Tropical Fukaya Algebras

Sushmita Venugopalan

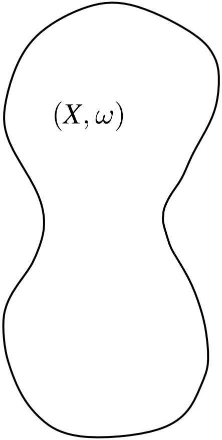
November 16, 2020

arXiv:2004.14314. Joint work with Chris Woodward.

Table of Contents

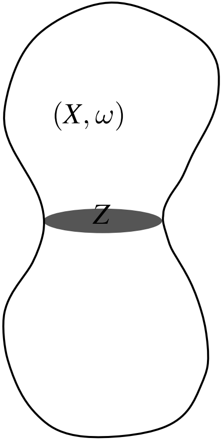
1. Introducing the problem
2. Single cut : Unbroken to broken
3. Multiple cut : Unbroken to broken
4. Degenerating matching conditions

Symplectic cut



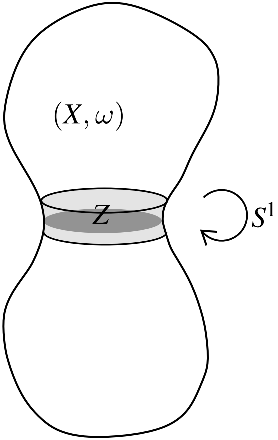
Given a symplectic manifold (X, ω),

Symplectic cut



a separating hypersurface $Z \subset X$,

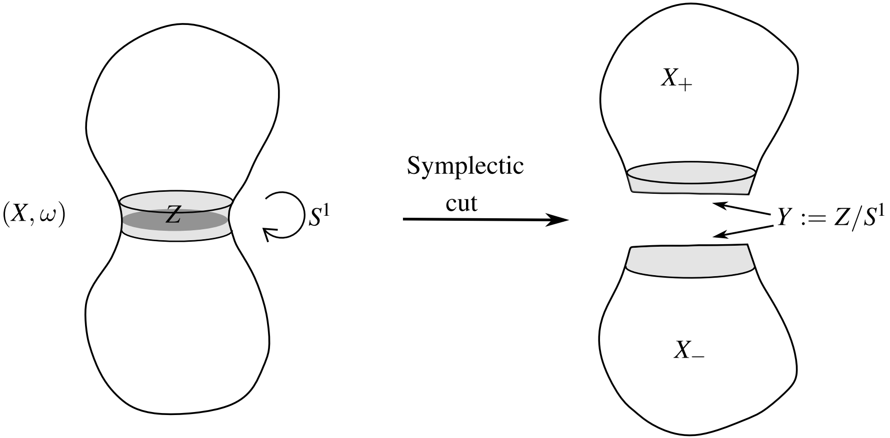
Symplectic cut



and a Hamiltonian S^1-action in a neighborhood of Z,

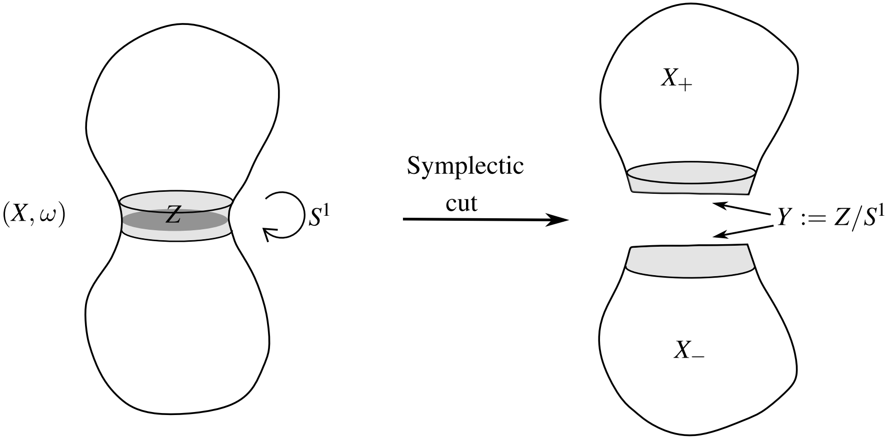
Symplectic cut

a symplectic cut produces two symplectic manifolds X_+, X_-.



Symplectic cut

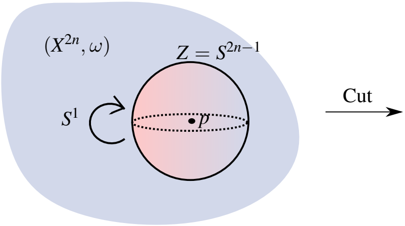
a symplectic cut produces two symplectic manifolds X_+, X_-.



Both X_+, X_- contain $Y := Z/S^1$ as a *relative divisor*.

Symplectic cut : examples

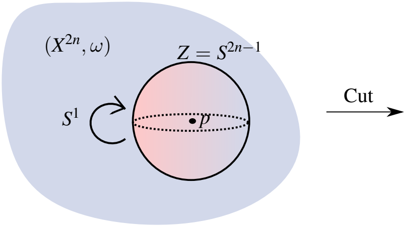
1. Blowing up a point.



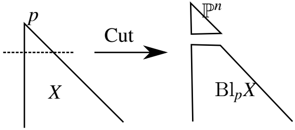
$$X_+ = \mathrm{Bl}_p X \text{ (outside } Z)$$
$$X_- = \mathbb{CP}^n \text{ (inside } Z)$$
$$Y = \mathbb{CP}^{n-1} \simeq Z/S^1$$

Symplectic cut : examples

1. Blowing up a point.



$$X_+ = \mathrm{Bl}_p X \text{ (outside } Z)$$
$$X_- = \mathbb{CP}^n \text{ (inside } Z)$$
$$Y = \mathbb{CP}^{n-1} \simeq Z/S^1$$



Symplectic cut

A cut space can be viewed as the degeneration of the smooth manifold X into a space $X_+ \cup_Y X_-$ with a normal crossing singularity along Y.

Symplectic cut

A cut space can be viewed as the degeneration of the smooth manifold X into a space $X_+ \cup_Y X_-$ with a normal crossing singularity along Y.
That is, there is a fibration

$$\mathcal{X} \to \Delta, \quad \Delta \subset \mathbb{C}$$

with $\mathcal{X}_t \simeq X$ for all $t \neq 0$, and $\mathcal{X}_0 \simeq X_+ \cup_Y X_-$.

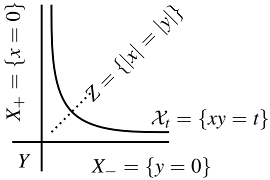
Symplectic cut

A cut space can be viewed as the degeneration of the smooth manifold X into a space $X_+ \cup_Y X_-$ with a normal crossing singularity along Y.
That is, there is a fibration

$$\mathcal{X} \to \Delta, \quad \Delta \subset \mathbb{C}$$

with $\mathcal{X}_t \simeq X$ for all $t \neq 0$, and $\mathcal{X}_0 \simeq X_+ \cup_Y X_-$.

Example : $\mathcal{X}$ is a Lefschetz fibration and the singular fiber $\mathcal{X}_0$ is disconnected by the singular point. The neighborhood of the singularity is as in the figure.



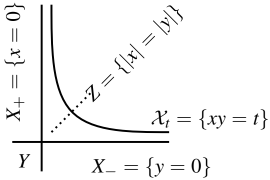
Symplectic cut

A cut space can be viewed as the degeneration of the smooth manifold X into a space $X_+ \cup_Y X_-$ with a normal crossing singularity along Y.
That is, there is a fibration

$$\mathcal{X} \to \Delta, \quad \Delta \subset \mathbb{C}$$

with $\mathcal{X}_t \simeq X$ for all $t \neq 0$, and $\mathcal{X}_0 \simeq X_+ \cup_Y X_-$.

Example : $\mathcal{X}$ is a Lefschetz fibration and the singular fiber $\mathcal{X}_0$ is disconnected by the singular point. The neighborhood of the singularity is as in the figure.



