

Tangent cones of Hermitian Yang–Mills connections with isolated singularities

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2018-01-26

Abstract

We give a simple direct proof of uniqueness of tangent cones for singular projectively Hermitian Yang–Mills connections on reflexive sheaves at isolated singularities modelled on a sum of μ -stable holomorphic bundles over \mathbf{P}^{n-1} .

Keywords: *holomorphic vector bundles, reflexive sheaves, Hermitian Yang–Mills metrics, tangent cones*

MSC2010: 53C07; 14J60, 32Q15

1 Introduction

A **projectively Hermitian Yang–Mills (PHYM)** connection A over a Kähler manifold X is a unitary connection A on a Hermitian vector bundle (E, H) over X satisfying

$$(1.1) \quad F_A^{0,2} = 0 \quad \text{and} \quad i\Lambda F_A - \frac{\text{tr}(i\Lambda F_A)}{\text{rk } E} \cdot \text{id}_E = 0.$$

Since $F_A^{0,2} = 0$, $\mathcal{E} := (E, \bar{\partial}_A)$ is a holomorphic vector bundle, and A is the Chern connection of H . A Hermitian metric H on a holomorphic vector bundle is called **PHYM** if its Chern connection A_H is PHYM. The celebrated Donaldson–Uhlenbeck–Yau Theorem [Don85; Don87; UY86] asserts that a holomorphic vector bundle \mathcal{E} on a compact Kähler manifold admits a PHYM metric if and only if it is μ -polystable; moreover, any two PHYM metrics are related by an automorphism of \mathcal{E} and by multiplication with a conformal factor. If H is a PHYM metric, then the connection A° on $\text{PU}(E, H)$, the principal $\text{PU}(r)$ -bundle associated with (E, H) , induced by A_H is **Hermitian Yang–Mills (HYM)**, that is, it satisfies $F_{A^\circ}^{0,2} = 0$ and $i\Lambda F_{A^\circ} = 0$; it depends only on the conformal class of H . Conversely, any HYM connection A°

on $\text{PU}(E, H)$ can be lifted to a PHYM connection A ; any two choices of lifts lead to isomorphic holomorphic vector bundles \mathcal{E} and conformal metrics H .

An **admissible PHYM connection** is a PHYM connection A on a Hermitian vector bundle (E, H) over $X \setminus \text{sing}(A)$ with $\text{sing}(A)$ a closed subset with locally finite $(2n - 4)$ -dimensional Hausdorff measure and $F_A \in L^2_{\text{loc}}(X)$.¹ Bando [Ban91] proved that if A is an admissible PHYM connection, then $(E, \bar{\partial}_A)$ extends to X as a reflexive sheaf \mathcal{E} with $\text{sing}(\mathcal{E}) \subset \text{sing}(A)$. Bando and Siu [BS94] proved that a reflexive sheaf on a compact Kähler manifold admits an admissible PHYM metric if and only if it is μ -polystable.

The technique used by Bando and Siu does not yield any information on the behaviour of the admissible PHYM connection A_H near the singularities of the reflexive sheaf \mathcal{E} —not even at isolated singularities. The simplest example of a reflexive sheaf on \mathbf{C}^n with an isolated singularity at 0 is $i_*\sigma^*\mathcal{F}$ with \mathcal{F} a holomorphic vector bundle over \mathbf{P}^{n-1} ; cf. Hartshorne [Har80, Example 1.9.1]. Here we use the obvious maps summarised in the following diagram:

$$\begin{array}{ccccc} \mathbf{C}^n & \xleftarrow{i} & \mathbf{C}^n \setminus \{0\} & \xrightarrow{\pi} & S^{2n-1} & \xrightarrow{\rho} & \mathbf{P}^{n-1}. \\ & & & \searrow & \sigma & \nearrow & \end{array}$$

The main result of this article gives a description of PHYM connections near singularities modelled on $i_*\sigma^*\mathcal{F}$ with \mathcal{F} a sum of μ -stable holomorphic vector bundles.

Theorem 1.2. *Let $\omega = \frac{1}{2i}\bar{\partial}\partial|z|^2 + O(|z|^2)$ be a Kähler form on $\bar{B}_R(0) \subset \mathbf{C}^n$. Let A be an admissible PHYM connection on a Hermitian vector bundle (E, H) over $B_R(0) \setminus \{0\}$ with $\text{sing}(A) = \{0\}$ and $(E, \bar{\partial}_A) \cong \sigma^*\mathcal{F}$ for some holomorphic vector bundle \mathcal{F} over \mathbf{P}^{n-1} . Denote by F the complex vector bundle underlying \mathcal{F} .*

If \mathcal{F} is sum of μ -stable holomorphic vector bundles, then there exist a Hermitian metric K on F , a connection A_ on $\sigma^*(F, K)$ which is the pullback of a connection on $\rho^*(F, K)$, and an isometry $(E, H) \cong \sigma^*(F, K)$ such that with respect to this isometry we have*

$$|z|^{k+1} |\nabla_{A_*}^k (A^\circ - A_*^\circ)| \leq C_k |z|^\alpha \quad \text{for each } k \geq 0.$$

The constants $C_k, \alpha > 0$ depend on $\omega, \mathcal{F}, A|_{B_R(0) \setminus B_{R/2}(0)}$, and $\|F_A\|_{L^2(B_R(0))}$.

