  is a unitary connection on  $\rho^*(F, K)$  such that  $A_* := \pi^*B$  is HYM with respect to  $\omega_0 := \frac{1}{2i}\bar{\partial}\partial|z|^2$ , then there is a  $k \in \mathbf{N}$  and, for each  $j \in \{1, \dots, k\}$ , there are  $\mu_j \in \mathbf{R}$ , a Hermitian vector bundle  $(F_j, K_j)$  on  $\mathbf{P}^{n-1}$ , and an irreducible unitary connection  $B_j$  on  $F_j$  satisfying*

$$F_{B_j}^{0,2} = 0 \quad \text{and} \quad i\Lambda F_{B_j} = (2n - 2)\pi\mu_j \cdot \text{id}_{F_j}$$

such that

$$F = \bigoplus_{j=1}^k F_j \quad \text{and} \quad B = \bigoplus_{j=1}^k \rho^* B_j + i\mu_j \text{id}_{\rho^* F_j} \cdot \theta.$$

Here  $\theta$  denotes the standard contact structure<sup>2</sup> on  $S^{2n-1}$ . In particular,

$$\mathcal{E} = (\sigma^* F, \bar{\partial}_{A_*}) \cong \bigoplus_{j=1}^k \sigma^* \mathcal{F}_j$$

with  $\mathcal{F}_j = (F_j, \bar{\partial}_{B_j})$   $\mu$ -stable.

To conclude the introduction we discuss two concrete examples in which Theorem 1.2 can be applied.

**Example 1.5** (Okonek, Schneider, and Spindler [OSS11, Example 1.1.13]). It follows from the Euler sequence that  $H^0(\mathcal{T}_{\mathbf{P}^3}(-1)) \cong \mathbf{C}^4$ . Denote by  $s_v \in H^0(\mathcal{T}_{\mathbf{P}^3}(-1))$  the section corresponding to  $v \in \mathbf{C}^4$ . If  $v \neq 0$ , then the rank two sheaf  $\mathcal{E} = \mathcal{E}_v$  defined by

$$0 \rightarrow \mathcal{O}_{\mathbf{P}^3} \xrightarrow{s_v} \mathcal{T}_{\mathbf{P}^3}(-1) \rightarrow \mathcal{E}_v \rightarrow 0$$

is reflexive and  $\text{sing}(\mathcal{E}) = \{[v]\}$ .

$\mathcal{E}$  is  $\mu$ -stable. To see this, because  $\mu(\mathcal{E}) = 1/2$ , it suffices to show that

$$\text{Hom}(\mathcal{O}_{\mathbf{P}^3}(k), \mathcal{E}) = H^0(\mathcal{E}(-k)) = 0 \quad \text{for each } k \geq 1.$$

However, by inspection of the Euler sequence,  $H^0(\mathcal{E}(-k)) \cong H^0(\mathcal{T}_{\mathbf{P}^3}(-k-1)) = 0$ . It follows that  $\mathcal{E}$  admits a PHYM metric  $H$  with  $F_H \in L^2$  and a unique singular point at  $[v] \in \mathbf{P}^3$ . To see that Theorem 1.2 applies, pick a standard affine neighborhood  $U \cong \mathbf{C}^3$  in which  $[v]$  corresponds to 0. In  $U$ , the Euler sequence becomes

$$0 \rightarrow \mathcal{O}_{\mathbf{C}^3} \xrightarrow{(1, z_1, z_2, z_3)} \mathcal{O}_{\mathbf{C}^3}^{\oplus 4} \rightarrow \mathcal{T}_{\mathbf{P}^3}(-1)|_U \rightarrow 0,$$

and  $s_v = [(1, 0, 0, 0)]$ ; hence,

$$0 \rightarrow \mathcal{O}_{\mathbf{C}^3} \xrightarrow{(z_1, z_2, z_3)} \mathcal{O}_{\mathbf{C}^3}^{\oplus 3} \rightarrow \mathcal{E}_v|_U \rightarrow 0.$$

On  $\mathbf{C}^3 \setminus \{0\}$ , this is the pullback of the Euler sequence on  $\mathbf{P}^2$ ; therefore,  $\mathcal{E}_v|_U \cong i_* \sigma^* \mathcal{T}_{\mathbf{P}^2}$ .

<sup>2</sup>With respect to standard coordinates on  $\mathbf{C}^n$ , the standard contact structure  $\theta$  on  $S^{2n-1}$  is such that  $\pi^* \theta = \sum_{j=1}^n (\bar{z}_j dz_j - z_j d\bar{z}_j) / 2i|z|^2$ .