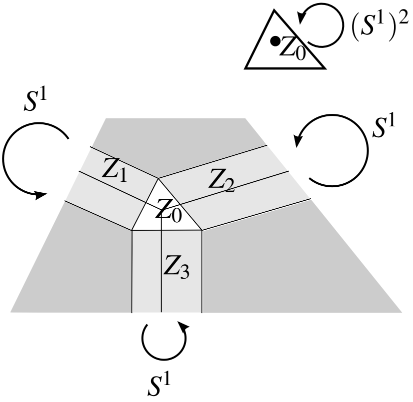
In general the degeneration corresponding to a symplectic cut is a family version of the above example, where the family is parametrized by Y.

Multiple cut

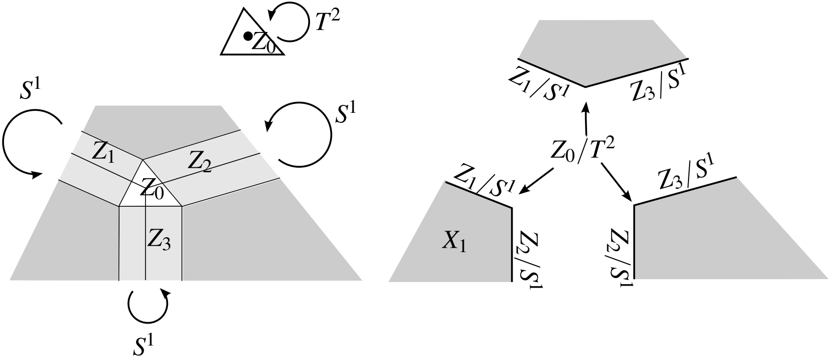
Inputs :

- A symplectic manifold (X, ω),
- hypersurfaces $Z_i \subset X$, $i = 1, 2, \ldots,$
- S^1-action on the neighborhoods of hypersurfaces,
- on neighborhoods of intersections $\cap_i Z_i$, the S^1-actions fit together into a Hamiltonian torus action.

Multiple cut

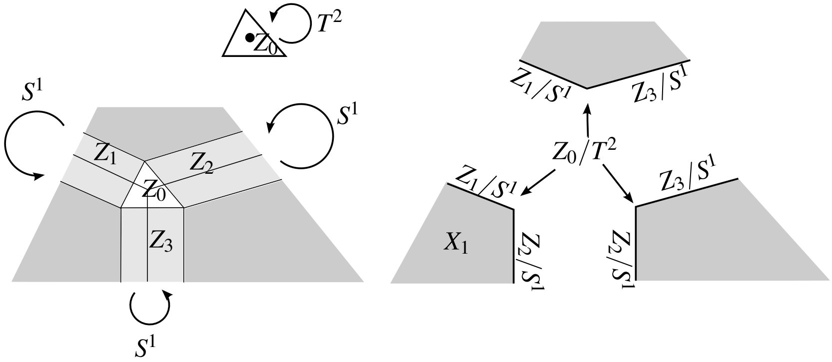


Multiple cut



Â

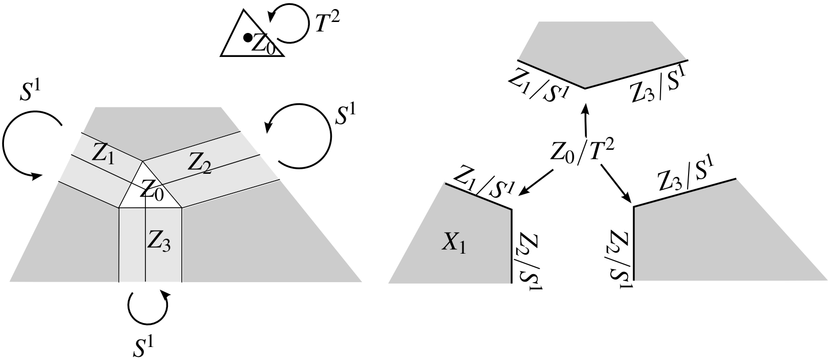
Multiple cut



Â

Output : Collection of symplectic manifolds, called **cut spaces**, with relative normal crossing divisors.

Multiple cut



Â

Output : Collection of symplectic manifolds, called **cut spaces**, with relative normal crossing divisors.

Example : In the cut space X_1, Z_1/S^1 and Z_2/S^1 are relative divisors whose intersection is Z_0/T^2.

Questions

Can counts of holomorphic curves in the unbroken manifold X be expressed in terms of

- counts of holomorphic curves in the broken manifold $\mathfrak{X}$?

Questions

Can counts of holomorphic curves in the unbroken manifold X be expressed in terms of

- counts of holomorphic curves in the broken manifold $\mathcal{X}$?
- a sum of products of curves in pieces of the broken manifold $\mathcal{X}$?

Objects of interest

- **Unbroken maps** $u : C \to X$: these are the standard pseudoholomorphic maps living in X, whose domain C is a nodal curve.

Objects of interest

- **Unbroken maps** $u : C \to X$: these are the standard pseudoholomorphic maps living in X, whose domain C is a nodal curve.
- **Broken maps** $u : C \to \mathcal{X}$: the domain is a nodal curve, different components of C map to different components in the broken manifold $\mathcal{X}$ and there is a matching condition at nodes. Broken maps have an underlying tropical graph.

Objects of interest

- **Unbroken maps** $u : C \to X$: these are the standard pseudoholomorphic maps living in X, whose domain C is a nodal curve.
- **Broken maps** $u : C \to \mathcal{X}$: the domain is a nodal curve, different components of C map to different components in the broken manifold $\mathcal{X}$ and there is a matching condition at nodes. Broken maps have an underlying tropical graph.
- **Split maps** $u : C \to \mathcal{X}$: a variant of a broken map in which the matching condition at the nodes is degenerated into a combinatorial condition.

Questions, restated

Can counts of holomorphic curves in the unbroken manifold X (unbroken maps) be expressed in terms of

- counts of holomorphic curves in the broken manifold $\mathcal{X}$ (broken maps)?

Questions, restated

Can counts of holomorphic curves in the unbroken manifold X (unbroken maps) be expressed in terms of

- counts of holomorphic curves in the broken manifold $\mathcal{X}$ (broken maps)?
- a sum of products of curves in pieces of the broken manifold $\mathcal{X}$ (split maps)?

Plan for the talk

- Part 1: Unbroken to Broken.
- Part 2 : Broken to Split.

Table of Contents

1 Introducing the problem

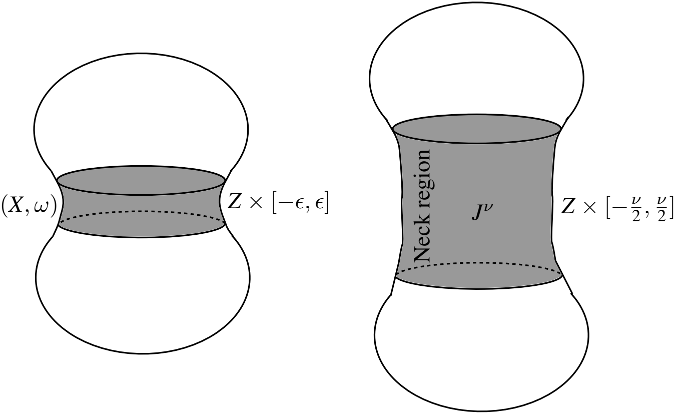
2 Single cut : Unbroken to broken

3 Multiple cut : Unbroken to broken

4 Degenerating matching conditions

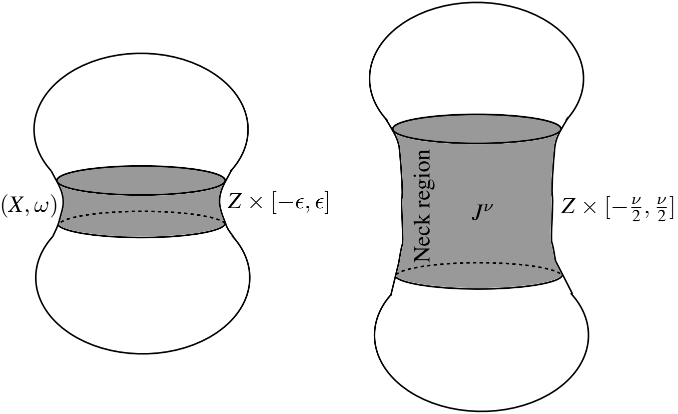
Single cut : stretching necks

For any $\nu > 0$, we equip the symplectic manifold (X, ω) with a tamed almost structure J^ν so that (X, J^ν) has a **neck** of length ν.