Remark 1.3. Using a gauge theoretic Łojasiewicz–Simon gradient inequality, Yang [Yano3, Theorem 1] proved that the tangent cone to a stationary Yang–Mills connection—in particular, a $\text{PU}(r)$ HYM connection—with an isolated singularity at x is unique provided

$$|F_A| \lesssim d(x, \cdot)^{-2}.$$

¹It should be pointed out that our notion of admissible PHYM connection follows Bando and Siu [BS94] and not Tian [Tiao00]. The notion of admissible Yang–Mills connection introduced by Tian is stronger: it assumes that the Hermitian vector bundle extends to all of X .

In our situation, such a curvature bound can be obtained from [Theorem 1.2](#). Our proof of this result, however, proceeds more directly—without making use of Yang’s theorem.

The hypothesis that \mathcal{F} be a sum of μ -stable holomorphic vector bundles is optimal. This is a consequence of the following observation, which will be proved in [Section 6](#).

Proposition 1.4. *Let (F, K) be a Hermitian vector bundle over \mathbf{P}^{n-1} . If B is a unitary connection on $\rho^*(F, K)$ such that $A_* := \pi^*B$ is HYM with respect to $\omega_0 := \frac{1}{2i}\bar{\partial}\partial|z|^2$, then there is a $k \in \mathbf{N}$ and, for each $j \in \{1, \dots, k\}$, there are $\mu_j \in \mathbf{R}$, a Hermitian vector bundle (F_j, K_j) on \mathbf{P}^{n-1} , and an irreducible unitary connection B_j on F_j satisfying*

$$F_{B_j}^{0,2} = 0 \quad \text{and} \quad i\Lambda F_{B_j} = (2n - 2)\pi\mu_j \cdot \text{id}_{F_j}$$

such that

$$F = \bigoplus_{j=1}^k F_j \quad \text{and} \quad B = \bigoplus_{j=1}^k \rho^*B_j + i\mu_j \text{id}_{\rho^*F_j} \cdot \theta.$$

Here θ denotes the standard contact structure² on S^{2n-1} . In particular,

$$\mathcal{E} = (\sigma^*F, \bar{\partial}_{A_*}) \cong \bigoplus_{j=1}^k \sigma^*\mathcal{F}_j$$

with $\mathcal{F}_j = (F_j, \bar{\partial}_{B_j})$ μ -stable.

To conclude the introduction we discuss two concrete examples in which [Theorem 1.2](#) can be applied.

Example 1.5 (Okonek, Schneider, and Spindler [[OSS11](#), Example 1.1.13]). It follows from the Euler sequence that $H^0(\mathcal{T}_{\mathbf{P}^3}(-1)) \cong \mathbf{C}^4$. Denote by $s_v \in H^0(\mathcal{T}_{\mathbf{P}^3}(-1))$ the section corresponding to $v \in \mathbf{C}^4$. If $v \neq 0$, then the rank two sheaf $\mathcal{E} = \mathcal{E}_v$ defined by

$$0 \rightarrow \mathcal{O}_{\mathbf{P}^3} \xrightarrow{s_v} \mathcal{T}_{\mathbf{P}^3}(-1) \rightarrow \mathcal{E}_v \rightarrow 0$$

is reflexive and $\text{sing}(\mathcal{E}) = \{[v]\}$.

\mathcal{E} is μ -stable. To see this, because $\mu(\mathcal{E}) = 1/2$, it suffices to show that

$$\text{Hom}(\mathcal{O}_{\mathbf{P}^3}(k), \mathcal{E}) = H^0(\mathcal{E}(-k)) = 0 \quad \text{for each } k \geq 1.$$

However, by inspection of the Euler sequence, $H^0(\mathcal{E}(-k)) \cong H^0(\mathcal{T}_{\mathbf{P}^3}(-k-1)) = 0$. It follows that \mathcal{E} admits a PHYM metric H with $F_H \in L^2$ and a unique singular point

²With respect to standard coordinates on \mathbf{C}^n , the standard contact structure θ on S^{2n-1} is such that $\pi^*\theta = \sum_{j=1}^n (\bar{z}_j dz_j - z_j d\bar{z}_j) / 2i|z|^2$.

at $[v] \in \mathbf{P}^3$. To see that [Theorem 1.2](#) applies, pick a standard affine neighborhood $U \cong \mathbf{C}^3$ in which $[v]$ corresponds to 0. In U , the Euler sequence becomes

$$0 \rightarrow \mathcal{O}_{\mathbf{C}^3} \xrightarrow{(1, z_1, z_2, z_3)} \mathcal{O}_{\mathbf{C}^3}^{\oplus 4} \rightarrow \mathcal{T}_{\mathbf{P}^3}(-1)|_U \rightarrow 0,$$

and $s_v = [(1, 0, 0, 0)]$; hence,

$$0 \rightarrow \mathcal{O}_{\mathbf{C}^3} \xrightarrow{(z_1, z_2, z_3)} \mathcal{O}_{\mathbf{C}^3}^{\oplus 3} \rightarrow \mathcal{E}_v|_U \rightarrow 0.$$

On $\mathbf{C}^3 \setminus \{0\}$, this is the pullback of the Euler sequence on \mathbf{P}^2 ; therefore, $\mathcal{E}_v|_U \cong i_* \sigma^* \mathcal{T}_{\mathbf{P}^2}$.

