# Spin(7)–instantons, Cayley submanifolds, and Fueter sections

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**Abstract**

We prove an existence theorem for Spin(7)–instantons, which are highly concentrated near a Cayley submanifold; thus giving a partial converse to Tian's foundational compactness theorem [Tia00]. As an application, we show how to construct Spin(7)–instantons on Spin(7)–manifolds with suitable local $K3$ Cayley fibrations. This recovers an example constructed by Lewis [Lew98].

## 1 Introduction

In this article we study some aspects of gauge theory on Spin(7)**–manifolds**, i.e., compact Riemannian 8–manifolds with holonomy contained in the exceptional Lie group $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$. Every Spin(7)–manifold $X$ comes equipped with a 4–form $\Phi$, which is a calibration in the sense of Harvey and Lawson [HL82]. Submanifolds $Q \subset X$ which are calibrated by $\Phi$ are called **Cayley submanifolds**. The linear operator $*(\cdot \wedge \Phi)\colon \Lambda^2 \to \Lambda^2$ has eigenvalues $-1$ and $3$ and with eigenspaces of dimension 21 and 7 respectively; and, in analogy with gauge theory on 4–manifolds, we consider connections $A$ whose curvature satisfies the "anti-self-duality" condition

$$* (F_A \wedge \Phi) = -F_A. \tag{1.1}$$

After gauge fixing, (1.1) becomes elliptic. Solutions to (1.1), commonly called Spin(7)**–instantons**, are absolute minimisers of the Yang–Mills functional. These equations play an important rôle in the Donaldson–Thomas programme [DT98] to develop gauge theory in higher dimensions and, by dimensional reduction, give rise to a plethora of interesting gauge theoretical equations in dimensions less than eight.

Tian [Tia00] discovered that there is an interesting relation between gauge theory in higher dimension and calibrated geometry. In particular, his foundational compactness result—extending work of Price [Pri83], Uhlenbeck [Uhl82a], and Nakajima [Nak88]—predicts that a sequence $(A_i)$ of Spin(7)–instantons *could* degenerate by "bubbling off ASD instantons transversely to a Cayley submanifold $Q$". More precisely, outside $Q$ the sequence $(A_i)$ converges smoothly (possibly after passing to a subsequence and changing gauge) and for each $x \in Q$ there exists a non-trivial ASD instanton $\mathfrak{I}(x)$ on $N_xQ \coloneqq T_xQ^\perp$ whose pullback to $T_xX$ is the limit of a blowing up of the sequence $(A_i)$ around the point $x$. The main result of this article gives sufficient conditions under which this phenomenon *will* appear.

**Theorem 1.2.** *Let $(X, \Phi)$ be a compact* Spin(7)*−manifold. Suppose we are given:*

- *an (irreducible and) unobstructed* Spin(7)*−instanton $A_0$ on a $G$−bundle $E_0$ over $X$,*
- *an unobstructed Cayley submanifold $Q$ and*
- *an unobstructed Fueter section $\mathfrak{I}$ of an instanton moduli bundle $\mathfrak{M} \to Q$ associated with $Q$ and $E_0|_Q$.*

*Then there exists a constant $\Lambda > 0$ and a $G$−bundle $E$ together with a family of (irreducible and) unobstructed* Spin(7)*−instantons $(A_\lambda)_{\lambda \in (0,\Lambda]}$ on $E$. Moreover, as $\lambda$ tends to zero $A_\lambda$ converges to $A_0$ on the complement of $Q$ and at each point $x \in Q$ an ASD instanton in the equivalence class given by $\mathfrak{I}(x)$ bubbles off transversely.*

*Remark* 1.3. We define the concepts of instanton moduli bundles and Fueter sections thereof in Section 4. For now, it shall suffice to say that $\mathfrak{M}$ is a bundle of moduli spaces and a Fueter section of $\mathfrak{M}$ is a section which satisfies a non-linear p.d.e. similar to a Dirac equation.

Unobstructedness is best understood as a notion of being in general position; see Definition 2.27, Definition 2.39 and Definition 4.11.

The proof of Theorem 1.2 is based on combining a gluing construction with adiabatic limit techniques. The analysis involved is similar to unpublished work by Brendle on the Yang–Mills equation in higher dimension [Bre03b; Bre03a] and Pacard–Ritoré's work on the Allen–Cahn equation [PR03]. The basic ideas, which are discussed briefly at the beginning of Section 5 and Section 6, are quite simple; however, the reader should be warned that some of the precise technical details are quite delicate.

Theorem 1.2 can be used as a tool to construct examples of Spin(7)−instantons. A particularly interesting situation, where our result can be applied, is if $X$ has a suitable local $K3$ Cayley fibration.

**Theorem 1.4.** *Let $X$ be a compact* Spin(7)*−manifold with holonomy equal to* Spin(7)*. Suppose that $Q$ is a Cayley submanifold in $X$ which has self-intersection number zero, is diffeomorphic to a $K3$ surface whose induced metric is sufficiently close to a hyperkähler metric and suppose that the induced connection on $NQ$ is almost flat. Then there exists a 5−dimensional family of* Spin(7)*−instantons on a* SU(2)*−bundle $E$ over $X$ with $c_2(E) = \mathrm{PD}[Q]$.*

*Moreover, if $Q_1, \ldots, Q_k$ is a collection of $k$ disjoint Cayley submanifolds as above, then there exists a $(8k-3)$−dimensional family of* Spin(7)*−instantons on a* SU(2)*−bundle $E$ over $X$ with $c_2(E) = \sum_{i=1}^k \mathrm{PD}[Q_i]$.*

Here is a concrete example.

**Example 1.5.** Joyce [Joy00, Example 14.3.3] gives an example of a Spin(7)–manifold which contains two disjoint Cayley submanifolds $Q_1$ and $Q_2$ of the kind required by above. Applying Theorem 1.4 in this situation recovers the example of a Spin(7)–instanton described in Lewis' DPhil thesis [Lew98]. In fact, it produces examples with $c_2(E) = n\,\mathrm{PD}[Q_1] + m\,\mathrm{PD}[Q_2]$ for arbitrary $n, m \in \mathbf{N}$ by taking the $Q_3, \ldots$ to be slight perturbations of $Q_1$ and $Q_2$ (which exist because $X$ is locally fibred near $Q_1$ and $Q_2$).

Every Cayley submanifold as above gives rise to a local fibration of $X$ by Cayley submanifolds, see Proposition 2.43; hence, we can use Theorem 1.4 to produce large families of Spin(7)–instantons. This can be compared with the situation on negative definite four-manifolds [Tau82], in which one can construct ASD instantons concentrated around any finite number of points.

Let us end the introduction on a speculative remark. Suppose that $X$ is a compact Spin(7)–manifold together with a fibration $\pi\colon X \to B$ to a compact base whose generic fibre is a $K3$ Cayley submanifold. In view of the above one could hope (very optimistically) that one can show that the moduli space $\mathscr{M}$ of Spin(7)–instantons on the SU(2)–bundle $E$ obtained by applying Theorem 1.4 to a generic fibre of $\pi$ is smooth (or only mildly singular), 5–dimensional and can be compactified by adding $B$ to the boundary. Then we can use $\mathscr{M}$ to construct a cobordism between $B$ and the link of the singular set of $\mathscr{M}$ much as in the original proof of Donaldson's theorem [Don83]. In particular, if $\mathscr{M} \cup B$ is smooth and compact, then $B$ is null-cobordant and, hence, $\sigma(B) = 0$. Although there are currently no known examples of Spin(7)–manifolds with (singular) $K3$ Cayley fibrations, the above might serve as an indication of what could be achieved using gauge theory on Spin(7)–manifolds.

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## 2 Review of Spin(7)–geometry

We begin with a crash course in Spin(7)–geometry, touching upon the basic concepts and facts relevant for this article. For a more thorough and comprehensive discussion we refer the reader to Joyce's book [Joy00], specifically Chapter 10.

### 2.1 Spin(7)–manifolds

In this section we approach Spin(7)–geometry by thinking of the 4–form $\Phi$, and not the metric, as the defining structure. However, both points of view are essentially equivalent.

**Definition 2.1.** A 4–form $\Phi$ on an 8–dimensional vector space $W$ is called **admissible** if there exists a basis of $W$ in which it is identified with the 4–form $\Phi_0$ on $\mathbf{R}^8$ defined by

$$\begin{aligned}\Phi_0 := {} & e^{0123} - e^{0145} - e^{0167} - e^{0246} + e^{0257} - e^{0347} \\ & + e^{4567} - e^{2367} - e^{2345} - e^{1357} + e^{1346} - e^{1256}.\end{aligned} \tag{2.2}$$

Here we denote the standard basis of $(\mathbf{R}^8)^*$ by $(e^0, \dots, e^7)$. The space of admissible forms on $W$ is denoted by $\mathscr{A}(W)$.

*Remark* 2.3. An intrinsic characterisation of admissible forms can be found in [SW17, Theorem 7.4 and Definition 7.5].

We use the following slightly unconventional definition, see Remark 2.13 for the relation with the usual definition.

**Definition 2.4.** Spin(7) is the subgroup of $\mathrm{GL}(\mathbf{R}^8)$ preserving the 4–form $\Phi_0$ defined in (2.2).

**Definition 2.5.** A Spin(7)–**structure** on an 8–dimensional manifold $X$ is an admissible 4–form $\Phi \in \Gamma(\mathscr{A}(TX)) \subset \Omega^4(X)$. An 8–manifold together with a Spin(7)–structure is called an **almost** Spin(7)–**manifold**.

**Proposition 2.6** ([SW17, Theorem 9.1 and Theorem 7.4]). Spin(7) *is a simple, compact, connected and simply connected Lie group of dimension* 21. Spin(7) *is a subgroup of* SO(8)*.*

It follows that each almost Spin(7)–manifold is canonically equipped with a metric $g_\Phi$ and an orientation.

**Definition 2.7.** Let $(X, \Phi)$ be an almost Spin(7)–manifold. The **torsion** of the Spin(7)–structure $\Phi$ is defined to be

$$\nabla_{g_\Phi}\Phi.$$

If $\nabla_{g_\Phi}\Phi = 0$, then $\Phi$ is called **torsion-free** and $(X, \Phi)$ is called a Spin(7)–**manifold**.

Compact Spin(7)–manifolds with $\mathrm{Hol}(g_\Phi) = \mathrm{Spin}(7)$ are difficult to come by. Joyce has developed two construction techniques, which yield a good number of examples, see [Joy96; Joy99; Joy00].

A very simple example of a Spin(7)–manifold is $(\mathbf{R}^8, \Phi_0)$. We will use this as a local model and it will be useful to realise it as a special case of the following examples.

**Example 2.8.** If $(S, \omega_1, \omega_2, \omega_3)$ and $(T, \mu_1, \mu_2, \mu_3)$ are a pair of hyperkähler surfaces, then $(S \times T, \Phi)$ with

$$\Phi := \mathrm{vol}_S + \mathrm{vol}_T - \sum_{i=1}^{3} \omega_i \wedge \mu_i \tag{2.9}$$

is a Spin(7)–manifold.

**Example 2.10.** If $(Y, \phi)$ is a $\mathrm{G}_2$–manifold, then $(\mathbf{R} \times Y, \Phi)$ with

$$\Phi := \mathrm{d}t \wedge \phi + \psi \tag{2.11}$$

and $\psi := \Theta(\phi) = *_\phi \phi$ is a Spin(7)–manifold.

Taking $S = T = \mathbf{R}^4$ with $(\omega_1, \omega_2, \omega_3) = (\mu_1, \mu_2, \mu_3)$ a positive orthonormal basis of $\Lambda^+ := \Lambda^+(\mathbf{R}^4)^*$ in Example 2.8 and $Y = \mathbf{R}^7$ with $\phi = e^{123} - e^{145} - e^{167} - e^{246} + e^{257} - e^{347}$ in Example 2.10 both recover $(\mathbf{R}^8, \Phi_0)$.

The following linear algebra fact can be seen as the Spin(7)–analogue of $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$, the splitting into (anti)-self-dual two-forms on $\mathbf{R}^4$.

**Proposition 2.12** ([SW17, Theorem 9.5]). *Let $\Phi$ be an admissible 4–form on an 8–dimensional vector space $W$. Then $\Lambda^2 W^*$ splits as follows*

$$\Lambda^2 W^* = \Lambda^2_7 \oplus \Lambda^2_{21}$$

*with*

$$\Lambda^2_7 := \{\alpha : *(\alpha \wedge \Phi) = 3\alpha\} \quad \textit{and}$$
$$\Lambda^2_{21} := \{\alpha : *(\alpha \wedge \Phi) = -\alpha\} \cong \mathfrak{spin}(7).$$

*Remark* 2.13. The action of Spin(7) on $\Lambda^2_7$ gives rise to a double cover Spin(7) $\to$ SO(7); hence, the above definition of Spin(7) agrees with the usual definition as the universal cover of SO(7).

Proposition 2.12 induces an analogous splitting of $\Lambda^2 T^*X$ for every almost Spin(7)–manifold. By slight abuse of notation we will denote the corresponding summands by $\Lambda^2_d$ as well. We denote the projection onto $\Lambda^2_d$ by

$$\pi_d \colon \Lambda^2 T^*X \to \Lambda^2_d.$$

The following propositions are easy to check via straight-forward computation.

**Proposition 2.14**. *If $\Phi$ is the admissible 4–form on a product of two quaternionic lines $S$ and $T$ defined as in (2.9), then $\Lambda^2_7$ splits as*

$$\Lambda^2_7 = \Lambda^2_3 \oplus \Lambda^2_4,$$

*where*

$$\Lambda^2_3 = \bigoplus_{i=1}^{3} \langle \omega_i - \mu_i \rangle \quad \textit{and}$$
$$\Lambda^2_4 = \left\{ \langle L\cdot, \cdot \rangle \in S^* \otimes T^* : L \in \mathrm{Hom}(S, T) \textit{ satisfying } \sum_i J_i L I_i = -3L \right\}.$$

*Here $I_i$ and $J_i$ denote the complex structures on $S$ and $T$ corresponding to $\omega_i$ and $\mu_i$ respectively.*

**Proposition 2.15**. *If $\Phi$ is the admissible 4–form on a product of $\mathbf{R}$ with a 7–dimensional vector space $V$ equipped with a non-degenerate 3–form $\phi$ defined as in (2.11), then $\Lambda^2_7$ can be written as*

$$\Lambda^2_7 = \{\mathrm{d}t \wedge v^* + i(v)\phi : v \in V\}$$

*and* $\Lambda^2_{21}$ *can be written as*

$$\Lambda^2_{21} = \{\mathrm{d}t \wedge *_V(\alpha \wedge \psi) - \alpha : \alpha \in \Lambda^2 V^*\}$$

*where* $\psi := \Theta(\phi)$.

**Proposition 2.16** ([Joy00, Proposition 10.5.6]). *If* $\Phi$ *is a* Spin(7)*–structure on* $X$*, then* $X$ *is spin and has a canonical spin structure with*

$$\$^+ = \Lambda^0 \oplus \Lambda^2_7 \quad \textit{and} \quad \$^- = \Lambda^1_8.$$

*Moreover, if* $\Phi$ *is torsion-free, then* $X$ *admits a non-trivial parallel spinor.*

## 2.2 Spin(7)–instantons

Throughout the remainder of this section we fix a Spin(7)–manifold $(X, \Phi)$. Also, let $G$ be a (compact semi-simple) Lie group and $E$ a $G$–bundle over a Spin(7)–manifold.

**Definition 2.17.** A connection $A \in \mathscr{A}(E)$ on $E$ is called a Spin(7)**–instanton** if it satisfies

$$*(F_A \wedge \Phi) = -F_A$$

or equivalently

$$\pi_7(F_A) = 0. \tag{2.18}$$

This equation originated in the physics literature [CDFN83] and was introduced to a wider mathematical audience by Donaldson–Thomas [DT98, Section 3]. Spin(7)–instantons were the topic of Lewis' DPhil thesis [Lew98]; in particular, he proposed the construction of one non-trivial example on a SU(2)–bundle over a Spin(7)–manifold with full holonomy Spin(7), cf. Section 8. Recently, a construction for Spin(7)–instantons on Spin(7)–manifolds arising from [Joy99] was given by Tanaka [Tan12].

For us the following "trivial" examples will play an important role.

**Example 2.19.** In the situation of Example 2.8 if $I$ is an ASD instanton over $T$, then its pullback to $S \times T$ is a Spin(7)–instanton.

**Example 2.20.** In the situation of Example 2.10 if $A$ is a $G_2$–instanton over $Y$, then its pullback to $\mathbf{R} \times Y$ is a Spin(7)–instanton.

If $A \in \mathscr{A}(E)$ is a connection on $E$, we define $L_A \colon \Omega^1(X, \mathfrak{g}_E) \to \Omega^0(X, \mathfrak{g}_E) \oplus \Omega^2_7(X, \mathfrak{g}_E)$ by

$$L_A(a) := \left(\mathrm{d}_A^* a, \pi_7(\mathrm{d}_A a)\right). \tag{2.21}$$

This is the linearisation of (2.18) supplemented with the Coulomb gauge condition; it also agrees with the negative Dirac operator on $X$ twisted by $\mathfrak{g}_E$.

*Remark* 2.22. In the situation of Example 2.20 denote the pullback of $A$ by $\mathbf{A}$. Identifying $\Omega^1(X, \mathfrak{g}_E)$ with $\Omega^0(\mathbf{R}\times Y, \mathbf{R}\oplus p_2^* T^* Y)$ and $\Omega^0(X, \mathfrak{g}_E)\oplus\Omega^2_7(X, \mathfrak{g}_E)$ with $\Omega^0(\mathbf{R}\times Y, \mathbf{R}\oplus p_2^* T^* Y)$ using Proposition 2.15, we can write

$$L_{\mathbf{A}} = \partial_t - \begin{pmatrix} 0 & \mathrm{d}_A^* \\ \mathrm{d}_A & *_Y(\psi \wedge \mathrm{d}_A) \end{pmatrix}.$$

Note that the second term is nothing but the linearisation of the $G_2$–instanton equation at $A$, see [Wal13a, Section 3].