Single cut : stretching necks

For any $\nu > 0$, we equip the symplectic manifold (X, ω) with a tamed almost structure J^ν so that (X, J^ν) has a **neck** of length ν.

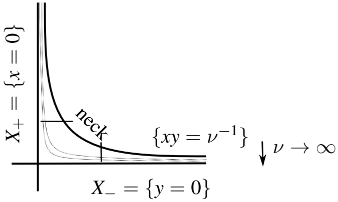


The neck region is a fibration $Z \times [-\frac{\nu}{2}, \frac{\nu}{2}] \to Y$, and the fibers are holomorphic cylinders $S^1 \times [-\frac{\nu}{2}, \frac{\nu}{2}]$.

Stretching the neck : example

A conic with neck length ν is

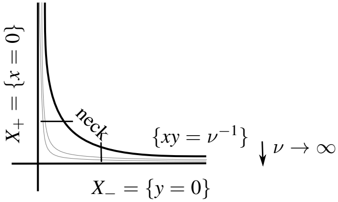
$$\{xy = \nu^{-1}\} \subset \mathbb{P}^2.$$



Stretching the neck : example

A conic with neck length ν is

$$\{xy = \nu^{-1}\} \subset \mathbb{P}^2.$$



Note : The symplectic form is unchanged on the family of neck-stretched manifolds (X, J_ν).

Convergence to broken maps

The moduli spaces of J^ν-holomorphic curves are homotopy equivalent for all ν.

Convergence to broken maps

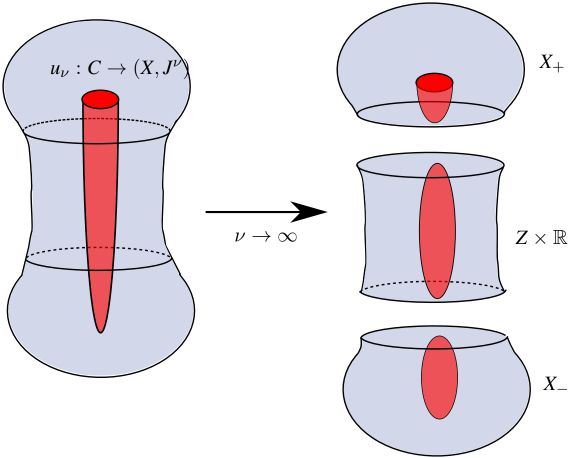
The moduli spaces of J^ν-holomorphic curves are homotopy equivalent for all ν.

In the limit $\nu \to \infty$, we obtain broken maps.

Theorem (Hofer et al, Ionel-Parker etc.)

Suppose $u_\nu : C \to (X, J^\nu)$ is a sequence of pseudoholomorphic maps with uniformly bounded ω-area. Then a subsequence converges to a broken map.

Convergence to broken maps



In the limit some curve components collapse into the relative divisor. Think of these as lying in the 'neck piece' $Z \times \mathbb{R}$.

Broken maps

- The target space of a broken map is a **broken manifold**

$$\mathcal{X} = X_+ \cup (\overline{Z \times \mathbb{R}}) \cup X_-$$

Broken maps

- The target space of a broken map is a **broken manifold**

$$\mathcal{X} = X_+ \cup (\overline{Z \times \mathbb{R}}) \cup X_-$$

- The space $\overline{Z \times \mathbb{R}}$ is called the **neck piece**. It is the compactification of $Z \times \mathbb{R}$ by adding divisors at $Z \times \{\pm\infty\}$. It is a $\mathbb{P}^1$-bundle over the relative divisor $Y := Z/S^1$.

Broken maps

- The target space of a broken map is a **broken manifold**

$$\mathcal{X} = X_+ \cup (\overline{Z \times \mathbb{R}}) \cup X_-$$

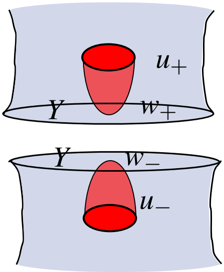
- The space $\overline{Z \times \mathbb{R}}$ is called the **neck piece**. It is the compactification of $Z \times \mathbb{R}$ by adding divisors at $Z \times \{\pm\infty\}$. It is a $\mathbb{P}^1$-bundle over the relative divisor $Y := Z/S^1$.
- A broken map consists of components in X_+, X_- and the neck piece $\overline{Z \times \mathbb{R}}$ satisfying a matching condition at nodes :

Broken maps

The matching condition at a node is

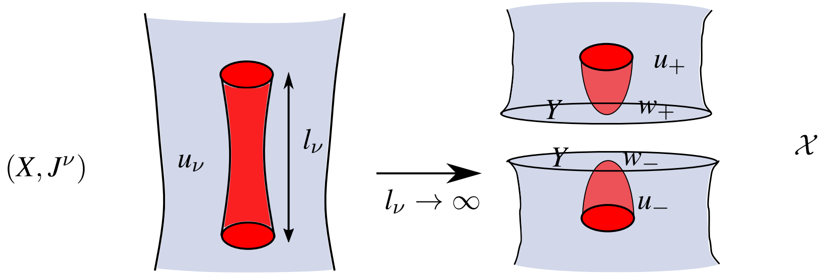
- $u_+(w_+)$, $u_-(w_-)$ are the same points in the relative divisor Y,
- The intersection multiplicities of u_+, u_- with Y at the nodal point are equal.

$Y := Z/S^1$
is the relative divisor

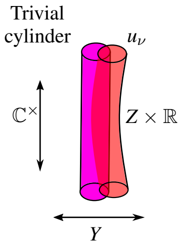


Justifying the convergence

Nodes are formed by the convergence of long cylinders with small area. This leads to equal intersection multiplicities and matching on the divisor Y :



Justifying convergence : Long cylinders with small area

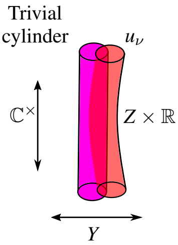


- Let

$$u_\nu : [0, l_\nu] \times S^1 \to (X, J^\nu)$$

be a sequence of cylinders with uniformly bounded Hofer energy whose projections to Y have small enough area.

Justifying convergence : Long cylinders with small area

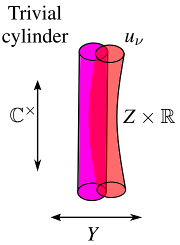


- Let

$$u_\nu : [0, l_\nu] \times S^1 \to (X, J^\nu)$$

  be a sequence of cylinders with uniformly bounded Hofer energy whose projections to Y have small enough area.
- Then, u_ν is asymptotically close to a 'trivial cylinder' in $Z \times \mathbb{R}$.

Justifying convergence : Long cylinders with small area



- Let
$$u_\nu : [0, l_\nu] \times S^1 \to (X, J^\nu)$$
be a sequence of cylinders with uniformly bounded Hofer energy whose projections to Y have small enough area.
- Then, u_ν is asymptotically close to a 'trivial cylinder' in $Z \times \mathbb{R}$.
- A trivial cylinder in the $\mathbb{C}^\times$-bundle $Z \times \mathbb{R} \to Y$ projects to a constant on Y, and is therefore an n-cover of a fiber.

Breaking annulus lemma

The precise statement for the phenomenon explained in the last slide is the following:

Theorem (Breaking annulus lemma)

Let $l_\nu \to 0$, and $u_\nu : [-l_\nu, l_\nu] \times S^1 \to Z \times \mathbb{R}$ be a sequence of holomorphic cylinders satisfying

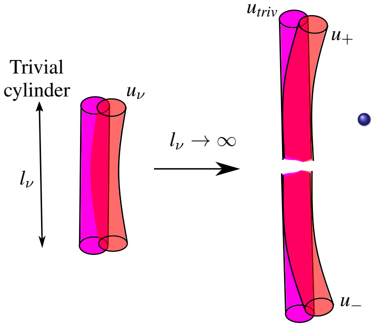
$$\omega_Y(u_\nu) < \hbar, \quad \sup_\nu E_{Hofer}(u_\nu) < \infty.$$

Then there is a subsequence of $(u_\nu)_\nu$ and constants C, $\gamma > 0$, $\mu \in \mathbb{Z}_+$ such that

$$d(u_\nu(s,t), u_\nu^{\mathrm{triv}}(s,t)) \leq C(e^{-\gamma|l_\nu - s|} + e^{-\gamma|-l_\nu - s|})$$

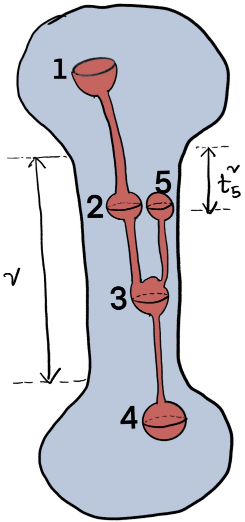
where u_ν^{triv} is a trivial cylinder defined as $u_\nu^{\mathrm{triv}}(s,t) = e^{\mu(s+it)}u_\nu(0,0)$.

