Tangent cones of Hermitian Yang–Mills connections with isolated singularities

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Abstract

We give a simple direct proof of uniqueness of tangent cones for singular projectively Hermitian Yang–Mills connections on reflexive sheaves at isolated singularities modelled on a sum of μ–stable holomorphic bundles over $\mathbf{P}^{n-1}$.

1 Introduction

A **projectively Hermitian Yang–Mills (PHYM)** connection A over a Kähler manifold X is a unitary connection A on a Hermitian vector bundle (E, H) over X satisfying

$$\mathrm{F}_A^{0,2} = 0 \quad \text{and} \quad i\Lambda \mathrm{F}_A - \frac{\mathrm{tr}(i\Lambda \mathrm{F}_A)}{\mathrm{rk}\, E} \cdot \mathrm{id}_E = 0. \tag{1.1}$$

Since $\mathrm{F}_A^{0,2} = 0$, $\mathscr{E} := (E, \bar\partial_A)$ is a holomorphic vector bundle, and A is the Chern connection of H. A Hermitian metric H on a holomorphic vector bundle is called **PHYM** if its Chern connection A_H is **P**HYM. The celebrated Donaldson–Uhlenbeck–Yau Theorem [Don85; Don87; UY86] asserts that a holomorphic vector bundle $\mathscr{E}$ on a compact Kähler manifold admits a **P**HYM metric if and only if it is μ–polystable; moreover, any two **P**HYM metrics are related by an automorphism of $\mathscr{E}$ and by multiplication with a conformal factor. If H is a **P**HYM metric, then the connection A° on $\mathbf{P}\mathrm{U}(E, H)$, the principal $\mathbf{P}\mathrm{U}(r)$–bundle associated with (E, H), induced by A_H is **Hermitian Yang–Mills (HYM)**, that is, it satisfies $\mathrm{F}_{A^\circ}^{0,2} = 0$ and $i\Lambda \mathrm{F}_{A^\circ} = 0$; it depends only on the conformal class of H. Conversely, any HYM connection A° on $\mathbf{P}\mathrm{U}(E, H)$ can be lifted to a **P**HYM connection A; any two choices of lifts lead to isomorphic holomorphic vector bundles $\mathscr{E}$ and conformal metrics H.

An **admissible** **P**HYM connection is a **P**HYM connection A on a Hermitian vector bundle (E, H) over $X \backslash \mathrm{sing}(A)$ with $\mathrm{sing}(A)$ a closed subset with locally finite $(2n-4)$–dimensional Hausdorff measure and $\mathrm{F}_A \in L^2_{\mathrm{loc}}(X)$.[1] Bando [Ban91] proved that if A is an admissible **P**HYM connection,

[1]It should be pointed out that our notion of admissible **P**HYM connection follows Bando and Siu [BS94] and not Tian [Tia00]. The notion of admissible Yang–Mills connection introduced by Tian is stronger: it assumes that the Hermitian vector bundle extends to all of X.

then $(E, \bar\partial_A)$ extends to X as a reflexive sheaf $\mathscr{E}$ with $\operatorname{sing}(\mathscr{E}) \subset \operatorname{sing}(A)$. Bando and Siu [BS94] proved that a reflexive sheaf on a compact Kähler manifold admits an admissible **P**HYM metric if and only if it is μ–polystable.

The technique used by Bando and Siu does not yield any information on the behaviour of the admissible **P**HYM connection A_H near the singularities of the reflexive sheaf $\mathscr{E}$—not even at isolated singularities. The simplest example of a reflexive sheaf on $\mathbf{C}^n$ with an isolated singularity at 0 is $i_*\sigma^*\mathscr{F}$ with $\mathscr{F}$ a holomorphic vector bundle over $\mathbf{P}^{n-1}$; cf. Hartshorne [Har80, Example 1.9.1]. Here we use the obvious maps summarised in the following diagram:

$$\begin{tikzcd} \mathbf{C}^n & \mathbf{C}^n\backslash\{0\} \arrow[l, hook', "i"'] \arrow[r, two heads, "\pi"] \arrow[rr, bend right, "\sigma"'] & S^{2n-1} \arrow[r, two heads, "\rho"] & \mathbf{P}^{n-1}. \end{tikzcd}$$

The main result of this article gives a description of **P**HYM connections near singularities modelled on $i_*\sigma^*\mathscr{F}$ with $\mathscr{F}$ a sum of μ–stable holomorphic vector bundles.

Theorem 1.2. *Let $\omega = \frac{1}{2i}\bar\partial\partial|z|^2 + O(|z|^2)$ be a Kähler form on $\bar B_R(0) \subset \mathbf{C}^n$. Let A be an admissible* **P***HYM connection on a Hermitian vector bundle (E, H) over $B_R(0)\backslash\{0\}$ with $\operatorname{sing}(A) = \{0\}$ and $(E, \bar\partial_A) \cong \sigma^*\mathscr{F}$ for some holomorphic vector bundle $\mathscr{F}$ over* $\mathbf{P}^{n-1}$*. Denote by F the complex vector bundle underlying $\mathscr{F}$.*

If $\mathscr{F}$ is a sum of μ–stable holomorphic vector bundles, then there exist a Hermitian metric K on F, a connection A_* on $\sigma^*(F, K)$ which is the pullback of a connection on $\rho^*(F, K)$, and an isometry $(E, H) \cong \sigma^*(F, K)$ such that with respect to this isometry we have

$$|z|^{k+1}|\nabla^k_{A_*}(A^\circ - A^\circ_*)| \leqslant C_k|z|^\alpha \quad \text{for each } k \geqslant 0.$$

The constants $C_k, \alpha > 0$ depend on ω, $\mathscr{F}$, $A|_{B_R(0)\backslash B_{R/2}(0)}$, and $\|\mathrm{F}_A\|_{L^2(B_R(0))}$.

