Super-rigidity and Castelnuovo's bound

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Abstract

This paper is concerned with the question of whether an energy bound implies a genus bound for pseudo-holomorphic curves in almost complex manifolds. After reviewing what is known in dimensions other than 6, we establish a new result in this direction in dimension 6; in particular, for symplectic Calabi–Yau 6–manifolds. The proof relies on compactness and regularity theorems for J–holomorphic currents.

1 Introduction

In 1889, Castelnuovo [Cas89] found a sharp upper bound for the genus of an irreducible degree d curve in $\mathbf{P}^n$; see [ACGH85, Chapter III Section 2] for a proof in modern language. A corollary of this result is that for every projective variety there is an upper bound for the genus of an irreducible curve representing a given homology class. Our starting point is the question:

Are there analogues of Castelnuovo's bound in almost complex geometry?

For plane curves Castelnuovo's bound reduces to the degree-genus formula. The latter is a consequence of the adjunction formula, which generalizes to an inequality for almost complex 4–manifolds [MS12, Theorem 2.6.4]. The adjunction inequality directly implies the following well-known genus bound.

Proposition 1.1. *Suppose that (M, J) is an almost complex 4–manifold. If there exists a simple J–holomorphic map $u\colon \Sigma \to M$ representing $A \in H_2(M)$, then the genus $g(\Sigma)$ satisfies*

$$g(\Sigma) \leqslant \frac{1}{2}\left(A \cdot A - \langle c_1(M), A\rangle\right) + 1. \tag{1.2}$$

We are not aware of any *universal* genus bounds in higher dimensions. There are, however, genus bounds for *generic* almost complex structures. Here is a simple example.

Proposition 1.3. *Let M be a manifold of dimension $2n \geqslant 8$. Denote by $\mathscr{J}$ the space of almost complex structures of class C^2 on M. There is a residual subset $\mathscr{J}_\clubsuit \subset \mathscr{J}$ such that for every $J \in \mathscr{J}_\clubsuit$ the following holds: if there exists a simple J–holomorphic map $u\colon \Sigma \to M$ representing $A \in H_2(M)$, then*

$$g(\Sigma) \leqslant \frac{\langle c_1(M), A\rangle}{n-3} + 1. \tag{1.4}$$

Proof. Denote by $\mathscr{M}$ the component of the universal moduli space of simple pseudo-holomorphic maps into M of genus g. This is a separable Banach manifold and the projection map $\pi\colon \mathscr{M} \to \mathscr{J}$ is a Fredholm map of class C^1 and index

$$(n-3)(2-2g) + 2\langle c_1(M), A\rangle;$$

see, e.g., [Wen10, Theorem 0; IP18, Proposition 5.1]. If (1.4) is violated, then this index is negative. The result thus follows from the Sard–Smale theorem [Sma65]. □

Remark 1.5. If M carries as symplectic form ω, then Proposition 1.3 holds with $\mathscr{J}$ replaced by the space $\mathscr{J}(\omega)$ of almost complex structures of class C^2 compatible with ω.

Remark 1.6. The proof of Proposition 1.3 also shows that if $n = 3$ and $\langle c_1(M), A\rangle < 0$, then for a generic almost complex structure J there are no J–holomorphic maps representing A whatsoever.

The preceding discussion leaves open the case of almost complex manifolds of dimension six and homology classes satisfying $\langle c_1(M), A\rangle \geqslant 0$. In this article we focus on the case

$$\langle c_1(M), A\rangle = 0,$$

that is: on classes for which the corresponding moduli space of J–holomorphic maps has dimension zero. This includes all homology classes in symplectic Calabi–Yau 3–folds. Our motivation for considering this case comes from our project to construct a symplectic analogue of the Pandharipande–Thomas invariants of projective Calabi–Yau 3–folds [DW17, Section 9]. Another motivation comes from the Gopakumar–Vafa conjecture. Bryan and Pandharipande [BP01] defined the Gopakumar–Vafa BPS invariants $n_A^g(M)$ of a symplectic Calabi–Yau 3–fold M in terms of its Gromov–Witten partition function. They conjectured that the BPS invariants $n_A^g(M)$ are integers and vanish for large genus g [BP01, Conjecture 1.2]. The integrality conjecture has been proved by Ionel and Parker [IP18]. The vanishing conjecture remains open and is closely related to the question about the existence of genus bounds for symplectic Calabi–Yau 3–folds.

Bryan and Pandharipande introduced the notion of *super-rigidity* for almost complex structures; see Definition 2.9. They conjectured that it is a generic property. This conjecture has now been proved by Wendl [Wen16]; see also [DW18] for a concise version of his proof. The main result of this paper shows that super-rigidity implies a Castelnuovo bound.

Theorem 1.7. *Let (M, J, g) be a compact almost Hermitian 6–manifold with a super-rigid almost complex structure J. Let $A \in H_2(M)$ be such that $\langle c_1(M), A\rangle = 0$. Given any $\Lambda > 0$, there are only finitely many simple J–holomorphic maps representing A with energy at most Λ.*

Remark 1.8. If J is tamed by a symplectic form ω, then imposing an upper bound on the energy is superfluous since the energy of any J–holomorphic map representing A is $\langle[\omega], A\rangle$.