Example 1.6. For $t \in \mathbf{C}$, define $f_t: \mathcal{O}_{\mathbf{P}^3}(-2)^{\oplus 2} \rightarrow \mathcal{O}_{\mathbf{P}^3}(-1)^{\oplus 5}$ by

$$f_t := \begin{pmatrix} z_0 & 0 \\ z_1 & z_0 \\ z_2 & z_1 \\ t \cdot z_3 & z_2 \\ 0 & z_3 \end{pmatrix},$$

and denote by \mathcal{E}_t the cokernel of f_t , i.e.,

$$(1.7) \quad 0 \rightarrow \mathcal{O}_{\mathbf{P}^3}(-2)^{\oplus 2} \xrightarrow{f_t} \mathcal{O}_{\mathbf{P}^3}(-1)^{\oplus 5} \rightarrow \mathcal{E}_t \rightarrow 0.$$

If $t \neq 0$, then \mathcal{E}_t is locally free; \mathcal{E}_0 is reflexive with $\text{sing}(\mathcal{E}_0) = \{[0 : 0 : 0 : 1]\}$. The proof of this is analogous to that of the reflexivity of \mathcal{E}_v from [Example 1.5](#) given in [\[OSS11, Example 1.1.13\]](#).

For each t , $H^0(\mathcal{E}_t) = H^0(\mathcal{E}_t^*(-1)) = 0$; hence, \mathcal{E}_t is μ -stable according to the criterion of Okonek, Schneider, and Spindler [\[OSS11, Remark 1.2.6\(b\)\]](#). The former vanishing is obvious since $H^0(\mathcal{O}_{\mathbf{P}^3}(-1)) = H^1(\mathcal{O}_{\mathbf{P}^3}(-2)) = 0$. The latter follows by dualising (1.7), twisting by $\mathcal{O}_{\mathbf{P}^3}(-1)$ and observing that the induced map $H^0(f_0^*): H^0(\mathcal{O}_{\mathbf{P}^3})^{\oplus 5} \rightarrow H^0(\mathcal{O}_{\mathbf{P}^3}(1))^{\oplus 2}$, which is given by

$$\begin{pmatrix} z_0 & z_1 & z_2 & t \cdot z_3 & 0 \\ 0 & z_0 & z_1 & z_2 & z_3 \end{pmatrix},$$

is injective.

In a standard affine neighborhood $U \cong \mathbf{C}^3$ of $[0 : 0 : 0 : 1]$, we have $\mathcal{E}_0|_U \cong i_* \sigma^*(\mathcal{T}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2}(1))$. To see this, note that the cokernel of the map $g: \mathcal{O}_{\mathbf{P}^2}^{\oplus 2} \rightarrow \mathcal{O}_{\mathbf{P}^2}(1)^{\oplus 4} \oplus \mathcal{O}_{\mathbf{P}^2}$ defined by

$$g := \begin{pmatrix} z_0 & 0 \\ z_1 & z_0 \\ z_2 & z_1 \\ 0 & z_2 \\ 0 & 1 \end{pmatrix}$$

is $\mathcal{T}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2}(1)$.

Conventions and notation. Set $B_r := B_r(0)$ and $\dot{B}_r := B_r(0) \setminus \{0\}$. We denote by $c > 0$ a generic constant, which depends only on \mathcal{F} , ω , $s|_{B_1 \setminus B_{1/2}}$, H_\diamond , and $\|F_H\|_{L^2(B_R(0))}$ (which will be introduced in the next section). Its value might change from one occurrence to the next. Should c depend on further data we indicate this by a subscript. We write $x \lesssim y$ for $x \leq cy$. The expression $O(x)$ denotes a quantity y with $|y| \lesssim x$. Since reflexive sheaves are locally free away from a closed subset of complex codimension three, without loss of generality, we will assume throughout that $n \geq 3$.

Acknowledgements. HSE and TW were partially supported by São Paulo State Research Council (FAPESP) grant 2015/50368-0 and the MIT–Brazil Lemann Seed Fund for Collaborative Projects. HSE is also funded by FAPESP grant 2014/24727-0 and Brazilian National Research Council (CNPq) grant PQ2 - 312390/2014-9.

2 Reduction to the metric setting

In the situation of [Theorem 1.2](#), the Hermitian metric H on \mathcal{E} corresponds to a PHYM metric on $\sigma^*\mathcal{F}$ via the isomorphism $(E, \bar{\partial}_A) \cong \sigma^*\mathcal{F}$. By slight abuse of notation, we will denote this metric by H as well.

Denote by $\mathcal{F}_1, \dots, \mathcal{F}_k$ the μ -stable summands of \mathcal{F} . Denote by K_j the PHYM metric on \mathcal{F}_j with

$$i\Lambda_{\omega_{FS}} F_{K_j} = \frac{2\pi}{(n-2)! \text{vol}(\mathbf{P}^{n-1})} \mu_j \cdot \text{id}_{F_j} = (2n-2)\pi \mu_j \cdot \text{id}_{F_j}$$

with ω_{FS} denoting the integral Fubini study form and for $\mu_j := \mu(\mathcal{F}_j)$. The Kähler form ω_0 associated with the standard Kähler metric on \mathbf{C}^n can be written as

$$(2.1) \quad \omega_0 = \frac{1}{2i} \bar{\partial} \partial |z|^2 = \pi r^2 \sigma^* \omega_{FS} + r dr \wedge \pi^* \theta$$

with θ as in [Proposition 1.4](#). Therefore, we have

$$i\Lambda_{\omega_0} F_{\sigma^* K_j} = (2n-2)\mu_j r^{-2} \cdot \text{id}_{\sigma^* F_j},$$

and $H_{\diamond, j} := r^{2\mu_j} \cdot \sigma^* K_j$ satisfies

$$\begin{aligned} i\Lambda_{\omega_0} F_{H_{\diamond, j}} &= i\Lambda_{\omega_0} F_{\sigma^* K_j} + i\Lambda_{\omega_0} \bar{\partial} \partial \log r^{2\mu_j} \cdot \text{id}_{\sigma^* F_j} \\ &= i\Lambda_{\omega_0} F_{\sigma^* K_j} + \frac{1}{2} \Delta \log r^{2\mu_j} \cdot \text{id}_{\sigma^* F_j} = 0. \end{aligned}$$