Justifying convergence : Long cylinders with small area



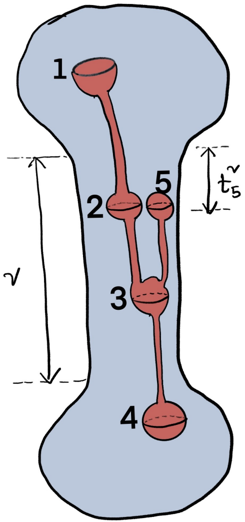
- A subsequence of $(u_\nu)_\nu$ converges to a node with intersection multiplicity n.

Justifying the convergence



- A converging sequence of maps $u_\nu : C \to X^\nu$ consists of pockets of high area separated by long cylinders :

Justifying the convergence



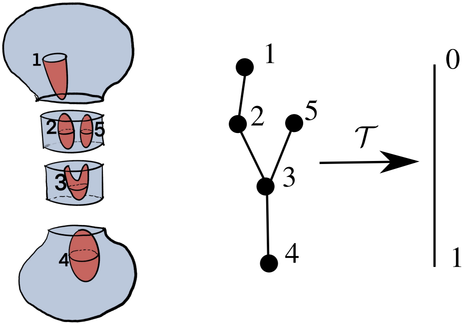
- A converging sequence of maps $u_\nu : C \to X^\nu$ consists of pockets of high area separated by long cylinders :
- A pocket of high area converges modulo a translation in the target space.
  For example $e^{t_5^\nu} u_\nu$ converges to a limit component of the broken map.

The idea of a tropical graph

For each component i of the domain of the limit broken map the limit $\lim_{\nu\to\infty} \frac{t^i_\nu}{\nu}$ gives a relative position of the map in the neck region.

The idea of a tropical graph

For each component i of the domain of the limit broken map the limit $\lim_{\nu\to\infty} \frac{t^i_\nu}{\nu}$ gives a relative position of the map in the neck region.



We obtain a map

$$\mathcal{T} : \text{Graph of domain curve} \to [0, 1]$$

called the *tropical graph*.

Table of Contents

1. Introducing the problem
2. Single cut : Unbroken to broken
3. Multiple cut : Unbroken to broken
4. Degenerating matching conditions

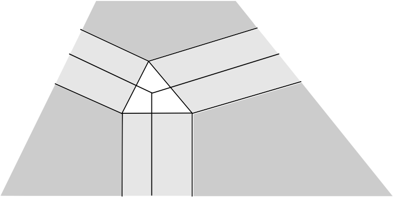
Tropical Hamiltonian action

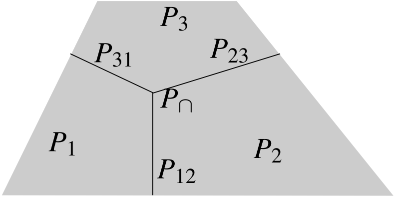
1. A multiple cut has an underlying **polytopal decomposition** of $\mathfrak{t}^\vee \simeq \mathbb{R}^n$ into a collection $\mathcal{P}^0$ of top-dimensional Delzant polytopes $P \subset \mathfrak{t}^\vee$.

Tropical Hamiltonian action

1. A multiple cut has an underlying **polytopal decomposition** of $\mathfrak{t}^\vee \simeq \mathbb{R}^n$ into a collection $\mathcal{P}^0$ of top-dimensional Delzant polytopes $P \subset \mathfrak{t}^\vee$.
2. Let $\mathcal{P}$ be the closure of $\mathcal{P}^0$ under intersections.

The polytopal decomposition for our first example of a multiple cut is :



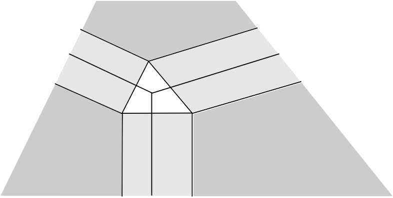


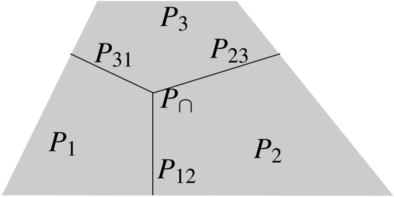
Here $P_{ij} = P_i \cap P_j$, $P_\cap = P_1 \cap P_2 \cap P_3$.

Tropical Hamiltonian action

1. A multiple cut has an underlying **polytopal decomposition** of $\mathfrak{t}^\vee \simeq \mathbb{R}^n$ into a collection $\mathcal{P}^0$ of top-dimensional Delzant polytopes $P \subset \mathfrak{t}^\vee$.
2. Let $\mathcal{P}$ be the closure of $\mathcal{P}^0$ under intersections.

The polytopal decomposition for our first example of a multiple cut is :





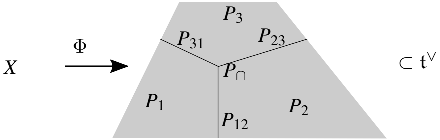
Here $P_{ij} = P_i \cap P_j$, $P_\cap = P_1 \cap P_2 \cap P_3$.

The set of polytopes is $\mathcal{P} = \{P_i\}_i \cup \{P_{ij}\}_{i,j} \cup \{P_\cap\}$.

Tropical Hamiltonian action

The input for a multiple cut consists of (more precise than earlier)

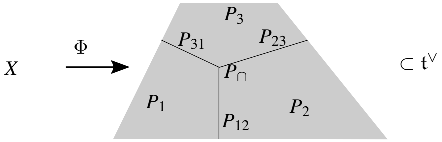
- a decomposition of $\mathfrak{t}^\vee \simeq \mathbb{R}^n$ into Delzant polytopes
- and a **tropical moment map** $\Phi : (X, \omega) \to \mathfrak{t}^\vee$.



Definition (Tropical moment map for a decomposition $\mathcal{P}$)

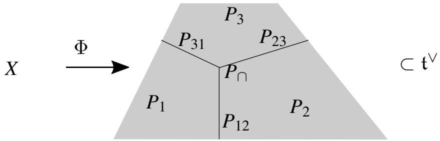
is a map $\Phi : X \to \mathfrak{t}^\vee$ that generates a Hamiltonian T_P-action in a neighborhood of $\Phi^{-1}(P)$, where $\mathfrak{t}_P := \mathrm{ann}(TP)$.

Tropical Hamiltonian action



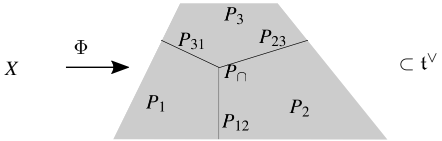
- In our example $\mathfrak{t}^\vee \simeq \mathbb{R}^2$, $T_{P_i} = \{\mathrm{Id}\}$, $T_{P_{ij}} \simeq S^1$, $T_{P_\cap} \simeq (S^1)^2$.

Tropical Hamiltonian action



- In our example $\mathfrak{t}^\vee \simeq \mathbb{R}^2$, $T_{P_i} = \{\mathrm{Id}\}$, $T_{P_{ij}} \simeq S^1$, $T_{P_\cap} \simeq (S^1)^2$.
- Thus there is an S^1-action in the neighborhood of $\Phi^{-1}(P_i)$ and a $(S^1)^2$-action in a neighborhood of $\Phi^{-1}(P_\cap)$.