**Example 1.6.** For  $t \in \mathbb{C}$ , define  $f_t: \mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus 2} \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 5}$  by

$$f_t := \begin{pmatrix} z_0 & 0 \\ z_1 & z_0 \\ z_2 & z_1 \\ t \cdot z_3 & z_2 \\ 0 & z_3 \end{pmatrix},$$

and denote by  $\mathcal{E}_t$  the cokernel of  $f_t$ , i.e.,

$$(1.7) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus 2} \xrightarrow{f_t} \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 5} \rightarrow \mathcal{E}_t \rightarrow 0.$$

If  $t \neq 0$ , then  $\mathcal{E}_t$  is locally free;  $\mathcal{E}_0$  is reflexive with  $\text{sing}(\mathcal{E}_0) = \{[0 : 0 : 0 : 1]\}$ . The proof of this is analogous to that of the reflexivity of  $\mathcal{E}_v$  from Example 1.5 given in [OSS11, Example 1.1.13].

For each  $t$ ,  $H^0(\mathcal{E}_t) = H^0(\mathcal{E}_t^*(-1)) = 0$ ; hence,  $\mathcal{E}_t$  is  $\mu$ -stable according to the criterion of Okonek, Schneider, and Spindler [OSS11, Remark 1.2.6(b)]. The former vanishing is obvious since  $H^0(\mathcal{O}_{\mathbb{P}^3}(-1)) = H^1(\mathcal{O}_{\mathbb{P}^3}(-2)) = 0$ . The latter follows by dualising (1.7), twisting by  $\mathcal{O}_{\mathbb{P}^3}(-1)$  and observing that the induced map  $H^0(f_0^*): H^0(\mathcal{O}_{\mathbb{P}^3})^{\oplus 5} \rightarrow H^0(\mathcal{O}_{\mathbb{P}^3}(1))^{\oplus 2}$ , which is given by

$$\begin{pmatrix} z_0 & z_1 & z_2 & t \cdot z_3 & 0 \\ 0 & z_0 & z_1 & z_2 & z_3 \end{pmatrix},$$

is injective.

In a standard affine neighborhood  $U \cong \mathbb{C}^3$  of  $[0 : 0 : 0 : 1]$ , we have  $\mathcal{E}_0|_U \cong i_*\sigma^*(\mathcal{T}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$ . To see this, note that the cokernel of the map  $g: \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \rightarrow \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^2}$  defined by

$$g := \begin{pmatrix} z_0 & 0 \\ z_1 & z_0 \\ z_2 & z_1 \\ 0 & z_2 \\ 0 & 1 \end{pmatrix}$$

is  $\mathcal{T}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)$ .

**Conventions and notation.** Set  $B_r := B_r(0)$  and  $\dot{B}_r := B_r(0) \setminus \{0\}$ . We denote by  $c > 0$  a generic constant, which depends only on  $\mathcal{F}$ ,  $\omega$ ,  $s|_{B_1 \setminus B_{1/2}}$ ,  $H_\diamond$ , and  $\|F_H\|_{L^2(B_R(0))}$  (which will be introduced in the next section). Its value might change from one occurrence to the next. Should  $c$  depend on further data we indicate this by a subscript. We write  $x \lesssim y$  for  $x \leq cy$ . The expression  $O(x)$  denotes a quantity  $y$  with  $|y| \lesssim x$ . Since reflexive sheaves are locally free away from a closed subset of complex codimension three, without loss of generality, we will assume throughout that  $n \geq 3$ .

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## 2 Reduction to the metric setting

In the situation of Theorem 1.2, the Hermitian metric  $H$  on  $\mathcal{E}$  corresponds to a PHYM metric on  $\sigma^*\mathcal{F}$  via the isomorphism  $(E, \bar{\partial}_A) \cong \sigma^*\mathcal{F}$ . By slight abuse of notation, we will denote this metric by  $H$  as well.

Denote by  $\mathcal{F}_1, \dots, \mathcal{F}_k$  the  $\mu$ -stable summands of  $\mathcal{F}$ . Denote by  $K_j$  the PHYM metric on  $\mathcal{F}_j$  with

$$i\Lambda_{\omega_{FS}} F_{K_j} = \frac{2\pi}{(n-2)! \text{vol}(\mathbf{P}^{n-1})} \mu_j \cdot \text{id}_{F_j} = (2n-2)\pi \mu_j \cdot \text{id}_{F_j}$$

with  $\omega_{FS}$  denoting the integral Fubini study form and for  $\mu_j := \mu(\mathcal{F}_j)$ . The Kähler form  $\omega_0$  associated with the standard Kähler metric on  $\mathbf{C}^n$  can be written as

$$(2.1) \quad \omega_0 = \frac{1}{2i} \bar{\partial} \partial |z|^2 = \pi r^2 \sigma^* \omega_{FS} + r dr \wedge \pi^* \theta$$

with  $\theta$  as in Proposition 1.4. Therefore, we have

$$i\Lambda_{\omega_0} F_{\sigma^* K_j} = (2n-2)\mu_j r^{-2} \cdot \text{id}_{\sigma^* F_j},$$

and  $H_{\diamond, j} := r^{2\mu_j} \cdot \sigma^* K_j$  satisfies

$$\begin{aligned} i\Lambda_{\omega_0} F_{H_{\diamond, j}} &= i\Lambda_{\omega_0} F_{\sigma^* K_j} + i\Lambda_{\omega_0} \bar{\partial} \partial \log r^{2\mu_j} \cdot \text{id}_{\sigma^* F_j} \\ &= i\Lambda_{\omega_0} F_{\sigma^* K_j} + \frac{1}{2} \Delta \log r^{2\mu_j} \cdot \text{id}_{\sigma^* F_j} = 0. \end{aligned}$$