**Proposition 2.23.** *If $A$ is a* Spin(7)*–instanton, then there is an open subset $U \subset \ker L_A$ and a smooth map $\kappa\colon U \to \operatorname{coker} L_A$ such that the moduli space of* Spin(7)*–instantons near $A$ is homeomorphic to $\kappa^{-1}(0)/\Gamma_A$. Here $\Gamma_A \subset \mathscr{G}(E)$ is the group of gauge transformations fixing $A$. The index of $L_A$ is given by*

$$\begin{aligned} \operatorname{index} L_A &= \dim \mathfrak{g} \cdot (b^1 - b^0 - b^2_7) \\ &\quad + \frac{1}{24}\int_X p_1(X)p_1(\mathfrak{g}_E) - \frac{1}{12}\int_X p_1(\mathfrak{g}_E)^2 - 2p_2(\mathfrak{g}_E). \end{aligned} \tag{2.24}$$

*If $E$ is a* SU$(r)$*–bundle, then*

$$\begin{aligned} \operatorname{index} L_A &= (r^2-1)(b^1 - b^0 - b^2_7) \\ &\quad - \frac{r}{12}\int_X p_1(X)c_2(E) - \int_X \left(1+\frac{r}{6}\right)c_2(E)^2 - \frac{r}{3}c_4(E). \end{aligned} \tag{2.25}$$

*Here $b^2_7$ is the refined second Betti number corresponding to $\Lambda^2_7$ in Proposition 2.12, see [Joy00, Definition 10.6.3].*

*Remark* 2.26. The index formula given by Lewis [Lew98, Theorem 3.2] is incorrect. He mistakenly couples the Dirac operator to $E$ instead of $\mathfrak{g}_E$.

*Proof of the index formula.* The existence of the Kuranishi map $\kappa$ is standard (see, e.g., [DK90, Section 4.2]); we only prove the index formula. Using

$$\mathrm{ch}_2(\mathfrak{g}_E\otimes\mathbf{C}) = -c_2(\mathfrak{g}_E\otimes\mathbf{C}) \quad \text{and}$$
$$\mathrm{ch}_4(\mathfrak{g}_E\otimes\mathbf{C}) = \frac{1}{12}(c_2(\mathfrak{g}_E\otimes\mathbf{C})^2 - 2c_4(\mathfrak{g}_E\otimes\mathbf{C}))$$

the index theorem yields

$$\begin{aligned}\operatorname{index} L_A &= -\int_X \hat{A}(X)\operatorname{ch}(\mathfrak{g}_E \otimes \mathbf{C}) \\ &= -\int_X \left(1 - \frac{p_1(X)}{24} + \frac{7p_1(X)^2 - 4p_2(X)}{5670}\right) \\ &\qquad \cdot \left(\dim \mathfrak{g} + p_1(\mathfrak{g}_E) + \frac{p_1(\mathfrak{g}_E)^2 - 2p_2(\mathfrak{g}_E)}{12}\right) \\ &= \dim \mathfrak{g} \cdot (b^1 - b^0 - b^2_7) \\ &\quad + \frac{1}{24}\int_X p_1(X)p_1(\mathfrak{g}_E) - \frac{1}{12}\int_X p_1(\mathfrak{g}_E)^2 - 2p_2(\mathfrak{g}_E).\end{aligned}$$

In the last step, we applied the identity derived up to this point with $E$ the trivial line bundle to obtain

$$b^0 - b^1 + b^2_7 = \int_X \frac{7p_1(X)^2 - 4p_2(X)}{5670}.$$

If $E$ is a $\mathrm{SU}(r)$–bundle, then we can use

$$\operatorname{ch}(\mathfrak{g}_E \otimes \mathbf{C}) = \operatorname{ch}(E \otimes E^*) - 1 = r^2 - 1 - 2rc_2(E) + \frac{6+r}{6}c_2(E)^2 - \frac{r}{3}c_4(E). \qquad \square$$

**Definition 2.27.** If $A$ is a $\mathrm{Spin}(7)$–instanton, then we denote by

$$\begin{aligned}\mathcal{H}^0_A &:= \ker L_A^* \cap \Omega^0(X, \mathfrak{g}_E), \\ \mathcal{H}^1_A &:= \ker L_A \cap \Omega^1(X, \mathfrak{g}_E) \quad \text{and} \\ \mathcal{H}^2_{7;A} &:= \ker L_A^* \cap \Omega^2_7(X, \mathfrak{g}_E)\end{aligned}$$

the **space of infinitesimal automorphisms**, the **space of infinitesimal deformations** and the **space of infinitesimal obstructions** respectively. $A$ is called **irreducible** if $\mathcal{H}^0_A = 0$ and **unobstructed** if $\mathcal{H}^2_{7;A} = 0$.

*Remark* 2.28. The above spaces can also be seen as the cohomology groups of the deformation complex

$$0 \to \Omega^0(X, \mathfrak{g}_E) \xrightarrow{\mathrm{d}_A} \Omega^1(X, \mathfrak{g}_E) \xrightarrow{\pi_7 \circ \mathrm{d}_A} \Omega^2_7(X, \mathfrak{g}_E) \to 0.$$

## 2.3 Cayley submanifolds

**Theorem 2.29** (Harvey and Lawson [HL82, Chapter IV Theorem 1.24]). *If $(X, \Phi)$ is a* $\mathrm{Spin}(7)$*–manifold, then $\Phi$ is a calibration. Moreover, $Q \subset X$ is calibrated by $\Phi$ if and only of at each point $x \in Q$ there exists a basis $(e_0, \dots, e_7)$ of $T_xX$ with respect to which $\Phi$ is given by (2.2) and $(e_0, \dots, e_3)$ is a positive basis of $T_xQ$.*

*Remark* 2.30. Recall that a differential $k$–form $\alpha$ on a Riemannian manifold $(M, g)$ is called a calibration if it is closed and has comass at most 1, that is, $\mathrm{d}\alpha = 0$ and for all orthogonal subset $\{e_1, \dots, e_k\} \subset T_x M$ we have $\alpha(e_1, \dots, e_k) \leqslant 1$.

**Definition 2.31**. Let $(X, \Phi)$ be a Spin(7)–manifold. Then $\Phi$ is called the **Cayley calibration**. An oriented 4–dimensional submanifold $Q \subset X$ that is calibrated by $\Phi$ is called a **Cayley submanifold**.

If $Q \subset (X, \Phi)$ is a Cayley submanifold, then it follows from Theorem 2.29 that there is a natural identification

$$\Lambda^+ T^* Q \cong \Lambda^+ N^* Q. \tag{2.32}$$

We define a subbundle $\mathrm{Hom}_\Phi(TQ, NQ) \subset \mathrm{Hom}(TQ, NQ)$ by decreeing that $L \in \mathrm{Hom}_\Phi(TQ, NQ)$ if and only if

$$\sum_i I_i L I_i = -3L,$$

cf. Proposition 2.14. Here $I_i$ runs through a local orthonormal basis of $\Lambda^+ T^* Q \cong \Lambda^+ N^* Q$, which we can identify with subsets of $\mathfrak{so}(TQ)$ and $\mathfrak{so}(NQ)$. Up to multiplication by $\frac{1}{4}$

$$\gamma L := L - \sum_i I_i L I_i$$

defines a projection of $\mathrm{Hom}(TQ, NQ)$ onto $\mathrm{Hom}_\Phi(TQ, NQ)$.

**Definition 2.33**. The **Fueter operator** $F_Q\colon \Gamma(Q, NQ) \to \Gamma(Q, \mathrm{Hom}_\Phi(TQ, NQ))$ associated with $Q$ is defined by

$$F_Q(n) := \gamma(\bar{\nabla} n).$$

*Remark* 2.34. If $e_0$ is a vector in $TQ$, then one can compose $F_Q$ with evaluation on $e_0$ to obtain the operator

$$\mathrm{ev}_{e_0} \circ F_Q(n) = \bar{\nabla}_{e_0} n - \sum_i I_i \bar{\nabla}_{e_i} n$$

where $e_i := I_i e_0$. It is therefore appropriate to think of $F$ as a Dirac-type operator.

*Remark* 2.35. Suppose that $Q$ is spin and $\mathfrak{s}$ is a spin structure on $Q$. Then the normal bundle $NQ$ is also spin, since $X$ is; moreover, there is a spin structure $\mathfrak{u}$ on $NQ$ such that $\$^+_Q = \$^+_{NQ}$ because of (2.32). If we set $U := \$^-_{NQ}$, then it can be seen that $\mathrm{Re}(\$^+_Q \otimes U) = NQ$, $\mathrm{Re}(\$^-_Q \otimes U) = \mathrm{Hom}_\Phi(TQ, NQ)$ and that $F_Q$ agrees with the twisted Dirac operator $\not{D}\colon \Gamma(\mathrm{Re}(\$^+_Q \otimes U)) \to \Gamma(\mathrm{Re}(\$^-_Q \otimes U))$. For more details we refer the reader to [McL98, Section 6] and [Hay12, Section 3.2].

**Theorem 2.36** (McLean [McL98, Section 6]). *Let $(X, \Phi)$ be a compact* Spin(7)*–manifold and let $Q \subset X$ be a compact Cayley submanifold. Then there is an open subset $\mathcal{O} \subset \ker F_Q$ and a smooth* *map $\kappa\colon \mathcal{O} \longrightarrow \operatorname{coker} F_Q$ such that the moduli space of Cayley submanifolds near $Q$ is homeomorphic to $\kappa^{-1}(0)$. The index of $F_Q$ is given by*

$$\operatorname{index} F_Q = \frac{\sigma(Q) + \chi(Q)}{2} - [Q] \cdot [Q]. \tag{2.37}$$

*Here $\sigma(Q) := b^+(Q) - b^-(Q)$ denotes the signature of $Q$.*

*Remark* 2.38. The index formula given by Joyce [Joy00, Equation (10.32)] is incorrect and likely a misprint as it also contradicts his remarks at the bottom of p. 267.

**Definition 2.39.** A Cayley submanifold $Q$ is called **unobstructed** if $F_Q$ is surjective.

*Proof of the index formula.* We can assume that $Q$ is spin. Then the index of $F_Q$ agrees with the index of the twisted Dirac operator $\not{D}_U$. By the Atiyah–Singer index theorem

$$\operatorname{index} \not{D}_U = \int_Q \hat{A}(Q)\mathrm{ch}_2(U) = -\frac{1}{4}\sigma(Q) - \int_Q c_2(U).$$

This is the formula given by McLean. In order to obtain a more useful expression, we make use of the fact that if $E$ and $F$ are a pair of SU(2)–bundles over a 4–manifold and $V = \mathrm{Re}(E \otimes F)$, then

$$\begin{aligned} e(V) &= c_2(F) - c_2(E) \quad \text{and} \\ p_1(V) &= -2(c_2(E) + c_2(F)). \end{aligned} \tag{2.40}$$

To see this, note that there must be universal formulas of the form $e(V) = \alpha(c_2(E) - c_2(F))$ and $p_1(V) = \beta(c_2(E) + c_2(F))$, because $e(V)$ changes sign when $E$ and $F$ are interchanged since this changes the orientation on $V$, and $p_1(V)$ is independent of the order of $E$ and $F$. The constants can be determined by a simple explicit computation for the spin bundles over $K3$. From these formulae it follows that

$$c_2(U) = -\frac{1}{4}\left(p_1(NQ) - 2e(NQ)\right).$$

To compute $p_1(NQ)$, we combine $\$^+_Q = \$^+_{NQ}$ and (2.40) to obtain

$$p_1(NQ) + 2e(NQ) = -4c_2(\$^+_{NQ}) = -4c_2(\$^+_Q) = p_1(Q) + 2e(Q); \tag{2.41}$$

hence,

$$\int_Q p_1(NQ) = 3\sigma(Q) + 2\chi(Q) - 2[Q] \cdot [Q]. \tag{2.42}$$

Therefore,

$$\int_Q c_2(U) = -\frac{3}{4}\sigma(Q) - \frac{1}{2}\chi(Q) + [Q] \cdot [Q],$$

which implies the claimed index formula. □

**Proposition 2.43.** *Let $X$ be a compact* Spin(7)*–manifold. Suppose that $Q$ is a compact Cayley submanifold in $X$ which has self-intersection number zero, is diffeomorphic to a $K3$ surface whose induced metric is sufficiently close to a hyperkähler metric and suppose that the induced connection on $NQ$ is almost flat. Then $X$ is locally fibred by Cayley $K3$ surfaces near $Q$.*

*Proof.* Using the fact that $Q$ and hence $NQ$ is spin as well as (2.40) one can show that $NQ$ is trivial. The Fueter operator $F_Q$ thus agrees with the Dirac operator $\not{D}_U \colon \Gamma(\mathrm{Re}(\$^+ \otimes U)) \to \Gamma(\mathrm{Re}(\$^- \otimes U))$. On a hyperkähler $K3$ surface the untwisted Dirac operator $\not{D}$ is surjective, has a four-dimensional kernel, and every non-zero element of $\ker \not{D}$ is nowhere vanishing; hence, the same is true for $\not{D}_U$ because the metric on $Q$ is sufficiently close to a hyperkähler metric and the connection on $U$ is almost flat. The existence of the local fibration now follows from (the proof of) Theorem 2.36. □

# 3 Moduli spaces of ASD instantons over $\mathbf{R}^4$

This section is intended to remind the reader of some basic facts about ASD instantons over $\mathbf{R}^4$, all of which are completely classical and most of which can be found in Donaldson–Segal [DS11, Section 6.1].

Fix a $G$–bundle $E$ over $S^4 = \mathbf{R}^4 \cup \{\infty\}$. Denote by $M$ the moduli space of ASD instantons on $E$ framed over the point at infinity, i.e.,

$$M(E) := \{A \in \mathscr{A}(E) : F_A^+ = 0\}/\mathscr{G}_0.$$

Here $\mathscr{A}(E)$ denotes the space of connections on $E$ and

$$\mathscr{G}_0(E) := \{g \in \mathscr{G}(E) : g|_{E_\infty} = \mathrm{id}\}$$

denotes the based gauge group. These moduli spaces are smooth manifolds, because ASD instantons over $S^4$ are always unobstructed as a consequence of the Weitzenböck formula, see, e.g., [Tau82, Proposition 2.2]. By Uhlenbeck's removable singularities theorem [Uhl82b, Theorem 4.1] we can think of $M$ as a **moduli space of framed finite energy ASD instantons** on $\mathbf{R}^4$. In a suitable functional analytic setup incorporating decay conditions at infinity, see, e.g., [Tau83] or [Nak90], the infinitesimal deformation theory of a framed ASD instanton $I$ over $\mathbf{R}^4$ is governed by the linear operator $\delta_I \colon \Omega^1(\mathbf{R}^4, \mathfrak{g}_E) \to \Omega^0(\mathbf{R}^4, \mathfrak{g}_E) \oplus \Omega^+(\mathbf{R}^4, \mathfrak{g}_E)$ defined by

$$\delta_I a := (\mathrm{d}_I^* a, \mathrm{d}_I^+ a). \tag{3.1}$$

From the work of Taubes [Tau83] it is known that $\delta_I$ is always surjective and that its kernel lies in $L^2$. More precisely, we have the following result whose proof can be found, e.g., in [Wal13a, Proposition 5.10].

**Proposition 3.2.** *Let $E$ be a $G$–bundle over $\mathbf{R}^4$ and let $I \in \mathscr{A}(E)$ be a finite energy ASD instanton on $E$. Then the following holds.*

1. *If $a \in \ker \delta_I$ decays to zero at infinity, that is to say $\lim_{r\to\infty} \sup_{\partial B_r(0)} |a| = 0$, then $|\nabla^k a| = O(r^{-3-k})$ for $k \geqslant 0$. Here $r\colon \mathbf{R}^4 \to [0,\infty)$ denotes the radius function $r(x) := |x|$.*

2. *If $(\xi,\omega) \in \ker \delta_I^*$ decays to zero at infinity, then $(\xi,\omega) = 0$.*

In particular, this implies $M$ can be equipped with the $L^2$–metric arising from the standard metric on $\mathbf{R}^4$. Any self-dual 2–form $\omega \in S(\Lambda^+)$ of unit length, determines a complex structure $J_\omega$ on $\mathbf{R}^4$ via $\Lambda^2(\mathbf{R}^4)^* \cong \mathfrak{so}(4)$. This makes $\mathbf{R} \oplus \Lambda^+$ into an algebra, which is abstractly isomorphic to the quaternions $\mathbf{H}$. A key fact is that $\delta_I$ commutes with the action of this algebra [Tau83, Proof of Theorem 3.2]; hence, $T_{[I]}M = \ker \delta_I \subset \Omega^1(\mathbf{R}^4, \mathfrak{g}_E)$ is preserved.

**Proposition 3.3.** *The $L^2$–metric and the complex structures $\{J_\omega : \omega \in S(\Lambda^+)\}$ define a hyperkähler structure on $M$.*

This structure is SO(4)–equivariant. $M$ carries an action of $\mathbf{R}^4 \rtimes \mathbf{R}^+$ where $\mathbf{R}^4$ acts by translation and $\mathbf{R}^+$ acts by dilation, i.e., by pullback via $s_\lambda$ where

$$s_\lambda(x) := \lambda x$$

for $\lambda \in \mathbf{R}^+$. Since the centre of mass of the measure $|F_I|^2\mathrm{vol}$ is equivariant with respect to the $\mathbf{R}^4$–action, we can write

$$M = M^\circ \times \mathbf{R}^4$$

where $M^\circ$ is the space of instantons centred at zero. The action of $\Lambda^+ \subset \Lambda^2 \cong \mathfrak{so}(4)$ preserves this product structure and $\Lambda^+$ acts on the factor $\mathbf{R}^4$ in the usual way.