In the situation of Theorem 1.7, Gromov's compactness theorem [Gro85; PW93; Ye94; Hum97] shows that there are only finitely many J–holomorphic maps representing A from Riemann surfaces of *fixed genus*. It is thus not of much use for proving Theorem 1.7. Instead, we use the following compactness result for *J–holomorphic cycles*, that is: formal sums of simple J–holomorphic curves, with respect to *geometric convergence*; see Definition 3.1 and Definition 3.2.

Lemma 1.9. *Let M be a manifold and let $(J_n, g_n)_{n\in\mathbf{N}}$ be a sequence of almost Hermitian structures converging to an almost Hermitian structure (J, g). Let $K \subset M$ be a compact subset and let $\Lambda > 0$. For each $n \in \mathbf{N}$ let C_n be a J_n–holomorphic cycle with support contained in K and of mass at most Λ. Then a subsequence of $(C_n)_{n\in\mathbf{N}}$ geometrically converges to a J–holomorphic cycle C.*

The proof of Lemma 1.9 relies on results in geometric measure theory; in particular, the recent work of De Lellis, Spadaro, and Spolaor [DSS17b; DSS18; DSS17a; DSS15] on regularity of semi-calibrated currents.

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2 Super-rigidity of J–holomorphic maps

We begin by briefly recalling the notion of super-rigidity as defined by Wendl. For a more detailed discussion we refer the reader to [Wen16, Section 2.1; DW18, Section 7]. Throughout, let (M, J, g) be an almost Hermitian $2n$–manifold.

Definition 2.1. A **J–holomorphic map** $u\colon (\Sigma, j) \to (M, J)$ is a pair consisting of a connected, closed Riemann surface (Σ, j) and a smooth map $u\colon \Sigma \to M$ satisfying the non-linear Cauchy–Riemann equation

$$\bar\partial_J(u, j) \coloneqq \frac{1}{2}(\mathrm{d}u + J(u) \circ \mathrm{d}u \circ j) = 0. \tag{2.2}$$

Definition 2.3. Let $u\colon (\Sigma, j) \to (M, J)$ be a J–holomorphic map. Let $\phi \in \mathrm{Diff}(\Sigma)$ be a diffeomorphism. The **reparametrization** of u by ϕ is the J–holomorphic map $u \circ \phi^{-1}\colon (\Sigma, \phi_* j) \to (M, J)$.

Definition 2.4. Let $u\colon (\Sigma, j) \to (M, J)$ be a J–holomorphic map and let $\pi\colon (\tilde\Sigma, \tilde j) \to (\Sigma, j)$ be a holomorphic map of degree $\deg(\pi) \geqslant 2$. The composition $u \circ \pi\colon (\tilde\Sigma, \tilde j) \to (M, J)$ is said to be a **multiple cover of** u. A J–holomorphic map is **simple** if it is not constant and not a multiple cover.

Super-rigidity is a condition on the infinitesimal deformation theory of J–holomorphic curves up to reparametrization. We will have to briefly review parts of this theory.

The index of a J–holomorphic map $u\colon (\Sigma, j) \to (M, J)$ is defined as

$$\mathrm{index}(u) \coloneqq (n - 3)\chi(\Sigma) + 2\langle[\Sigma], u^* c_1(M, J)\rangle. \tag{2.5}$$

This is the Fredholm index of the linearization of (2.2). The restriction of this linearization to $\Gamma(u^*TM)$ is given by

$$\xi \mapsto \frac{1}{2}\left(\nabla\xi + J \circ (\nabla\xi) \circ j + (\nabla_\xi J) \circ \mathrm{d}u \circ j\right). \tag{2.6}$$

Here ∇ denotes any torsion-free connection on TM and also the induced connection on u^*TM. Since (u,j) is a J–holomorphic map, the right-hand side of (2.6) does not depend on the choice of ∇; see [MS12, Proposition 3.1.1].

Let $u\colon (\Sigma, j) \to (M, J)$ be a non-constant J–holomorphic map. There exists a unique complex subbundle

$$Tu \subset u^*TM$$

of rank one containing $\mathrm{d}u(T\Sigma)$; see [IS99, Section 1.3; Wen10, Section 3.3; DW18, Appendix A]. The generalized normal bundle of u is defined as

$$Nu := u^*TM/Tu.$$

If u is an immersion, then Nu is the usual normal bundle; if $\tilde{u} = u \circ \pi$ is a multiple cover of an immersion, then $N\tilde{u} = \pi^*Nu$. Define the normal Cauchy–Riemann operator

$$\mathfrak{d}^N_{u,J}\colon \Gamma(Nu) \to \Omega^{0,1}(Nu) \tag{2.7}$$

by the formula (2.6). The non-zero elements of the kernel of $\mathfrak{d}^N_{u,J}$ correspond to infinitesimal deformations of u which deform the image $u(\Sigma)$.

Definition 2.8. A non-constant J–holomorphic map u is called **rigid** if $\ker \mathfrak{d}^N_{u,J} = 0$. A simple J–holomorphic map $u\colon (\Sigma, j) \to (M, J)$ is called **super-rigid** if it is rigid and all of its multiple covers are rigid.

It follows from [IS99, Lemma 1.5.1; Wen10, Theorem 3] that $\dim\ker \mathfrak{d}^N_{u,J} \geqslant \mathrm{index}(u)$. Consequently, a super-rigid J–holomorphic map must have $\mathrm{index}(u) \leqslant 0$.