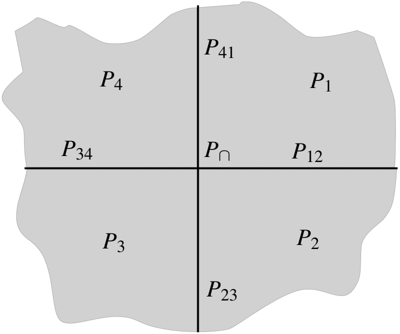
Tropical Hamiltonian action



- In our example $\mathfrak{t}^\vee \simeq \mathbb{R}^2$, $T_{P_i} = \{\mathrm{Id}\}$, $T_{P_{ij}} \simeq S^1$, $T_{P_\cap} \simeq (S^1)^2$.
- Thus there is an S^1-action in the neighborhood of $\Phi^{-1}(P_i)$ and a $(S^1)^2$-action in a neighborhood of $\Phi^{-1}(P_\cap)$.
- In general $P_0 \subset P_1 \implies T_{P_1} \subset T_{P_0}$.

Tropical Hamiltonian action

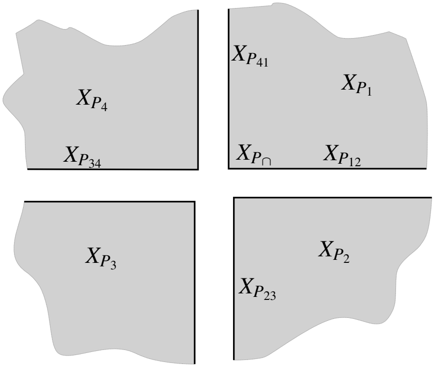
Running example for multiple cut : Two orthogonal single cuts form a multiple cut with polytopal decomposition as below.



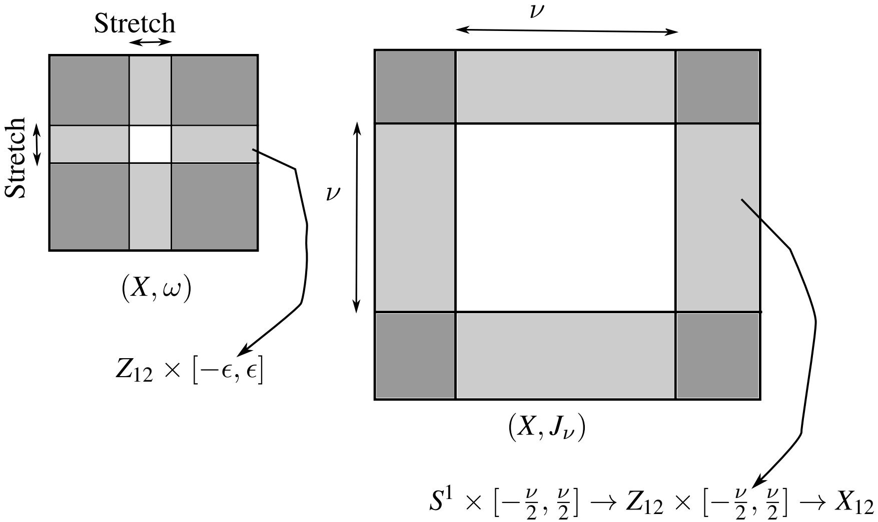
The set of polytopes is $\mathcal{P} = \{P_1, P_2, P_3, P_4, P_{12}, P_{23}, P_{34}, P_{41}, P_\cap\}$.

Tropical Hamiltonian action

- Given a tropical Hamiltonian action $(X, \mathcal{P}, \Phi)$ a multiple cut produces a **cut space** X_P corresponding to each top-dimensional polytope $P \in \mathcal{P}$.
- A cut space X_P has **relative divisors** X_Q for every codimension one polytope $Q \subset P$.
- For example, $X_{P_{12}}, X_{P_{23}}$ are relative divisors in X_{P_2}.



Neck-stretched almost complex structure



Convergence for a multiple cut

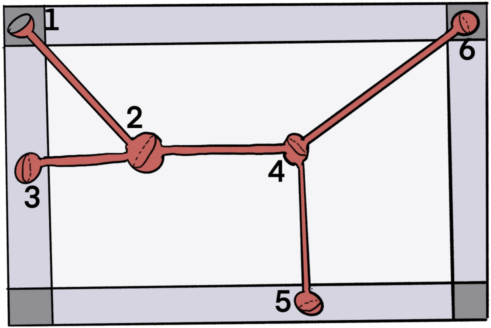
Theorem (VW)

A sequence of pseudoholomorphic maps $u_\nu : C_\nu \to (X, J_\nu)$ with bounded ω-area has a subsequence that converges to a broken map in $\mathcal{X}$.

Similar results : Eleny Ionel, Brett Parker, Mohammad F. Tehrani.
Idea of a tropical graph : Brett Parker.

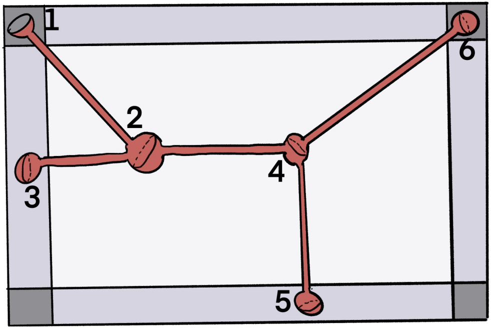
Convergence for a multiple cut

The following is a picture to have in mind for convergence in a multiple cut. Elements in a convergent sequence of holomorphic maps $u_\nu : C \to (X, J_\nu)$ are of the following form.



Convergence for a multiple cut

The following is a picture to have in mind for convergence in a multiple cut. Elements in a convergent sequence of holomorphic maps $u_\nu : C \to (X, J_\nu)$ are of the following form.



There are pockets of high area separated by long cylinders with small area.

Broken manifold

The target space of a broken map is a **broken manifold**.

Broken manifold

The target space of a broken map is a **broken manifold**.

The broken manifold $\mathcal{X}$ is the disjoint union

$$\mathcal{X} := \bigsqcup_{P \in \mathcal{P}} X_{\overline{P}}$$

where $X_{\overline{P}} \to X_P$ is a T_P-toric fibration.

Broken manifold

The target space of a broken map is a **broken manifold**.

The broken manifold $\mathcal{X}$ is the disjoint union

$$\mathcal{X} := \bigsqcup_{P \in \mathcal{P}} X_{\overline{P}}$$

where $X_{\overline{P}} \to X_P$ is a T_P-toric fibration.

Thus relative submanifolds in cut spaces are thickened into top-dimensional pieces in the broken manifold.

Broken manifold

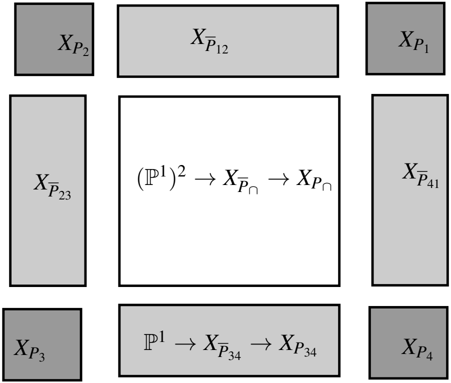
The target space of a broken map is a **broken manifold**.

The broken manifold $\mathcal{X}$ is the disjoint union

$$\mathcal{X} := \bigsqcup_{P \in \mathcal{P}} X_{\overline{P}}$$

where $X_{\overline{P}} \to X_P$ is a T_P-toric fibration.

Thus relative submanifolds in cut spaces are thickened into top-dimensional pieces in the broken manifold.

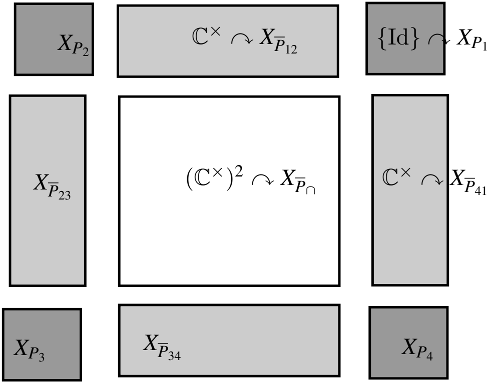


Broken manifold

A component $X_{\overline{P}}$ of the broken manifold has a natural action of the complex torus $T_{P,\mathbb{C}}$.