Denote by  $A_{\diamond, j}$  the Chern connection associated with  $H_{\diamond, j}$  and by  $B_j$  the Chern connection associated with  $K_j$ . The isometry  $r^{\mu_j} : (\sigma^* F_j, H_{\diamond, j}) \rightarrow \sigma^*(F_j, K_j)$  transforms  $A_{\diamond, j}$  into

$$A_{*, j} := (r^{\mu_j})_* A_{\diamond, j} = \sigma^* B_j + i\mu_j \text{id}_{\sigma^* F_j} \cdot \pi^* \theta.$$

In particular,

$$A_* := \bigoplus_{j=1}^k A_{*, j}$$

is the pullback of a connection  $B$  on  $S^{2n-1}$ . Moreover,  $A_*$  is unitary with respect to

$$H_* := \bigoplus_{j=1}^k \sigma^* K_j.$$

**Proposition 2.2.** *Assume the above situation. Set  $H_\diamond := \bigoplus_{j=1}^k H_{\diamond,j}$  and fix  $R > 0$ . We have*

$$(2.3) \quad \left\| |z|^{2+\ell} \nabla_{H_\diamond}^\ell F_{H_\diamond} \right\|_{L^\infty(B_R)} < \infty \quad \text{for each } \ell \geq 0.$$

*Proof.* Using the isometry  $g := \bigoplus_{j=1}^k r^{\mu_j}$  both assertions can be translated to corresponding statements for  $A_*$ . The first assertion then follows since  $A_*$  is the pullback of a connection  $B$  on  $S^{2n-1}$ .  $\square$

In the situation of Theorem 1.2, after a conformal change, which does not affect  $A^\circ$ , we can assume that  $\det H = \det H_\diamond$ . Setting

$$s := \log(H_\diamond^{-1}H) \in C^\infty(\dot{B}_r, \text{isu}(\sigma^*F, H_\diamond))^3$$

$$\text{and } \Upsilon(s) := \frac{e^{\text{ad}_s} - 1}{\text{ad}_s},$$

we have

$$e_*^{s/2}H = H_\diamond \quad \text{and} \quad e_*^{s/2}A = A_\diamond + a$$

$$\text{with } a := \frac{1}{2}\Upsilon(-s/2)\partial_{A_\diamond}s - \frac{1}{2}\Upsilon(s/2)\bar{\partial}_{A_\diamond}s;$$

see, e.g., [JW16, Appendix A]. Moreover, with  $g := \bigoplus_{j=1}^k r^{\mu_j}$  we have

$$g_*e_*^{s/2}A = A_* + gag^{-1}.$$

Since

$$|\nabla_{A_*}^k gag^{-1}|_{H_*} = |\nabla_{H_\diamond}^k a|_{H_\diamond} \quad \text{for each } k \geq 0,$$

Theorem 1.2 will be a consequence of Proposition 2.2 and the following result.

**Theorem 2.4.** *Suppose  $\omega = \frac{1}{2i}\bar{\partial}\partial|z|^2 + O(|z|^2)$  is a Kähler form on  $\bar{B}_R \subset \mathbb{C}^n$ ,  $\mathcal{E}$  is a holomorphic vector bundle over  $\dot{B}_R$ , and  $H_\diamond$  is a Hermitian metric on  $\mathcal{E}$  which is HYM with respect to  $\omega_0$  and satisfies (2.3). If  $H$  is an admissible HYM metric on  $\mathcal{E}$  with  $\text{sing}(A_H) = \{0\}$  and  $\det H = \det H_\diamond$ , then*

$$s := \log(H_\diamond^{-1}H) \in C^\infty(\dot{B}_R, \text{isu}(\pi^*F, H_\diamond))$$

satisfies

$$|s| \leq C_0 \quad \text{and} \quad |z|^k |\nabla_{H_\diamond}^k s| \leq C_k |z|^\alpha \quad \text{for each } k \geq 1.$$

The constants  $C_k, \alpha > 0$  depend on  $\omega, H_\diamond, s|_{B_R \setminus B_{R/2}}$ , and  $\|F_H\|_{L^2(B_R)}$ .

The next three sections of this paper are devoted to proving Theorem 2.4. Without loss of generality, we will assume that the radius  $R$  is one. We set  $B := B_1$  and  $\dot{B} := \dot{B}_1$ .