Denote by $A_{\diamond, j}$ the Chern connection associated with $H_{\diamond, j}$ and by B_j the Chern connection associated with K_j . The isometry $r^{\mu_j} : (\sigma^* F_j, H_{\diamond, j}) \rightarrow \sigma^*(F_j, K_j)$ transforms $A_{\diamond, j}$ into

$$A_{*, j} := (r^{\mu_j})_* A_{\diamond, j} = \sigma^* B_j + i\mu_j \text{id}_{\sigma^* F_j} \cdot \pi^* \theta.$$

In particular,

$$A_* := \bigoplus_{j=1}^k A_{*,j}$$

is the pullback of a connection B on S^{2n-1} . Moreover, A_* is unitary with respect to

$$H_* := \bigoplus_{j=1}^k \sigma^* K_j.$$

Proposition 2.2. *Assume the above situation. Set $H_\diamond := \bigoplus_{j=1}^k H_{\diamond,j}$ and fix $R > 0$. We have*

$$(2.3) \quad \left\| |z|^{2+\ell} \nabla_{H_\diamond}^\ell F_{H_\diamond} \right\|_{L^\infty(B_R)} < \infty \quad \text{for each } \ell \geq 0.$$

Proof. Using the isometry $g := \bigoplus_{j=1}^k r^{\mu_j}$ both assertions can be translated to corresponding statements for A_* . The first assertion then follows since A_* is the pullback of a connection B on S^{2n-1} . \square

In the situation of Theorem 1.2, after a conformal change, which does not affect A^\diamond , we can assume that $\det H = \det H_\diamond$. Setting

$$s := \log(H_\diamond^{-1}H) \in C^\infty(\dot{B}_r, \text{isu}(\sigma^*F, H_\diamond))^3$$

$$\text{and } \Upsilon(s) := \frac{e^{\text{ad}_s} - 1}{\text{ad}_s},$$

we have

$$e_*^{s/2} H = H_\diamond \quad \text{and} \quad e_*^{s/2} A = A_\diamond + a$$

$$\text{with } a := \frac{1}{2} \Upsilon(-s/2) \partial_{A_\diamond} s - \frac{1}{2} \Upsilon(s/2) \bar{\partial}_{A_\diamond} s;$$

see, e.g., [JW16, Appendix A]. Moreover, with $g := \bigoplus_{j=1}^k r^{\mu_j}$ we have

$$g_* e_*^{s/2} A = A_* + gag^{-1}.$$

Since

$$|\nabla_{A_*}^k gag^{-1}|_{H_*} = |\nabla_{H_\diamond}^k a|_{H_\diamond} \quad \text{for each } k \geq 0,$$

Theorem 1.2 will be a consequence of Proposition 2.2 and the following result.

³If H, K are two Hermitian inner products on a complex vector space V , then there is a unique endomorphism $T \in \text{End}(V)$ which is self-adjoint with respect to H and K , has positive spectrum, and satisfies $H(Tv, w) = K(v, w)$. It is customary to denote T by $H^{-1}K$, and thus $\log(H^{-1}K) = \log(T)$.

Theorem 2.4. *Suppose $\omega = \frac{1}{2i} \bar{\partial} \partial |z|^2 + O(|z|^2)$ is a Kähler form on $\bar{B}_R \subset \mathbb{C}^n$, \mathcal{E} is a holomorphic vector bundle over \bar{B}_R , and H_\circ is a Hermitian metric on \mathcal{E} which is HYM with respect to ω_0 and satisfies (2.3). If H is an admissible HYM metric on \mathcal{E} with $\text{sing}(A_H) = \{0\}$ and $\det H = \det H_\circ$, then*

$$s := \log(H_\circ^{-1}H) \in C^\infty(\dot{B}_R, \text{isu}(\pi^*F, H_\circ))$$

satisfies

$$|s| \leq C_0 \quad \text{and} \quad |z|^k |\nabla_{H_\circ}^k s| \leq C_k |z|^\alpha \quad \text{for each } k \geq 1.$$

The constants $C_k, \alpha > 0$ depend on $\omega, H_\circ, s|_{B_R \setminus B_{R/2}}$, and $\|F_H\|_{L^2(B_R)}$.

The next three sections of this paper are devoted to proving Theorem 2.4. Without loss of generality, we will assume that the radius R is one. We set $B := B_1$ and $\dot{B} := \dot{B}_1$.

3 A priori C^0 estimate

As a first step towards proving Theorem 2.4 we bound $|s|$, using an argument which is essentially contained in Bando and Siu [BS94, Theorem 2(a) and (b)].

Proposition 3.1. *We have $|s| \in L^\infty(B)$ and $\|s\|_{L^\infty(B)} \leq c$.*

Proof. The proof relies on the differential inequality

$$(3.2) \quad \Delta \log \text{tr } H_0^{-1} H_1 \lesssim |K_{H_1} - K_{H_0}|$$

for Hermitian metrics H_0 and H_1 with $\det H_0 = \det H_1$, and with

$$K_H := i\Delta F_H - \frac{\text{tr}(i\Delta F_H)}{\text{rk } E} \cdot \text{id}_E;$$

see [Siu87, p. 13] for a proof.