**Example 3.4.** If $E$ is the unique SU(2)–bundle over $S^4$ with $c_2(E) = 1$, then $E$ carries a single ASD instanton $I$, commonly called "the one-instanton", unique up to scaling, translation and changing the framing at infinity. We can naturally write the corresponding moduli space as

$$M = M^\circ \times \mathbf{R}^4 = (\mathrm{Re}(\mathrm{Hom}(\mathbf{C}^2, \mathbb{S}^+))\backslash\{0\})/\mathbf{Z}_2 \times \mathbf{R}^4.$$

Here $\mathbb{S}^+$ is the positive spin representation associated with $\mathbf{R}^4$ and $\mathbf{C}^2$ has to be thought of as a SU(2) representation. In this situation both $\mathbf{C}^2$ and $\mathbb{S}^+$ have canonical quaternionic structures and thus $\mathrm{Hom}(\mathbf{C}^2, \mathbb{S}^+)$ inherits a real structure. The real part are simply the quaternionic-linear homomorphisms. The reader can consult [DK90, Section 3.1] for a more extensive discussion.

**Example 3.5.** In general, if $E$ is an SU(2)–bundle over $S^4$, then $M$ can be understood rather explicitly in terms of the ADHM construction [DK90, Section 3.3].

**Proposition 3.6.** *There exists a $G$–bundle $\mathbf{E}$ over $M \times S^4$ together with a framing $\mathbf{E}|_{M\times\{\infty\}} \to G$ and a tautological connection $\mathbf{A} \in \mathscr{A}(\mathbf{E})$ on $\mathbf{E}$ such that:*

- *$\mathbf{E}|_{\{[I]\}\times S^4} \cong E$ and*
- *$\mathbf{A}$ restricted to $\{[I]\} \times \mathbf{R}^4$ is equivalent to $[I]$ via $\mathscr{G}_0(E)$.*

*If we decompose the curvature of the tautological connection* $\mathbf{A}$ *over* $M\times\mathbf{R}^4$ *according to the bi-grading on* $\Lambda^*T^*(M\times\mathbf{R}^4)$ *induced by* $T(M\times\mathbf{R}^4)=\pi_1^*TM\oplus\pi_2^*T\mathbf{R}^4$*, then its components satisfy the following:*

- $F_{\mathbf{A}}^{2,0}=-2\Delta_I^{-1}\langle[a,b]\rangle$.
- $F_{\mathbf{A}}^{1,1}\in\Gamma(\mathrm{Hom}(\pi_1^*TM,\pi_2^*T\mathbf{R}^4\otimes\mathfrak{g}_{\mathbf{E}}))$ *at* $([I],x)$ *is the evaluation of* $a\in T_{[I]}M=\ker\delta_I$ *at* $x$*; in particular, it is* $(\mathbf{R}\oplus\Lambda^+)$*–linear.*
- $F_{\mathbf{A}}^{0,2}\in\Gamma(\pi_2^*\Lambda^-(\mathbf{R}^4)^*\otimes\mathfrak{g}_{\mathbf{E}})$.

*Proof sketch.* There is a tautological connection on the pullback of $E$ to $\mathscr{A}(E)\times S^4$. It is flat in the $\mathscr{A}(E)$–direction. It is $\mathscr{G}_0$–equivariant, but not basic; hence, induces a connection on $M\times S^4$ after choosing a connection on $\mathscr{A}(E)\to\mathscr{A}(E)/\mathscr{G}_0(E)$. We chose the connection given whose horizontal distribution is given by the Coulomb gauge with respect to the metric on $\mathbf{R}^4$; that is, the connection with connection 1–form $\theta(a)=\Delta_I^{-1}\mathrm{d}_I^*a$ for $a\in T_I\mathscr{A}=\Omega^1(\mathbf{R}^n,\mathfrak{g}_E)$. The $(2,0)$–component of the curvature of $\mathbf{A}$ arises from the curvature of this connection. The second two bullets are tautological. □

# 4 Fueter sections of instanton moduli bundles over Cayley submanifolds

We now discuss models of Spin(7)–instantons which are highly concentrated near a Cayley submanifold $Q$ in a Spin(7)–manifold $(X,\Phi)$.

## 4.1 The flat model

We begin with studying the situation on $\mathbf{R}^8=\mathbf{R}^4\times\mathbf{R}^4$. Fix a basis $(\omega_1,\omega_2,\omega_3)$ of $\Lambda^+:=\Lambda^+(\mathbf{R}^4)^*$ satisfying

$$\omega_i\wedge\omega_j=2\delta_{ij}\mathrm{vol}$$

with vol denoting the standard volume form on $\mathbf{R}^4$. Set $J_i:=J_{\omega_i}$. The standard Spin(7)–structure $\Phi$ on $\mathbf{R}^8=\mathbf{R}^4\times\mathbf{R}^4$ can be written as

$$\Phi:=\pi_1^*\mathrm{vol}+\pi_2^*\mathrm{vol}-\sum_{i=1}^3\pi_1^*\omega_i\wedge\pi_2^*\omega_i.$$

It is a straight-forward computation, using Proposition 2.14, to check that:

**Proposition 4.1.** *A connection* $A$ *on a* $G$*–bundle* $\pi_2^*E$ *is a* Spin(7)*–instanton if and only if:*

- $\left(F_A^{2,0}\right)^+=\left(F_A^{0,2}\right)^+$ *and*

- $F_A^{1,1}$ *thought of as map* $L\colon T\mathbf{R}^4 \to \Omega^1(\mathbf{R}^4, \mathfrak{g}_E)$ *satisfies*

$$L - \sum_{i=1}^{3} J_i \circ L \circ J_i = 0. \tag{4.2}$$

Let $U$ be an open subset of $\mathbf{R}^4$. Suppose $A_i$ is a sequence of Spin(7)–instantons on $U \times \mathbf{R}^4$ on $\pi_2^* E$ concentrating along $U \times \{0\}$ and $(\lambda_i)$ is a null-sequence such that $[(x, y) \mapsto (x, \lambda_i y)]^* A_i$ converges to $A$. Then it follows from (4.1) that

- $\left(F_A^{0,2}\right)^+ = 0$ and
- $F_A^{1,1}$ satisfies (4.2).

By the first bullet, such an $A$ determines a map $\mathfrak{I}\colon U \to M$ and by the second bullet this map satisfies the Fueter equation

$$\nabla\mathfrak{I} - \sum_{i=1}^{3} J_i \circ \nabla\mathfrak{I} \circ J_i = 0.$$

Up to gauge equivalence, $A$ can be reconstructed from $\mathfrak{I}$ by pulling back the tautological connection on $M \times \mathbf{R}^4$ via $\mathfrak{I} \times \mathrm{id}_{\mathbf{R}^4}$. Thus, Fueter maps into $M$ can serve as models for highly concentrated Spin(7)–instantons on $U \times \mathbf{R}^4$.

## 4.2 The model on $NQ$

We now globalise the above discussion. Fix a moduli space $M$ of framed ASD instantons on a $G$–bundle $E$ over $\mathbf{R}^4$, as in Section 3 and denote by $E_\infty$ a $G$–bundle over $Q$ together with a connection $A_\infty$.

**Definition 4.3.** The **instanton moduli bundle** $\mathfrak{M} \to Q$ associated with $Q$, $E_\infty$ and $M$ is defined by

$$\mathfrak{M} := (\mathrm{Fr}(NQ) \times E_\infty) \times_{\mathrm{SO}(4)\times G} M.$$

**Example 4.4.** If $M = (\mathrm{Re}(\mathrm{Hom}(\mathbf{C}^2, \$^+))\backslash\{0\})/\mathbf{Z}_2 {\times} \mathbf{R}^4$, as in Example 3.4, and we pick spin structures $\mathfrak{s}$ and $\mathfrak{u}$ as in Remark 2.35, then

$$\mathfrak{M} = (\mathfrak{s} \times \mathfrak{u} \times E_\infty) \times_{\mathrm{Spin}(4)\times G} M = (\mathrm{Re}(\mathrm{Hom}(\mathbf{C}^2, \$^+))\backslash\{0\})/\mathbf{Z}_2 \times NQ.$$

Denote by $N_\infty Q := \mathrm{Fr}(NQ) \times_{\mathrm{SO}(4)} S^4$ the sphere-bundle obtained from $NQ$ by adjoining a section at infinity.

**Theorem 4.5** (Donaldson–Segal [DS11] and Haydys [Hay12]). *To each section $\mathfrak{I} \in \Gamma(\mathfrak{M})$ we can assign a $G$–bundle $E = E(\mathfrak{I})$ over $N_\infty Q$ together with a connection $I = I(\mathfrak{I})$ and a framing $f\colon E|_\infty \to E_\infty$ such that:*

- *For each $x \in Q$ the restriction of $I$ to $N_x Q$ represents $\mathfrak{I}(x)$.*

- *The framing $f$ identifies the restriction of $I$ to the section at infinity with $A_\infty$.*

We set $I_\lambda \coloneqq I(s^*_{1/\lambda}\mathfrak{I})$ and impose the condition that

$$\lim_{\lambda\to 0} s^*_\lambda \pi^0_7(F_{I_\lambda}) = 0 \tag{4.6}$$

where $\pi^0_7$ denotes the zeroth order Taylor expansion of $\pi_7$ off $Q$. As before, this condition can be phrased in terms of a p.d.e. on $\mathfrak{I}$. Define the vertical tangent bundle $V\mathfrak{M}$ to $\mathfrak{M}$ by

$$V\mathfrak{M} \coloneqq (\mathrm{Fr}(NQ) \times E_\infty) \times_{\mathrm{SO}(4)\times G} TM.$$

If $\mathfrak{I}$ is a section of $\mathfrak{M}$, then $\Phi$ selects a subbundle

$$\mathrm{Hom}_\Phi(TQ, \mathfrak{I}^*V\mathfrak{M}) \subset \mathrm{Hom}(TQ, \mathfrak{I}^*V\mathfrak{M})$$

and there is a "Clifford multiplication" map

$$\gamma\colon\ \mathrm{Hom}(TQ, \mathfrak{I}^*V\mathfrak{M}) \to \mathrm{Hom}_\Phi(TQ, \mathfrak{I}^*V\mathfrak{M})$$

as discussed before. Moreover, the connections on $NQ$ and $E_\infty$ induce a connection on $\mathfrak{M}$ assigning to each section $\mathfrak{I}$ its covariant derivative $\nabla\mathfrak{I} \in \Omega^1(\mathfrak{I}^*V\mathfrak{M})$.

**Definition 4.7.** The **Fueter operator** $\mathfrak{F} = \mathfrak{F}_\Phi$ associated with $\mathfrak{M}$ is defined by

$$\mathfrak{I} \in \Gamma(\mathfrak{M}) \mapsto \mathfrak{F}_\Phi\mathfrak{I} \coloneqq \gamma(\nabla\mathfrak{I}) \in \Gamma(\mathrm{Hom}_\Phi(TQ, \mathfrak{I}^*V\mathfrak{M})).$$

A section $\mathfrak{I} \in \Gamma(\mathfrak{M})$ is called a **Fueter section** if it satisfies

$$\mathfrak{F}\mathfrak{I} = 0.$$

**Example 4.8.** If $M$ is as in Example 3.4, then the Fueter operator $\mathfrak{F}$ lifts to the twisted Dirac operator

$$\not{D}\colon\ \Gamma(\mathrm{Re}(\mathrm{Hom}(E_\infty, \$^+) \oplus \$^+ \otimes U)) \to \Gamma(\mathrm{Re}(\mathrm{Hom}(E_\infty, \$^-) \oplus \$^- \otimes U)).$$

The Fueter operator $\mathfrak{F}$ is compatible with the product structure on

$$\mathfrak{M} = \mathring{\mathfrak{M}} \times NQ$$

corresponding to $M = M^\circ \times \mathbf{R}^4$ with $M^\circ$ denoting the space of instantons centred at zero. Its restriction to the second factor is given by the Fueter operator $F_Q$ associated with $Q$.

**Theorem 4.9** (Donaldson–Segal [DS11] and Haydys [Hay12]). *If $\mathfrak{I} \in \Gamma(\mathfrak{M})$, then we can identify $\Gamma(\mathrm{Hom}_\Phi(TQ, \mathfrak{I}^*V\mathfrak{M}))$ with a subspace of $\Omega^2\left(NQ, \mathfrak{g}_{E(\mathfrak{I})}\right)$. With respect to this identification we have the identity*

$$\mathfrak{F}\mathfrak{I} = \pi^0_7\left(F^{1,1}_{I(\mathfrak{I})}\right).$$

*In particular, $I(\mathfrak{I})$ satisfies equation (4.6) if and only if $\mathfrak{I}$ is a Fueter section.*

**Definition 4.10.** The **linearised Fueter operator**

$$F_{\mathfrak{I}} = F_{\mathfrak{I},\Phi}\colon\ \Gamma(\mathfrak{I}^*V\mathfrak{M}) \to \Gamma(\mathrm{Hom}_\Phi(TQ, \mathfrak{I}^*V\mathfrak{M}))$$

for $\mathfrak{I} \in \Gamma(\mathfrak{M})$ is defined by

$$F_{\mathfrak{I},\Phi}(\hat{\mathfrak{I}}) \coloneqq \gamma(\nabla\hat{\mathfrak{I}}) \in \Gamma(\mathrm{Hom}_\Phi(TQ, \mathfrak{I}^*V\mathfrak{M})).$$

**Definition 4.11.** A Fueter section $\mathfrak{I}$ is called **unobstructed** if the linearised Fueter operator $F_{\mathfrak{I}}$ is surjective.

**Example 4.12.** If $M$ is as in Example 3.4, then the linearised Fueter operator $F_{\mathfrak{I}}$ lifts to the twisted Dirac operator $\not{D}\colon\ \Gamma(\mathrm{Re}(\mathrm{Hom}(E_\infty, \$^+) \oplus \$^+ \otimes U)) \to \Gamma(\mathrm{Re}(\mathrm{Hom}(E_\infty, \$^-) \oplus \$^- \otimes U))$. In particular, it only depends on the spin structure $\mathfrak{s}$ and not on $\mathfrak{I}$. Using the Atiyah–Singer index theorem we can compute that in the current situation

$$\operatorname{index} \mathring{F}_{\mathfrak{I}} = -\frac{1}{4}\sigma(Q) - \int_Q c_2(E_\infty) \tag{4.13}$$

where $\mathring{F}_{\mathfrak{I}}$ is the restriction of $F_{\mathfrak{I}}$ to $V\mathring{\mathfrak{M}}$.

# 5 Approximate Spin(7)–instantons

Throughout the next three sections we assume the hypotheses of Theorem 1.2. For each sufficiently small gluing parameter $\lambda > 0$ we first construct a connection $A_\lambda$ by grafting $I_\lambda = I(\mathfrak{I}_\lambda)$ into $A_0$ by hand. $A_\lambda$ will not quite be a Spin(7)–instanton; however, $\pi_7(F_{A_\lambda})$, the failure of being a Spin(7)–instanton, can be made very small. We are then left with solving the mildly non-linear p.d.e.

$$\left(\mathrm{d}^*_{A_\lambda} a, \pi_7(F_{A_\lambda+a})\right) = L_\lambda a + Q(a) + \pi_7(F_{A_\lambda}) = 0 \tag{5.1}$$

with

$$L_\lambda \coloneqq L_{A_\lambda} = \begin{pmatrix} \mathrm{d}^*_{A_\lambda} \\ \pi_7 \mathrm{d}_{A_\lambda} \end{pmatrix},$$

see (2.21), and

$$Q(a) \coloneqq \frac{1}{2}\pi_7([a \wedge a])$$

for $a = a(\lambda) \in \Omega^1(X, \mathfrak{g}_{E_\lambda})$. Given suitable control on $L_\lambda$ and $Q$, (5.1) can be solved by appealing to Banach's fixed-point theorem.

*Remark* 5.2. If $A_0$ is reducible, we might not be able to construct $a$ such that $\mathrm{d}^*_{A_\lambda} a = 0$ on the nose, but only "modulo $H^0_{A_0}$". For the purpose of proving Theorem 1.2 it is not important to have $\mathrm{d}^*_{A_\lambda} a = 0$. If $A_0$ is reducible then, in order to achieve surjectivity, one has to work with $\bar{L}_\lambda \colon \Omega^1(X, \mathfrak{g}_{E_\lambda}) \oplus H^0_{A_0} \to \Omega^0(X, \mathfrak{g}_{E_\lambda}) \oplus \Omega^2_7(X, \mathfrak{g}_{E_\lambda})$ defined by

$$\bar{L}_\lambda(a, o) = L_\lambda(a) + \iota_\lambda(o)$$

where $\iota_\lambda \colon H^0_{A_0} \to \Omega^0(X, \mathfrak{g}_{E_\lambda})$ is a inclusion map constructed by first cutting of $o$ to zero near $Q$ and then thinking of it as a section of $\mathfrak{g}_{E_\lambda}$. In order to not clutter the exposition any further, we assume in the following that $A_0$ is irreducible.

*Convention* 5.3. We fix a constant $\Lambda > 0$ such that all of the statements of the kind "if $\lambda \in (0, \Lambda]$, then ..." appearing in the following are valid. This is possible since there are only a finite number of these statements and each one of them is valid provided $\Lambda$ is sufficiently small. By $c > 0$ we will denote a generic constant whose value does not depend on $\lambda \in (0, \Lambda]$ but may change from one occurrence to the next.