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<sup>3</sup>If  $H, K$  are two Hermitian inner products on a complex vector space  $V$ , then there is a unique endomorphism  $T \in \text{End}(V)$  which is self-adjoint with respect to  $H$  and  $K$ , has positive spectrum, and satisfies  $H(Tv, w) = K(v, w)$ . It is customary to denote  $T$  by  $H^{-1}K$ , and thus  $\log(H^{-1}K) = \log(T)$ .

### 3 A priori $C^0$ estimate

As a first step towards proving Theorem 2.4 we bound  $|s|$ , using an argument which is essentially contained in Bando and Siu [BS94, Theorem 2(a) and (b)].

**Proposition 3.1.** *We have  $|s| \in L^\infty(B)$  and  $\|s\|_{L^\infty(B)} \leq c$ .*

*Proof.* The proof relies on the differential inequality

$$(3.2) \quad \Delta \log \operatorname{tr} H_0^{-1} H_1 \lesssim |K_{H_1} - K_{H_0}|$$

for Hermitian metrics  $H_0$  and  $H_1$  with  $\det H_0 = \det H_1$ , and with

$$K_H := i\Delta F_H - \frac{\operatorname{tr}(i\Delta F_H)}{\operatorname{rk} E} \cdot \operatorname{id}_E;$$

see [Siu87, p. 13] for a proof.

**Step 1.** *We have  $\log \operatorname{tr} e^s \in W^{1,2}(B)$  and  $\|\log \operatorname{tr} e^s\|_{W^{1,2}(B)} \leq c$ .*

Choose  $1 \leq i < j \leq n$  and define the projection  $\pi: B \rightarrow \mathbb{C}^{n-2}$  by

$$\pi(z) := (z_1, \dots, \hat{z}_i, \dots, \hat{z}_j, \dots, z_n).$$

For  $\zeta \in \mathbb{C}^{n-2}$ , denote by  $\nabla_\zeta$  and  $\Delta_\zeta$  the derivative and the Laplacian on the slice  $\pi^{-1}(\zeta)$  respectively. Set  $f_\zeta := \log \operatorname{tr} e^s|_{\pi^{-1}(\zeta)}$ . Applying (3.2) to  $H|_{\pi^{-1}(\zeta)}$  and  $H_\circ|_{\pi^{-1}(\zeta)}$  we obtain

$$\Delta_\zeta f_\zeta \lesssim |F_H| + |F_{H_\circ}|.$$

Fix  $\chi \in C^\infty(\mathbb{C}^2; [0, 1])$  such that  $\chi(\eta) = 1$  for  $|\eta| \leq 1/2$  and  $\chi(\eta) = 0$  for  $|\eta| \geq 1/\sqrt{2}$ . For  $0 < |\zeta| \leq 1/\sqrt{2}$  and  $\varepsilon > 0$ , we have

$$\begin{aligned} \int_{\pi^{-1}(\zeta)} |\nabla_\zeta(\chi f_\zeta)|^2 &\lesssim \int_{\pi^{-1}(\zeta)} \chi^2 f_\zeta^2 (|F_H| + |F_{H_\circ}|) + 1 \\ &\leq \varepsilon \int_{\pi^{-1}(\zeta)} |\chi f_\zeta|^2 + \varepsilon^{-1} \int_{\pi^{-1}(\zeta)} |F_H|^2 + |F_{H_\circ}|^2 + 1. \end{aligned}$$

Using the Dirichlet–Poincaré inequality and rearranging, we obtain

$$\int_{\pi^{-1}(\zeta)} |\chi f_\zeta|^2 + |\nabla_\zeta(\chi f_\zeta)|^2 \lesssim \int_{\pi^{-1}(\zeta)} |F_H|^2 + |F_{H_\circ}|^2 + 1.$$

Integrating over  $0 < |\zeta| \leq 1/\sqrt{2}$  yields

$$\int_B |\log \operatorname{tr} e^s|^2 + |\nabla' \log \operatorname{tr} e^s|^2 \lesssim \int_B |F_H|^2 + |F_{H_\circ}|^2 + 1$$

with  $\nabla'$  denoting the derivative along the fibres of  $\pi$ . Using (2.3) and  $n \geq 3$ ,  $F_{H_\circ} \in L^2(B)$ . Since the choice of  $i, j$  defining  $\pi$  was arbitrary, the asserted inequality follows.