Definition 2.9. An almost complex structure J is called **super-rigid** if:

1. Every simple J–holomorphic map of index zero is super-rigid.
2. Every simple J–holomorphic map has non-negative index.
3. Every simple J–holomorphic map of index zero is an embedding, and every two simple J–holomorphic maps of index zero either have disjoint images or are related by a reparametrization.

Remark 2.10. In dimension four, one should weaken (3) and require only that every simple J–holomorphic map of index zero is an immersion with transverse self-intersections, and that two such maps are either transverse to one another or are related by reparametrization. However, we will only be concerned with dimension (at least) six.

Definition 2.11. Let (M, ω) be a symplectic manifold. Denote by $\mathcal{J}(\omega)$ the separable Banach manifold[1] of almost complex structures on M compatible with ω. Denote by $\mathcal{J}_\diamond(\omega)$ the subset of those almost complex structures $J \in \mathcal{J}(\omega)$ which are super-rigid.

[1]The cognisant reader will know that the space almost complex structures compatible with ω is naturally a Fréchet manifold. To obtain a separable Banach we work with Floer's C^∞_ε topology; see [Flo88, Section 5; MS12, Remark 3.2.7].

Definition 2.12. Let X be a topological space. A subset $A \subset X$ is called **residual** if it is the intersection of countably many dense open subsets. Recall that a residual subset of a complete metric space is dense.

The following result confirms Bryan and Pandharipande's super-rigidity conjecture.

Theorem 2.13 (Wendl [Wen16, Theorem A]; see also [DW18, Theorem 7.16]). *Let (M, ω) be a symplectic manifold. If $\dim M \geqslant 6$, then $\mathscr{J}_\diamond(\omega) \subset \mathscr{J}(\omega)$ is a residual subset.*

3 J–holomorphic cycles and geometric convergence

In this section we introduce the notions of J–holomorphic cycles and geometric convergence. We then compare these with the notions of closed J–holomorphic integral currents and weak convergence. This comparison result together with a classical compactness result in geometric measure theory then imply Lemma 1.9. Throughout, let (M, J, g) be an almost Hermitian manifold. Denote by $\sigma \coloneqq g(J\cdot, \cdot)$ the corresponding Hermitian form.

Definition 3.1. A **J–holomorphic curve** is a subset of M which is the image of a simple J–holomorphic map $u\colon (\Sigma, j) \to (M, J)$. A **$J$–holomorphic cycle** C is a formal linear combination

$$C = \sum_{k=1}^{K} m_k C_k$$

of J–holomorphic curves $C_1, \dots, C_K$ with coefficients $m_1, \dots, m_K \in \mathbf{N}$. The **homology class** represented by C is

$$[C] \coloneqq \sum_{k=1}^{K} m_k (u_k)_*[\Sigma_k].$$

We denote by $[C]_{\mathbf{R}}$ the image of $[C]$ under the map $H_2(M, \mathbf{Z}) \to H_2(M, \mathbf{R})$. The **support** of C is the subset

$$\operatorname{supp}(C) \coloneqq \bigcup_{k=1}^{k} C_k.$$

The **current** associated with C is defined by

$$\delta_C(\alpha) \coloneqq \sum_{k=1}^{K} m_k \int_{\Sigma_k} u_k^* \alpha \quad \text{for} \quad \alpha \in \Omega_c^2(M).$$

The **mass** of C is

$$\mathbf{M}(C) \coloneqq \sum_{k=1}^{K} \operatorname{area}(C_k) = \delta_C(\sigma).$$

We say that C is **smooth** if the J–holomorphic curves $C_1, \dots, C_K$ are embedded and pairwise disjoint.

Definition 3.2 (Taubes [Tau98, Definition 3.1]). Let M be a manifold and let $(J_n, g_n)_{n\in\mathbf{N}}$ be a sequence of almost Hermitian structures converging to an almost Hermitian structure (J, g). For every $n \in \mathbf{N}$ let C_n be a J_n–holomorphic cycle. We say that $(C_n)_{n\in\mathbf{N}}$ **geometrically converges** to a J–holomorphic cycle C if:

1. $(\delta_{C_n})_{n\in\mathbf{N}}$ weakly converges to δ_C; that is,

$$\lim_{n\to\infty} \delta_{C_n}(\alpha) = \delta_C(\alpha) \quad \text{for all} \quad \alpha \in \Omega^2_c(M)$$

and

2. $(\mathrm{supp}(C_n))_{n\in\mathbf{N}}$ converges to $\mathrm{supp}(C)$ in the Hausdorff distance; that is:

$$\lim_{n\to\infty} d_H(\mathrm{supp}(C), \mathrm{supp}(C_n)) \to 0. \tag{3.3}$$

Let us remind the reader that the Hausdorff distance between two closed sets X and Y is defined by

$$d_H(X, Y) := \max\left\{\sup_{x\in X} d(x, Y), \sup_{y\in Y} d(y, X)\right\}.$$

The following results compare J–holomorphic cycles and geometric convergence with closed integral currents on M which are calibrated by σ and weak convergence. We refer the reader to the lecture notes [Lan05] for background on geometric measure theory.

Proposition 3.4. *If δ_C is a closed integral current which is calibrated by σ, then there exist: a J–holomorphic map $u\colon (\Sigma, j) \to (M, J)$, with a possibly disconnected domain Σ, and a locally constant function $k\colon \Sigma \to \mathbf{N}$ such that*

$$\delta_C(\alpha) = \int_\Sigma k \cdot u^*\alpha \quad \textit{for all} \quad \alpha \in \Omega^2_c(M).$$

In particular, there exists a J–holomorphic cycle C whose associated current is δ_C.