## 5.1 Pregluing construction

**Construction 5.4.** *For each $\lambda \in (0, \Lambda]$ we construct a $G$–bundle $E_\lambda$ together with a connection $A_\lambda = A \#_\lambda \mathfrak{I}$ from $E_0$, $A_0 \in \mathscr{A}(E_0)$ and $\mathfrak{I}$. The bundles $E_\lambda$ are pairwise isomorphic.*

Let us set up some notation. Fix a constant $\zeta > 0$ such that the exponential map identifies a tubular neighbourhood of width $10\zeta$ of $Q$ in $X$ with a neighbourhood of the zero section in $NQ$. For $I \subset \mathbf{R}$ we set

$$U_I := \{v \in NQ : |v| \in I\} \quad \text{and} \quad V_I := \{x \in X : r(x) \in I\}.$$

Here

$$r := d(\cdot, Q) \colon X \to [0, \infty)$$

denotes the distance from $Q$. Fix a smooth-cut off function $\chi \colon [0, \infty) \to [0, 1]$ which vanishes on $[0, 1]$ and is equal to one on $[2, \infty)$. For $\lambda \in (0, \Lambda]$ we define $\chi^-_\lambda \colon X \to [0, 1]$ and $\chi^+ \colon X \to [0, 1]$ by

$$\chi^-_\lambda(x) := \chi(r(x)/\lambda) \quad \text{and} \quad \chi^+(x) := 1 - \chi(r(x)/2\zeta),$$

respectively.

Using radial parallel transport we can identify $E(\mathfrak{I})$ over $U_{(R,\infty)}$ for some $R > 0$ with the pullback of $E(\mathfrak{I})|_\infty$ to said region and similarly we can identify $E_0$ over $V_{[0,\zeta)}$ with the pullback of $E_0|_Q$. Hence, via the framing $\Phi$ we can identify $s^*_{1/\lambda} E(\mathfrak{I})$ with $E_0$ on the overlap $V_{(\lambda,\sigma)}$ for $\lambda \in (0, \Lambda]$. Patching both bundles via this identification yields $E_\lambda$.

To construct a connection on $E_\lambda$ note that on the overlap $I_\lambda := s^*_{1/\lambda} I(\mathfrak{I})$ and $A_0$ can be written as

$$I_\lambda = A_0|_Q + i_\lambda \quad \text{and} \quad A_0 = A_0|_Q + a.$$

Here and in the following, by a slight abuse of notation, we denote by $A_0|_Q$ the pullback of $A_0|_Q$ to the overlap. We define $A_\lambda$ by interpolating between $I_\lambda$ and $A$ on the overlap as follows

$$A_\lambda \coloneqq A_0|_Q + \chi_\lambda^- a + \chi^+ i_\lambda. \tag{5.5}$$

This completes the construction. □

## 5.2 Weighted Hölder spaces

In order to quantify to what extent $\pi_7(A_\lambda)$ is small, we introduce certain norms, which are especially adapted to the geometric situation at hand.

**Definition 5.6.** For $\lambda \in (0, \Lambda]$ we define a family of weight functions $w_{\ell,\delta;\lambda}$ on $X$ depending on two additional parameters $\ell \in \mathbf{R}$ and $\delta \in \mathbf{R}$ as follows

$$w_{\ell,\delta;\lambda}(x) \coloneqq \begin{cases} \lambda^\delta(\lambda + r(x))^{-\ell-\delta} & \text{if } r(x) \leqslant \sqrt{\lambda} \\ r(x)^{-\ell+\delta} & \text{if } r(x) > \sqrt{\lambda} \end{cases}$$

and set $w_{\ell,\delta;\lambda}(x, y) \coloneqq \min\{w_{\ell,\delta;\lambda}(x), w_{\ell,\delta;\lambda}(y)\}$. For a Hölder exponent $\alpha \in (0, 1)$ and $\ell, \delta \in \mathbf{R}$ we define (semi-)norms

$$\begin{aligned} \|f\|_{L^\infty_{\ell,\delta;\lambda}(U)} &\coloneqq \|w_{\ell,\delta;\lambda} f\|_{L^\infty(U)}, \\ [f]_{C^{0,\alpha}_{\ell,\delta;\lambda}(U)} &\coloneqq \sup_{\substack{x \neq y \in U: \\ d(x,y) \leqslant \lambda + \min\{r(x), r(y)\}}} w_{\ell-\alpha,\delta;\lambda}(x, y) \frac{|f(x) - f(y)|}{d(x, y)^\alpha} \quad \text{and} \\ \|f\|_{C^{k,\alpha}_{\ell,\delta;\lambda}(U)} &\coloneqq \sum_{j=0}^{k} \|\nabla^k f\|_{L^\infty_{\ell-j,\delta;\lambda}(U)} + [\nabla^k f]_{C^{0,\alpha}_{\ell-j,\delta;\lambda}}. \end{aligned}$$

Here $f$ is a section of a vector bundle over $U \subset X$ equipped with an inner product and a compatible connection. We use parallel transport to compare the values of $f$ at different points. If $U$ is not specified, then we take $U = X$.

We will primarily use these norms for $\mathfrak{g}_{E_\lambda}$–valued tensor fields.

*Remark* 5.7. The reader may find the following heuristic useful. Let $f$ be a $k$–form on $X$. Fix a small ball centred at a point $x \in Q$, identify it with a small ball in $T_xX = T_xQ \oplus N_xQ$ and rescale this ball by a factor $1/\lambda$. Upon pulling everything back to this rescaled ball the weight function $w_{-k,\delta;\lambda}$ becomes essentially $\lambda^k(1 + |y|)^{k-\delta}$, where $y$ denotes the $N_xQ$–coordinate. Thus as $\lambda$ goes to zero a uniform bound $\|f_\lambda\|_{L^\infty_{-k,\delta;\lambda}}$ on a family $(f_\lambda)$ of $k$–forms ensures that the pullbacks of $f_\lambda$ decay like $|y|^{-k+\delta}$ in the direction of $N_xQ$. At the same time it forces $f_\lambda$ not to blowup at a rate faster than $r^{-k-\delta}$ along $Q$. The "discrepancy" in the exponents can be seen to be rather natural by considering the action of the inversion $y \mapsto \lambda y/|y|^2$.

**Proposition 5.8.** *If $(f,g)\mapsto f\cdot g$ is a bilinear form satisfying $|f\cdot g|\leqslant |f||g|$, then*

$$\|f\cdot g\|_{C^{k,\alpha}_{\ell_1+\ell_2,\delta_1+\delta_2;\lambda}}\leqslant \|f\|_{C^{k,\alpha}_{\ell_1,\delta_1;\lambda}}\|g\|_{C^{k,\alpha}_{\ell_2,\delta_2;\lambda}}.$$

*Proof.* This follows immediately from the above definition. □

**Corollary 5.9.** *If $\delta<0$, then there is a constant $c>0$ which is independent of $\lambda\in(0,\Lambda]$ such that*

$$\|f\|_{C^{k,\alpha}_{\ell,\delta;\lambda}}\leqslant c\lambda^{\delta/2}\|f\|_{C^{k,\alpha}_{\ell,0;\lambda}}\quad\textit{and}\quad \|f\|_{C^{k,\alpha}_{\ell,0;\lambda}}\leqslant c\|f\|_{C^{k,\alpha}_{\ell,\delta;\lambda}}$$

*Proof.* Use $\|1\|_{C^{k,\alpha}_{0,\delta;\lambda}}\leqslant c\lambda^{\delta/2}$ and $\|1\|_{C^{k,\alpha}_{0,-\delta;\lambda}}\leqslant c$ for $\delta<0$. □

There are certain components of $\Omega^1(X,\mathfrak{g}_{E_\lambda})$ and $\Omega^2_7(X,\mathfrak{g}_{E_\lambda})$, which need to be treated separately. The following definition identifies these components.

**Definition 5.10.** Define $\mu_\lambda\colon\ \Gamma(\mathfrak{I}^*V\mathfrak{M})\to\Omega^1(X,\mathfrak{g}_{E_\lambda})$ by

$$\mu_\lambda\hat{\mathfrak{I}}\coloneqq \chi^+ s^*_{1/\lambda}\hat{\mathfrak{I}}$$

and $\nu_\lambda\colon\ \Gamma\left(\mathrm{Hom}_\Phi(TQ,\mathfrak{I}^*V\mathfrak{M})\right)\to\Omega^2_7(X,\mathfrak{g}_{E_\lambda})$ by

$$\nu_\lambda\hat{\mathfrak{T}}\coloneqq \pi_7(\chi^+ s^*_{1/\lambda}\hat{\mathfrak{T}}).$$

Here we first identify $\hat{\mathfrak{I}}\in\Gamma(\mathfrak{I}^*V\mathfrak{M})$ with an element of $\Omega^1\left(NQ,E(\mathfrak{I})\right)$, then view the restriction of its pullback via $s_{1/\lambda}$ to $U_{[0,\sigma)}$ as lying in $\Omega^1(V_{[0,\sigma)},\mathfrak{g}_{E_\lambda})$ and finally extended it to all of $X$ by multiplication with $\chi^+$; similarly we proceed with $\hat{\mathfrak{T}}$.

Define $\pi_\lambda\colon\ \Omega^1(X,\mathfrak{g}_{E_\lambda})\to\Gamma(\mathfrak{I}^*V\mathfrak{M})$ by

$$(\pi_\lambda a)(x)\coloneqq\sum_\kappa\int_{N_xQ}\langle a,\mu_\lambda\kappa\rangle\,\kappa$$

and $\sigma_\lambda\colon\ \Omega^2_7(X,\mathfrak{g}_{E_\lambda})\to\Gamma\left(\mathrm{Hom}_\Phi(TQ,\mathfrak{I}^*V\mathfrak{M})\right)$ by

$$(\sigma_\lambda\alpha)(x)\coloneqq\sum_\beta\int_{N_xQ}\langle\alpha,\nu_\lambda\beta\rangle\,\beta,$$

Here $\kappa$ runs through an orthonormal basis of $V\mathfrak{M}_{\mathfrak{I}(x)}$ with respect to the inner product $\langle\mu_\lambda\cdot,\mu_\lambda\cdot\rangle$ and $\beta$ runs through an orthonormal basis of $\mathrm{Hom}_\Phi\left(T_xQ,V\mathfrak{M}_{\mathfrak{I}(x)}\right)$ with respect to the inner product $\langle\nu_\lambda\cdot,\nu_\lambda\cdot\rangle$.

Clearly, $\pi_\lambda\mu_\lambda=\mathrm{id}$ and $\sigma_\lambda\nu_\lambda=\mathrm{id}$; hence,

$$\bar\pi_\lambda\coloneqq\mu_\lambda\pi_\lambda\quad\text{and}\quad\bar\sigma_\lambda\coloneqq\nu_\lambda\sigma_\lambda$$

are projections. We denote the complementary projections by

$$\rho_\lambda\coloneqq\mathrm{id}-\bar\pi_\lambda\quad\text{and}\quad\tau_\lambda\coloneqq\mathrm{id}-\bar\sigma_\lambda.$$

**Proposition 5.11.** *For $\ell \leqslant -1$ and $\delta \in \mathbf{R}$ such that $\ell + \delta \in (-3, -1)$ there is a constant $c > 0$ such that for all and $\lambda \in (0, \Lambda]$ we have*

$$\|\mu_\lambda \hat{\mathfrak{I}}\|_{C^{0,\alpha}_{\ell,\delta;\lambda}} \leqslant c\lambda^{-1-\ell}\|\hat{\mathfrak{I}}\|_{C^{0,\alpha}} \quad \textit{and} \quad \|\pi_\lambda a\|_{C^{0,\alpha}} \leqslant c\lambda^{1+\ell-\alpha}\|a\|_{C^{0,\alpha}_{\ell,\delta;\lambda}(V_{[0,\sigma)})}$$

*as well as*

$$\|\nu_\lambda \hat{\mathfrak{T}}\|_{C^{0,\alpha}_{\ell,\delta;\lambda}} \leqslant c\lambda^{-1-\ell}\|\hat{\mathfrak{T}}\|_{C^{k,\alpha}} \quad \textit{and} \quad \|\sigma_\lambda \alpha\|_{C^{0,\alpha}} \leqslant c\lambda^{1+\ell-\alpha}\|\alpha\|_{C^{0,\alpha}_{\ell,\delta;\lambda}(V_{[0,\sigma)})}.$$

*In particular, $\bar{\pi}_\lambda$, $\rho_\lambda$, $\bar{\sigma}_\lambda$ and $\tau_\lambda$ are bounded by $c\lambda^{-\alpha}$ with respect to the $C^{0,\alpha}_{\ell,\delta;\lambda}$–norms.*

*Proof.* We only prove the first two estimates; the last two are identical up to a change in notation. From Proposition 3.2 it follows at once that

$$\|s^*_{1/\lambda}\hat{\mathfrak{I}}\|_{C^{0,\alpha}_{-3,0;\lambda}(V_{[0,\sigma)})} \leqslant c\lambda^2\|\hat{\mathfrak{I}}\|_{C^{0,\alpha}}.$$

The first inequality thus is a consequence of Proposition 5.8 since $\|\chi^+_t\|_{C^{0,\alpha}_{3+\ell,\delta;\lambda}} \leqslant c\lambda^{-3-\ell}$ for $\ell + \delta > -3$.

To prove the second inequality, note that by Proposition 3.2 for $\kappa \in (V\mathfrak{M}_t)_{\mathfrak{I}_t(x)}$ we have $|s^*_{1/\lambda}\kappa|(y) \leqslant c\lambda^2/(\lambda + |y|)^3\|\kappa\|_{L^2}$ and thus

$$\begin{aligned}\int_{N_xQ}\left\langle a, \chi^+ s^*_{1/\lambda}\kappa\right\rangle &\leqslant c\int_0^{\sqrt{\lambda}} \lambda^{2-\delta}(\lambda + r)^{\ell+\delta-3}r^3\mathrm{d}r \cdot \|a\|_{L^\infty_{\ell,\delta;\lambda}}\|\kappa\|_{L^2} \\ &\quad + c\int_{\sqrt{\lambda}}^{\sigma} \lambda^2 r^{\ell-\delta}(\lambda + r)^{-3}r^3\mathrm{d}r \cdot \|a\|_{L^\infty_{\ell,\delta;\lambda}}\|\kappa\|_{L^2} \\ &\leqslant c\lambda^{3+\ell}\|a\|_{L^\infty_{\ell,\delta;\lambda}}\|\kappa\|_{L^2}\end{aligned}$$

since $\ell \leqslant -1$ and $\ell + \delta < -1$. If $\kappa$ is an element of an orthonormal basis of $(V\mathfrak{M})_{\mathfrak{I}(x)}$ with respect to $\langle \mu_\lambda \cdot, \mu_\lambda \cdot \rangle$, then $\|\kappa\|_{L^2} \leqslant c/\lambda$ since for $\kappa_1, \kappa_2 \in (V\mathfrak{M})_{\mathfrak{I}(x)}$

$$\lambda^2\langle \kappa_1, \kappa_2\rangle_{L^2} \sim \left\langle \chi^+ s^*_{1/\lambda}\kappa_1, \chi^+ s^*_{1/\lambda}\kappa_2\right\rangle_{L^2}$$

where $\sim$ means comparable uniformly in $\lambda$. Therefore,

$$\|\pi_\lambda a\|_{L^\infty} \leqslant c\lambda^{1+\ell}\|a\|_{L^\infty_{\ell,\delta;\lambda}}.$$

The estimates on the Hölder norms follow by the same kind of argument. □

Ultimately, we will be working with the following function spaces.

**Definition 5.12.** Denote by $\mathfrak{X}_\lambda$ and $\mathfrak{Y}_\lambda$ the Banach spaces $C^{1,\alpha}\Omega^1(X, \mathfrak{g}_{E_\lambda})$ and $C^{0,\alpha}\Omega^0(X, \mathfrak{g}_{E_\lambda}) \oplus C^{0,\alpha}\Omega^2_7(X, \mathfrak{g}_{E_\lambda})$ equipped with the norms

$$\|a\|_{\mathfrak{X}_\lambda} := \lambda^{-\delta/2}\|\rho_\lambda a\|_{C^{1,\alpha}_{-1,\delta;\lambda}} + \lambda\|\pi_\lambda a\|_{C^{1,\alpha}} \quad \text{and}$$
$$\|(\xi, \alpha)\|_{\mathfrak{Y}_\lambda} := \lambda^{-\delta/2}\|\xi\|_{C^{0,\alpha}_{-2,\delta;\lambda}} + \lambda^{-\delta/2}\|\tau_\lambda \alpha\|_{C^{0,\alpha}_{-2,\delta;\lambda}} + \lambda\|\sigma_\lambda \alpha\|_{C^{0,\alpha}},$$

respectively. Here we fix $\delta \in (-1, 0)$ and $0 < \alpha \ll |\delta|$; for concreteness, let us take $\delta = -\frac{1}{2}$ and $\alpha = \frac{1}{256}$.

*Remark* 5.13. We choose the factor $\lambda^{-\delta/2}$ in view of Corollary 5.9.

## 5.3 Error estimate

**Proposition 5.14.** *There exists a constant $c > 0$ such that for all $\lambda \in (0, \Lambda]$*

$$\|\pi_7(F_{A_\lambda})\|_{C^{0,\alpha}_{-2,0;\lambda}} \leqslant c\lambda^2;$$

*in particular,*

$$\|\pi_7(F_{A_\lambda})\|_{\mathfrak{Y}_\lambda} \leqslant c\lambda^{2-\alpha}.$$

*Remark* 5.15. With more work the exponent can be improved from $2 - \alpha$ to 2.

The proof of this result requires some preparation.