**Step 2.** *The differential inequality*

$$\Delta \log \operatorname{tr} e^s \lesssim |K_{H_\circ}|$$

*holds on  $B$  in the sense of distributions.*

Fix a smooth function  $\chi : [0, \infty) \rightarrow [0, 1]$  which vanishes on  $[0, 1]$  and is equal to one on  $[2, \infty)$ . Set  $\chi_\varepsilon := \chi(|\cdot|/\varepsilon)$ . By (3.2), for  $\phi \in C_0^\infty(B)$ , we have

$$\begin{aligned} \int_B \Delta \phi \cdot \log \operatorname{tr} e^s &= \lim_{\varepsilon \rightarrow 0} \int_B \chi_\varepsilon \cdot \Delta \phi \cdot \log \operatorname{tr} e^s \\ &\lesssim \int_B \phi \cdot |K_{H_\circ}| + \lim_{\varepsilon \rightarrow 0} \int_B \phi \cdot (\Delta \chi_\varepsilon \cdot \log \operatorname{tr} e^s - 2\langle \nabla \chi_\varepsilon, \nabla \log \operatorname{tr} e^s \rangle). \end{aligned}$$

Since  $n \geq 3$ , we have  $\|\chi_\varepsilon\|_{W^{2,2}(B)} \lesssim \varepsilon^2$ . Because  $\log \operatorname{tr} e^s \in W^{1,2}(B)$  this shows that the limit vanishes.

**Step 3.** *We have  $\log \operatorname{tr} e^s \in L^\infty(B)$  and  $\|\log \operatorname{tr} e^s\|_{L^\infty(B)} \leq c$ .*

Since  $\operatorname{tr} s = 0$ , we have  $|s| \leq \operatorname{rk}(\mathcal{E}) \cdot \log \operatorname{tr} e^s$ ; in particular,  $\log \operatorname{tr} e^s$  is non-negative. By hypothesis  $K_H = 0$ . Since  $H_\circ$  is HYM with respect to  $\omega_0$  and  $|F_{H_\circ}| \lesssim |z|^{-2}$  by hypothesis (2.3), we have  $|K_{H_\circ}| \leq c$ . The asserted inequality thus follows from Step 2 via Moser iteration; see [GT01, Theorem 8.1].  $\square$

## 4 A priori Morrey estimates

The following decay estimate is the crucial ingredient of the proof of Theorem 2.4.

**Proposition 4.1.** *There is a constant  $\alpha > 0$ , such that for  $r \in [0, 1]$  we have*

$$\int_{B_r} |\nabla_{H_\circ} s|^2 \lesssim r^{2n-2+2\alpha}.$$

The proof of this proposition relies on a Neumann–Poincaré type inequality, which we describe in what follows. Denote by  $\nabla_{T,r}$  the connection on  $i\operatorname{su}(E, H_\circ)|_{\partial B_r}$  induced by  $\nabla_{H_\circ}$ . The linear operator  $\nabla_{T,r} : \Gamma(\partial B_r, i\operatorname{su}(E, H_\circ)) \rightarrow \Omega^1(\partial B_r, i\operatorname{su}(E, H_\circ))$  has a finite dimensional kernel. Since  $\nabla_{H_\circ}$  is conical, we can identify<sup>4</sup>

$$\ker \nabla_{T,r} = \ker \nabla_{T,1} =: K.$$

Moreover, we can regard  $K$  as a subset of constant sections:  $K \subset \Gamma(\dot{B}_r, i\operatorname{su}(E, H_\circ))$ . Denote by  $\pi_r : \Gamma(\partial B_r, i\operatorname{su}(E, H_\circ)) \rightarrow K$  the  $L^2$ -orthogonal projection onto  $K$  and define  $\Pi_r : \Gamma(\dot{B}_{2r}, i\operatorname{su}(E, H_\circ)) \rightarrow K$  by

$$\Pi_r s := \frac{1}{r} \int_r^{2r} \pi_t(s|_{\partial B_t}) dt.$$

<sup>4</sup> $K$  can be determined explicitly from the from the decomposition of  $\mathcal{F}$  into  $\mu$ -stable summands, but we will not need a precise description of  $K$ .



**Proposition 4.2.** *We have*

$$\int_{B_{2r} \setminus B_r} |s - \Pi_r s|^2 \lesssim r^2 \int_{B_{2r} \setminus B_r} |\nabla_{H_c} s|^2.$$

*Proof.* The asserted estimate is scale-invariant; hence, we may assume  $r = 1/2$ . To prove the estimate in this case it suffices to prove the cylindrical estimate

$$\int_{1/2}^1 \int_{\partial B} |s(t, \hat{x}) - \Pi s(t, \cdot)|^2 d\hat{x} dt \lesssim \int_{1/2}^1 \int_{\partial B} |\partial_t s(t, \hat{x})|^2 + |\nabla_T s(t, \hat{x})|^2 d\hat{x} dt$$

with  $s$  denoting a section over  $[1/2, 1] \times \partial B$ ,  $\pi := \pi_1$ ,  $\Pi s := 2 \int_{1/2}^1 \pi s(t, \cdot) dt$ , and  $\nabla_T := \nabla_{T,1}$ .