Proposition 3.5. *In the situation of Definition 3.2, if condition (1) holds and there exists a compact subset containing $\mathrm{supp}(C_n)$ for every $n \in \mathbf{N}$, then condition (2) holds as well.*

We prove the second result first, since it is more elementary.

Proof of Proposition 3.5. This is contained in the proof of [Tau98, Proposition 3.3] and also a well-known fact in geometric measure theory. Let us explain the proof nevertheless. The salient point is the monotonicity formula for J–holomorphic curves; see, e.g., [PW93, Corollary 3.2; Tau98, Lemma 3.4]. It states that there are constants $c, r_0 > 0$ such that for every J–holomorphic curve C, every $x \in C$, and every $r \in [0, r_0]$

$$\mathbf{M}(\delta_C|_{B_r(x)}) = \delta_C|_{B_r(x)}(\sigma) \geqslant cr^2. \tag{3.6}$$

Moreover, it follows from the proof of the monotonicity formula that these constants can be chosen such that (3.6) holds for all almost Hermitian structures in the sequence $(J_n, g_n)_{n\in\mathbf{N}}$ as well as the limit (J, g).

Condition (3.3) is equivalent to

$$\lim_{n\to\infty} \sup\{d(x, \operatorname{supp}(C)) : x \in \operatorname{supp}(C_n)\} = 0 \quad \text{and} \tag{3.7}$$

$$\lim_{n\to\infty} \sup\{d(x, \operatorname{supp}(C_n)) : x \in \operatorname{supp}(C)\} = 0. \tag{3.8}$$

If (3.7) fails, then after passing to a subsequence there exists an $\varepsilon > 0$ and a sequence of points (x_n) with

$$x_n \in \operatorname{supp}(C_n) \quad \text{but} \quad d(x_n, \operatorname{supp}(C)) \geqslant \varepsilon.$$

After passing to a further subsequence (x_n) converges to a limit $x \in M$ with $d(x, \operatorname{supp}(C)) \geqslant \varepsilon$. Fix $0 < r \leqslant \min\{\varepsilon/2, r_0\}$. Let $\chi \in C^\infty(M, [0, 1])$ be supported in $B_{2r}(x)$ and equal to one in $B_r(x)$. By (3.6), for $n \gg 1$

$$c_0 r^2 \leqslant \mathbf{M}(\delta_{C_n}|_{B_{2r}(x)}) \leqslant \delta_{C_n}(\chi\sigma).$$

This contradicts the weak convergence condition (1), because

$$\delta_C(\chi\sigma) = 0.$$

If (3.8) fails, then a slight variation of this argument derives another contradiction to (1). □

Proof of Proposition 3.4. For symplectic 4–manifolds this was proved by Taubes [Tau96a, Proposition 6.1]. Taubes' argument and the work of Rivière and Tian [RT09] establish the result for general symplectic manifolds. The extension to almost Hermitian manifolds relies on the work of De Lellis, Spadaro, and Spolaor [DSS18; DSS17b; DSS17a; DSS15]. Their main result [DSS18, Theorem 0.2] implies that the singular set of δ_C is finite (since it is discrete and since δ_C is closed and of finite mass and thus compact support). We need not just their main result but also the following intermediate result.

Definition 3.9. Given $k \in \mathbf{N}$, set

$$\tilde{D}^k \coloneqq \left\{(z, w) \in \mathbf{C}^2 : z = w^k \text{ and } |z| < 1\right\}.$$

We consider $\tilde{D}^k \backslash \{0\}$ as oriented smooth manifold such that the map $(z, w) \mapsto z$ is an orientation-preserving local diffeomorphism. We equip it with the pull-back of the flat metric.

Definition 3.10. Let $k \in \mathbf{N}$ and $\alpha \in (0, 1)$. Let $f\colon \tilde{D}^k \to \mathbf{R}^{2n-2}$ be a continuous injective map which is of class $C^{3,\alpha}$ on $\tilde{D}^k \backslash \{0\}$ and satisfies $|\mathrm{d}f| \lesssim |z|^\alpha$. Define $\underline{f}\colon \tilde{D}^k \to \mathbf{R}^{2n}$ by

$$\underline{f}(z, w) \coloneqq (z, f(w)).$$

Let $U \subset \mathbf{R}^{2n}$ be an open subset and let $\phi\colon U \to M$ be a chart. The **graph** of f with respect to ϕ is the integral current $G_{f,\phi}$ defined by

$$G_{f,\phi}(\alpha) \coloneqq \int_{\tilde{D}^k\setminus\{0\}} \underline{f}^*\phi^*\alpha \quad \text{for} \quad \alpha \in \Omega^2_c(M).$$