Step 1. *We have $\log \text{tr } e^s \in W^{1,2}(B)$ and $\|\log \text{tr } e^s\|_{W^{1,2}(B)} \leq c$.*

Choose $1 \leq i < j \leq n$ and define the projection $\pi: B \rightarrow \mathbb{C}^{n-2}$ by

$$\pi(z) := (z_1, \dots, \hat{z}_i, \dots, \hat{z}_j, \dots, z_n).$$

For $\zeta \in \mathbb{C}^{n-2}$, denote by ∇_ζ and Δ_ζ the derivative and the Laplacian on the slice $\pi^{-1}(\zeta)$ respectively. Set $f_\zeta := \log \text{tr } e^s|_{\pi^{-1}(\zeta)}$. Applying (3.2) to $H|_{\pi^{-1}(\zeta)}$ and $H_\circ|_{\pi^{-1}(\zeta)}$ we obtain

$$\Delta_\zeta f_\zeta \lesssim |F_H| + |F_{H_\circ}|.$$

Fix $\chi \in C^\infty(\mathbb{C}^2; [0, 1])$ such that $\chi(\eta) = 1$ for $|\eta| \leq 1/2$ and $\chi(\eta) = 0$ for $|\eta| \geq 1/\sqrt{2}$. For $0 < |\zeta| \leq 1/\sqrt{2}$ and $\varepsilon > 0$, we have

$$\begin{aligned} \int_{\pi^{-1}(\zeta)} |\nabla_\zeta(\chi f_\zeta)|^2 &\lesssim \int_{\pi^{-1}(\zeta)} \chi^2 f_\zeta (|F_H| + |F_{H_o}|) + 1 \\ &\leq \varepsilon \int_{\pi^{-1}(\zeta)} |\chi f_\zeta|^2 + \varepsilon^{-1} \int_{\pi^{-1}(\zeta)} |F_H|^2 + |F_{H_o}|^2 + 1. \end{aligned}$$

Using the Dirichlet–Poincaré inequality and rearranging, we obtain

$$\int_{\pi^{-1}(\zeta)} |\chi f_\zeta|^2 + |\nabla_\zeta(\chi f_\zeta)|^2 \lesssim \int_{\pi^{-1}(\zeta)} |F_H|^2 + |F_{H_o}|^2 + 1.$$

Integrating over $0 < |\zeta| \leq 1/\sqrt{2}$ yields

$$\int_B |\log \operatorname{tr} e^s|^2 + |\nabla' \log \operatorname{tr} e^s|^2 \lesssim \int_B |F_H|^2 + |F_{H_o}|^2 + 1$$

with ∇' denoting the derivative along the fibres of π . Using (2.3) and $n \geq 3$, $F_{H_o} \in L^2(B)$. Since the choice of i, j defining π was arbitrary, the asserted inequality follows.

Step 2. The differential inequality

$$\Delta \log \operatorname{tr} e^s \lesssim |K_{H_o}|$$

holds on B in the sense of distributions.

Fix a smooth function $\chi: [0, \infty) \rightarrow [0, 1]$ which vanishes on $[0, 1]$ and is equal to one on $[2, \infty)$. Set $\chi_\varepsilon := \chi(\cdot/|\varepsilon)$. By (3.2), for $\phi \in C_0^\infty(B)$, we have

$$\begin{aligned} \int_B \Delta \phi \cdot \log \operatorname{tr} e^s &= \lim_{\varepsilon \rightarrow 0} \int_B \chi_\varepsilon \cdot \Delta \phi \cdot \log \operatorname{tr} e^s \\ &\lesssim \int_B \phi \cdot |K_{H_o}| + \lim_{\varepsilon \rightarrow 0} \int_B \phi \cdot (\Delta \chi_\varepsilon \cdot \log \operatorname{tr} e^s - 2\langle \nabla \chi_\varepsilon, \nabla \log \operatorname{tr} e^s \rangle). \end{aligned}$$

Since $n \geq 3$, we have $\|\chi_\varepsilon\|_{W^{2,2}(B)} \lesssim \varepsilon^2$. Because $\log \operatorname{tr} e^s \in W^{1,2}(B)$ this shows that the limit vanishes.

Step 3. We have $\log \operatorname{tr} e^s \in L^\infty(B)$ and $\|\log \operatorname{tr} e^s\|_{L^\infty(B)} \leq c$.

Since $\operatorname{tr} s = 0$, we have $|s| \leq \operatorname{rk}(\mathcal{E}) \cdot \log \operatorname{tr} e^s$; in particular, $\log \operatorname{tr} e^s$ is non-negative. By hypothesis $K_H = 0$. Since H_o is HYM with respect to ω_0 and $|F_{H_o}| \lesssim |z|^{-2}$ by hypothesis (2.3), we have $|K_{H_o}| \leq c$. The asserted inequality thus follows from Step 2 via Moser iteration; see [GT01, Theorem 8.1]. \square

4 A priori Morrey estimates

The following decay estimate is the crucial ingredient of the proof of Theorem 2.4.