We compute

$$\begin{aligned} & \int_{1/2}^1 \int_{\partial B} |s(t, \hat{x}) - \Pi s(t, \cdot)|^2 d\hat{x} dt \\ &= 4 \int_{1/2}^1 \int_{\partial B} \left| \int_{1/2}^1 s(t, \hat{x}) - \pi s(u, \cdot) du \right|^2 d\hat{x} dt \\ &\lesssim \int_{1/2}^1 \int_{1/2}^1 \int_{\partial B} |s(t, \hat{x}) - \pi s(u, \cdot)|^2 d\hat{x} du dt \\ &\lesssim \int_{1/2}^1 \int_{1/2}^1 \int_{\partial B} |s(t, \hat{x}) - \pi s(t, \cdot)|^2 + |\pi s(t, \cdot) - \pi s(u, \cdot)|^2 d\hat{x} du dt. \end{aligned}$$

The first summand can be bounded as follows

$$\begin{aligned} \int_{1/2}^1 \int_{1/2}^1 \int_{\partial B} |s(t, \hat{x}) - \pi s(t, \cdot)|^2 d\hat{x} dt du &\lesssim \int_{1/2}^1 \int_{1/2}^1 \int_{\partial B} |\nabla_T s(t, \hat{x})|^2 d\hat{x} dt du \\ &\lesssim \int_{1/2}^1 \int_{\partial B} |\nabla_T s(t, \hat{x})|^2 d\hat{x} dt. \end{aligned}$$

The second summand can be controlled as in the usual proof of the Neumann–Poincaré inequality. We have

$$\begin{aligned} |\pi s(t, \cdot) - \pi s(u, \cdot)| &= \left| \int_0^1 \partial_v \pi s(t + v(t-u), \cdot) dv \right| \\ &\leq \left| \int_0^1 \pi(\partial_t s)(t + v(t-u), \cdot) dv \right| \\ &\lesssim \left( \int_0^1 \int_{\partial B} |(\partial_t s)(t + v(t-u), \hat{x})|^2 d\hat{x} dv \right)^{1/2}. \end{aligned}$$

Plugging this into the second summand and symmetry considerations yield

$$\begin{aligned}
& \int_{1/2}^1 \int_{1/2}^1 \int_{\partial B} |\pi s(t, \cdot) - \pi s(u, \cdot)|^2 \, d\hat{x} \, du \, dt \\
& \lesssim \int_{1/2}^1 \int_{1/2}^1 \int_0^1 \int_{\partial B} |(\partial_t s)(t + v(t - u), \hat{x})|^2 \, d\hat{x} \, dv \, du \, dt \\
& \lesssim \int_{1/2}^1 \int_{\partial B} |\partial_t s(t, \hat{x})|^2 \, d\hat{x} \, dt.
\end{aligned}$$

This finishes the proof the proof.  $\square$

The proof of Proposition 4.1 also uses the following observation about

$$\hat{s}_r := \log(e^s e^{-\Pi_r s}).$$

**Proposition 4.3.** *We have*

$$|\nabla_{H_\circ} s| \lesssim |\nabla_{H_\circ} \hat{s}_r|, \quad |\hat{s}_r| \lesssim |s - \Pi_r s|, \quad \text{and} \quad |\nabla_{H_\circ} \hat{s}_r|^2 \lesssim 1 - \Delta |\hat{s}_r|^2.$$

*Proof.* The first two inequalities follow by elementary considerations.

Since  $s$  is bounded in  $L^\infty(B)$ ,  $\Pi_r s$  is uniformly bounded and, consequently, so is  $\hat{s}_r$ . By [JW16, Proposition A.8], we have

$$\Delta |\hat{s}_r|^2 + 2|v(-\hat{s}_r) \nabla_{H_\circ} \hat{s}_r|^2 \lesssim |\mathbb{K}_{H_\circ}| + |\mathbb{K}_{H_\circ e^{\hat{s}_r}}|$$

with

$$v(-\hat{s}_r) = \sqrt{\frac{1 - e^{-\text{ad}_{\hat{s}_r}}}{\text{ad}_{\hat{s}_r}}} \in \text{End}(\mathfrak{gl}(E));$$

see, e.g., [JW16, Proposition A.6]. We have

$$\mathbb{K}_{H_\circ e^{\hat{s}_r}} = \text{Ad}(e^{-\Pi_r s}) \mathbb{K}_{H_\circ e^s} = 0$$

because  $H_\circ e^s$  is HYM. Given this, the third inequality follows using

$$\sqrt{\frac{1 - e^{-x}}{x}} \gtrsim \frac{1}{\sqrt{1 + |x|}},$$

$\|\mathbb{K}_{H_\circ}\|_{L^\infty} \leq c$ , which is a consequence of (2.3), and the fact that  $H_\circ$  is HYM with respect to  $\omega_0$ , and the bound on  $|s|$  established in Proposition 3.1.  $\square$

*Proof of Proposition 4.1.* Given the above discussion, the proof is very similar to that of [JW16, Proposition C.2]. Nevertheless, for the reader's convenience we provide the necessary details.