Remark 1.3. Using a gauge theoretic Łojasiewicz–Simon gradient inequality, Yang [Yan03, Theorem 1] proved that the tangent cone to a stationary Yang–Mills connection—in particular, a $\mathbf{P}\mathrm{U}(r)$ HYM connection—with an isolated singularity at x is unique provided

$$|\mathrm{F}_A| \lesssim d(x, \cdot)^{-2}.$$

In our situation, such a curvature bound can be obtained from Theorem 1.2. Our proof of this result, however, proceeds more directly—without making use of Yang's theorem.

The hypothesis that $\mathscr{F}$ be a sum of μ–stable holomorphic vector bundles is optimal. This is a consequence of the following observation, which will be proved in Section 6.

Proposition 1.4. *Let (F, K) be a Hermitian vector bundle over $\mathbf{P}^{n-1}$. If B is a unitary connection on $\rho^*(F, K)$ such that $A_* := \pi^*B$ is HYM with respect to $\omega_0 := \frac{1}{2i}\bar\partial\partial|z|^2$, then there is a $k \in \mathbf{N}$ and, for each $j \in \{1, \dots, k\}$, there are $\mu_j \in \mathbf{R}$, a Hermitian vector bundle (F_j, K_j) on $\mathbf{P}^{n-1}$, and an irreducible unitary connection B_j on F_j satisfying*

$$\mathrm{F}^{0,2}_{B_j} = 0 \quad \text{and} \quad i\Lambda\mathrm{F}_{B_j} = (2n-2)\pi\mu_j \cdot \mathrm{id}_{F_j}$$

such that

$$F = \bigoplus_{j=1}^{k} F_j \quad \textit{and} \quad B = \bigoplus_{j=1}^{k} \rho^* B_j + i\mu_j \,\mathrm{id}_{\rho^* F_j} \cdot \theta.$$

Here θ denotes the standard contact structure[2] *on S^{2n-1}. In particular,*

$$\mathscr{E} = (\sigma^* F, \bar{\partial}_{A_*}) \cong \bigoplus_{j=1}^{k} \sigma^* \mathscr{F}_j$$

with $\mathscr{F}_j = (F_j, \bar{\partial}_{B_j})$ μ–stable.

To conclude the introduction we discuss two concrete examples in which Theorem 1.2 can be applied.

Example 1.5 (Okonek, Schneider, and Spindler [OSS11, Example 1.1.13]). It follows from the Euler sequence that $H^0(\mathscr{T}_{\mathbf{P}^3}(-1)) \cong \mathbf{C}^4$. Denote by $s_v \in H^0(\mathscr{T}_{\mathbf{P}^3}(-1))$ the section corresponding to $v \in \mathbf{C}^4$. If $v \neq 0$, then the rank two sheaf $\mathscr{E} = \mathscr{E}_v$ defined by

$$0 \to \mathscr{O}_{\mathbf{P}^3} \xrightarrow{s_v} \mathscr{T}_{\mathbf{P}^3}(-1) \to \mathscr{E}_v \to 0$$

is reflexive and $\mathrm{sing}(\mathscr{E}) = \{[v]\}$.

$\mathscr{E}$ is μ–stable. To see this, because $\mu(\mathscr{E}) = 1/2$, it suffices to show that

$$\mathrm{Hom}(\mathscr{O}_{\mathbf{P}^3}(k), \mathscr{E}) = H^0(\mathscr{E}(-k)) = 0 \quad \text{for each } k \geqslant 1.$$

However, by inspection of the Euler sequence, $H^0(\mathscr{E}(-k)) \cong H^0(\mathscr{T}_{\mathbf{P}^3}(-k-1)) = 0$. It follows that $\mathscr{E}$ admits a PHYM metric H with $\mathrm{F}_H \in L^2$ and a unique singular point at $[v] \in \mathbf{P}^3$. To see that Theorem 1.2 applies, pick a standard affine neighborhood $U \cong \mathbf{C}^3$ in which $[v]$ corresponds to 0. In U, the Euler sequence becomes

$$0 \to \mathscr{O}_{\mathbf{C}^3} \xrightarrow{(1, z_1, z_2, z_3)} \mathscr{O}_{\mathbf{C}^3}^{\oplus 4} \to \mathscr{T}_{\mathbf{P}^3}(-1)|_U \to 0,$$

and $s_v = [(1,0,0,0)]$; hence,

$$0 \to \mathscr{O}_{\mathbf{C}^3} \xrightarrow{(z_1, z_2, z_3)} \mathscr{O}_{\mathbf{C}^3}^{\oplus 3} \to \mathscr{E}_v|_U \to 0.$$

On $\mathbf{C}^3 \backslash \{0\}$, this is the pullback of the Euler sequence on $\mathbf{P}^2$; therefore, $\mathscr{E}_v|_U \cong i_* \sigma^* \mathscr{T}_{\mathbf{P}^2}$.

[2]With respect to standard coordinates on $\mathbf{C}^n$, the standard contact structure θ on S^{2n-1} is such that $\pi^*\theta = \sum_{j=1}^n (\bar{z}_j \mathrm{d}z_j - z_j \mathrm{d}\bar{z}_j)/2i|z|^2$.

Example 1.6. For $t \in \mathbf{C}$, define $f_t\colon \mathscr{O}_{\mathbf{P}^3}(-2)^{\oplus 2} \to \mathscr{O}_{\mathbf{P}^3}(-1)^{\oplus 5}$ by

$$f_t := \begin{pmatrix} z_0 & 0 \\ z_1 & z_0 \\ z_2 & z_1 \\ t\cdot z_3 & z_2 \\ 0 & z_3 \end{pmatrix},$$

and denote by $\mathscr{E}_t$ the cokernel of f_t, i.e.,

$$0 \to \mathscr{O}_{\mathbf{P}^3}(-2)^{\oplus 2} \xrightarrow{f_t} \mathscr{O}_{\mathbf{P}^3}(-1)^{\oplus 5} \to \mathscr{E}_t \to 0. \tag{1.7}$$

If $t \neq 0$, then $\mathscr{E}_t$ is locally free; $\mathscr{E}_0$ is reflexive with $\operatorname{sing}(\mathscr{E}_0) = \{[0:0:0:1]\}$. The proof of this is analogous to that of the reflexivity of $\mathscr{E}_v$ from Example 1.5 given in [OSS11, Example 1.1.13].