Proposition 4.1. *There is a constant $\alpha > 0$, such that for $r \in [0, 1]$ we have*

$$\int_{B_r} |\nabla_{H_\circ} s|^2 \lesssim r^{2n-2+2\alpha}.$$

The proof of this proposition relies on a Neumann–Poincaré type inequality, which we describe in what follows. Denote by $\nabla_{T,r}$ the connection on $\text{isu}(E, H_\circ)|_{\partial B_r}$ induced by ∇_{H_\circ} . The linear operator $\nabla_{T,r} : \Gamma(\partial B_r, \text{isu}(E, H_\circ)) \rightarrow \Omega^1(\partial B_r, \text{isu}(E, H_\circ))$ has a finite dimensional kernel. Since ∇_{H_\circ} is conical, we can identify⁴

$$\ker \nabla_{T,r} = \ker \nabla_{T,1} =: K.$$

Moreover, we can regard K as a subset of constant sections: $K \subset \Gamma(\dot{B}_r, \text{isu}(E, H_\circ))$. Denote by $\pi_r : \Gamma(\partial B_r, \text{isu}(E, H_\circ)) \rightarrow K$ the L^2 -orthogonal projection onto K and define $\Pi_r : \Gamma(\dot{B}_{2r}, \text{isu}(E, H_\circ)) \rightarrow K$ by

$$\Pi_r s := \frac{1}{r} \int_r^{2r} \pi_t(s|_{\partial B_t}) dt.$$

Proposition 4.2. *We have*

$$\int_{B_{2r} \setminus B_r} |s - \Pi_r s|^2 \lesssim r^2 \int_{B_{2r} \setminus B_r} |\nabla_{H_\circ} s|^2.$$

Proof. The asserted estimate is scale-invariant; hence, we may assume $r = 1/2$. To prove the estimate in this case it suffices to prove the cylindrical estimate

$$\int_{1/2}^1 \int_{\partial B} |s(t, \hat{x}) - \Pi s(t, \cdot)|^2 d\hat{x} dt \lesssim \int_{1/2}^1 \int_{\partial B} |\partial_t s(t, \hat{x})|^2 + |\nabla_T s(t, \hat{x})|^2 d\hat{x} dt$$

with s denoting a section over $[1/2, 1] \times \partial B$, $\pi := \pi_1$, $\Pi s := 2 \int_{1/2}^1 \pi s(t, \cdot) dt$, and $\nabla_T := \nabla_{T,1}$.

⁴ K can be determined explicitly from the decomposition of \mathcal{F} into μ -stable summands, but we will not need a precise description of K .

We compute

$$\begin{aligned}
& \int_{1/2}^1 \int_{\partial B} |s(t, \hat{x}) - \Pi s(t, \cdot)|^2 d\hat{x} dt \\
&= 4 \int_{1/2}^1 \int_{\partial B} \left| \int_{1/2}^1 s(t, \hat{x}) - \pi s(u, \cdot) du \right|^2 d\hat{x} dt \\
&\lesssim \int_{1/2}^1 \int_{1/2}^1 \int_{\partial B} |s(t, \hat{x}) - \pi s(u, \cdot)|^2 d\hat{x} du dt \\
&\lesssim \int_{1/2}^1 \int_{1/2}^1 \int_{\partial B} |s(t, \hat{x}) - \pi s(t, \cdot)|^2 + |\pi s(t, \cdot) - \pi s(u, \cdot)|^2 d\hat{x} du dt.
\end{aligned}$$

The first summand can be bounded as follows

$$\begin{aligned}
\int_{1/2}^1 \int_{1/2}^1 \int_{\partial B} |s(t, \hat{x}) - \pi s(t, \cdot)|^2 d\hat{x} dt du &\lesssim \int_{1/2}^1 \int_{1/2}^1 \int_{\partial B} |\nabla_T s(t, \hat{x})|^2 d\hat{x} dt du \\
&\lesssim \int_{1/2}^1 \int_{\partial B} |\nabla_T s(t, \hat{x})|^2 d\hat{x} dt.
\end{aligned}$$

The second summand can be controlled as in the usual proof of the Neumann–Poincaré inequality. We have

$$\begin{aligned}
|\pi s(t, \cdot) - \pi s(u, \cdot)| &= \left| \int_0^1 \partial_v \pi s(t + v(t-u), \cdot) dv \right| \\
&\leq \left| \int_0^1 \pi(\partial_t s)(t + v(t-u), \cdot) dv \right| \\
&\lesssim \left(\int_0^1 \int_{\partial B} |(\partial_t s)(t + v(t-u), \hat{x})|^2 d\hat{x} dv \right)^{1/2}.
\end{aligned}$$

Plugging this into the second summand and symmetry considerations yield

$$\begin{aligned}
& \int_{1/2}^1 \int_{1/2}^1 \int_{\partial B} |\pi s(t, \cdot) - \pi s(u, \cdot)|^2 d\hat{x} du dt \\
&\lesssim \int_{1/2}^1 \int_{1/2}^1 \int_0^1 \int_{\partial B} |(\partial_t s)(t + v(t-u), \hat{x})|^2 d\hat{x} dv du dt \\
&\lesssim \int_{1/2}^1 \int_{\partial B} |\partial_t s(t, \hat{x})|^2 d\hat{x} dt.
\end{aligned}$$

This finishes the proof the proof. □

The proof of Proposition 4.1 also uses the following observation about

$$\hat{s}_r := \log(e^s e^{-\Pi r^s}).$$

Proposition 4.3. *We have*

$$|\nabla_{H_\diamond} s| \lesssim |\nabla_{H_\diamond} \hat{s}_r|, \quad |\hat{s}_r| \lesssim |s - \Pi_r s|, \quad \text{and} \quad |\nabla_{H_\diamond} \hat{s}_r|^2 \lesssim 1 - \Delta |\hat{s}_r|^2.$$

Proof. The first two inequalities follow by elementary considerations.