**Proposition 5.16.** *In the tubular neighbourhood $V_{[0,\zeta)}$ of $Q$ we can write the Taylor expansion of $\pi_7$ in the direction transverse to $Q$ as*

$$\pi_7 = \pi_7^0 + \pi_7^1 + \pi_7^{\geqslant 2}$$

*where $\pi_7^0$ denotes the zeroth order term, $\pi_7^1$ denotes the first order term and vanishes on $\Lambda^- N^*Q$ and $\pi_7^{\geqslant 2}$ denotes the remainder term; moreover, there is a constant $c > 0$ which is independent of $\lambda \in (0, \Lambda]$ such that*

$$\|\pi_7^0\|_{C^{0,\alpha}_{0,0;\lambda}(V_{[0,\zeta)})} + \|\pi_7^1\|_{C^{0,\alpha}_{1,0;\lambda}(V_{[0,\zeta)})} + \|\pi_7^{\geqslant 2}\|_{C^{0,\alpha}_{2,0;\lambda}(V_{[0,\zeta)})} \leqslant c.$$

*Proof.* If we pull the identity map of a tubular neighbourhood of $Q$ back to a tubular neighbourhood of the zero section of $NQ$ via the exponential map, then the Taylor expansion of its derivative around $Q$ can be expressed in the splitting $TNQ = \pi_1^*TQ \oplus \pi_2^*NQ$ as

$$(x, y) \mapsto (x, y) + \left(\mathrm{II}_y(x), y\right) + O\left(|y|^2\right)$$

where II is the second fundamental form of $Q$ in $X$, which we think of as a map from $NQ$ to $\mathrm{End}(TQ)$. This immediately yields the desired expansion of $\pi_7$ near $Q$ with $\pi_7^1$ vanishing on $\Lambda^- N^*Q$. □

**Proposition 5.17**. *There is a constant $c > 0$ such that for all $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ we have*

$$\begin{aligned} \left\| F^{2,0}_{I_\lambda} - F_{A_0|Q} \right\|_{C^{0,\alpha}_{-2,0;\lambda}(V_{[0,\sigma)})} &\leqslant c\lambda^2, \\ \left\| F^{1,1}_{I_\lambda} \right\|_{C^{0,\alpha}_{-3,0;\lambda}(V_{[0,\sigma)})} &\leqslant c\lambda^2 \quad \text{and} \\ \left\| F^{0,2}_{I_{t,\lambda}} \right\|_{C^{0,\alpha}_{-4,0;\lambda}(V_{[0,\sigma)})} &\leqslant c\lambda^2. \end{aligned}$$

*Proof.* Theorem 4.5 asserts that the restriction of $I = I(\mathfrak{I})$ to the section at infinity agrees with $A_0|_Q$. For a local coordinate system $(z_1, \ldots, z_4, w_1, \ldots, w_4)$ based at a point on the section at infinity and with $z_i$ denoting the coordinates along $Q$ and $w_i$ denote transverse coordinates we can write

$$I = A_0|_Q + \sum_{i,j=1}^{4} w_i(\xi_{ij}\mathrm{d}z_j + \eta_{ij}\mathrm{d}w_j) + O(|w|^2)$$

for $\xi_{ij}, \eta_{ij} \in \mathfrak{g}$. It follows that $F_I^{1,1} = -\sum_{i,j=1}^4 \xi_{ij}\mathrm{d}z_i \wedge \mathrm{d}w_j + O(|w|)$. However, by Proposition 3.6 and Proposition 3.2, when viewed from the zero section the curvature component $F_I^{1,1}$ decays like $r^{-3}$. This translates into $\xi_{ij} = 0$, and we can write

$$I = A_0|_Q + \sum_{i,j=1}^{4} \eta_{ij} w_i \mathrm{d}w_j + O(|w|^2). \tag{5.18}$$

This means that, $F_I^{2,0} - F_{A_0|Q}$ vanishes to first order along the section at infinity which when viewed from the zero section in $NQ$ means that

$$\left|F_I^{2,0} - F_{A_0|Q}\right| \leqslant \frac{c}{1 + |w|^2}.$$

The first estimate now follows from a simple scaling consideration.

The last two estimates follow from simple scaling considerations using Proposition 3.2 and Theorem 4.5 together with the fact that the curvature of a finite energy ASD instanton decays at least like $|y|^{-4}$. ◻

**Proposition 5.19**. *There is a constant $c > 0$ such that for all $\lambda \in (0, \Lambda]$ we have*

$$\begin{aligned} \|i_\lambda\|_{C^{0,\alpha}_{-3,0;\lambda}(V_{(\lambda,\sigma)})} + \|\mathrm{d}_{I_\lambda} i_\lambda\|_{C^{0,\alpha}_{-4,0;\lambda}(V_{(\lambda,\sigma)})} &\leqslant c\lambda^2 \quad \text{and} \\ \|a\|_{C^{0,\alpha}_{1,0;\lambda}(V_{[0,\sigma)})} + \|\mathrm{d}_{A_0|Q} a\|_{C^{0,\alpha}_{0,0;\lambda}(V_{[0,\sigma)})} &\leqslant c. \end{aligned}$$

*Proof.* The first estimate follows from (5.18) and a simple scaling consideration, while the last follows from the fact that we put $A_0$ into radial gauge from zero section in $NQ$. ◻

*Proof of Proposition 5.14.* We proceed in four steps. First we estimate an approximation $\tilde{e}_\lambda$ of

$$e_\lambda := \pi_7(F_{A_\lambda}).$$

Then we estimate the difference $e_\lambda - \tilde{e}_\lambda$ separately in the three subsets $V_{[0,\lambda)}$, $V_{[\lambda,\sigma/2)}$ and $V_{[\sigma/2,\sigma)}$ constituting $V_{[0,\sigma)}$ which contains the support of $e_\lambda$.

It will be convenient to use the following shorthand notation

$$\|f\|_{\ell,U} := \|f\|_{C^{0,\alpha}_{\ell,0;\lambda}(U)}.$$

Note that if $(f,g) \mapsto f \cdot g$ is a bilinear map satisfying $|f \cdot g| \leqslant |f||g|$, then it follows from Proposition 5.8 that $\|f \cdot g\|_{\ell_1+\ell_2,U} \leqslant \|f\|_{\ell_1,U} \cdot \|g\|_{\ell_2,U}$.

**Step 1.** *The term*

$$\tilde{e}_\lambda := \pi_7\left(F_{I_\lambda} - F_{A_0|_Q}\right)$$

*satisfies* $\|\tilde{e}_\lambda\|_{-2,V_{[0,\sigma)}} \leqslant c\lambda^2$.

Because of Theorem 4.9, the fact that $F^{0,2}_{I_\lambda}$ is anti-self-dual and Proposition 5.16 we can write $\tilde{e}_\lambda$ on $V_{[0,\sigma)}$ as

$$\pi_7\left(F^{2,0}_{I_\lambda} - F_{A|_Q}\right) + (\pi^1_7 + \pi^{\geqslant 2}_7)\left(F^{1,1}_{I_\lambda}\right) + \pi^{\geqslant 2}_7\left(F^{0,2}_{I_\lambda}\right).$$

Using Proposition 5.16 and Proposition 5.17 as well as $\|1\|_{-1,V_{[0,\sigma)}} \leqslant c$ we estimate $\|\tilde{e}_\lambda\|_{-2,V_{[0,\sigma)}}$ by

$$\begin{aligned}
&\left\|F^{(2,0)}_{I_\lambda} - F_{A_0|_Q}\right\|_{-2,V_{[0,\sigma)}} \cdot \|\pi_7\|_{0,V_{[0,\sigma)}} \\
&\quad + \left\|F^{1,1}_{I_\lambda}\right\|_{-3,V_{[0,\sigma)}} \cdot \left(\left\|\pi^1_7\right\|_{1,V_{[0,\sigma)}} + \|1\|_{-1,V_{[0,\sigma)}} \cdot \left\|\pi^{\geqslant 2}_7\right\|_{2,V_{[0,\sigma)}}\right) \\
&\quad + \left\|F^{0,2}_{I_\lambda}\right\|_{-4,V_{[0,\sigma)}} \cdot \|\pi^{\geqslant 2}_7\|_{2,V_{[0,\sigma)}} \leqslant c\lambda^2.
\end{aligned}$$

**Step 2.** *We prove that* $\|e_\lambda - \tilde{e}_\lambda\|_{V_{[0,2\lambda)}} \leqslant c\lambda^2$.

Since

$$\left\|\pi_7(F_{A_0|_Q})\right\|_{-2,V_{[0,2\lambda)}} \leqslant \|1\|_{-2,V_{[0,2\lambda)}} \cdot \left\|\pi_7(F_{A_0|_Q})\right\|_{0,V_{[0,2\lambda)}} \leqslant c\lambda^2,$$

it suffices to estimate $F_{A_\lambda} - F_{I_\lambda}$ in $V_{[0,2\lambda)}$. Now, in $V_{[0,2\lambda)}$ the curvature of $A_\lambda$ is given by

$$F_{A_\lambda} = F_{I_\lambda} + \chi^-_\lambda \mathrm{d}_{I_\lambda} a + \frac{1}{2}(\chi^-_\lambda)^2[a \wedge a] + \mathrm{d}\chi^-_\lambda \wedge a.$$

Using Proposition 5.19 and the fact that the cut-off functions $\chi_\lambda^-$ where constructed so that $\|\chi_\lambda^-\|_{0,V_{[0,\sigma)}} + \|d\chi_\lambda^-\|_{-1,V_{[0,\sigma)}} \leqslant c$ we obtain

$$\begin{aligned}
&\|F_{A_\lambda} - F_{I_\lambda}\|_{-2,V_{[0,2\lambda)}} \\
&\quad \leqslant \|1\|_{-2,V_{[0,2\lambda)}} \cdot \|\chi_\lambda^-\|_{0,V_{[0,2\lambda)}} \cdot \|d_{A|_Q} a\|_{0,V_{[0,2\lambda)}} \\
&\qquad + \|\chi_\lambda^-\|_{0,V_{[0,2\lambda)}} \cdot \|i_\lambda\|_{-3,V_{[\lambda,\sigma)}} \cdot \|a\|_{1,V_{[0,2\lambda)}} \\
&\qquad + \frac{1}{2}\|1\|_{-4,V_{[0,2\lambda)}} \cdot \|\chi_\lambda^-\|^2_{0,V_{[0,2\lambda)}} \cdot \|a\|^2_{1,V_{[0,2\lambda)}} \\
&\qquad + \|1\|_{-2,V_{[0,2\lambda)}} \cdot \|d\chi_\lambda^-\|_{-1,V_{[0,2\lambda)}} \cdot \|a\|_{1,V_{[0,2\lambda)}} \leqslant c\lambda^2.
\end{aligned}$$

**Step 3.** *We prove that* $\|e_\lambda - \tilde{e}_\lambda\|_{V_{(2\lambda,\sigma/2)}} \leqslant c\lambda^2$.

This is an immediate consequence of $\pi_7(F_{A_0}) = 0$ and Proposition 5.19 since in $V_{[2\lambda,\sigma/2)}$ the curvature of $A_\lambda$ is given by $F_{A_\lambda} = F_{A_0} + [i_\lambda \wedge a] + F_{I_\lambda} - F_{A_0|_Q}$.

**Step 4.** *We prove that* $\|e_\lambda - \tilde{e}_\lambda\|_{V_{[\sigma/2,\sigma)}} \leqslant c\lambda^2$.

In $V_{[\sigma/2,\sigma)}$ the curvature of $A_\lambda$ is given by

$$F_{A_\lambda} = F_{A_0} + \chi^+ d_{A_0} i_\lambda + \frac{1}{2}(\chi^+)^2[i_\lambda \wedge i_\lambda] + d\chi^+ \wedge i_\lambda.$$

Since $\|\chi^+\|_{\ell,V_{[\sigma/2,\sigma)}} + \|d\chi^+\|_{\ell,V_{[\sigma/2,\sigma)}} \leqslant c$, it follows that

$$\begin{aligned}
&\|F_{A_\lambda} - F_{A_0}\|_{-2,V_{[\sigma/2,\sigma)}} \\
&\quad \leqslant \|\chi^+\|_{2,V_{[\sigma/2,\sigma)}} \cdot \|d_{I_\lambda} i_\lambda\|_{-4,V_{[\sigma/2,\sigma)}} \\
&\qquad + \|\chi^+\|_{0,V_{[\sigma/2,\sigma)}} \cdot \|a\|_{1,V_{[\sigma/2,\sigma)}} \cdot \|i_\lambda\|_{-3,V_{[\sigma/2,\sigma)}} \\
&\qquad + \frac{1}{2}\|\chi^+\|^2_{2,V_{[\sigma/2,\sigma)}} \cdot \|i_\lambda\|^2_{-3,V_{[\sigma/2,\sigma)}} \\
&\qquad + \|d\chi^+\|_{1,V_{[\sigma/2,\sigma)}} \cdot \|i_\lambda\|_{-3,V_{[\sigma/2,\sigma)}} \leqslant c\lambda^2.
\end{aligned}$$

This completes the estimate. □

# 6 Linear analysis

**Proposition 6.1.** *For* $\lambda \in (0,\Lambda]$ *the linear operator* $L_\lambda\colon \mathfrak{X}_\lambda \to \mathfrak{Y}_\lambda$ *has a right inverse* $R_\lambda\colon \mathfrak{Y}_\lambda \to \mathfrak{X}_\lambda$ *and there exists a constant* $c > 0$ *which is independent of* $\lambda \in (0,\Lambda]$ *such that*

$$\|R_\lambda(\xi,\alpha)\|_{\mathfrak{X}_\lambda} \leqslant c\|(\xi,\alpha)\|_{\mathfrak{Y}_\lambda}.$$

This is the key to proving Theorem 1.2. We produce $R_\lambda$ by gluing various local right inverses "by hand". We decompose $L_\lambda$ as

$$L_\lambda = \begin{pmatrix} \mathfrak{R}_\lambda & \mathfrak{p}_\lambda \\ \mathfrak{q}_\lambda & \mathfrak{L}_\lambda \end{pmatrix}$$

where

$$\mathfrak{R}_\lambda := \bar{\sigma}_\lambda L_\lambda \bar{\pi}_\lambda, \quad \mathfrak{L}_\lambda := \tau_\lambda L_\lambda \rho_\lambda,$$
$$\mathfrak{p}_\lambda := \bar{\sigma}_\lambda L_\lambda \rho_\lambda, \quad \text{and} \quad \mathfrak{q}_\lambda := \tau_\lambda L_\lambda \bar{\pi}_\lambda.$$

In the course of this section we will show that $\mathfrak{R}_\lambda$ is essentially the linearised Fueter operator $F_{\mathfrak{I}}$, which has a right inverse by assumption, and that local right inverses for $\mathfrak{L}_\lambda$ can be seen to exist by considerations of model operators on $\mathbf{R}^8$ and on the complement of $Q$, while $\mathfrak{p}_\lambda$ and $\mathfrak{q}_\lambda$ are negligibly small terms. An approximate right inverse $\tilde{R}_\lambda$ can then be constructed by carefully patching together the local right inverses. Finally, a simple deformation argument will yield $R_\lambda$.

## 6.1 The model operator on $\mathbf{R}^8$

Let $I$ be a finite energy ASD instanton on a $G$–bundle $E$ over $\mathbf{R}^4$. By a slight abuse of notation we denote the pullbacks of $I$ and $E$ to $\mathbf{R}^8 = \mathbf{R}^4 \times \mathbf{R}^4$ by $I$ and $E$ as well. We define $\mathbf{L}_I \colon \Omega^0(\mathbf{R}^8, \mathfrak{g}_E) \to \Omega^0(\mathbf{R}^8, \mathfrak{g}_E) \oplus \Omega^2_7(\mathbf{R}^8, \mathfrak{g}_E)$ by

$$\mathbf{L}_I(a) := (\mathrm{d}_A^* a, \pi_7 \mathrm{d}_A a).$$

Here $\pi_7$ is taken with respect to the standard Spin(7)–structure $\Phi_0$ on $\mathbf{R}^8$, see (2.2).

By Remark 2.22 we can, with the appropriate identifications being made, write

$$\mathbf{L}_I = \partial_t - L_I$$

where we think of $I$ as a $\mathrm{G}_2$–instanton on $\{0\} \times \mathbf{R}^3 \times \mathbf{R}^4$ and $L_I$ is as in

$$L_{A,\phi} := \begin{pmatrix} 0 & \mathrm{d}_A^* \\ \mathrm{d}_A & *(\psi \wedge \mathrm{d}_A) \end{pmatrix}.$$

In particular, using [Wal13a, Proposition 7.1] we see that

$$\mathbf{L}_I \mathbf{L}_I^* = \mathbf{L}_I^* \mathbf{L}_I = \Delta_{\mathbf{R}^4} + \begin{pmatrix} \delta_I \delta_I^* & \\ & \delta_I^* \delta_I \end{pmatrix} \tag{6.2}$$

and, hence, we can argue as in [Wal13a, Section 7].

*Remark* 6.3. In the above situation thinking of $\mathbf{R}^8$ as $\mathbf{R}^4 \times \mathbf{R}^4$ as in Example 2.8 and at the same time as $\mathbf{R} \times (\mathbf{R}^3 \times \mathbf{R}^4)$ as in Example 2.10, the summands $\Lambda^2_3$ and $\Lambda^2_4$ in Proposition 2.14 are identified, via Proposition 2.15, with $\mathbf{R}^3$ and $\mathbf{R}^4$ respectively.

**Definition 6.4.** Define weight functions $w\colon \mathbf{R}^8 \to [0,\infty)$ and, by slight abuse of notation, $w\colon (\mathbf{R}^8)^2 \to [0,\infty)$ by

$$w(x) \coloneqq 1 + |\pi_2(x)| \quad \text{and} \quad w(x,y) \coloneqq \min\{w(x), w(y)\}.$$

Here $\pi_2\colon \mathbf{R}^8 = \mathbf{R}^4 \times \mathbf{R}^4 \to \mathbf{R}^4$ is the projection to the second factor. For a Hölder exponent $\alpha \in (0,1)$ and a weight parameter $\beta \in \mathbf{R}$ we define

$$\begin{aligned}
[f]_{C^{0,\alpha}_\beta(U)} &\coloneqq \sup_{d(x,y)\leqslant w(x,y)} w(x,y)^{\alpha-\beta} \frac{|f(x)-f(y)|}{d(x,y)^\alpha},\\
\|f\|_{L^\infty_\beta(U)} &\coloneqq \left\|w^{-\beta} f\right\|_{L^\infty(U)} \quad \text{and}\\
\|f\|_{C^{k,\alpha}_\beta(U)} &\coloneqq \sum_{j=0}^{k} \left\|\nabla^j f\right\|_{L^\infty_{\beta-j}(U)} + \left[\nabla^j f\right]_{C^{0,\alpha}_{\beta-j}(U)}.
\end{aligned}$$

Here $f$ is a section of a vector bundle over $U \subset \mathbf{R}^8$ equipped with an inner product and a compatible connection. We use parallel transport to compare the values of $f$ at different points. If $U$ is not specified, then we take $U = \mathbf{R}^8$. We denote by $C^{k,\alpha}_\beta$ the subspace of elements $f$ of the Banach space $C^{k,\alpha}$ with $\|f\|_{C^{k,\alpha}_\beta} < \infty$ and equip it with the norm $\|\cdot\|_{C^{k,\alpha}_\beta}$.