Lemma 3.11 (De Lellis, Spadaro, and Spolaor [DSS17a, Section 1]). *For every $x \in \operatorname{supp}(\delta_C)$ there are: a neighborhood U of x, finite collections of maps $f_1, \dots, f_m$ and charts $\phi_1, \dots, \phi_m$ as in Definition 3.10, and weights $\ell_1, \dots, \ell_m \in \mathbf{N}$ such that*

$$\delta_C|_U = \sum_{i=1}^m \ell_i G_{f_i,\phi_i}.$$

Denote by $\mathring{\Sigma}$ the regular part of $\operatorname{supp}(\delta_C)$. Since δ_C is calibrated, the tangent spaces to $\mathring{\Sigma}$ are J–invariant. Therefore, $\mathring{\Sigma}$ canonically is a Riemann surface. As mentioned earlier, the singular set $\operatorname{sing}(\delta_C) \coloneqq \operatorname{supp}(C)\setminus\mathring{\Sigma}$ is finite. Lemma 3.11 shows that each $x \in \operatorname{sing}(\delta_C)$ has a neighborhood U such that

$$\mathring{\Sigma} \cap U \cong \mathbf{C}^* \sqcup \cdots \sqcup \mathbf{C}^*.$$

Thus, $\mathring{\Sigma}$ can be compactified to a Riemann surface Σ by adding finitely many points.

The Riemann surface Σ comes with a continuous map $u\colon \Sigma \to M$. Its restriction to $\mathring{\Sigma}$ is smooth and J–holomorphic. It follows from elliptic regularity that u is, in fact, smooth and J–holomorphic on all of Σ. The above discussion shows that

$$\delta_C(\alpha) = \int_\Sigma k \cdot u^*\alpha$$

for some locally constant function $k\colon \Sigma \to \mathbf{N}$. □

Proof of Lemma 1.9. The sequence of closed integral currents $(\delta_{C_n})_{n\in\mathbf{N}}$ has uniformly bounded mass. Therefore, there exists a subsequence which weakly converges to a closed integral current δ_C calibrated by σ; see, e.g., [Fed69, Theorem 4.2.17; Sim83, Theorem 27.3; Lan05, Theorem 3.7]. By Proposition 3.4, δ_C is the current associated with a J–holomorphic cycle C. By Proposition 3.5, the sequence of pseudo-holomorphic cycles (C_n) geometrically converges to C. □

4 Real Cauchy–Riemann operators and almost complex structures

We will show that associated with every real Cauchy–Riemann operator defined on a vector bundle there is a natural almost complex structure on the total space of that bundle. This construction is inspired by related ideas from [Tau96b, p. 825–826]; similar material can be found in [Wen16, Appendix B].

Definition 4.1. Let (Σ, j) be a Riemann surface. Let $\pi\colon E \to \Sigma$ be a Hermitian vector bundle over Σ. A real first order linear differential operator $\mathfrak{d}\colon \Gamma(E) \to \Omega^{0,1}(\Sigma, E)$ is called a **real Cauchy–Riemann operator** if

$$\mathfrak{d}(fs) = (\bar{\partial} f)s + f\mathfrak{d}s \tag{4.2}$$

for all $f \in C^\infty(M, \mathbf{R})$. The **anti-linear part** of $\mathfrak{d}$ is defined as

$$\mathfrak{n} = \mathfrak{n}_{\mathfrak{d}} := \frac{1}{2}(\mathfrak{d} + J\mathfrak{d}J) \in \Gamma(\mathrm{Hom}(E, \overline{\mathrm{Hom}}_{\mathbf{C}}(T\Sigma, E))).$$

Every real Cauchy–Riemann operator can be written as

$$\mathfrak{d} = \bar{\partial}_\nabla + \mathfrak{n}$$

where $\bar{\partial}_\nabla := \nabla^{0,1}$ is the Dolbeault operator associated with a Hermitian connection ∇ on E. Denote by $H_\nabla \subset TE$ the horizontal distribution of ∇. It induces an isomorphism

$$TE = H_\nabla \oplus \pi^* E \cong \pi^* T\Sigma \oplus \pi^* E. \tag{4.3}$$

Definition 4.4. The **complex structure** J_∇ on E associated with ∇ is defined by pulling back the standard complex structure $j \oplus i$ on $\pi^* T\Sigma \oplus \pi^* E$ by the isomorphism (4.3).

It is well-known that a section $s \in \Gamma(E)$ satisfies $\bar{\partial}_\nabla s = 0$ if and only if the map $s\colon \Sigma \to E$ is J_∇–holomorphic. The following proposition extends this to real Cauchy–Riemann operators.

Definition 4.5. Let $\mathfrak{d} = \bar{\partial}_\nabla + \mathfrak{n}$ be a real Cauchy–Riemann operator. Define $L_{\mathfrak{d}}\colon TE \to TE$ by

$$L_{\mathfrak{d}} = -2\mathfrak{n}(v) j \pi_*$$

at $v \in E$. The **almost complex structure** $J_{\mathfrak{d}}$ on E associated with $\mathfrak{d}$ is defined by

$$J_{\mathfrak{d}} := J_\nabla + L_{\mathfrak{d}}.$$

Proposition 4.6. *For every real Cauchy–Riemann operator $\mathfrak{d}\colon \Gamma(E) \to \Omega^{0,1}(E)$ the following hold:*

1. *$J_{\mathfrak{d}}$ is an almost complex structure..*
2. *The projection $\pi\colon E \to \Sigma$ is holomorphic with respect to $J_{\mathfrak{d}}$.*
3. *For every $x \in \Sigma$ the fiber $E_x = \pi^{-1}(x)$ is a $J_{\mathfrak{d}}$–holomorphic submanifold of E.*
4. *A section $s \in \Gamma(E)$ satisfies $\mathfrak{d}s = 0$ if and only if $s\colon \Sigma \to E$ is a $J_{\mathfrak{d}}$–holomorphic map.*
5. *There exists a symplectic form ω on the unit disc bundle $B_1(\Sigma) \subset N\Sigma$ which tames $J_{\mathfrak{d}}$.*

Proof. With respect to (4.3) we have

$$J_{\mathfrak{d}} = \begin{pmatrix} j & 0 \\ -2\mathfrak{n}(v)j & i \end{pmatrix} \tag{4.7}$$

at $v \in E$. Since $\mathfrak{n}(v)$ is anti-linear,

$$\mathfrak{n}(v)j^2 + i\mathfrak{n}(v)j = 0.$$

Therefore,

$$J_{\mathfrak{d}}^2 = -\mathrm{id};$$

that is, (1) holds.