Since s is bounded in $L^\infty(B)$, $\Pi_r s$ is uniformly bounded and, consequently, so is \hat{s}_r . By [JW16, Proposition A.8], we have

$$\Delta |\hat{s}_r|^2 + 2|v(-\hat{s}_r)\nabla_{H_\diamond} \hat{s}_r|^2 \lesssim |\mathbb{K}_{H_\diamond}| + |\mathbb{K}_{H_\diamond e^{\hat{s}_r}}|$$

with

$$v(-\hat{s}_r) = \sqrt{\frac{1 - e^{-\text{ad}_{\hat{s}_r}}}{\text{ad}_{\hat{s}_r}}} \in \text{End}(\mathfrak{gl}(E));$$

see, e.g., [JW16, Proposition A.6]. We have

$$\mathbb{K}_{H_\diamond e^{\hat{s}_r}} = \text{Ad}(e^{-\Pi_r s})\mathbb{K}_{H_\diamond e^s} = 0$$

because $H_\diamond e^s$ is HYM. Given this, the third inequality follows using

$$\sqrt{\frac{1 - e^{-x}}{x}} \gtrsim \frac{1}{\sqrt{1 + |x|}},$$

$\|\mathbb{K}_{H_\diamond}\|_{L^\infty} \leq c$, which is a consequence of (2.3), and the fact that H_\diamond is HYM with respect to ω_0 , and the bound on $|s|$ established in Proposition 3.1. \square

Proof of Proposition 4.1. Given the above discussion, the proof is very similar to that of [JW16, Proposition C.2]. Nevertheless, for the reader's convenience we provide the necessary details.

Define $g: [0, 1/2] \rightarrow [0, \infty]$ by

$$g(r) := \int_{B_r} |z|^{2-2n} |\nabla_{H_\diamond} s|^2.$$

We will show that

$$g(r) \leq cr^{2\alpha},$$

which implies the asserted inequality.

Step 1. *We have $g \leq c$.*

Fix a smooth function $\chi: [0, \infty) \rightarrow [0, 1]$ which is equal to one on $[0, 1]$ and vanishes outside $[0, 2]$. Set $\chi_r(\cdot) := \chi(|\cdot|/r)$. For $r > \varepsilon > 0$, using Proposition 4.3

and Proposition 3.1, and with G denoting Green's function on B centered at 0, we have

$$\begin{aligned}
\int_{B_r \setminus B_\varepsilon} |z|^{2-2n} |\nabla_{H_\circ} s|^2 &\lesssim \int_{B_r \setminus B_\varepsilon} |z|^{2-2n} |\nabla_{H_\circ} \hat{s}_r|^2 \\
&\lesssim \int_{B_{2r} \setminus B_{\varepsilon/2}} \chi_r (1 - \chi_{\varepsilon/2}) G(1 - \Delta |\hat{s}_r|^2) \\
&\lesssim \int_{B_{2r} \setminus B_r} |z|^{-2n} |s - \Pi_r s|^2 + r^2 + \varepsilon^{-2n} \int_{B_\varepsilon \setminus B_{\varepsilon/2}} |s - \Pi_r s|^2 \\
&\leq c.
\end{aligned}$$

Step 2. *There are constants $\gamma \in [0, 1)$ and $A > 0$ such that*

$$g(r) \leq \gamma g(2r) + Ar^2.$$

Continuing the inequality from Step 1 using Proposition 4.2, we have

$$\begin{aligned}
\int_{B_r \setminus B_\varepsilon} |z|^{2-2n} |\nabla_{H_\circ} s|^2 &\lesssim \int_{B_{2r} \setminus B_r} |z|^{2-2n} |\nabla_{H_\circ} s|^2 + r^2 + \varepsilon^{2-2n} \int_{B_\varepsilon \setminus B_{\varepsilon/2}} |\nabla_{H_\circ} s|^2 \\
&\lesssim g(2r) - g(r) + r^2 + g(\varepsilon).
\end{aligned}$$

By Lebesgue's monotone convergence theorem, the last term vanishes as ε tends to zero; hence, the asserted inequality follows with $\gamma = \frac{c}{c+1}$ and $A = c$.

Step 3. *We have $g \leq cr^{2\alpha}$ for some $\alpha \in (0, 1)$.*

This follows from Step 1 and Step 2 and as in [JW16, Step 3 in the proof of Proposition C.2]. \square

5 Proof of Theorem 2.4

For $r > 0$, define $m_r : \mathbf{C}^n \rightarrow \mathbf{C}^n$ by $m_r(z) := rz$. Set

$$s_r := m_r^*(s|_{B_{4r} \setminus B_{r/2}}) \in C^\infty(B_4 \setminus B_{1/2}, \text{isu}(E, H_*)) \quad \text{and} \quad H_{\circ, r} := m_r^* H_\circ.$$

The metric $H_{\circ, r} e^{s_r}$ is HYM with respect to $\omega_r := r^{-2} m_r^* \omega$ and $\|F_{H_{\circ, r}}\|_{C^k(B_4 \setminus B_{1/2})} \leq c_k$.