Broken manifold

A component $X_{\overline{P}}$ of the broken manifold has a natural action of the complex torus $T_{P,\mathbb{C}}$.



Symmetry group on $X_{\overline{P}_\cap}$
$= T_{P_\cap} := \mathbb{C}^\times$

Symmetry group on $X_{\overline{P}_{ij}}$
$= T_{P_{ij}} := \mathbb{C}^\times$

Symmetry group on X_{P_i}
$= T_{P_i} := \{\mathrm{Id}\}$

Broken manifold

In the broken manifold $\mathcal{X}$ the piece $X_{\overline{P}}$ is a toric fibration

$$V_{P^\vee} \to X_{\overline{P}} \to X_P$$

whose fiber is a toric variety with moment polytope $P^\vee$ satisfying

Broken manifold

In the broken manifold $\mathcal{X}$ the piece $X_{\overline{P}}$ is a toric fibration

$$V_{P^\vee} \to X_{\overline{P}} \to X_P$$

whose fiber is a toric variety with moment polytope $P^\vee$ satisfying

- $P^\vee$ is complementary dimensional : $\dim(P^\vee) = \dim(\mathfrak{t}^\vee) - \dim(P)$,

Broken manifold

In the broken manifold $\mathcal{X}$ the piece $X_{\overline{P}}$ is a toric fibration

$$V_{P^\vee} \to X_{\overline{P}} \to X_P$$

whose fiber is a toric variety with moment polytope $P^\vee$ satisfying

- $P^\vee$ is complementary dimensional : $\dim(P^\vee) = \dim(\mathfrak{t}^\vee) - \dim(P)$,
- if $Q \subset P$ is a face, then $P^\vee \subset Q^\vee$ is a face.

Broken manifold

In the broken manifold $\mathcal{X}$ the piece $X_{\overline{P}}$ is a toric fibration

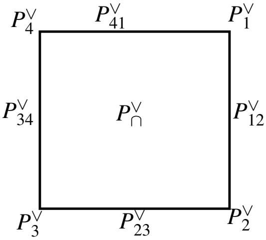
$$V_{P^\vee} \to X_{\overline{P}} \to X_P$$

whose fiber is a toric variety with moment polytope $P^\vee$ satisfying

- $P^\vee$ is complementary dimensional : $\dim(P^\vee) = \dim(\mathfrak{t}^\vee) - \dim(P)$,
- if $Q \subset P$ is a face, then $P^\vee \subset Q^\vee$ is a face.

The **dual poltyopes** $P^\vee$ fit into a **dual complex**

$$B^\vee := (\cup_{P \in \mathcal{P}} P^\vee)/\sim.$$



Broken manifold

Some remarks :

- We will see that the tropical graph lies in the dual complex.

Broken manifold

Some remarks :

- We will see that the tropical graph lies in the dual complex.
- There are different choices of the dual complex. In our example, we may vary the side lengths of the rectangle.

Broken manifold

Some remarks :

- We will see that the tropical graph lies in the dual complex.
- There are different choices of the dual complex. In our example, we may vary the side lengths of the rectangle.
- We will see that the moduli space of broken maps $\mathcal{M}(\mathcal{X})$ depends on the dual complex.

Broken manifold

Some remarks :

- We will see that the tropical graph lies in the dual complex.
- There are different choices of the dual complex. In our example, we may vary the side lengths of the rectangle.
- We will see that the moduli space of broken maps $\mathcal{M}(\mathcal{X})$ depends on the dual complex.
- The dual complex is part of the datum required to construct neck-stretched almost complex structures. Thus the moduli space of maps $\mathcal{M}(X, J^{\nu})$ depends on the dual complex.

Broken manifold

Since the piece $X_{\overline{P}}$ is a toric fibration

$$V_{P^\vee} \to X_{\overline{P}} \xrightarrow{\pi_P} X_P$$

it has two kinds of **relative divisors** :

- **horizontal relative divisors**, which are inverse images of relative divisors $X_Q \subset X_P$, $Q \in \mathcal{P}$ namely

$$\pi_P^{-1}(X_Q),$$

- and **vertical relative divisors**, which are torus-invariant divisors of the fiber $V_{P^\vee}$.

Broken manifold

Since the piece $X_{\overline{P}}$ is a toric fibration

$$V_{P^\vee} \to X_{\overline{P}} \xrightarrow{\pi_P} X_P$$

it has two kinds of **relative divisors** :

- **horizontal relative divisors**, which are inverse images of relative divisors $X_Q \subset X_P$, $Q \in \mathcal{P}$ namely

$$\pi_P^{-1}(X_Q),$$

- and **vertical relative divisors**, which are torus-invariant divisors of the fiber $V_{P^\vee}$.

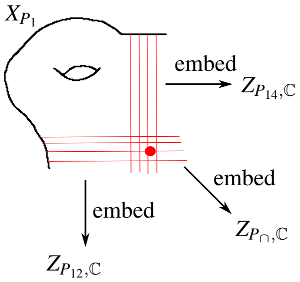
We introduce notation for the $T_{P,\mathbb{C}}$-principal bundle

$$Z_{P,\mathbb{C}} := X_{\overline{P}} \backslash \{\text{vertical divisors}\},$$

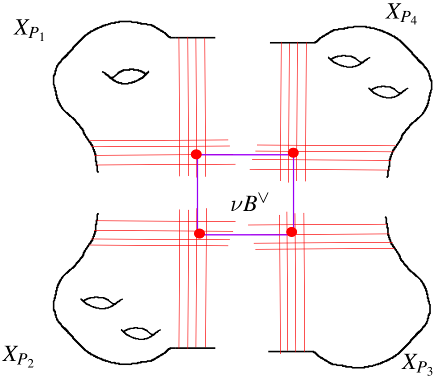
and the complement of all relative divisors

$$X_{\overline{P}}^{\circ} := X_{\overline{P}} \backslash \{\text{relative divisors}\}.$$

Broken manifold



Cylindrical coordinates
in (X_{P_1} – relative divisors)



(X, J^ν)

Neck stretching using dual polytope

Broken map

A broken map $u : C \to \mathcal{X}$ consists of

- a domain nodal curve C modelled on a graph Γ,
- for each component $C_v \subset C$, a map $u_v : C_v \to X_{\overline{P}(v)}$ to a component of $\mathcal{X}$ such that
$$u_v(C_v \backslash \{nodes\}) \subset X^{\circ}_{\overline{P}(v)},$$
- and a *tropical graph* $\Gamma \to B^{\vee}$.

For every edge e there are holomorphic coordinates in neighborhoods of the nodal lifts $w_e^{\pm}$ for which the *matching condition* is satisfied.

Broken map

A broken map $u : C \to \mathcal{X}$ consists of

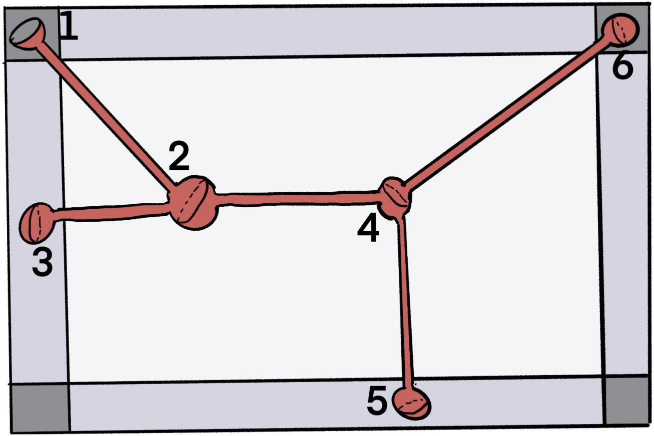
- a domain nodal curve C modelled on a graph Γ,
- for each component $C_v \subset C$, a map $u_v : C_v \to X_{\overline{P}(v)}$ to a component of $\mathcal{X}$ such that
$$u_v(C_v \backslash \{nodes\}) \subset X^{\circ}_{\overline{P}(v)},$$
- and a *tropical graph* $\Gamma \to B^{\vee}$.

For every edge e there are holomorphic coordinates in neighborhoods of the nodal lifts $w_e^{\pm}$ for which the *matching condition* is satisfied.