Define  $g: [0, 1/2] \rightarrow [0, \infty]$  by

$$g(r) := \int_{B_r} |z|^{2-2n} |\nabla_{H_\circ} s|^2.$$

We will show that

$$g(r) \leq cr^{2\alpha},$$

which implies the asserted inequality.

**Step 1.** *We have  $g \leq c$ .*

Fix a smooth function  $\chi: [0, \infty) \rightarrow [0, 1]$  which is equal to one on  $[0, 1]$  and vanishes outside  $[0, 2]$ . Set  $\chi_r(\cdot) := \chi(|\cdot|/r)$ . For  $r > \varepsilon > 0$ , using Proposition 4.3 and Proposition 3.1, and with  $G$  denoting Green's function on  $B$  centered at 0, we have

$$\begin{aligned} \int_{B_r \setminus B_\varepsilon} |z|^{2-2n} |\nabla_{H_\circ} s|^2 &\lesssim \int_{B_r \setminus B_\varepsilon} |z|^{2-2n} |\nabla_{H_\circ} \hat{s}_r|^2 \\ &\lesssim \int_{B_{2r} \setminus B_{\varepsilon/2}} \chi_r(1 - \chi_{\varepsilon/2}) G(1 - \Delta |\hat{s}_r|^2) \\ &\lesssim \int_{B_{2r} \setminus B_r} |z|^{-2n} |s - \Pi_r s|^2 + r^2 + \varepsilon^{-2n} \int_{B_\varepsilon \setminus B_{\varepsilon/2}} |s - \Pi_r s|^2 \\ &\leq c. \end{aligned}$$

**Step 2.** *There are constants  $\gamma \in [0, 1)$  and  $A > 0$  such that*

$$g(r) \leq \gamma g(2r) + Ar^2.$$

Continuing the inequality from Step 1 using Proposition 4.2, we have

$$\begin{aligned} \int_{B_r \setminus B_\varepsilon} |z|^{2-2n} |\nabla_{H_\circ} s|^2 &\lesssim \int_{B_{2r} \setminus B_r} |z|^{2-2n} |\nabla_{H_\circ} s|^2 + r^2 + \varepsilon^{2-2n} \int_{B_\varepsilon \setminus B_{\varepsilon/2}} |\nabla_{H_\circ} s|^2 \\ &\lesssim g(2r) - g(r) + r^2 + g(\varepsilon). \end{aligned}$$

By Lebesgue's monotone convergence theorem, the last term vanishes as  $\varepsilon$  tends to zero; hence, the asserted inequality follows with  $\gamma = \frac{c}{c+1}$  and  $A = c$ .

**Step 3.** *We have  $g \leq cr^{2\alpha}$  for some  $\alpha \in (0, 1)$ .*

This follows from Step 1 and Step 2 and as in [JW16, Step 3 in the proof of Proposition C.2].  $\square$

## 5 Proof of Theorem 2.4

For  $r > 0$ , define  $m_r : \mathbb{C}^n \rightarrow \mathbb{C}^n$  by  $m_r(z) := rz$ . Set

$$s_r := m_r^*(s|_{B_{4r} \setminus B_{r/2}}) \in C^\infty(B_4 \setminus B_{1/2}, \text{isu}(E, H_*)) \quad \text{and} \quad H_{\diamond, r} := m_r^* H_\diamond.$$

The metric  $H_{\diamond, r} e^{s_r}$  is HYM with respect to  $\omega_r := r^{-2} m_r^* \omega$  and  $\|F_{H_{\diamond, r}}\|_{C^k(B_4 \setminus B_{1/2})} \leq c_k$ .

Proposition 3.1, (2.3) and interior estimates for HYM metrics [JW16, Theorem C.1] imply that

$$\|s_r\|_{C^k(B_3 \setminus B_{3/4})} \leq c_k.$$

By Proposition 4.1, we have

$$\|\nabla_{H_{\diamond, r}} s_r\|_{L^2(B_4 \setminus B_{1/2})} \lesssim r^\alpha.$$

Schematically,  $K_{H_{\diamond, r} e^{s_r}} = 0$  can be written as

$$\nabla_{H_{\diamond, r}}^* \nabla_{H_{\diamond, r}} s_r + B(\nabla_{H_{\diamond, r}} s \otimes \nabla_{H_{\diamond, r}} s_r) = C(K_{H_{\diamond, r}}),$$

where  $B$  and  $C$  are linear with coefficients depending on  $s$ , but not on its derivatives; see, e.g., [JW16, Proposition A.1]. Since  $\|K_{H_{\diamond, r}}\|_{C^k(B_3 \setminus B_{3/4})} \leq c_k r^2$ , as in [JW16, Step 3 in the proof of Proposition 5.1], standard interior estimates imply that

$$\|\nabla_{H_{\diamond, r}}^k s_r\|_{L^2(B_2 \setminus B_1)} \leq c_k r^\alpha$$

and, hence, the asserted inequalities, for each  $k \geq 1$ . (The asserted inequality for  $k = 0$  has already been proven in Proposition 3.1.)  $\square$

## 6 Proof of Proposition 1.4

We will make use of the following general fact about connections over manifolds with free  $S^1$ -actions.