Proposition 3.1, (2.3) and interior estimates for HYM metrics [JW16, Theorem C.1] imply that

$$\|s_r\|_{C^k(B_3 \setminus B_{3/4})} \leq c_k.$$

By Proposition 4.1, we have

$$\|\nabla_{H_{\circ, r}} s_r\|_{L^2(B_4 \setminus B_{1/2})} \lesssim r^\alpha.$$

Schematically, $K_{H_{\circ,r}} e^{s_r} = 0$ can be written as

$$\nabla_{H_{\circ,r}}^* \nabla_{H_{\circ,r}} s_r + B(\nabla_{H_{\circ,r}} s \otimes \nabla_{H_{\circ,r}} s_r) = C(K_{H_{\circ,r}}),$$

where B and C are linear with coefficients depending on s , but not on its derivatives; see, e.g., [JW16, Proposition A.1]. Since $\|K_{H_{\circ,r}}\|_{C^k(B_3 \setminus B_{3/4})} \leq c_k r^2$, as in [JW16, Step 3 in the proof of Proposition 5.1], standard interior estimates imply that

$$\|\nabla_{H_{\circ,r}}^k s_r\|_{L^2(B_2 \setminus B_1)} \leq c_k r^\alpha$$

and, hence, the asserted inequalities, for each $k \geq 1$. (The asserted inequality for $k = 0$ has already been proven in Proposition 3.1.) \square

6 Proof of Proposition 1.4

We will make use of the following general fact about connections over manifolds with free S^1 -actions.

Proposition 6.1. *Let M be a manifold with a free S^1 -action. Denote the associated Killing field by $\xi \in \text{Vect}(M)$ and let $q: M \rightarrow M/S^1$ be the canonical projection. Suppose $\theta \in \Omega^1(M)$ is such that $\theta(\xi) = 1$ and $\mathcal{L}_\xi \theta = 0$. Let A be a unitary connection on a Hermitian vector bundle (E, H) over M . If $i(\xi)F_A = 0$, then there is a $k \in \mathbf{N}$ and, for each $j \in \{1, \dots, k\}$, a Hermitian vector bundles (F_j, K_j) over M/S^1 such that*

$$E = \bigoplus_{j=1}^k E_j \quad \text{and} \quad H = \bigoplus_{j=1}^k H_j$$

with $E_j := q^* F_j$ and $H_j := q^* K_j$; moreover, the bundles E_j are parallel and, for each $j \in \{1, \dots, k\}$, there are a unitary connection B_j on F_j and $\mu_j \in \mathbf{R}$ such that

$$A = \bigoplus_{j=1}^k q^* B_j + i\mu_j \text{id}_{E_j} \cdot \theta.$$

Proof. Denote by $\tilde{\xi} \in \text{Vect}(U(E))$ the A -horizontal lift of ξ . This vector field integrates to an \mathbf{R} -action on $U(E)$. Thinking of A as an $\mathfrak{u}(r)$ -valued 1-form on $U(E)$ and F_A as an $\mathfrak{u}(r)$ -valued 2-form on $U(E)$, we have

$$\mathcal{L}_{\tilde{\xi}} A = i(\tilde{\xi})F_A = 0;$$

hence, A is invariant with respect to the \mathbf{R} -action on $U(E)$.

The obstruction to the \mathbf{R} -action on $U(E)$ inducing an S^1 -action is the action of $1 \in \mathbf{R}$ and corresponds to a gauge transformation $\mathfrak{g}_A \in \mathcal{G}(U(E))$ fixing A . If this

obstruction vanishes, i.e., $\mathfrak{g}_A = \text{id}_{U(E)}$, then $E \cong q^*F$ with $F = E/S^1$ and there is a connection A_0 on F such that $A = q^*A_0$.

If the obstruction does not vanish, we can decompose E into pairwise orthogonal parallel subbundles E_j such that \mathfrak{g}_A acts on E_j as multiplication with $e^{i\mu_j}$ for some $\mu_j \in \mathbf{R}$. Set $\tilde{A} := A - \bigoplus_{j=1}^k i\mu_j \text{id}_{E_j} \cdot \theta$. This connection also satisfies $i(\tilde{\xi})F_{\tilde{A}} = 0 \in \Omega^1(M, \mathfrak{g}_E)$ and the subbundles E_j are also parallel with respect to E_j . Since $\mathfrak{g}_{\tilde{A}} = \text{id}_E$, the assertion follows. \square

In the situation of Proposition 1.4, with $\xi \in S^{2n-1}$ denoting the Killing field for the S^1 -action we have $i(\xi)F_{A_0} = 0$; c.f., Tian [Tia00, discussion after Conjecture 2]. Therefore, we can write

$$A_* = \bigoplus_{j=1}^k \sigma^* B_j + i\mu_j \text{id}_{E_j} \cdot \pi^* \theta.$$

Since $d\theta = 2\pi\rho^* \omega_{FS}$, we have

$$F_{A_*} = \bigoplus_{j=1}^k \sigma^* F_{B_j} + 2\pi i\mu_j \text{id}_{E_j} \cdot \sigma^* \omega_{FS}.$$

Using (2.1), A_* being HYM with respect to ω_0 can be seen to be equivalent to

$$F_{B_j}^{0,2} = 0 \quad \text{and} \quad i\Lambda F_{B_j} = (2n-2)\pi\mu_j \cdot \text{id}_{E_j}.$$

The isomorphism $\mathcal{E} = (E, \bar{\partial}_{A_*}) \cong \bigoplus_{j=1}^k \rho^* \mathcal{F}_j$ with $\mathcal{F}_j = (F_j, \bar{\partial}_{B_j})$ is given by g^{-1} with $g := \bigoplus_{j=1}^k r^{\mu_j}$. \square

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