For each t, $H^0(\mathscr{E}_t) = H^0(\mathscr{E}_t^*(-1)) = 0$; hence, $\mathscr{E}_t$ is μ-stable according to the criterion of Okonek, Schneider, and Spindler [OSS11, Remark 1.2.6(b)]. The former vanishing is obvious since $H^0(\mathscr{O}_{\mathbf{P}^3}(-1)) = H^1(\mathscr{O}_{\mathbf{P}^3}(-2)) = 0$. The latter follows by dualising (1.7), twisting by $\mathscr{O}_{\mathbf{P}^3}(-1)$ and observing that the induced map $H^0(f_0^*)\colon H^0(\mathscr{O}_{\mathbf{P}^3})^{\oplus 5} \to H^0(\mathscr{O}_{\mathbf{P}^3}(1))^{\oplus 2}$, which is given by

$$\begin{pmatrix} z_0 & z_1 & z_2 & t\cdot z_3 & 0 \\ 0 & z_0 & z_1 & z_2 & z_3 \end{pmatrix},$$

is injective.

In a standard affine neighborhood $U \cong \mathbf{C}^3$ of $[0:0:0:1]$, we have $\mathscr{E}_0|_U \cong i_*\sigma^*(\mathscr{T}_{\mathbf{P}^2} \oplus \mathscr{O}_{\mathbf{P}^2}(1))$. To see this, note that the cokernel of the map $g\colon \mathscr{O}_{\mathbf{P}^2}^{\oplus 2} \to \mathscr{O}_{\mathbf{P}^2}(1)^{\oplus 4} \oplus \mathscr{O}_{\mathbf{P}^2}$ defined by

$$g := \begin{pmatrix} z_0 & 0 \\ z_1 & z_0 \\ z_2 & z_1 \\ 0 & z_2 \\ 0 & 1 \end{pmatrix}$$

is $\mathscr{T}_{\mathbf{P}^2} \oplus \mathscr{O}_{\mathbf{P}^2}(1)$.

Conventions and notation. Set $B_r := B_r(0)$ and $\dot{B}_r := B_r(0)\backslash\{0\}$. We denote by $c > 0$ a generic constant, which depends only on $\mathscr{F}$, ω, $s|_{B_1\backslash B_{1/2}}$, $H_\diamond$, and $\|\mathrm{F}_H\|_{L^2(B_R(0))}$ (which will be introduced in the next section). Its value might change from one occurrence to the next. Should c depend on further data we indicate this by a subscript. We write $x \lesssim y$ for $x \leqslant cy$. The expression $O(x)$ denotes a quantity y with $|y| \lesssim x$. Since reflexive sheaves are locally free away from a closed subset of complex codimension three, without loss of generality, we will assume throughout that $n \geqslant 3$.

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2 Reduction to the metric setting

In the situation of Theorem 1.2, the Hermitian metric H on $\mathscr{E}$ corresponds to a $\mathbf{P}$HYM metric on $\sigma^*\mathscr{F}$ via the isomorphism $(E, \bar\partial_A) \cong \sigma^*\mathscr{F}$. By slight abuse of notation, we will denote this metric by H as well.

Denote by $\mathscr{F}_1, \dots, \mathscr{F}_k$ the μ–stable summands of $\mathscr{F}$. Denote by K_j the $\mathbf{P}$HYM metric on $\mathscr{F}_j$ with

$$i\Lambda_{\omega_{FS}} \mathrm{F}_{K_j} = \frac{2\pi}{(n-2)!\mathrm{vol}(\mathbf{P}^{n-1})}\mu_j \cdot \mathrm{id}_{F_j} = (2n-2)\pi\mu_j \cdot \mathrm{id}_{F_j}$$

with ω_{FS} denoting the integral Fubini study form and for $\mu_j \coloneqq \mu(\mathscr{F}_j)$. The Kähler form ω_0 associated with the standard Kähler metric on $\mathbf{C}^n$ can be written as

$$\omega_0 = \frac{1}{2i}\bar\partial\partial|z|^2 = \pi r^2 \sigma^*\omega_{FS} + r\mathrm{d}r \wedge \pi^*\theta \tag{2.1}$$

with θ as in Proposition 1.4. Therefore, we have

$$i\Lambda_{\omega_0}\mathrm{F}_{\sigma^*K_j} = (2n-2)\mu_j r^{-2} \cdot \mathrm{id}_{\sigma^*F_j},$$

and $H_{\diamond,j} \coloneqq r^{2\mu_j} \cdot \sigma^*K_j$ satisfies

$$\begin{aligned} i\Lambda_{\omega_0}\mathrm{F}_{H_{\diamond,j}} &= i\Lambda_{\omega_0}\mathrm{F}_{\sigma^*K_j} + i\Lambda_{\omega_0}\bar\partial\partial\log r^{2\mu_j} \cdot \mathrm{id}_{\sigma^*F_j} \\ &= i\Lambda_{\omega_0}\mathrm{F}_{\sigma^*K_j} + \frac{1}{2}\Delta\log r^{2\mu_j} \cdot \mathrm{id}_{\sigma^*F_j} = 0. \end{aligned}$$