The linear operator $\mathbf{L}_I$ can serve as a model for $L_\lambda$ in the following sense: Fix $x \in Q$. Set $I \coloneqq I(\mathfrak{I})|_{N_xQ}$ and $E \coloneqq E(\mathfrak{I})|_{N_xQ}$. Identify $T_xX = T_xQ \times N_xQ$ with $\mathbf{R}^8 = \mathbf{R}^4 \times \mathbf{R}^4$ in such a way that the summands are preserved and $\Phi|_{T_xX}$ is identified with $\Phi_0$. For $\varepsilon_1, \varepsilon_2 > 0$ we define

$$V_{\varepsilon_1,\varepsilon_2} \coloneqq B_{\varepsilon_1}(x) \cap V_{[0,\varepsilon_2)}.$$

Using the exponential map we can identify $V_{\varepsilon_1,\varepsilon_2}$ with a small neighbourhood $\tilde{U}_{\varepsilon_1,\varepsilon_2}$ of the origin in $\mathbf{R}^8$. With respect to this identification a $\mathfrak{g}_{E_\lambda}$–valued tensor field $f$ on $V_{\varepsilon_1,\varepsilon_2}$ is identified with a $s^*_{1/\lambda}\mathfrak{g}_E$–valued tensor field $\tilde{f}$ on $\tilde{U}_{\varepsilon_1,\varepsilon_2;\lambda}$, and if $k \in \mathbf{N}$ is a scaling parameter, then with $f$ we can associate a $\mathfrak{g}_E$–valued tensor field $s_{d,\lambda} f$ on

$$U_{\varepsilon_1,\varepsilon_2;\lambda} \coloneqq \lambda^{-1}\tilde{U}_{\varepsilon_1,\varepsilon_2} = \lambda^{-1}\exp_x^{-1}(V_{\varepsilon_1,\varepsilon_2})$$

defined by

$$(s_{d,\lambda} f)(x,y) \coloneqq \lambda^d \tilde{f}(\lambda x, \lambda y) = \lambda^d f \circ \exp(\lambda(x,y)).$$

**Proposition 6.5.** *There are constants $c, \varepsilon_0 > 0$ such that for $\varepsilon \in (0,\varepsilon_0]$ and $\lambda \in (0,\Lambda]$ we have*

$$\begin{aligned}
\frac{1}{c}\lambda^{d+\ell}\|f\|_{C^{k,\alpha}_{\ell,\delta;\lambda}\left(V_{\varepsilon,N\sqrt{\lambda}}\right)} &\leqslant \|s_{d,\lambda} f\|_{C^{k,\alpha}_{\ell+\delta}\left(U_{\varepsilon,N\sqrt{\lambda};\lambda}\right)}\\
&\leqslant cN^{-2\delta}\lambda^{d+\ell}\|f\|_{C^{k,\alpha}_{\ell,\delta;\lambda}\left(V_{\varepsilon,N\sqrt{\lambda}}\right)}
\end{aligned}$$

*and*

$$\left\|L_\lambda a - s^{-1}_{2,\lambda}\mathbf{L}_I s_{1,\lambda} a\right\|_{C^{0,\alpha}_{-2,\delta;\lambda}\left(V_{\varepsilon,N\sqrt{\lambda}}\right)} \leqslant c(\varepsilon + \sqrt{\lambda})\|a\|_{C^{1,\alpha}_{-1,\delta;\lambda}\left(V_{\varepsilon,N\sqrt{\lambda}}\right)}.$$

*Here, in the first estimate, we also allow $k = \alpha = 0$, thus making a statement about weighted $L^\infty$–norms.*

For $\beta < -1$ we define $\pi_I\colon C^{k,\alpha}_\beta \to C^{k,\alpha}(\mathbf{R}^4, \ker \delta_I)$ by

$$\pi_I(a)(x) := \sum_\kappa \langle a(x,\cdot), \kappa\rangle_{L^2(\mathbf{R}^4)}\,\kappa$$

where $\kappa$ runs through an $L^2$ orthonormal basis of $\ker\delta_I$ and set

$$\mathfrak{A}^{k,\alpha}_\beta := \ker \pi_I \cap C^{k,\alpha}_\beta.$$

The projection operators $\pi_\lambda$ and $\sigma_\lambda$ can be viewed as "global versions" of $\pi_I$. It follows from the discussion following Proposition 3.2 that $\mathbf{L}_I$ defines a linear $\mathbf{L}_I\colon \mathfrak{A}^{1,\alpha}_\beta \to \mathfrak{A}^{0,\alpha}_{\beta-1}$.

The key result of this section is the following.

**Proposition 6.6.** *For $\beta \in (-2,-1)$ the linear operator $\mathbf{L}_I\colon \mathfrak{A}^{1,\alpha}_\beta \to \mathfrak{A}^{0,\alpha}_{\beta-1}$ is invertible.*

The proof rests on the following estimate.

**Proposition 6.7.** *For $\beta \in (-3,-1)$ there is a constant $c > 0$ such that for all $a \in \mathfrak{A}^{1,\alpha}_\beta$ the following holds*

$$\|a\|_{C^{1,\alpha}_\beta} \leqslant c\|\mathbf{L}_I a\|_{C^{0,\alpha}_{\beta-1}} \quad \textit{and} \quad \|a\|_{C^{1,\alpha}_\beta} \leqslant c\|\mathbf{L}_I^* a\|_{C^{0,\alpha}_{\beta-1}}.$$

*Proof of Proposition 6.6 assuming Proposition 6.7.* From Proposition 6.7 it follows that $\mathbf{L}_I\colon \mathfrak{A}^{1,\alpha}_\beta \to \mathfrak{A}^{0,\alpha}_{\beta-1}$ is injective and its image is closed. Thus we can identify its cokernel with the kernel of $\mathbf{L}_I^*\colon \left(\mathfrak{A}^{0,\alpha}_{\beta-1}\right)^* \to \left(\mathfrak{A}^{1,\alpha}_\beta\right)^*$. Since $\beta > -2$, the image of $\pi_I$ is contained in $C^{0,\alpha}_{\beta-1}$ and thus $C^{0,\alpha}_{\beta-1} = \mathfrak{A}^{0,\alpha}_{\beta-1} \oplus \operatorname{im}\pi_I$. Via this splitting we can extend any $b \in \ker \mathbf{L}_I^*$ to an element of $\left(C^{0,\alpha}_{\beta-1}\right)^*$ which still satisfies $\mathbf{L}_I^* b = 0$. By elliptic regularity $b$ is smooth and it follows from Lemma 6.8 that $b$ is invariant under translations in the $\mathbf{R}^4$–direction. Now, $b$ must be contained in $C^{1,\alpha}_{-3-\beta}$. Since $-3-\beta \in (-3,-1)$, it follows that $b = 0$ by Proposition 6.7. Therefore $\mathbf{L}_I$ is also surjective; hence, invertible. □

**Lemma 6.8** ([Wal13a, Lemma A.1]). *Let $E$ be a vector bundle of bounded geometry over a Riemannian manifold $X$ of bounded geometry and with subexponential volume growth, and suppose that $D\colon C^\infty(X,E) \to C^\infty(X,E)$ is a uniformly elliptic operator of second order whose coefficients and their first derivatives are uniformly bounded, that is non-negative, such that $\langle Da, a\rangle \geqslant 0$ for all $a \in W^{2,2}(X,E)$, and formally self-adjoint. If $a \in C^\infty(\mathbf{R}^n \times X, E)$ satisfies*

$$(\Delta_{\mathbf{R}^n} + D)a = 0$$

*and $\|a\|_{L^\infty}$ is finite, then $a$ is constant in the $\mathbf{R}^n$–direction, that is $a(x,y) = a(y)$. Here, by slight abuse of notation, we denote the pullback of $E$ to $\mathbf{R}^n \times X$ by $E$ as well.*

*Proof of Proposition 6.7.* We restrict to the case of $\mathbf{L}_I$ as the case $\mathbf{L}_I^*$ differs only by a slight change in notation. First, it is easy to see that there are Schauder estimates, cf. [Wal13a, Proposition 7.6],

$$\|a\|_{C^{1,\alpha}_\beta} \leqslant c\left(\|\mathbf{L}_I a\|_{C^{0,\alpha}_{\beta-1}} + \|a\|_{L^\infty_\beta}\right)$$

with $c = c(\beta) > 0$. The crucial step is then to show that if $\beta \in (-3,-1)$ there is a constant $c > 0$ such that for all $a \in \mathfrak{A}^{1,\alpha}_\beta$ we have

$$\|a\|_{L^\infty_\beta} \leqslant c\|\mathbf{L}_I a\|_{C^{0,\alpha}}.$$

This is proved by contradiction: Suppose the estimate does not hold. Then there exists a sequence $a_i \in \mathfrak{A}^{1,\alpha}_\beta$ such that

$$\|a_i\|_{L^\infty_\beta} = 1 \quad \text{and} \quad \|\mathbf{L}_I a_i\|_{C^{0,\alpha}_{\beta-1}} \leqslant \frac{1}{i}.$$

Hence, by the above Schauder estimate

$$\|a_i\|_{C^{1,\alpha}_\beta} \leqslant 2c.$$

Pick $(x_i, y_i) \in \mathbf{R}^4 \times \mathbf{R}^4$ such that

$$w(x_i,y_i)^{-\beta}|a_i(x_i,y_i)| = 1.$$

By translation we can assume that $x_i = 0$. Without loss of generality one of the following two cases must occur. We rule out both of them thus proving the estimate.

**Case 1.** *The sequence $|y_i|$ stays bounded.*

Let $K$ be a compact subset of $\mathbf{R}^8$. When restricted to $K$, the elements $a_i$ are uniformly bounded in $C^{1,\alpha}$. Thus, by Arzelà–Ascoli, we can assume (after passing to a subsequence) that $a_i$ converges to a limit $a$ in $C^{1,\alpha/2}$. Since $K$ was arbitrary, this yields $a \in \Omega^1(\mathbf{R}^8, \mathfrak{g}_E)$ satisfying

$$|a|(x,y) < c(1+|y|)^\beta$$

as well as

$$\mathbf{L}_I a = 0 \quad \text{and} \quad \pi_I a = 0.$$

It follows from Lemma 6.8 that $a = 0$. On the other hand we can assume that $y_i$ converges to some point $y \in \mathbf{R}^4$ for which we would have $|a|(0,y) = w(0,y)^\beta \neq 0$. This is a contradiction.

**Case 2.** *The sequence $|y_i|$ goes to infinity.*

Define a rescaled sequence $\tilde{a}_i$ by

$$\tilde{a}_i(x,y) := |y_i|^{-\beta}(\xi_i, a_i)(|y_i|x, |y_i|y)$$

and set $\tilde{y}_i = y_i/|y_i|$. The rescaled sequence then satisfies

$$\|\tilde{a}_i\|_{\tilde{C}^{1,\alpha}_{\beta}} \leqslant 2c, \quad \|\mathbf{L}\tilde{a}_i\|_{\tilde{C}^{0,\alpha}_{\beta-1}} \leqslant 2/i \quad \text{and} \quad \tilde{w}(0,\tilde{y}_i)^{-\beta}|\tilde{a}_i(0,\tilde{y}_i)| \geqslant 1/2$$

where the norms $\|\cdot\|_{\tilde{C}^{k,\alpha}_{\beta}}$ are defined as those in Definition 6.4, but with weight function $w(x) = |\pi_2(x)|$ instead of $w(x) = 1 + |\pi_2(x)|$, and where $L$ is defined by

$$\mathbf{L} := \partial_t - L$$

with

$$L(\xi, a) := (\mathrm{d}^* a, \mathrm{d}\xi + *(\psi \wedge \mathrm{d}a)).$$

We can now pass to a limit using Arzelà–Ascoli as before to obtain $\tilde{a}$ defined over $\mathbf{R}^4 \times (\mathbf{R}^4 \backslash \{0\})$ satisfying

$$|\tilde{a}|(x,y) < c|y|^{\beta} \quad \text{and} \quad L\tilde{a} = 0.$$

Since $\beta > -3$, $L\tilde{a} = 0$ holds on all of $\mathbf{R}^8$ in the sense of distributions. Hence, by standard elliptic theory, $\tilde{a}$ extends to a bounded smooth solution of $L\tilde{a} = 0$ on $\mathbf{R}^8$. Since $L^*L = \Delta_{\mathbf{R}^4} + \Delta_{\mathbf{R}^4}$, it follows from Lemma 6.8 that $\tilde{a}$ is invariant in the $\mathbf{R}^4$–direction. Therefore, we can think of the components of $\tilde{a}$ as harmonic functions on $\mathbf{R}^4$. These decay to zero at infinity as $\beta < 0$ and, hence, must vanish identically. On the other hand we know that $|\tilde{y}_i| = 1$ and thus without loss of generality $\tilde{y}_i$ converges to some point $\tilde{y}$ in the unit sphere for which $|\tilde{a}|(0,\tilde{y}) \geqslant \frac{1}{2}$, contradicting $\tilde{a} = 0$. □

## 6.2 The model away from $Q$

**Definition 6.9.** Define weighted Hölder norms $\|\cdot\|_{C^{k,\alpha}_{\beta}}$ for tensor fields (with values in $\mathfrak{g}_E$) on $X\backslash Q$ by

$$[f]_{C^{0,\alpha}_{\beta}} := \sup_{d(x,y)\leqslant w(x,y)} w(x,y)^{\alpha-\beta}\frac{|f(x)-f(y)|}{d(x,y)^{\alpha}}.$$

$$\|f\|_{L^{\infty}_{\beta}} := \|w^{-\beta}f\|_{L^{\infty}} \quad \text{and}$$

$$\|f\|_{C^{k,\alpha}_{\beta}} := \sum_{j=0}^{k}\|\nabla^j f\|_{L^{\infty}_{\beta-j}} + [\nabla^j f]_{C^{0,\alpha}_{\beta-j}}.$$

with weight functions given by

$$w(x) := r(x) \quad \text{and} \quad w(x,y) := \min\{w(x), w(y)\}.$$

(Recall, that $r\colon X \to [0,\infty)$ is defined by $r(x) = d(\cdot, Q)$.

If we fix a constant $N > 0$, then over $V_{[\sqrt{\lambda}/N,\infty)}$ we can view a tensor field $f$ with values in $\mathfrak{g}_{E_\lambda}$ as one which takes values in $\mathfrak{g}_E$ and vice versa.

**Proposition 6.10.** *There is a constant $c > 0$ such that for $\lambda \in (0, \Lambda]$ with respect to the above identification we have*

$$\frac{1}{c}\|a\|_{C^{k,\alpha}_{-\ell+\delta}\left(V_{[\sqrt{\lambda}/N,\infty)}\right)} \leqslant \|a\|_{C^{k,\alpha}_{\ell,\delta,\lambda}\left(V_{[\sqrt{\lambda}/N,\infty)}\right)} \leqslant cN^{-2\delta}\|a\|_{C^{k,\alpha}_{-\ell+\delta}\left(V_{[\sqrt{\lambda}/N,\infty)}\right)}$$

*and*

$$\|L_\lambda a - L_{A_0}a\|_{C^{0,\alpha}_{-2,\delta,\lambda}\left(V_{[\sqrt{\lambda}/N,\infty)}\right)} \leqslant c\sqrt{\lambda}/N|a\|_{C^{1,\alpha}_{-1,\delta,\lambda}\left(V_{[\sqrt{\lambda}/N,\infty)}\right)}.$$

**Proposition 6.11.** *For $\beta \in (-3, 0)$ the operator $L_{A_0}\colon\ C^{1,\alpha}_\beta \to C^{0,\alpha}_{\beta-1}$ has a right inverse $R_{A_0}$.*

*Proof.* Denote by $\pi\colon\ C^{1,\alpha}_\beta \to \ker L_A$ the $L^2$–projection to the (smooth) kernel of $L_A$. This is well defined, because $\beta > -3$. We will shortly prove the estimates

$$\|a\|_{C^{1,\alpha}_\beta} \leqslant c\left(\|L_A a\|_{C^{0,\alpha}_{\beta-1}} + \|\pi a\|_{L^\infty_\beta}\right)$$
$$\text{and} \quad \|a\|_{C^{1,\alpha}_\beta} \leqslant c\|L^*_A a\|_{C^{0,\alpha}_{\beta-1}}.$$

From the first estimate it follows immediately that the image of $L_A\colon\ C^{1,\alpha}_\beta \to C^{0,\alpha}_{\beta-1}$ is closed and its kernel is finite-dimensional (in fact, it can be seen to agree with the smooth kernel of $L_A$). To show that $L_A$ has a right inverse it suffices to prove that $\operatorname{coker} L_A = 0$. Let $b \in \ker L^*_A \cong \operatorname{coker} L_A$. Then using elliptic regularity it can be seen that $b$ represents an element in the kernel of $L^*_A\colon\ C^{1,\alpha}_{-3-\beta} \to C^{0,\alpha}_{-4-\beta}$. But then $b = 0$ by the second estimate.