Both (2) and (3) immediately follow from (4.7).

We prove (4). Let $s \colon \Sigma \to E$ be a section. The projection of $\mathrm{d}s$ to the first factor of (4.3) is $\pi_* \circ \mathrm{d}s = \mathrm{id}_{T\Sigma}$ and thus j–linear. The projection of $\mathrm{d}s \colon T\Sigma \to s^*TE$ to the second factor is its covariant derivative $\nabla s \colon T\Sigma \to s^*E$. Therefore, the J_∇–antilinear part of $\mathrm{d}s$ is

$$\frac{1}{2}(\mathrm{d}s + J_\nabla \circ \mathrm{d}s \circ j) = (\nabla s)^{0,1} = \bar{\partial}_\nabla s.$$

The $J_{\mathfrak{d}}$–antilinear part of $\mathrm{d}s$ is

$$\begin{aligned} \frac{1}{2}(\mathrm{d}s + J_{\mathfrak{d}} \circ \mathrm{d}s \circ j) &= \frac{1}{2}(\mathrm{d}s + J_\nabla \circ \mathrm{d}s \circ j + L_{\mathfrak{d}} \circ \mathrm{d}s \circ j) \\ &= \bar{\partial}_\nabla s + L_{\mathfrak{d}} \circ \mathrm{d}s \circ j \\ &= \bar{\partial}_\nabla s + \mathfrak{n}_{\mathfrak{d}} s = \mathfrak{d}s. \end{aligned}$$

Therefore, $\mathrm{d}s \colon T\Sigma \to TE$ is $J_{\mathfrak{d}}$–linear if and only if $\mathfrak{d}s = 0$.

The proof of (5) is standard; see, e.g., [Wen16, Lemma B.2]. Nevertheless, we include it here for completeness. Let ω_Σ be an area form on Σ. Let ω_E be any closed 2–form on $B_1(\Sigma)$ which is positive when restricted to the fibers of E; that is, for all vertical tangent vectors v_E

$$\omega_E(v_E, J_\nabla v_E) \gtrsim |v_E|^2. \tag{4.8}$$

Such a form can be constructed by choosing local unitary trivializations of $E|_{U_i} \cong U_i \times \mathbf{C}^k$, denoting by λ_i the corresponding Liouville 1–forms on $\mathbf{C}^k$ vanishing at zero, and setting

$$\omega_E = \mathrm{d}\left(\sum_i \chi_i \circ \pi \cdot \lambda_i\right)$$

for a partition of unity (χ_i). This form satisfies (4.8) on E. It remains to show that for $\tau \gg 1$ the closed 2–form $\omega = \tau\omega_\Sigma + \omega_E$ tames J_u on $B_1(\Sigma)$. For a tangent vector w to E at a point $(x, v) \in B_1(\Sigma)$ denote by w_H and w_E its horizontal and vertical parts in the decomposition (4.3). We have

$$\begin{aligned} \omega(w, J_{\mathfrak{d}} w) &= (\tau\omega_\Sigma + \omega_E)(w, (J_\nabla + L_{\mathfrak{d}})w) \\ &= \tau\omega_\Sigma(w_H, jw_H) + \omega_E(w_E, J_\nabla w_E) + \omega_E(w_E, L_{\mathfrak{d}} w_H). \end{aligned}$$

From $|L_{\mathfrak{d}}(v)| \lesssim |v| < 1$ it follows that

$$|\omega_E(w_E, L_{\mathfrak{d}} w_H)| \lesssim |w_E||w_H|.$$

Since

$$\tau\omega_\Sigma(w_H, jw_H) + \omega_E(w_E, J_\nabla w_E) \gtrsim \tau|w_H|^2 + |v_E|,$$

it follows that ω tames J_u provided $\tau \gg 1$. $\square$

The next two propositions are concerned with the following situation. Let $u\colon (\Sigma, j) \to (M, J)$ be a J–holomorphic embedding. Denote by $Nu \to \Sigma$ its normal bundle and by $\mathfrak{d}^N_{u,J}$ the normal Cauchy–Riemann introduced in (2.7). Write

$$J_u \coloneqq J_{\mathfrak{d}^N_{u,J}} \tag{4.9}$$

for the almost complex structure on the total space of Nu associated with $\mathfrak{d}^N_{u,J}$.