Denote by $A_{\diamond,j}$ the Chern connection associated with $H_{\diamond,j}$ and by B_j the Chern connection associated with K_j. The isometry $r^{\mu_j} \colon (\sigma^*F_j, H_{\diamond,j}) \to \sigma^*(F_j, K_j)$ transforms $A_{\diamond,j}$ into

$$A_{*,j} \coloneqq (r^{\mu_j})_* A_{\diamond,j} = \sigma^*B_j + i\mu_j\,\mathrm{id}_{\sigma^*F_j} \cdot \pi^*\theta.$$

In particular,

$$A_* \coloneqq \bigoplus_{j=1}^k A_{*,j}$$

is the pullback of a connection B on S^{2n-1}. Moreover, A_* is unitary with respect to

$$H_* \coloneqq \bigoplus_{j=1}^k \sigma^*K_j.$$

Proposition 2.2. *Assume the above situation. Set* $H_\diamond := \bigoplus_{j=1}^k H_{\diamond,j}$ *and fix* $R > 0$. *We have*

$$\big\| |z|^{2+\ell} \nabla_{H_\diamond}^\ell \mathrm{F}_{H_\diamond} \big\|_{L^\infty(B_R)} < \infty \quad \textit{for each } \ell \geqslant 0. \tag{2.3}$$

Proof. Using the isometry $g := \bigoplus_{j=1}^k r^{\mu_j}$ both assertions can be translated to corresponding statements for A_*. The first assertion then follows since A_* is the pullback of a connection B on S^{2n-1}. □

In the situation of Theorem 1.2, after a conformal change, which does not affect A°, we can assume that $\det H = \det H_\diamond$. Setting

$$s := \log(H_\diamond^{-1} H) \in C^\infty(\dot{B}_r, i\mathfrak{su}(\sigma^* F, H_\diamond))[3]$$
$$\text{and} \quad \Upsilon(s) := \frac{e^{\mathrm{ad}_s} - 1}{\mathrm{ad}_s},$$

we have

$$e_*^{s/2} H = H_\diamond \quad \text{and} \quad e_*^{s/2} A = A_\diamond + a$$
$$\text{with} \quad a := \frac{1}{2}\Upsilon(-s/2)\partial_{A_\diamond} s - \frac{1}{2}\Upsilon(s/2)\bar{\partial}_{A_\diamond} s;$$

see, e.g., [JW18, Appendix A]. Moreover, with $g := \bigoplus_{j=1}^k r^{\mu_j}$ we have

$$g_* e_*^{s/2} A = A_* + gag^{-1}.$$

Since

$$|\nabla_{A_*}^k gag^{-1}|_{H_*} = |\nabla_{H_\diamond}^k a|_{H_\diamond} \quad \text{for each } k \geqslant 0,$$

Theorem 1.2 will be a consequence of Proposition 2.2 and the following result.

Theorem 2.4. *Suppose* $\omega = \frac{1}{2i}\bar{\partial}\partial|z|^2 + O(|z|^2)$ *is a Kähler form on* $\bar{B}_R \subset \mathbf{C}^n$, $\mathscr{E}$ *is a holomorphic vector bundle over* $\dot{B}_R$, *and* $H_\diamond$ *is a Hermitian metric on* $\mathscr{E}$ *which is HYM with respect to* ω_0 *and satisfies* (2.3). *If* H *is an admissible HYM metric on* $\mathscr{E}$ *with* $\mathrm{sing}(A_H) = \{0\}$ *and* $\det H = \det H_\diamond$, *then*

$$s := \log(H_\diamond^{-1} H) \in C^\infty(\dot{B}_R, i\mathfrak{su}(\pi^* F, H_\diamond))$$

satisfies

$$|s| \leqslant C_0 \quad \textit{and} \quad |z|^k |\nabla_{H_\diamond}^k s| \leqslant C_k |z|^\alpha \quad \textit{for each } k \geqslant 1.$$

The constants $C_k, \alpha > 0$ *depend on* ω, $H_\diamond$, $s|_{B_R \setminus B_{R/2}}$, *and* $\|\mathrm{F}_H\|_{L^2(B_R)}$.

The next three sections of this paper are devoted to proving Theorem 2.4. Without loss of generality, we will assume that the radius R is one. We set $B := B_1$ and $\dot{B} := \dot{B}_1$.

[3] If H, K are two Hermitian inner products on a complex vector space V, then there is a unique endomorphism $T \in \mathrm{End}(V)$ which is self-adjoint with respect to H and K, has positive spectrum, and satisfies $H(Tv, w) = K(v, w)$. It is customary to denote T by $H^{-1}K$, and thus $\log(H^{-1}K) = \log(T)$.

3 A priori C^0 estimate

As a first step towards proving Theorem 2.4 we bound $|s|$, using an argument which is essentially contained in Bando and Siu [BS94, Theorem 2(a) and (b)].

Proposition 3.1. *We have $|s| \in L^\infty(B)$ and $\|s\|_{L^\infty(B)} \leqslant c$.*

Proof. The proof relies on the differential inequality

$$\Delta \log \operatorname{tr} H_0^{-1} H_1 \lesssim |\mathrm{K}_{H_1} - \mathrm{K}_{H_0}| \tag{3.2}$$

for Hermitian metrics H_0 and H_1 with $\det H_0 = \det H_1$, and with

$$\mathrm{K}_H := i\Lambda \mathrm{F}_H - \frac{\operatorname{tr}(i\Lambda \mathrm{F}_H)}{\operatorname{rk} E} \cdot \mathrm{id}_E;$$

see [Siu87, p. 13] for a proof.