Remark : As in SFT, one may think of u as a map from the punctured curve $C - \{nodes\}$ to manifolds with cylindrical ends.

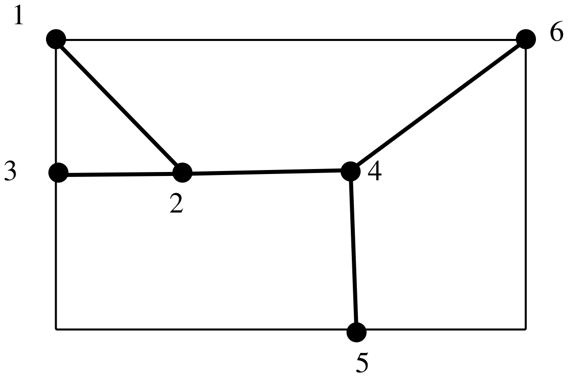
Broken map

For example, a sequence of maps $u_\nu : C \to (X, J_\nu)$ of the form



Broken map

converges to a broken map whose tropical graph is



Broken map : Tropical graph

A broken map u has a tropical graph

$$\mathcal{T} : \Gamma \to B^{\vee}$$

that satisfies the following.

1. (Vertex Polytope) For a vertex v of Γ

$$u(C_v) \subset X_{\overline{P(v)}} \implies \mathcal{T}(v) \in P^{\vee}$$

2. (Edge Slope) The node cooresponding to an edge e of Γ has intersection multiplicity $\mu := (\mu_1, \ldots, \mu_n) \in (Z_{>0})^n$ with relative divisors

$\implies \mathcal{T}(e)$ is a line segment with slope μ.

Broken map : Tropical graph

A broken map u has a tropical graph

$$\mathcal{T} : \Gamma \to B^\vee$$

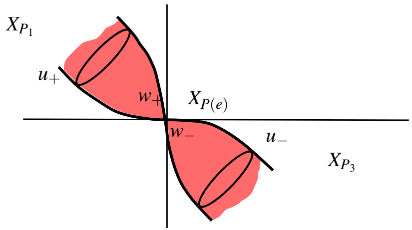
that satisfies the following.

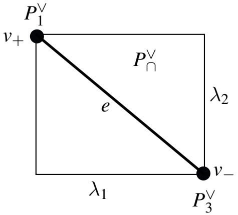
1. (Vertex Polytope) For a vertex v of Γ
$$u(C_v) \subset X_{\overline{P(v)}} \implies \mathcal{T}(v) \in P^\vee$$
2. (Edge Slope) The node cooresponding to an edge e of Γ has intersection multiplicity $\mu := (\mu_1, \ldots, \mu_n) \in (Z_{>0})^n$ with relative divisors
$$\implies \mathcal{T}(e) \text{ is a line segment with slope } \mu.$$

Remark : Two tropical graphs are isomorphic if they have the same edge slopes. Thus a tropical graph is a combinatorial invariant of broken maps.

Broken map : Example

- A broken map consists of a map and a tropical graph.

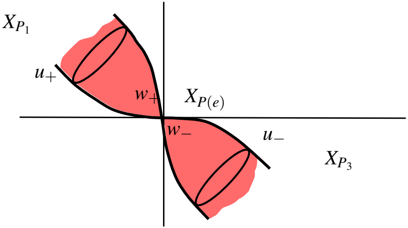


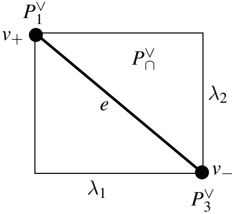


$\mu(e) :=$Slope of $e = (\mu_1, \mu_2)$

Broken map : Example

- A broken map consists of a map and a tropical graph.

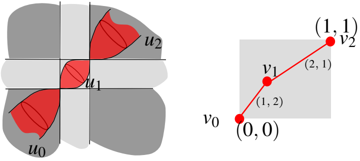


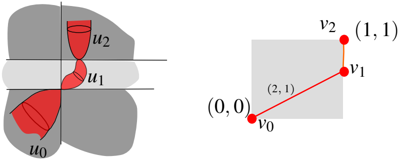


$\mu(e)$:=Slope of $e = (\mu_1, \mu_2)$

- At the node w the intersection multiplicity with relative divisors on both lifts of the node w are equal to the slope $\mu(e) = (\mu_1, \mu_2) \in (\mathbb{Z}_{>0})^2$.

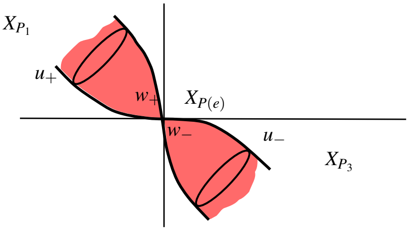
Broken map : Example

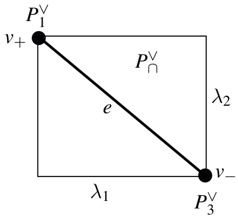




Broken map : Matching condition

We state the matching condition for the example.





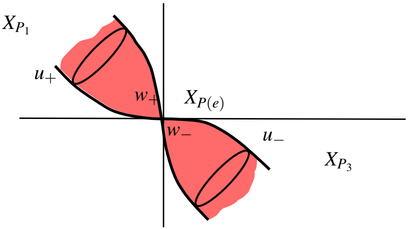
$\mu(e)$:=Slope of $e = (\mu_1, \mu_2)$

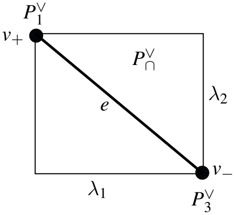
The matching condition at the node w is that

- (Horizontal) $u_+(w_+) = u_-(w_-)$ on $X_{P_\cap}$, and
- (Vertical) the leading Taylor coefficients of u_+, u_- in the directions normal to $X_{P_\cap}$ match.

Broken map : Matching condition

In general the matching condition is as follows:





$\mu(e)$:=Slope of $e = (\mu_1, \mu_2)$

The neighborhood of the node w_e lies in a region with a $T_{P(e),\mathbb{C}}$-action where $P(e) = P(v_+) \cap P(v_-)$. The matching condition is that

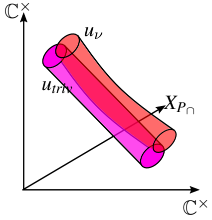
- (Horizontal) $u_+(w_{e,+}) = u_-(w_{e,-})$ on $X_{P(e)}$, and
- (Vertical) the leading Taylor coefficients of u_+, u_- match in the vertical directions of the fibration

$$T_{P(e),\mathbb{C}} \to Z_{P(e),\mathbb{C}} \to X_{P(e)}.$$

Broken map : Matching condition

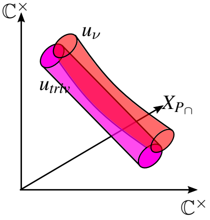
Remark : In the case of a single cut, the vertical matching condition is automatically satisfied by a suitable choice of holomorphic coordinates in neighborhoods of the nodal lifts.

Justifying the matching condition



- Suppose the node w is formed by the convergence of long cylinder $u_\nu : [0, l_\nu] \times S^1 \to (X, J_\nu)$ with small area.

Justifying the matching condition

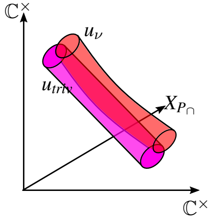


- Suppose the node w is formed by the convergence of long cylinder $u_\nu : [0, l_\nu] \times S^1 \to (X, J_\nu)$ with small area.
- We may view u_ν as lying in a $(\mathbb{C}^\times)^2$-fibration

$$(\mathbb{C}^\times)^2 \to Z_{P_\cap,\mathbb{C}} \xrightarrow{\pi_{P_\cap}} X_{P_\cap},$$

and the maps u_ν are asymptotically close to a trivial cylinder.