Now we are left with proving the above estimates. We will only prove the first estimate, since the proof of the second estimate is similar, but slightly easier. First of all we have the following Schauder estimate

$$\|a\|_{C^{1,\alpha}_{\beta,t}} \leqslant c(\|L_A a\|_{C^{0,\alpha}_{\beta-1,t}} + \|a\|_{L^\infty_{\beta,t}}).$$

To prove that

$$\|a\|_{L^\infty_{\beta,t}} \leqslant c\left(\|L_A a\|_{C^{0,\alpha}_{\beta-1,t}} + \|\pi a\|_{L^\infty_{\beta,t}}\right)$$

one argues by contradiction. If $a_i$ is a sequence of counterexamples as before, then we can assume that it either gives rise to a non-trivial element $a$ in the kernel of $L_A\colon\ C^{1,\alpha}_\beta \to C^{0,\alpha}_{\beta-1}$ which also satisfies $\pi a = 0$ or localises in smaller and smaller neighbourhoods of $Q$. To see that the first case cannot occur observe that if $a \in C^{1,\alpha}_\beta$ solves $L_A a = 0$ on $X\backslash Q$, then it follows that $L_A a = 0$ on all of $X$ in the sense of distributions and thus $a$ extends smoothly to $X$, since $\beta > -3$. This contradicts $\pi a = 0$. Thus we must be in the second case. Rescaling $a_i$ near $Q$ as before yields a non-trivial harmonic function on $\mathbf{R}^4 \times \mathbf{R}^4\backslash\{0\}$ which is bounded by a constant multiple of $|y|^\beta$.

Since $\beta > -3$ the function extends to $\mathbf{R}^8$ and by Lemma 6.8 it is invariant in the $\mathbf{R}^4$–direction. Hence, it corresponds to a decaying harmonic function on $\mathbf{R}^4$, since $\beta < 0$, and must vanish identically. So the second case does not occur either; thus proving that the claimed estimate must hold. □

### 6.3 Comparison of $\mathfrak{K}_\lambda$ with $F_\mathfrak{I}$

**Proposition 6.12.** *There is a constant $c > 0$ such that for all $\lambda \in (0, \Lambda]$ we have*

$$\|(L_\lambda \mu_\lambda - \nu_\lambda F_\mathfrak{I})\hat{\mathfrak{I}}\|_{C^{0,\alpha}_{-2,0;\lambda}} \leqslant c\lambda^2 \|\hat{\mathfrak{I}}\|_{C^{1,\alpha}}.$$

**Corollary 6.13.** *There is a constant $c > 0$ such that for all $\lambda \in (0, \Lambda]$ we have*

$$\|(\sigma_\lambda L_\lambda \mu_\lambda - F_\mathfrak{I})\hat{\mathfrak{I}}\|_{C^{0,\alpha}} \leqslant c\lambda^{1-\alpha} \|\hat{\mathfrak{I}}\|_{C^{1,\alpha}}.$$

*Proof of Proposition 6.12.* We use the model operator $\tilde{L}_\lambda$ defined by

$$\tilde{L}_\lambda a := \left(\mathrm{d}^*_{I_\lambda} a, \pi_7^0(\mathrm{d}_{I_\lambda} a)\right).$$

If we view $\Gamma(\mathfrak{I}^* V\mathfrak{M})$ as a subspace of $\Omega^1(NQ, \mathfrak{g}_E)$, then on this subspace $\tilde{L}_\lambda$ agrees with the linearised Fueter operator $F_\mathfrak{I}$. We thus have to estimate the terms in the expression

$$\begin{aligned} L_\lambda \mu_\lambda \hat{\mathfrak{I}} - \nu_\lambda F_\mathfrak{I} \hat{\mathfrak{I}} &= L_\lambda(\mu_\lambda \hat{\mathfrak{I}} - \hat{\mathfrak{I}}_\lambda) + (L_\lambda - \tilde{L}_\lambda)\hat{\mathfrak{I}} + s^*_{1/\lambda} F_\mathfrak{I} \hat{\mathfrak{I}} - \nu_\lambda F_\mathfrak{I} \hat{\mathfrak{I}} \\ &=: \mathrm{I} + \mathrm{II} + \mathrm{III} \end{aligned}$$

on $V_{[0,\zeta)}$. It is easy to see that

$$\|\mathrm{I}\|_{C^{0,\alpha}_{-2,0;\lambda}(V_{[0,\zeta)})} + \|\mathrm{III}\|_{C^{0,\alpha}_{-2,0;\lambda}(V_{[0,\zeta)})} \leqslant c\lambda^2 \|\hat{\mathfrak{I}}\|_{C^{1,\alpha}}.$$

by using that fact that I and III are supported in $V_{[\sigma/2,\sigma)}$ and the estimates

$$\|L_\lambda a\|_{C^{0,\alpha}_{-2,0;\lambda}(V_{[0,\sigma)})} \leqslant c\|a\|_{C^{1,\alpha}_{-1,0;\lambda}(V_{[0,\sigma)})} \quad \text{and} \quad \|F_\mathfrak{I} \hat{\mathfrak{I}}\|_{C^{0,\alpha}} \leqslant c\|\hat{\mathfrak{I}}\|_{C^{1,\alpha}}$$

as well as

$$\begin{aligned} \|\mu_\lambda \hat{\mathfrak{I}} - \hat{\mathfrak{I}}_\lambda\|_{C^{k,\alpha}_{-\ell,0,\lambda}(V_{[\sigma/2,\sigma)})} &\leqslant \|\chi^+ - 1\|_{C^{k,\alpha}_{\ell+3,0;\lambda}(V_{[\sigma/2,\sigma)})} \cdot \|\hat{\mathfrak{I}}_\lambda\|_{C^{k,\alpha}_{-3,0;\lambda}(V_{[\sigma/2,\sigma)})} \\ &\leqslant c\lambda^2 \|\hat{\mathfrak{I}}\|_{C^{k,\alpha}} \end{aligned}$$

and a similar estimate for $\nu_\lambda$.

The key for the estimate of II is to notice that

$$\pi_7^0\left((\mathrm{d}_{I_\lambda}\hat{\mathfrak{I}})^{0,2}\right) = \pi_7^1\left((\mathrm{d}_{I_\lambda}\hat{\mathfrak{I}})^{0,2}\right) = 0,$$

because $\delta_{\mathfrak{I}(x)}(\hat{\mathfrak{I}}|_{N_xQ}) = 0$ and $\pi_7^0$ and $\pi_7^1$ vanish on $\Lambda^- NQ$. Therefore,

$$\begin{aligned}\mathrm{II} &= \pi_7\left((A_\lambda - I_\lambda) \wedge \hat{\mathfrak{I}}_\lambda\right) + \pi_7^1\left((\mathrm{d}_{I_\lambda}\hat{\mathfrak{I}}_\lambda)^{2,0} + (\mathrm{d}_{I_\lambda}\hat{\mathfrak{I}}_\lambda)^{1,1}\right) + \pi_7^{\geqslant 2}(\mathrm{d}_{I_\lambda}\hat{\mathfrak{I}}_\lambda)\\ &=: \mathrm{II}_1 + \mathrm{II}_2 + \mathrm{II}_3.\end{aligned}$$

It follows from Proposition 5.19 that

$$\|A_\lambda - I_\lambda\|_{C^{0,\alpha}_{1,0;\lambda}(V_{[0,\sigma)})} = \|\chi_\lambda^- a + (\chi^+ - 1)i_\lambda\|_{C^{0,\alpha}_{1,0;\lambda}(V_{[0,\sigma)})} \leqslant c \tag{6.14}$$

which in conjunction with

$$\|\hat{\mathfrak{I}}_\lambda\|_{C^{k,\alpha}_{-3,0;\lambda}(V_{[0,\sigma)})} \leqslant c\lambda^2 \|\hat{\mathfrak{I}}\|_{C^{k,\alpha}} \tag{6.15}$$

yields

$$\|\mathrm{II}_1\|_{C^{0,\alpha}_{-2,0;\lambda}} \leqslant c\lambda^2\|\hat{\mathfrak{I}}\|_{C^{1,\alpha}}.$$

$\mathrm{II}_2$ and $\mathrm{II}_3$ can be estimated using Proposition 5.16, Proposition 5.19 and (6.15). □

## 6.4 Estimate of $\mathfrak{p}_\lambda$ and $\mathfrak{q}_\lambda$

**Proposition 6.16.** *For $\delta \in (-1, 0)$ there exists a constant $c > 0$ such that for all $\lambda \in (0, \Lambda]$ we have*

$$\begin{aligned}\|\sigma_\lambda \mathfrak{p}_\lambda a\|_{C^{0,\alpha}} &\leqslant c\lambda^{-\alpha}\|\rho_\lambda a\|_{C^{1,\alpha}_{-1,\delta;\lambda}} \quad \textit{and}\\ \|\mathfrak{q}_\lambda a\|_{C^{0,\alpha}_{-2,\delta;\lambda}} &\leqslant c\lambda^{2+\delta/2-\alpha}\|\pi_\lambda a\|_{C^{1,\alpha}}.\end{aligned}$$

*Proof.* First note that the second estimate is an immediate consequence of Proposition 5.11 and Proposition 6.12, because

$$\mathfrak{q}_\lambda a = \tau_\lambda(L_\lambda \mu_\lambda - \nu_\lambda F_{\mathfrak{I}})\mu_\lambda a,$$

since $\tau_\lambda \nu_\lambda = 0$. Now, to estimate $\mathfrak{p}_\lambda$ we define

$$\tilde{\pi}_\lambda \colon\ \Omega^1(NQ, \mathfrak{g}_{E(\mathfrak{I}_\lambda)}) \to \Gamma(\mathfrak{I}^* V\mathfrak{M}) \subset \Omega^1(NQ, \mathfrak{g}_{E(\mathfrak{I}_\lambda)})$$

by

$$(\tilde{\pi}_\lambda a)(x) \coloneqq \sum_\kappa \int_{N_xQ} \langle a, \kappa\rangle\, \kappa$$

and

$$\tilde{\sigma}_\lambda \colon\ \Omega^2(NQ, \mathfrak{g}_{E(\mathfrak{I}_\lambda)}) \to \Gamma(\mathrm{Hom}_\Phi(TQ, \mathfrak{I}^* V\mathfrak{M})) \subset \Omega^2(NQ, \mathfrak{g}_{E(\mathfrak{I}_\lambda)})$$

by

$$(\tilde{\sigma}_\lambda \alpha)(x) \coloneqq \sum_\beta \int_{N_xQ} \langle \alpha, \beta\rangle\, \beta.$$

Here, at each point $x \in Q$, $\kappa$ runs through an orthonormal basis of $V\mathfrak{M}_{\mathfrak{I}(x)}$ and $\beta$ runs through an orthonormal basis of $\mathrm{Hom}_\Phi(T_xQ, V\mathfrak{M}_{\mathfrak{I}(x)})$. We set $\tilde{\rho}_\lambda \coloneqq \mathrm{id} - \tilde{\pi}_\lambda$ and $\tilde{\tau}_\lambda \coloneqq \mathrm{id} - \tilde{\sigma}_\lambda$. One can check that $\tilde{\sigma}_\lambda \tilde{L}_\lambda \tilde{\rho}_\lambda = 0$. For $a$ supported in $V_{[0,\sigma)}$, which we can assume without loss of generality,

$$\begin{aligned}\mathfrak{p}_\lambda a &= \bar{\sigma}_\lambda (L_\lambda - \tilde{L}_\lambda)\rho_\lambda a + (\bar{\sigma}_\lambda - \tilde{\sigma}_\lambda)\tilde{L}_\lambda \rho_\lambda a + \tilde{\sigma}_\lambda \tilde{L}_\lambda (\rho_\lambda - \tilde{\rho}_\lambda) a \\ &= \bar{\sigma}_\lambda \mathrm{I} + \mathrm{II} + \tilde{\sigma}_\lambda \mathrm{III}.\end{aligned}$$

The terms II and III (resp. I) can be estimated similar to I and III (resp. II) in the proof of Proposition 6.12. □

## 6.5 Patching local inverses

*Proof of Proposition 6.1.* Fix $y \in \mathfrak{Y}_\lambda$ and set

$$u \coloneqq \bar{\sigma}_\lambda y \quad \text{and} \quad v \coloneqq \tau_\lambda y.$$

**Step 1.** *An approximate inverse for u.*

Denote by $G_{\mathfrak{I}}$ a fixed right inverse of $F_{\mathfrak{I}}$ and set

$$z \coloneqq \mu_\lambda G_{\mathfrak{I}} \sigma_\lambda u.$$

We have

$$\|z\|_{\mathfrak{X}_\lambda} \leqslant c\|y\|_{\mathfrak{Y}_\lambda}$$

and by Corollary 6.13 and Proposition 6.16 we have

$$\|L_\lambda z - u\|_{\mathfrak{Y}_\lambda} \leqslant c\lambda^{1-\alpha}\|y\|_{\mathfrak{Y}_\lambda}. \tag{6.17}$$

**Step 2.** *Choice of cut-off functions.*

We construct an approximate inverse for $v$ by finding local approximate inverses and then patching these together. This requires two kinds of cut-off functions. The first kind is constructed as follows: Let $\chi\colon [0,\infty) \to [0,1]$ denote the smooth-cut off function chosen in Section 5 which vanishes on $[0,1]$ and is equal to one on $[2,\infty)$. We define $\chi_\lambda\colon X \to [0,1]$ by

$$\chi_\lambda(x) \coloneqq \chi(r(x)/\sqrt{\lambda}).$$

Then

$$\|\chi_\lambda\|_{C^{0,\alpha}_{0,0;\lambda}} \leqslant c.$$

Fix a small constant $\varepsilon > 0$, a large constant $N \gg 1$, and note that in the following we can choose the constant $c > 0$ independent of $\varepsilon$ and $N$. Throughout, we will make use of $\lambda \ll \varepsilon$ and $\lambda \ll 1/N$. We can pick a finite number of points $\{x_\gamma : \gamma \in \Gamma\} \subset Q$ such that the balls $B_\varepsilon(x_\gamma)$ cover all of $Q$ and a partition of unity $1 = \sum_{\gamma\in\Gamma} \chi_\gamma$ subordinate to this cover such that

$$\|\chi_\gamma\|_{C^{0,\alpha}_{0,0;\lambda}(\mathrm{supp}(1-\chi_\lambda))} \leqslant c\varepsilon^{-\alpha}.$$

We can now write

$$v = \sum_{\gamma \in \Gamma} v_\gamma + v_0$$

with

$$v_\gamma \coloneqq (1 - \chi_\lambda)\chi_\gamma v \quad \text{and} \quad v_0 \coloneqq \chi_\lambda v.$$

Although $v_0$ and the $v_\gamma$ depend on $\lambda$ we choose not to make this dependence explicit in order not to clutter the notation any more. By construction we have

$$\sum_\gamma \|v_\gamma\|_{C^{0,\alpha}_{-2,\delta;\lambda}} + \|v_0\|_{C^{0,\alpha}_{-2,\delta;\lambda}} \leqslant c\varepsilon^{-\alpha}\|v\|_{C^{0,\alpha}_{-2,\delta;\lambda}}. \tag{6.18}$$

The second kind of cut-off functions is constructed as follows: We choose $\beta^\pm_{\lambda,N}\colon X \to [0,1]$ such that

$$\beta^+_{\lambda,N}(x) = \begin{cases} 1 & r(x) \leqslant 2\sqrt{\lambda} \\ 0 & r(x) \geqslant 2N\sqrt{\lambda} \end{cases}$$

and

$$\beta^-_{\lambda,N}(x) = \begin{cases} 0 & r(x) \leqslant \sqrt{\lambda}/N \\ 1 & r(x) \geqslant \sqrt{\lambda} \end{cases}$$

as well as

$$\|\mathrm{d}\beta^\pm_{\lambda,N}\|_{C^{0,\alpha}_{-1,0;\lambda}} \leqslant c/\log(N) \quad \text{and} \quad \|\beta^\pm_{\lambda,N}\|_{C^{0,\alpha}_{0,0;\lambda}} \leqslant c. \tag{6.19}$$

This can be arranged by interpolating between 0 and 1 logarithmically, i.e., by defining $\beta^+_{\lambda,N}$ as an appropriate smoothing of $\log(2N\sqrt{\lambda}/r)/\log(N)$ in the intermediate region and similarly $\beta^-_{\lambda,N}$ as a smoothing of $\log(Nr/\sqrt{\lambda})/\log(N)$. Moreover, we choose $\tilde\chi_\gamma\colon Q \to [0,1]$ such that $\tilde\chi_\gamma$ equals one on $B_\varepsilon(x_\gamma)$, $\tilde\chi_\gamma$ vanishes outside $B_{2\varepsilon}(x_\gamma)$ and satisfies

$$\|\mathrm{d}\tilde\chi_\gamma\|_{C^{0,\alpha}_{-1,0;\lambda}\left(\operatorname{supp}\beta^+_{\lambda,N}\right)} \leqslant cN\sqrt{\lambda}/\varepsilon^{1+\alpha} \quad \text{and} \quad \|\tilde\chi_\gamma\|_{C^{0,\alpha}_{0,0;\lambda}\left(\operatorname{supp}\beta^+_{\lambda,N}\right)} \leqslant c. \tag{6.20}$$