Step 1. *We have $\log \operatorname{tr} e^s \in W^{1,2}(B)$ and $\|\log \operatorname{tr} e^s\|_{W^{1,2}(B)} \leqslant c$.*

Choose $1 \leqslant i < j \leqslant n$ and define the projection $\pi \colon B \to \mathbf{C}^{n-2}$ by

$$\pi(z) := (z_1, \ldots, \hat{z}_i, \ldots \hat{z}_j, \ldots, z_n).$$

For $\zeta \in \mathbf{C}^{n-2}$, denote by ∇_ζ and Δ_ζ the derivative and the Laplacian on the slice $\pi^{-1}(\zeta)$ respectively. Set $f_\zeta := \log \operatorname{tr} e^s|_{\pi^{-1}(\zeta)}$. Applying (3.2) to $H|_{\pi^{-1}(\zeta)}$ and $H_\diamond|_{\pi^{-1}(\zeta)}$ we obtain

$$\Delta_\zeta f_\zeta \lesssim |\mathrm{F}_H| + |\mathrm{F}_{H_\diamond}|.$$

Fix $\chi \in C^\infty(\mathbf{C}^2; [0,1])$ such that $\chi(\eta) = 1$ for $|\eta| \leqslant 1/2$ and $\chi(\eta) = 0$ for $|\eta| \geqslant 1/\sqrt{2}$. For $0 < |\zeta| \leqslant 1/\sqrt{2}$ and $\varepsilon > 0$, we have

$$\begin{aligned}\int_{\pi^{-1}(\zeta)} |\nabla_\zeta(\chi f_\zeta)|^2 &\lesssim \int_{\pi^{-1}(\zeta)} \chi^2 f_\zeta (|\mathrm{F}_H| + |\mathrm{F}_{H_\diamond}|) + 1 \\ &\leqslant \varepsilon \int_{\pi^{-1}(\zeta)} |\chi f_\zeta|^2 + \varepsilon^{-1} \int_{\pi^{-1}(\zeta)} |\mathrm{F}_H|^2 + |\mathrm{F}_{H_\diamond}|^2 + 1.\end{aligned}$$

Using the Dirichlet–Poincaré inequality and rearranging, we obtain

$$\int_{\pi^{-1}(\zeta)} |\chi f_\zeta|^2 + |\nabla_\zeta(\chi f_\zeta)|^2 \lesssim \int_{\pi^{-1}(\zeta)} |\mathrm{F}_H|^2 + |\mathrm{F}_{H_\diamond}|^2 + 1.$$

Integrating over $0 < |\zeta| \leqslant 1/\sqrt{2}$ yields

$$\int_B |\log \operatorname{tr} e^s|^2 + |\nabla' \log \operatorname{tr} e^s|^2 \lesssim \int_B |\mathrm{F}_H|^2 + |\mathrm{F}_{H_\diamond}|^2 + 1$$

with ∇' denoting the derivative along the fibres of π. Using (2.3) and $n \geqslant 3$, $\mathrm{F}_{H_\diamond} \in L^2(B)$. Since the choice of i, j defining π was arbitrary, the asserted inequality follows.

Step 2. *The differential inequality*

$$\Delta \log \operatorname{tr} e^s \lesssim |\mathrm{K}_{H_\diamond}|$$

holds on B in the sense of distributions.

Fix a smooth function $\chi\colon [0,\infty) \to [0,1]$ which vanishes on $[0,1]$ and is equal to one on $[2,\infty)$. Set $\chi_\varepsilon \coloneqq \chi(|\cdot|/\varepsilon)$. By (3.2), for $\phi \in C_0^\infty(B)$, we have

$$\begin{aligned}\int_B \Delta\phi \cdot \log \operatorname{tr} e^s &= \lim_{\varepsilon\to 0} \int_B \chi_\varepsilon \cdot \Delta\phi \cdot \log \operatorname{tr} e^s \\ &\lesssim \int_B \phi \cdot |\mathrm{K}_{H_\diamond}| + \lim_{\varepsilon\to 0} \int_B \phi \cdot (\Delta\chi_\varepsilon \cdot \log \operatorname{tr} e^s - 2\langle \nabla\chi_\varepsilon, \nabla \log \operatorname{tr} e^s\rangle)\,.\end{aligned}$$

Since $n \geqslant 3$, we have $\|\chi_\varepsilon\|_{W^{2,2}(B)} \lesssim \varepsilon^2$. Because $\log \operatorname{tr} e^s \in W^{1,2}(B)$ this shows that the limit vanishes.

Step 3. *We have* $\log \operatorname{tr} e^s \in L^\infty(B)$ *and* $\|\log \operatorname{tr} e^s\|_{L^\infty(B)} \leqslant c$.

Since $\operatorname{tr} s = 0$, we have $|s| \leqslant \operatorname{rk}(\mathscr{E}) \cdot \log \operatorname{tr} e^s$; in particular, $\log \operatorname{tr} e^s$ is non-negative. By hypothesis $\mathrm{K}_H = 0$. Since $H_\diamond$ is HYM with respect to ω_0 and $|\mathrm{F}_{H_\diamond}| \lesssim |z|^{-2}$ by hypothesis (2.3), we have $|\mathrm{K}_{H_\diamond}| \leqslant c$. The asserted inequality thus follows from Step 2 via Moser iteration; see [GT01, Theorem 8.1]. □

4 A priori Morrey estimates

The following decay estimate is the crucial ingredient of the proof of Theorem 2.4.

Proposition 4.1. *There is a constant $\alpha > 0$, such that for $r \in [0,1]$ we have*

$$\int_{B_r} |\nabla_{H_\diamond} s|^2 \lesssim r^{2n-2+2\alpha}.$$

The proof of this proposition relies on a Neumann–Poincaré type inequality, which we describe in what follows. Denote by $\nabla_{T,r}$ the connection on $i\mathfrak{su}(E,H_\diamond)|_{\partial B_r}$ induced by $\nabla_{H_\diamond}$. The linear operator $\nabla_{T,r}\colon \Gamma(\partial B_r, i\mathfrak{su}(E,H_\diamond)) \to \Omega^1(\partial B_r, i\mathfrak{su}(E,H_\diamond))$ has a finite dimensional kernel. Since $\nabla_{H_\diamond}$ is conical, we can identify[4]

$$\ker \nabla_{T,r} = \ker \nabla_{T,1} =: K.$$

Moreover, we can regard K as a subset of constant sections: $K \subset \Gamma(\dot{B}_r, i\mathfrak{su}(E,H_\diamond))$. Denote by $\pi_r\colon \Gamma(\partial B_r, i\mathfrak{su}(E,H_\diamond)) \to K$ the L^2–orthogonal projection onto K and define $\Pi_r\colon \Gamma(\dot{B}_{2r}, i\mathfrak{su}(E,H_\diamond)) \to K$ by

$$\Pi_r s \coloneqq \frac{1}{r}\int_r^{2r} \pi_t(s|_{\partial B_t})\,\mathrm{d}t.$$

[4] K can be determined explicitly from the from the decomposition of $\mathscr{F}$ into μ–stable summands, but we will not need a precise description of K.