Proposition 4.10. *For every $\lambda > 0$ define $\sigma_\lambda\colon\ Nu \to Nu$ by*

$$\sigma_\lambda(v) \coloneqq \lambda v.$$

If $U \subset Nu$ is an open neighborhood of the zero section in Nu such that the exponential map $\exp\colon\ U \to M$ is an embedding, then

$$\sigma_\lambda^* \exp^* J \to J_u \quad \textit{as} \quad \lambda \to 0.$$

Proof. Denote by ∇ the connection on $Nu \to \Sigma$ induced by the Levi–Civita connection on M. Throughout this proof, we identify

$$TU = \pi^* T\Sigma \oplus \pi^* Nu$$

as in (4.3). The two almost complex structures J_∇ and $\exp^* J$ on $U \subset Nu$ agree along the zero section. The Taylor expansion of $\exp^* J$ is of the form

$$\exp^* J(x, v) = J_\nabla(x, 0) + \nabla_v J(x, 0) + O(|v|^2). \tag{4.11}$$

Set

$$L(x, v) \coloneqq \nabla_n J(x, 0).$$

We write L as the matrix

$$L(x, v) = \begin{pmatrix} L_{11}(x, v) & L_{12}(x, v) \\ L_{21}(x, v) & L_{22}(x, v) \end{pmatrix}.$$

Here each L_{ij} is linear in v. The derivative $\mathrm{d}\sigma_\lambda$ is given by

$$\mathrm{d}\sigma_\lambda = \begin{pmatrix} \mathrm{id} & \\ & \lambda \end{pmatrix}.$$

Therefore,

$$\begin{aligned}(\sigma_\lambda)^* L(x,v) &= \begin{pmatrix} \mathrm{id} & \\ & \lambda^{-1} \end{pmatrix} \begin{pmatrix} L_{11}(x,\lambda v) & L_{12}(x,\lambda v) \\ L_{21}(x,\lambda v) & L_{22}(x,\lambda v) \end{pmatrix} \begin{pmatrix} \mathrm{id} & \\ & \lambda \end{pmatrix} \\ &= \begin{pmatrix} \lambda L_{11}(x,v) & \lambda^2 L_{12}(x,v) \\ L_{21}(x,v) & \lambda L_{22}(x,v) \end{pmatrix}.\end{aligned}$$

As λ tend to zero, all but the bottom left entry tend to zero.

By construction, $\sigma_\lambda^* J_\nabla = J_\nabla$. As λ tends to zero, the rescalings of terms of second order and higher in (4.11) tend to zero. It remains to identify the term L_{21}. By definition,

$$L_{21}(x,v) = \pi_{Nu} \circ \nabla_v J(x,0) \circ \pi_*.$$

Comparing (2.6), Definition 4.1, and Definition 4.5, we see that $L_{21} = L_u$. This finishes the proof. □

Proposition 4.12. *If $\tilde{u}\colon (\tilde{\Sigma}, \tilde{j}) \to (Nu, J_u)$ is a simple J_u–holomorphic map whose image is not contained in the zero section, then:*

1. *the map $\varphi\colon (\tilde{\Sigma}, \tilde{j}) \to (\Sigma, j)$ given by $\varphi := \pi \circ \tilde{u}$ is non-constant and holomorphic, and*
2. *the J–holomorphic map $u \circ \varphi\colon (\tilde{\Sigma}, \tilde{j}) \to (M, J)$ is not rigid; in particular, the J–holomorphic map $u\colon (\Sigma, j) \to (M, J)$ is not super-rigid.*

Proof. By Proposition 4.6 (2), $\pi\colon Nu \to \Sigma$ is $J_\mathfrak{d}$–holomorphic. Therefore, φ is holomorphic. The map φ is constant if and only if the image of $\tilde{u}$ is contained in a fiber of π. This is impossible, because then $\tilde{u}$ would be constant. This proves (1).

The normal bundle of the J–holomorphic map $u \circ \varphi$ is $N_{u\circ\varphi} = \varphi^* Nu$. The corresponding normal Cauchy–Riemann operator is

$$\mathfrak{d}^N_{u\circ\varphi, J} = \varphi^* \mathfrak{d}^N_{u,J}. \tag{4.13}$$

Since $\tilde{u}$ takes values in Nu, for every $x \in \tilde{\Sigma}$ we have

$$\tilde{u}(x) \in Nu_{\pi(u(x))} = Nu_{\varphi(x)} = (\varphi^* Nu)_x.$$

This gives rise to the tautological section $s \in \Gamma(\tilde{\Sigma}, \varphi^* Nu)$ defined by

$$s(x) := \tilde{u}(x) \in (\varphi^* Nu)_x.$$

This section is not the zero section, because the image of $\tilde{u}$ is not contained in the zero section. The upcoming discussion will show that

$$\mathfrak{d}^N_{u\circ\varphi, J} s = 0.$$

In light of Proposition 4.6 (4), this will imply (2).

Denote by Z the finite set of critical values of φ. Set $\tilde{Z} := \varphi^{-1}(Z)$. The restriction $\tilde{u}\colon \tilde{\Sigma}\backslash\tilde{Z} \to Nu$ is a J_u–holomorphic embedding and $\varphi\colon \tilde{\Sigma}\backslash\tilde{Z} \to \Sigma\backslash Z$ is an unbranched holomorphic covering map. Hence, every $x \in \tilde{\Sigma}\backslash\tilde{Z}$ has an open neighborhood U such that $\tilde{u}|_U$ is an embedding and $\varphi|_U$ is biholomorphic. Therefore, $\pi|_{\tilde{u}(U)}$ maps $\tilde{u}(U)$ holomorphically to $\varphi(U)$. It follows that $\tilde{u}(U) \subset Nu$ is the graph of a J_u–holomorphic section $f\colon \varphi(U) \to Nu|_{\varphi(U)}$. By construction and by Proposition 4.6,

$$s|_U = \varphi^* f \quad \text{and} \quad \mathfrak{d}^N_{u,J} f = 0.$$

The relation (4.13) shows that $\mathfrak{d}^N_{u\circ\varphi,J} s = 0$ holds on U. Since $x \in \tilde{\Sigma}\backslash\tilde{Z}$ was arbitrary, it holds on all of $\tilde{\Sigma}\backslash\tilde{Z}$. In fact, since s is smooth, it holds on all of $\tilde{\Sigma}$. ◻