**Proposition 6.1.** *Let  $M$  be a manifold with a free  $S^1$ -action. Denote the associated Killing field by  $\xi \in \text{Vect}(M)$  and let  $q : M \rightarrow M/S^1$  be the canonical projection. Suppose  $\theta \in \Omega^1(M)$  is such that  $\theta(\xi) = 1$  and  $\mathcal{L}_\xi \theta = 0$ . Let  $A$  be a unitary connection on a Hermitian vector bundle  $(E, H)$  over  $M$ . If  $i(\xi)F_A = 0$ , then there is a  $k \in \mathbb{N}$  and, for each  $j \in \{1, \dots, k\}$ , a Hermitian vector bundles  $(F_j, K_j)$  over  $M/S^1$  such that*

$$E = \bigoplus_{j=1}^k E_j \quad \text{and} \quad H = \bigoplus_{j=1}^k H_j$$

with  $E_j := q^* F_j$  and  $H_j := q^* K_j$ ; moreover, the bundles  $E_j$  are parallel and, for each  $j \in \{1, \dots, k\}$ , there are a unitary connection  $B_j$  on  $F_j$  and  $\mu_j \in \mathbb{R}$  such that

$$A = \bigoplus_{j=1}^k q^* B_j + i\mu_j \text{id}_{E_j} \cdot \theta.$$

*Proof.* Denote by  $\tilde{\xi} \in \text{Vect}(U(E))$  the  $A$ -horizontal lift of  $\xi$ . This vector field integrates to an  $\mathbf{R}$ -action on  $U(E)$ . Thinking of  $A$  as an  $\mathfrak{u}(r)$ -valued 1-form on  $U(E)$  and  $F_A$  as an  $\mathfrak{u}(r)$ -valued 2-form on  $U(E)$ , we have

$$\mathcal{L}_{\tilde{\xi}}A = i(\tilde{\xi})F_A = 0;$$

hence,  $A$  is invariant with respect to the  $\mathbf{R}$ -action on  $U(E)$ .

The obstruction to the  $\mathbf{R}$ -action on  $U(E)$  inducing an  $S^1$ -action is the action of  $1 \in \mathbf{R}$  and corresponds to a gauge transformation  $\mathfrak{g}_A \in \mathcal{G}(U(E))$  fixing  $A$ . If this obstruction vanishes, i.e.,  $\mathfrak{g}_A = \text{id}_{U(E)}$ , then  $E \cong q^*F$  with  $F = E/S^1$  and there is a connection  $A_0$  on  $F$  such that  $A = q^*A_0$ .

If the obstruction does not vanish, we can decompose  $E$  into pairwise orthogonal parallel subbundles  $E_j$  such that  $\mathfrak{g}_A$  acts on  $E_j$  as multiplication with  $e^{i\mu_j}$  for some  $\mu_j \in \mathbf{R}$ . Set  $\tilde{A} := A - \bigoplus_{j=1}^k i\mu_j \text{id}_{E_j} \cdot \theta$ . This connection also satisfies  $i(\tilde{\xi})F_{\tilde{A}} = 0 \in \Omega^1(M, \mathfrak{g}_E)$  and the subbundles  $E_j$  are also parallel with respect to  $E_j$ . Since  $\mathfrak{g}_{\tilde{A}} = \text{id}_E$ , the assertion follows.  $\square$

In the situation of Proposition 1.4, with  $\xi \in S^{2n-1}$  denoting the Killing field for the  $S^1$ -action we have  $i(\xi)F_{A_0} = 0$ ; c.f., Tian [Tia00, discussion after Conjecture 2]. Therefore, we can write

$$A_* = \bigoplus_{j=1}^k \sigma^*B_j + i\mu_j \text{id}_{E_j} \cdot \pi^*\theta.$$

Since  $d\theta = 2\pi\rho^*\omega_{FS}$ , we have

$$F_{A_*} = \bigoplus_{j=1}^k \sigma^*F_{B_j} + 2\pi i\mu_j \text{id}_{E_j} \cdot \sigma^*\omega_{FS}.$$

Using (2.1),  $A_*$  being HYM with respect to  $\omega_0$  can be seen to be equivalent to

$$F_{B_j}^{0,2} = 0 \quad \text{and} \quad i\Lambda F_{B_j} = (2n-2)\pi\mu_j \cdot \text{id}_{E_j}.$$

The isomorphism  $\mathcal{E} = (E, \bar{\partial}_{A_*}) \cong \bigoplus_{j=1}^k \rho^*\mathcal{F}_j$  with  $\mathcal{F}_j = (F_j, \bar{\partial}_{B_j})$  is given by  $g^{-1}$  with  $g := \bigoplus_{j=1}^k r^{\mu_j}$ .  $\square$

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