Proposition 4.2. *We have*

$$\int_{B_{2r}\setminus B_r} |s - \Pi_r s|^2 \lesssim r^2 \int_{B_{2r}\setminus B_r} |\nabla_{H_\diamond} s|^2.$$

Proof. The asserted estimate is scale-invariant; hence, we may assume $r = 1/2$. To prove the estimate in this case it suffices to prove the cylindrical estimate

$$\int_{1/2}^{1}\int_{\partial B} |s(t,\hat{x}) - \Pi s(t,\cdot)|^2 \,\mathrm{d}\hat{x}\mathrm{d}t \lesssim \int_{1/2}^{1}\int_{\partial B} |\partial_t s(t,\hat{x})|^2 + |\nabla_T s(t,\hat{x})|^2 \,\mathrm{d}\hat{x}\mathrm{d}t$$

with s denoting a section over $[1/2,1]\times\partial B$, $\pi := \pi_1$, $\Pi s := 2\int_{1/2}^{1} \pi s(t,\cdot)\,\mathrm{d}t$, and $\nabla_T := \nabla_{T,1}$.

We compute

$$\begin{aligned}
&\int_{1/2}^{1}\int_{\partial B} |s(t,\hat{x}) - \Pi s(t,\cdot)|^2 \mathrm{d}\hat{x}\mathrm{d}t \\
&\quad = 4\int_{1/2}^{1}\int_{\partial B} \left|\int_{1/2}^{1} s(t,\hat{x}) - \pi s(u,\cdot)\,\mathrm{d}u\right|^2 \mathrm{d}\hat{x}\mathrm{d}t \\
&\quad \lesssim \int_{1/2}^{1}\int_{1/2}^{1}\int_{\partial B} |s(t,\hat{x}) - \pi s(u,\cdot)|^2 \,\mathrm{d}\hat{x}\mathrm{d}u\mathrm{d}t \\
&\quad \lesssim \int_{1/2}^{1}\int_{1/2}^{1}\int_{\partial B} |s(t,\hat{x}) - \pi s(t,\cdot)|^2 + |\pi s(t,\cdot) - \pi s(u,\cdot)|^2 \,\mathrm{d}\hat{x}\mathrm{d}u\mathrm{d}t.
\end{aligned}$$

The first summand can be bounded as follows

$$\begin{aligned}
\int_{1/2}^{1}\int_{1/2}^{1}\int_{\partial B} |s(t,\hat{x}) - \pi s(t,\cdot)|^2 \,\mathrm{d}\hat{x}\mathrm{d}t\mathrm{d}u &\lesssim \int_{1/2}^{1}\int_{1/2}^{1}\int_{\partial B} |\nabla_T s(t,\hat{x})|^2 \,\mathrm{d}\hat{x}\mathrm{d}t\mathrm{d}u \\
&\lesssim \int_{1/2}^{1}\int_{\partial B} |\nabla_T s(t,\hat{x})|^2 \,\mathrm{d}\hat{x}\mathrm{d}t.
\end{aligned}$$

The second summand can be controlled as in the usual proof of the Neumann–Poincaré inequality. We have

$$\begin{aligned}
|\pi s(t,\cdot) - \pi s(u,\cdot)| &= \left|\int_0^1 \partial_v \pi s(t + v(t-u),\cdot)\,\mathrm{d}v\right| \\
&\leqslant \left|\int_0^1 \pi(\partial_t s)(t + v(t-u),\cdot)\,\mathrm{d}v\right| \\
&\lesssim \left(\int_0^1\int_{\partial B} |(\partial_t s)(t + v(t-u),\hat{x})|^2 \,\mathrm{d}\hat{x}\mathrm{d}v\right)^{1/2}.
\end{aligned}$$

Plugging this into the second summand and symmetry considerations yield

$$\begin{aligned}
&\int_{1/2}^{1}\int_{1/2}^{1}\int_{\partial B}|\pi s(t,\cdot)-\pi s(u,\cdot)|^2\,\mathrm{d}\hat{x}\mathrm{d}u\mathrm{d}t \\
&\quad\lesssim \int_{1/2}^{1}\int_{1/2}^{1}\int_{0}^{1}\int_{\partial B}|(\partial_t s)(t+v(t-u),\hat{x})|^2\,\mathrm{d}\hat{x}\mathrm{d}v\mathrm{d}u\mathrm{d}t \\
&\quad\lesssim \int_{1/2}^{1}\int_{\partial B}|\partial_t s(t,\hat{x})|^2\,\mathrm{d}\hat{x}\mathrm{d}t.
\end{aligned}$$

This finishes the proof. □

The proof of Proposition 4.1 also uses the following observation about

$$\hat{s}_r := \log(e^s e^{-\Pi_r s}).$$

Proposition 4.3. *We have*

$$|\nabla_{H_\diamond} s| \lesssim |\nabla_{H_\diamond}\hat{s}_r|, \quad |\hat{s}_r| \lesssim |s-\Pi_r s|, \quad \textit{and} \quad |\nabla_{H_\diamond}\hat{s}_r|^2 \lesssim 1-\Delta|\hat{s}_r|^2.$$

Proof. The first two inequalities follow by elementary considerations.