5 Proof of Theorem 1.7

Suppose, by contradiction, that there are are infinitely many *distinct* J–holomorphic curves $C_n \subset M$ representing the class $A \in H_2(M, \mathbf{Z})$ and of energy at most Λ. By Lemma 1.9, after passing to a subsequence, the sequence (C_n) converges geometrically to a J–holomorphic cycle

$$C_\infty = \sum_{k=1}^{K} m_k C_\infty^k.$$

Proposition 5.1. *C_∞ is connected and smooth.*

Proof. By Definition 3.2 (1),

$$\langle c_1(M), [C_\infty]\rangle = \delta_{C_\infty}(c_1(M)) = \lim_{n\to\infty} \delta_{C_n}(c_1(M)) = \langle c_1(M), A\rangle = 0.$$

Let $u_k\colon\ \Sigma_k \to M$ be a simple J–holomorphic map whose image is C_∞^k. The index formula (2.5) yields

$$\sum_{k=1}^{K} m_k \operatorname{index}(u_k) = \sum_{k=1}^{K} 2m_k \langle c_1(M), [C_\infty^k]\rangle = 2\langle c_1(M), [C_\infty]\rangle = 0.$$

Since J is super-rigid, by Definition 2.11 (2), there are no J–holomorphic curves of negative index. Thus, we have $\operatorname{index}(u_k) \geqslant 0$ for all k and the above computation shows that

$$\operatorname{index}(u_1) = \cdots = \operatorname{index}(u_k) = 0.$$

Therefore, by Definition 2.11 (3), the J–holomorphic curves $C_\infty^1, \ldots, C_\infty^1$ are embedded and pairwise disjoint. This proves that C_∞ is smooth.

To see that C_∞ is connected, observe that if C_∞ where disconnected, then Definition 3.2 (2) would imply that C_n is disconnected for $n \gg 1$. However, C_n is a J–holomorphic curve and thus connected by definition. ◻

In the following, we rescale the sequence (C_n) and extract a further limit $\tilde{C}_\infty$. The properties of $\tilde{C}_\infty$ will give a contradiction to J being super-rigid.

Henceforth, we denote by C_∞ the J–holomorphic curve underlying the J–holomorphic cycle C_∞; that is, we ignore its multiplicity. Since the curves C_n are all distinct, we can assume that they are all distinct from C_∞. We can also assume that every C_n is contained in a sufficiently small tubular neighborhood of C_∞. By slight abuse of notation, we regard C_n as an $\exp^* J$–holomorphic curve in the normal bundle NC_∞ and C_∞ as the zero section in NC_∞.

For every $\lambda > 0$ let σ_λ be as in Proposition 4.10. Choose (λ_n) such that such that the sets

$$\tilde{C}_n \coloneqq \sigma_{\lambda_n}^{-1}(C_n)$$

satisfy

$$d_H(\tilde{C}_n, C_\infty) = 1/2. \tag{5.2}$$

Set

$$J_n \coloneqq \sigma_{\lambda_n}^* \exp^* J.$$

By construction, the $\tilde{C}_n$ are J_n–holomorphic. By Proposition 4.10, the sequence (J_n) converges to the almost complex structure J_u associated with the J–holomorphic map $u\colon C \hookrightarrow M$. The sequence $(\tilde{C}_n)$ is contained in the compact disc bundle $\bar{B}_{1/2}(C_\infty) \subset NC_\infty$. By Proposition 4.6 (5), J_u is tamed by a symplectic form ω on $B_1(C)$. Consequently, for all $n \gg 1$ the almost complex structure J_n is tamed by ω as well. Define a Riemannian metric g on $B_1(C_\infty)$ by

$$g \coloneqq \frac{1}{2}(\omega(J_u\cdot, \cdot) + \omega(\cdot, J_u\cdot)).$$

The analogously defined metrics g_n are Hermitian with respect to J_n and converge to g. By the energy identity [MS12, Lemma 2.2.1],

$$\lim_{n\to\infty} \mathbf{M}(\tilde{C}_n) = \lim_{n\to\infty} \delta_{\tilde{C}_n}(\omega) = \delta_{\tilde{C}}(\omega) < \infty.$$

Therefore, the mass of $\tilde{C}_n$ with respect to g_n (and thus also g) can be bounded independent of n.

By Lemma 1.9, a subsequence of $(\tilde{C}_n)$ geometrically converges to a J–holomorphic cycle

$$\tilde{C}_\infty = \sum_{k=1}^{K} \tilde{m}_k \tilde{C}_\infty^k.$$

Condition (5.2) guarantees that $\operatorname{supp}(\tilde{C}_\infty) \neq C$. Therefore, Proposition 4.12 applies to at least one of the $\tilde{C}_k$; but this contradicts C being super-rigid. ◻

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