**Step 3**. *Construction of local approximate inverses.*

Let $I_\gamma$ be the ASD instanton obtained by restricting $I = I(\mathfrak{I})$ to $N_{x_\gamma}Q$. Using the identifications and the notation of Section 6.1 we define

$$\tilde w_\gamma \coloneqq s_{1,\lambda}^{-1}\mathbf{L}_{I_\gamma}^{-1}\rho_{I_\gamma}s_{2,\lambda}v_\gamma \quad \text{and} \quad w_\gamma \coloneqq \rho_\lambda\tilde\chi_\gamma\beta^+_{\lambda,N}\tilde w_\gamma.$$

where $\rho_{I_\gamma} \coloneqq \mathrm{id} - \pi_{I_\gamma}$. Under the identifications employed in Section 6.1 the projections $\pi_\lambda$ and $\sigma_\lambda$ are identified. From $\sigma_\lambda v = 0$ one can deduce that

$$\|\pi_{I_\gamma}s_{2,\lambda}v_\gamma\|_{C^{0,\alpha}_{-2-\delta}} \leqslant c\varepsilon\|s_{2,\lambda}v_\gamma\|_{C^{0,\alpha}_{-2-\delta}} \leqslant c\varepsilon\|v_\gamma\|_{C^{0,\alpha}_{-2,\delta;\lambda}}.$$

Using Proposition 6.5 we conclude that

$$\|\tilde{w}_\gamma\|_{C^{1,\alpha}_{-1,\delta;\lambda}(V_{2\varepsilon,\zeta})} \leqslant c\|s_{1,\lambda}\tilde{w}_\gamma\|_{C^{1,\alpha}_{-1+\delta}(U_{2\varepsilon,\infty;\lambda})} \leqslant c\|v_\gamma\|_{C^{0,\alpha}_{-2,\delta;\lambda}} \tag{6.21}$$

and

$$\|L_\lambda \tilde{w}_\gamma - v_\gamma\|_{C^{0,\alpha}_{-2,\delta;\lambda}(V_{2\varepsilon,\zeta})} \leqslant c\varepsilon\|v_\gamma\|_{C^{0,\alpha}_{-2,\delta;\lambda}} \tag{6.22}$$

Since $\pi_{I_\gamma}(s_{1,\lambda}\tilde{w}_\gamma) = 0$, it follows that

$$\|\tilde{\pi}_{I_\gamma} s_{1,\lambda}\tilde{w}_\gamma\|_{C^{1,\alpha}_{-1+\delta}(U_{2\varepsilon,\infty;\lambda})} \leqslant c\varepsilon\|s_{1,\lambda}\tilde{w}_\gamma\|_{C^{1,\alpha}_{-1+\delta}(U_{2\varepsilon,\infty;\lambda})}$$

here $\tilde{\pi}_{I_\gamma}$ is defined like $\pi_{I_\gamma}$ but with $\ker \delta_{I|_{N_{\exp_{x_\gamma}(\lambda\cdot-)}Q}}$ instead of $\ker \delta_{I|_{N_{x_\gamma}Q}}$. Therefore,

$$\|\bar{\pi}_\lambda w_\gamma\|_{C^{1,\alpha}_{-1,\delta;\lambda}} \leqslant c\varepsilon\|v_\gamma\|_{C^{1,\alpha}_{-1,\delta;\lambda}} \tag{6.23}$$

and it follows that

$$\begin{aligned}\sum_\gamma \|w_\gamma\|_{C^{1,\alpha}_{-1,\delta;\lambda}} &\leqslant c(1 + N\sqrt{\lambda}/\varepsilon^{1+\alpha} + 1/\log(N))\sum_\gamma \|v_\gamma\|_{C^{0,\alpha}_{-2,\delta;\lambda}} \\ &\leqslant c\varepsilon^{-\alpha}(1 + N\sqrt{\lambda}/\varepsilon^{1+\alpha} + 1/\log(N))\|v\|_{C^{1,\alpha}_{-1,\delta;\lambda}}.\end{aligned}$$

By Proposition 6.10, $w_0 \coloneqq \beta^-_{\lambda,N} R_{A_0} v_0$, with $R_{A_0}$ as in Proposition 6.11, satisfies

$$\|w_0\|_{\mathfrak{X}_\lambda} \leqslant c\|v_0\|_{\mathfrak{Y}_\lambda}. \tag{6.24}$$

Combining all of the above we see that the $\tilde{R}_\lambda\colon \mathfrak{Y}_\lambda \to \mathfrak{X}_\lambda$ defined by

$$\tilde{R}_\lambda y \coloneqq z + \sum_\gamma w_\gamma + w_0.$$

is bounded by $c\varepsilon^{-\alpha}(1 + N\sqrt{\lambda}/\varepsilon^{1+\alpha} + 1/\log(N))$.

**Step 4.** *$\tilde{R}_\lambda$ is an approximate right inverse to $L_\lambda$.*

We need to estimate the three types of terms

$$\begin{aligned}\mathrm{I} &\coloneqq \|L_\lambda z - u\|_{\mathfrak{Y}_\lambda}, \\ \mathrm{II}_\gamma &\coloneqq \|L_\lambda w_\gamma - v_\gamma\|_{\mathfrak{Y}_\lambda} \quad \text{and} \\ \mathrm{III} &\coloneqq \|L_\lambda w_0 - v_0\|_{\mathfrak{Y}_\lambda}.\end{aligned}$$

We have already treated I with (6.17). Now,

$$\begin{aligned}\mathrm{II}_\gamma = \|L_\lambda w_\gamma - v_\gamma\|_{\mathfrak{Y}_\lambda} &\leqslant \lambda^{-\delta/2}\|L_\lambda \rho_\lambda \tilde{\chi}_\gamma \beta^+_{\lambda,N}\tilde{w}_\gamma - v_\gamma\|_{C^{0,\alpha}_{-2,\delta;\lambda}} \\ &\quad + \lambda\|\sigma_\lambda L\rho_\lambda\tilde{\chi}_\gamma\beta^+_{\lambda,N}\tilde{w}_\gamma - \sigma_\lambda v_\gamma\|_{C^{0,\alpha}}\end{aligned}$$

Using (6.21), Proposition 6.16 and the fact that $\pi_\lambda v = 0$ the last term can be seen to be bounded by $c\lambda^{1-\alpha}\|v_Y\|_{\mathfrak{Y}_\lambda}$. To control the first term use the fact that on the support of $v_Y$ we have $\tilde{\chi}_Y\beta^+_{\lambda,N} = 1$, (6.19), (6.20), (6.21), (6.22) and (6.23) to derive

$$\begin{aligned}\|L_\lambda\rho_\lambda\tilde{\chi}_Y\beta^+_{\lambda,N}\tilde{w}_Y - v_Y\|_{C^{0,\alpha}_{-2,\delta;\lambda}} &\leqslant c\|L_\lambda\tilde{w}_Y - v_Y\|_{C^{0,\alpha}_{-2,\delta;\lambda}(V_{2\varepsilon,\zeta})}\\ &\quad + c\|\mathrm{d}(\tilde{\chi}_\lambda\beta^+_{\lambda,N})\|_{C^{0,\alpha}_{-1,0;\lambda}(\operatorname{supp}\beta^+_{\lambda,N})}\|\tilde{w}_Y\|_{C^{0,\alpha}_{-1,\delta;\lambda}(V_{2\varepsilon,\zeta})}\\ &\quad + c\|\bar{\pi}_\lambda\tilde{\chi}_Y\beta^+_{\lambda,N}\tilde{w}_Y\|_{C^{1,\alpha}_{-1,\delta;\lambda}}\\ &\leqslant c(\varepsilon + 1/\log(N) + N\sqrt{\lambda}/\varepsilon)\|v_\lambda\|_{C^{0,\alpha}_{-2,\delta;\lambda}}.\end{aligned}$$

Similarly,

$$\mathrm{III} \leqslant c(\sqrt{\lambda} + 1/\log(N))\|y\|_{\mathfrak{Y}_\lambda}$$

Putting everything together we obtain

$$\|L_\lambda\tilde{R}_\lambda y - y\|_{\mathfrak{Y}_\lambda} \leqslant c\varepsilon^{-\alpha}(\varepsilon + 1/\log N + N\sqrt{\lambda}/\varepsilon)\|y\|_{\mathfrak{Y}_\lambda}.$$

By choosing $\varepsilon$ small enough, $N$ large enough and $\lambda$ small enough we can make the factor in front of $\|y\|_{\mathfrak{Y}_\lambda}$ arbitrarily small.

**Step 5**. *Construction of $R_\lambda$.*

We can arrange that

$$\|L_\lambda\tilde{R}_\lambda y - y\|_{\mathfrak{Y}_\lambda} \leqslant \frac{1}{2}\|y\|_{\mathfrak{Y}_\lambda}.$$

for all $\lambda \in (0,\Lambda]$; hence, the series

$$R_\lambda \coloneqq \tilde{R}_\lambda(L_\lambda\tilde{R}_\lambda)^{-1} = \tilde{R}_\lambda\sum_{k=0}^{\infty}\left(\mathrm{id} - L_\lambda\tilde{R}_\lambda\right)^k$$

converges and constitutes a right inverse for $L_\lambda$. Clearly, $R_\lambda$ is bounded uniformly with respect to $\lambda \in (0,\Lambda]$. □

## 7 Conclusion of the proof of Theorem 1.2

The last ingredient we need for the proof of Theorem 1.2 is the following estimate on the polarisation

$$Q(a_1, a_2) \coloneqq \frac{1}{2}\pi_7([a_1 \wedge a_2])$$

of the quadratic form $Q$ appearing in (5.1).

**Proposition 7.1.** *There is a constant $c > 0$ such that for all $\lambda \in (0, \Lambda]$ we have*

$$\begin{aligned}
&\|\tau_\lambda Q(a_1, a_2)\|_{C^{0,\alpha}_{-2,\delta;\lambda}} \\
&\quad \leqslant c\lambda^{-\alpha}\Big(\|\rho_\lambda a_1\|_{C^{0,\alpha}_{-1,\delta;\lambda}} \cdot \|\rho_\lambda a_2\|_{C^{0,\alpha}_{-1,\delta;\lambda}} + \|\rho_\lambda a_1\|_{C^{0,\alpha}_{-1,\delta;\lambda}} \cdot \|\pi_\lambda a_2\|_{C^{0,\alpha}} \\
&\qquad + \|\pi_\lambda a_1\|_{C^{0,\alpha}} \cdot \|\rho_\lambda a_2\|_{C^{0,\alpha}_{-1,\delta;\lambda}} + \|\pi_\lambda a_1\|_{C^{0,\alpha}} \|\pi_\lambda a_2\|_{C^{0,\alpha}}\Big)
\end{aligned}$$

*and*

$$\begin{aligned}
&\lambda\|\sigma_\lambda Q(a_1, a_2)\|_{C^{0,\alpha}} \\
&\quad \leqslant c\lambda^{-\alpha}\Big(\|\rho_\lambda a_1\|_{C^{0,\alpha}_{-1,\delta;\lambda}} \cdot \|\rho_\lambda a_2\|_{C^{0,\alpha}_{-1,\delta;\lambda}} + \|\rho_\lambda a_1\|_{C^{0,\alpha}_{-1,\delta;\lambda}} \cdot \|\pi_\lambda a_2\|_{C^{0,\alpha}} \\
&\qquad + \|\pi_\lambda a_1\|_{C^{0,\alpha}} \cdot \|\rho_\lambda a_2\|_{C^{0,\alpha}_{-1,\delta;\lambda}} + \lambda\|\pi_\lambda a_1\|_{C^{0,\alpha}} \cdot \|\pi_\lambda a_2\|_{C^{0,\alpha}}\Big).
\end{aligned}$$

*In particular,*

$$\|Q(a_1, a_2)\|_{\mathfrak{y}_\lambda} \leqslant c\lambda^{-2-\delta/2}\|a_1\|_{\mathfrak{x}_\lambda}\|a_2\|_{\mathfrak{x}_\lambda}$$

*Proof.* The first estimate is an immediate consequence of Proposition 5.8 and Proposition 5.11. For the second estimate we only have to explain why we get a factor $\lambda$ in front of $\|\pi_\lambda a_1\|_{C^{0,\alpha}} \cdot \|\pi_\lambda a_2\|_{C^{0,\alpha}}$. Note that

$$\tilde{\sigma}_\lambda \pi_7^0 \left(\mu_\lambda \hat{\mathfrak{I}}_1 \wedge \mu_\lambda \hat{\mathfrak{I}}_2\right) = 0$$

because of Proposition 2.14 (the $\Lambda^2_4$–component already vanishes). Arguing as in the proof of Proposition 6.12 we see that we gain a factor of $\lambda$. □

Setting $\tilde{Q}_\lambda = Q \circ R_\lambda$, (5.1) becomes

$$x + \tilde{Q}_\lambda(x) + \pi_7(F_{A_\lambda}) = 0.$$

In view of Proposition 5.14, Proposition 6.1 and Proposition 7.1, this equation can be solved by appealing to the following consequence of Banach's fixed-point theorem.

**Lemma 7.2** ([DK90, Lemma 7.2.23]). *Let $X$ be a Banach space and let $T\colon X \to X$ be a smooth map with $T(0) = 0$. Suppose there is a constant $c > 0$ such that*

$$\|Tx - Ty\| \leqslant c\,(\|x\| + \|y\|)\,\|x - y\|.$$

*Then if $y \in X$ satisfies $\|y\| \leqslant \frac{1}{10c}$, there exists a unique $x \in X$ with $\|x\| \leqslant \frac{1}{5c}$ solving*

$$x + Tx = y.$$

*The unique solution satisfies $\|x\| \leqslant 2\|y\|$.*

Elliptic regularity implies that $A_\lambda + a$ is smooth. Since $a$ is small, the existence of a right inverse of $L_{A_\lambda}$ guarantees the existence of a right inverse of $L_{A_\lambda + a}$; hence, $A_\lambda + a$ is irreducible and unobstructed. □

## 8 Proof of Theorem 1.4

Since $\mathrm{Hol}(g_\Phi) = \mathrm{Spin}(7)$, $b^1 = b^2_7 = 0$ [Joy00, Proposition 10.6.5] and thus the product connection $\theta$ on the trivial SU(2)–bundle is unobstructed. It is reducible; however, does not cause any problems, see Remark 5.2. We have index $L_\theta = -3$. If we choose $M$ as in Example 3.4, then

$$\mathfrak{M} = \left(\mathrm{Re}(\mathrm{Hom}(\mathbf{C}^2, \$^+))\backslash\{0\}\right)/\mathbf{Z}_2 \times \mathrm{Re}(\$^+ \otimes U)$$

By Example 4.8, the Fueter operator lifts to the Dirac operator

$$\not{D}\colon\ \Gamma(\mathrm{Re}(\mathrm{Hom}(\mathbf{C}^2, \$^+) \oplus \$^+ \otimes U)) \to \Gamma(\mathrm{Re}(\mathrm{Hom}(\mathbf{C}^2, \$^-) \oplus \$^- \otimes U)).$$

Arguing as in Proposition 2.43, we see that $\not{D}$ is surjective and has an 8–dimensional kernel and all non-zero elements of the kernel are no-where vanishing, provided the metric on $Q$ is sufficiently close to a hyperkähler metric and the induced connection on $NQ$ is almost flat. We can thus apply Theorem 1.2 and obtain a 5–dimensional family of Spin(7)–instantons over $X$. A similar argument also proves the last assertion of the theorem. □

## 9 Comparing index formulae

**Proposition 9.1.** *Let $(X, \Phi)$ be a compact* Spin(7)*–manifold, let $Q$ be a Cayley submanifold of $X$ and let $E_0$ and $E$ be* SU(2)*–bundles over $X$ which are related by*

$$c_2(E) = c_2(E_0) + \mathrm{PD}[Q].$$

*If $A$ is a connection on $E$ and $A_0$ is a connection on $E_0$, then*

$$\mathrm{index}\, L_A = \mathrm{index}\, L_{A_0} + \mathrm{index}\, F_Q + \mathrm{index}\, \mathring{F} + \frac{5}{3}\int_Q e(\mathrm{Re}(\mathrm{Hom}(E_0, \$^+_Q))) \tag{9.2}$$

*where $L_A$ and $L_{A_0}$ are as in (2.21) and*

$$\mathring{F} = \not{D}\colon\ \Gamma(\mathrm{Re}(\mathrm{Hom}(E_\infty, \$^+))) \to \Gamma(\mathrm{Re}(\mathrm{Hom}(E_\infty, \$^-)))$$

*as in Example 4.12.*

In the situation of Proposition 9.1 whenever Theorem 1.2 can be applied $e(\mathrm{Re}(\$^+_Q \otimes E_0))$ vanishes. This is because in those situation the Fueter section $\mathfrak{I}$ gives rise to a no-where vanishing section of $\mathrm{Re}(\$^+_Q \otimes E_0)$. Hence, (9.2) can be taken as evidence that Theorem 1.2 gives a description of an open subset of the moduli space of Spin(7)–instantons. (Note that the gluing parameter $\lambda$ is already contained in index $\mathring{F}_{\mathfrak{I}}$.)

*Proof of Proposition 9.1.* By Proposition 2.23 and (2.42) we have

$$\begin{aligned}\operatorname{index} L_{A_\lambda} - \operatorname{index} L_{A_0} &= -\frac{1}{6}\int_Q p_1(X) - \frac{4}{3}[Q]\cdot[Q] - \frac{8}{3}\int_Q c_2(E_0) \\ &= -\sigma(Q) - \frac{1}{3}\chi(Q) - [Q]\cdot[Q] - \frac{8}{3}\int_Q c_2(E_0).\end{aligned}$$

By (2.37) and (4.13) we have

$$\operatorname{index} F_Q + \operatorname{index} \mathring{F}_{\mathfrak{J}} = \frac{1}{4}\sigma(Q) + \frac{1}{2}\chi(Q) - [Q]\cdot[Q] - \int_Q c_2(E_0).$$

Using (2.40) and (2.41)

$$\begin{aligned}\int_Q e(\mathrm{Re}(\mathrm{Hom}(E_0, \$_Q^+)) &= \int_Q e(\mathrm{Re}(E_0^* \otimes \$_Q^+)) \\ &= \int_Q e(\mathrm{Re}(E_0 \otimes \$_Q^+)) \\ &= -\int_Q c_2(E_0) - \frac{3}{4}\sigma(Q) - \frac{1}{2}\chi(Q).\end{aligned}$$

Verifying (9.2) is now straight-forward. □

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