Since s is bounded in $L^\infty(B)$, $\Pi_r s$ is uniformly bounded and, consequently, so is $\hat{s}_r$. By [JW18, Proposition A.9], we have

$$\Delta|\hat{s}_r|^2 + 2|v(-\hat{s}_r)\nabla_{H_\diamond}\hat{s}_r|^2 \lesssim |\mathrm{K}_{H_\diamond}| + |\mathrm{K}_{H_\diamond e^{\hat{s}_r}}|$$

with

$$v(-\hat{s}_r) = \sqrt{\frac{1-e^{-\mathrm{ad}_{\hat{s}_r}}}{\mathrm{ad}_{\hat{s}_r}}} \in \mathrm{End}(\mathfrak{gl}(E)).$$

Since $H_\diamond e^s$ is HYM and $\Pi_r s$ is constant with respect to $\nabla_{H_\diamond}$, we have

$$\mathrm{K}_{H_\diamond e^{\hat{s}_r}} = \mathrm{K}_{H_\diamond e^s e^{-\Pi_r s}} = i\Lambda\bar{\partial}(e^{\Pi_r s}\bar{\partial}_{H_\diamond e^s} e^{-\Pi_r s}) = \mathrm{Ad}(e^{\Pi_r s})\mathrm{K}_{H_\diamond e^s} = 0.$$

Given this, the third inequality follows using

$$\sqrt{\frac{1-e^{-x}}{x}} \gtrsim \frac{1}{\sqrt{1+|x|}},$$

$\|\mathrm{K}_{H_\diamond}\|_{L^\infty} \leqslant c$, which is a consequence of (2.3), and the fact that $H_\diamond$ is HYM with respect to ω_0, and the bound on $|s|$ established in Proposition 3.1. □

Proof of Proposition 4.1. Given the above discussion, the proof is very similar to that of [JW18, Proposition C.2]. Nevertheless, for the reader's convenience we provide the necessary details.

Define $g\colon [0,1/2] \to [0,\infty]$ by

$$g(r) := \int_{B_r} |z|^{2-2n} |\nabla_{H_\diamond} s|^2.$$

We will show that

$$g(r) \leqslant c r^{2\alpha},$$

which implies the asserted inequality.

Step 1. *We have $g \leqslant c$.*

Fix a smooth function $\chi\colon [0,\infty) \to [0,1]$ which is equal to one on $[0,1]$ and vanishes outside $[0,2]$. Set $\chi_r(\cdot) := \chi(|\cdot|/r)$. For $r > \varepsilon > 0$, using Proposition 4.3 and Proposition 3.1, and with G denoting Green's function on B centered at 0, we have

$$\begin{aligned}
\int_{B_r \setminus B_\varepsilon} |z|^{2-2n} |\nabla_{H_\diamond} s|^2 &\lesssim \int_{B_r \setminus B_\varepsilon} |z|^{2-2n} |\nabla_{H_\diamond} \hat{s}_r|^2 \\
&\lesssim \int_{B_{2r} \setminus B_{\varepsilon/2}} \chi_r (1 - \chi_{\varepsilon/2}) G (1 - \Delta |\hat{s}_r|^2) \\
&\lesssim \int_{B_{2r} \setminus B_r} |z|^{-2n} |s - \Pi_r s|^2 + r^2 + \varepsilon^{-2n} \int_{B_\varepsilon \setminus B_{\varepsilon/2}} |s - \Pi_r s|^2 \\
&\leqslant c.
\end{aligned}$$

Step 2. *There are constants $\gamma \in [0,1)$ and $A > 0$ such that*

$$g(r) \leqslant \gamma g(2r) + A r^2.$$

Continuing the inequality from Step 1 using Proposition 4.2, we have

$$\begin{aligned}
\int_{B_r \setminus B_\varepsilon} |z|^{2-2n} |\nabla_{H_\diamond} s|^2 &\lesssim \int_{B_{2r} \setminus B_r} |z|^{2-2n} |\nabla_{H_\diamond} s|^2 + r^2 + \varepsilon^{2-2n} \int_{B_\varepsilon \setminus B_{\varepsilon/2}} |\nabla_{H_\diamond} s|^2 \\
&\lesssim g(2r) - g(r) + r^2 + g(\varepsilon).
\end{aligned}$$

By Lebesgue's monotone convergence theorem, the last term vanishes as ε tends to zero; hence, the asserted inequality follows with $\gamma = \frac{c}{c+1}$ and $A = c$.

Step 3. *We have $g \leqslant c r^{2\alpha}$ for some $\alpha \in (0,1)$.*

This follows from Step 1 and Step 2 and as in [JW18, Step 3 in the proof of Proposition C.2]. □

5 Proof of Theorem 2.4

For $r > 0$, define $m_r \colon \mathbf{C}^n \to \mathbf{C}^n$ by $m_r(z) \coloneqq rz$. Set

$$s_r \coloneqq m_r^*(s|_{B_{4r}\setminus B_{r/2}}) \in C^\infty(B_4\setminus B_{1/2}, i\mathfrak{su}(E,H_*)) \quad \text{and} \quad H_{\diamond,r} \coloneqq m_r^* H_\diamond.$$

The metric $H_{\diamond,r}e^{s_r}$ is HYM with respect to $\omega_r \coloneqq r^{-2}m_r^*\omega$ and $\|\mathrm{F}_{H_\diamond,r}\|_{C^k(B_4\setminus B_{1/2})} \leqslant c_k$.

Proposition 3.1, (2.3) and interior estimates for HYM metrics [JW18, Theorem C.1] imply that

$$\|s_r\|_{C^k(B_3\setminus B_{3/4})} \leqslant c_k.$$

By Proposition 4.1, we have

$$\|\nabla_{H_\diamond,r} s_r\|_{L^2(B_4\setminus B_{1/2})} \lesssim r^\alpha.$$

Schematically, $\mathrm{K}_{H_{\diamond,r}e^{s_r}} = 0$ can be written as

$$\nabla^*_{H_{\diamond,r}}\nabla_{H_{\diamond,r}} s_r + B(\nabla_{H_{\diamond,r}} s \otimes \nabla_{H_{\diamond,r}} s_r) = C(\mathrm{K}_{H_{\diamond,r}}),$$

where B and C are linear with coefficients depending on s, but not on its derivatives; see, e.g., [JW18, Proposition A.1]. Since $\|\mathrm{K}_{H_{\diamond,r}}\|_{C^k(B_3\setminus B_{3/4})} \leqslant c_k r^2$, as in [JW18, Step 3 in the proof of Proposition 5.1], standard interior estimates imply that

$$\|\nabla^k_{H_{\diamond,r}} s_r\|_{L^2(B_2\setminus B_1)} \leqslant c_k r^\alpha$$

and, hence, the asserted inequalities, for each $k \geqslant 1$. (The asserted inequality for $k = 0$ has already be proven in Proposition 3.1.) □

6 Proof of Proposition 1.4

We will make use of the following general fact about connections over manifolds with free S^1–actions.

Proposition 6.1. *Let M be a manifold with a free S^1–action. Denote the associated Killing field by $\xi \in \mathrm{Vect}(M)$ and let $q\colon M \to M/S^1$ be the canonical projection. Suppose $\theta \in \Omega^1(M)$ is such that $\theta(\xi) = 1$ and $\mathscr{L}_\xi\theta = 0$. Let A be a unitary connection on a Hermitian vector bundle (E,H) over M. If $i(\xi)\mathrm{F}_A = 0$, then there is a $k \in \mathbf{N}$ and, for each $j \in \{1,\dots,k\}$, a Hermitian vector bundles (F_j,K_j) over M/S^1 such that*

$$E = \bigoplus_{j=1}^k E_j \quad \textit{and} \quad H = \bigoplus_{j=1}^k H_j$$

with $E_j \coloneqq q^*F_j$ and $H_j \coloneqq q^*K_j$; moreover, the bundles E_j are parallel and, for each $j \in \{1,\dots,k\}$, there are a unitary connection B_j on F_j and $\mu_j \in \mathbf{R}$ such that

$$A = \bigoplus_{j=1}^k q^*B_j + i\mu_j\,\mathrm{id}_{E_j}\cdot\theta.$$

Proof. Denote by $\tilde{\xi} \in \mathrm{Vect}(\mathrm{U}(E))$ the A–horizontal lift of ξ. This vector field integrates to an $\mathbf{R}$–action on $\mathrm{U}(E)$. Thinking of A as an $\mathfrak{u}(r)$–valued 1–form on $\mathrm{U}(E)$ and F_A as an $\mathfrak{u}(r)$–valued 2–form on $\mathrm{U}(E)$, we have

$$\mathscr{L}_{\tilde{\xi}}A = i(\tilde{\xi})\mathrm{F}_A = 0;$$

hence, A is invariant with respect to the $\mathbf{R}$–action on $\mathrm{U}(E)$.

The obstruction to the $\mathbf{R}$–action on $U(E)$ inducing an S^1–action is the action of $1 \in \mathbf{R}$ and corresponds to a gauge transformation $\mathbf{g}_A \in \mathscr{G}(\mathrm{U}(E))$ fixing A. If this obstruction vanishes, i.e., $\mathbf{g}_A = \mathrm{id}_{\mathrm{U}(E)}$, then $E \cong q^*F$ with $F = E/S^1$ and there is a connection A_0 on F such that $A = q^*A_0$.

If the obstruction does not vanish, we can decompose E into pairwise orthogonal parallel subbundles E_j such that $\mathbf{g}_A$ acts on E_j as multiplication with $e^{i\mu_j}$ for some $\mu_j \in \mathbf{R}$. Set $\tilde{A} \coloneqq A - \bigoplus_{j=1}^k i\mu_j \,\mathrm{id}_{E_j} \cdot \theta$. This connection also satisfies $i(\tilde{\xi})\mathrm{F}_{\tilde{A}} = 0 \in \Omega^1(M, \mathfrak{g}_E)$ and the subbundles E_j are also parallel with respect to E_j. Since $\mathbf{g}_{\tilde{A}} = \mathrm{id}_E$, the assertion follows. □

In the situation of Proposition 1.4, with $\xi \in S^{2n-1}$ denoting the Killing field for the S^1–action we have $i(\xi)\mathrm{F}_{A_0} = 0$; c.f., Tian [Tia00, discussion after Conjecture 2]. Therefore, we can write

$$A_* = \bigoplus_{j=1}^k \sigma^* B_j + i\mu_j \,\mathrm{id}_{E_j} \cdot \pi^*\theta.$$

Since $\mathrm{d}\theta = 2\pi\rho^*\omega_{FS}$, we have

$$\mathrm{F}_{A_*} = \bigoplus_{j=1}^k \sigma^*\mathrm{F}_{B_j} + 2\pi i\mu_j \,\mathrm{id}_{E_j} \cdot \sigma^*\omega_{FS}.$$

Using (2.1), A_* being HYM with respect to ω_0 can be seen to be equivalent to

$$\mathrm{F}^{0,2}_{B_j} = 0 \quad \text{and} \quad i\Lambda\mathrm{F}_{B_j} = (2n-2)\pi\mu_j \cdot \mathrm{id}_{E_j}.$$

The isomorphism $\mathscr{E} = (E, \bar{\partial}_{A_*}) \cong \bigoplus_{j=1}^k \rho^*\mathscr{F}_j$ with $\mathscr{F}_j = (F_j, \bar{\partial}_{B_j})$ is given by g^{-1} with $g \coloneqq \bigoplus_{j=1}^k r^{\mu_j}$. □

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