Justifying the matching condition



- Suppose the node w is formed by the convergence of long cylinder $u_\nu : [0, l_\nu] \times S^1 \to (X, J_\nu)$ with small area.
- We may view u_ν as lying in a $(\mathbb{C}^\times)^2$-fibration

$$(\mathbb{C}^\times)^2 \to Z_{P_\cap,\mathbb{C}} \xrightarrow{\pi_{P_\cap}} X_{P_\cap},$$

and the maps u_ν are asymptotically close to a trivial cylinder.
- A trivial cylinder satisfies

$$\begin{aligned} (Base) \quad & \pi_{P_\cap} \circ u_{triv} = constt, \\ (Fiber) \quad & u_{triv}(z) = (az^{\mu_1}, bz^{\mu_2}) \end{aligned}$$

for some $\mu = (\mu_1, \mu_2) \in (\mathbb{Z}_+)^2$.

Justifying the matching condition

The breaking annulus lemma is the precise statement for 'asympytotic closeness to a trivial cylinder'. The result is stated for a $T_{P,\mathbb{C}}$-principal bundle

$$T_{P,\mathbb{C}} \to Z_{P,\mathbb{C}} \to X_P.$$

Theorem (Breaking annulus lemma)

Let $l_\nu \to 0$, and $u_\nu : [-l_\nu, l_\nu] \times S^1 \to Z_{P,\mathbb{C}}$ be a sequence of holomorphic cylinders satisfying

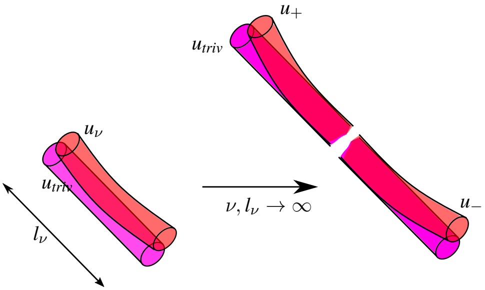
$$\omega_{X_P}(u_\nu) < \hbar, \quad \sup_\nu E_{Hofer}(u_\nu) < \infty.$$

Then there is a subsequence of $(u_\nu)_\nu$ and constants C, $\gamma > 0$, $\mu \in \mathfrak{t}_{P,\mathbb{Z}}$ such that

$$d(u_\nu(s,t), u_\nu^{\text{triv}}(s,t)) \leq C(e^{-\gamma|l_\nu - s|} + e^{-\gamma|-l_\nu - s|})$$

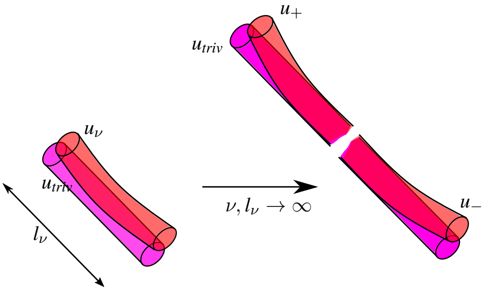
where u_ν^{triv} is a trivial cylinder defined as $u_\nu^{\text{triv}}(s,t) = e^{\mu(s+it)}u_\nu(0,0)$.

Justifying the matching condition



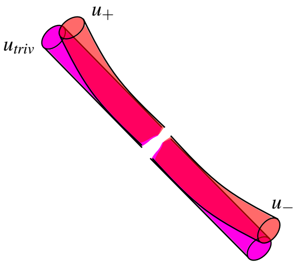
- Breaking annulus lemma $\Longrightarrow$ the punctured curves $u_+|C\backslash\{w_+\}$, $u_-|C\backslash\{w_-\}$ are asymptotic to the **same** trivial cylinder.

Justifying the matching condition



- Breaking annulus lemma $\implies$ the punctured curves $u_+|C\backslash\{w_+\}$, $u_-|C\backslash\{w_-\}$ are asymptotic to the **same** trivial cylinder.
- This is equivalent to the matching condition at w. Indeed $u^{\pm}_{\mathrm{triv}}$ is the leading order Taylor approximation of $u^{\pm}$.

Matching condition : an alternate statement

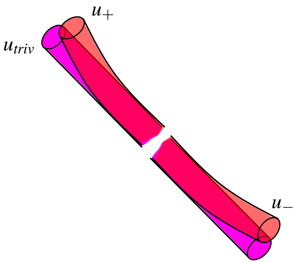


- The trivial cylinder u_{triv} is

$$\mathbb{C}^\times \simeq T_{\mu,\mathbb{C}} \subset (\mathbb{C}^\times)^2$$

the subtorus generated by the intersection multiplicity vector $\mu = (\mu_1, \mu_2) \in (\mathbb{Z}_+)^2$.

Matching condition : an alternate statement



- The trivial cylinder u_{triv} is

$$\mathbb{C}^\times \simeq T_{\mu,\mathbb{C}} \subset (\mathbb{C}^\times)^2$$

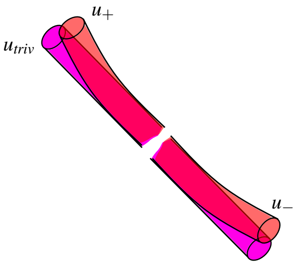
the subtorus generated by the intersection multiplicity vector $\mu = (\mu_1, \mu_2) \in (\mathbb{Z}_+)^2$.

- Matching condition :

$$(u_+ \mod T_{\mu,\mathbb{C}})(w_+) = (u_- \mod T_{\mu,\mathbb{C}})(w_-)$$

in $Z_{P_\cap,\mathbb{C}}/T_{\mu,\mathbb{C}}$.

Matching condition : an alternate statement



- The trivial cylinder u_{triv} is

$$\mathbb{C}^\times \simeq T_{\mu,\mathbb{C}} \subset (\mathbb{C}^\times)^2$$

  the subtorus generated by the intersection multiplicity vector $\mu = (\mu_1, \mu_2) \in (\mathbb{Z}_+)^2$.

- Matching condition :

$$(u_+ \mod T_{\mu,\mathbb{C}})(w_+) = (u_- \mod T_{\mu,\mathbb{C}})(w_-)$$

  in $Z_{P_\cap,\mathbb{C}}/T_{\mu,\mathbb{C}}$.

Remark : The matching condition has codimension $\dim(X) - 2$. Therefore the presence of a node does not decrease the index of the map.

Remarks about convergence and gluing

- The limit of a converging sequence $(u_\nu)_\nu$ of J_ν-holomorphic maps of index zero is a broken map $u : C \to \mathcal{X}$ of index 0.

Remarks about convergence and gluing

- The limit of a converging sequence $(u_\nu)_\nu$ of J_ν-holomorphic maps of index zero is a broken map $u : C \to \mathcal{X}$ of index 0.
- Conversely gluing an index 0 map produces an index zero J_ν-holomorphic map for all ν.

Remarks about convergence and gluing

- The limit of a converging sequence $(u_\nu)_\nu$ of J_ν-holomorphic maps of index zero is a broken map $u : C \to \mathcal{X}$ of index 0.
- Conversely gluing an index 0 map produces an index zero J_ν-holomorphic map for all ν.
- The convergence result includes a statement relating the neck lengths in the domains and target space, which is proved using the breaking annulus lemma. This statement is used in the gluing proof.

Tropical symmetry group

- Let Γ be the tropical graph of a broken map. The **tropical symmetry group** is the subgroup

$$T_{\mathrm{trop}}(\Gamma) \subset \Pi_{v \in \mathrm{Vert}(\Gamma)} T_{P(v),\mathbb{C}}$$

that preserves the matching condition at nodes,

Tropical symmetry group

- Let Γ be the tropical graph of a broken map. The **tropical symmetry group** is the subgroup

$$T_{\mathrm{trop}}(\Gamma) \subset \Pi_{v \in \mathrm{Vert}(\Gamma)} T_{P(v),\mathbb{C}}$$

that preserves the matching condition at nodes,

- and is given by the condition that

$$(t_v)_{v \in \mathrm{Vert}(\Gamma)} \in T_{\mathrm{trop}}(\Gamma)$$

iff for any edge $e = (v_+, v_-)$,

$$t_{v_+} t_{v_-}^{-1} \in T_{\mu(e),\mathbb{C}}.$$

Tropical symmetry group

- Let Γ be the tropical graph of a broken map. The **tropical symmetry group** is the subgroup

$$T_{\text{trop}}(\Gamma) \subset \Pi_{v \in \text{Vert}(\Gamma)} T_{P(v),\mathbb{C}}$$

that preserves the matching condition at nodes,

- and is given by the condition that

$$(t_v)_{v \in \text{Vert}(\Gamma)} \in T_{\text{trop}}(\Gamma)$$

iff for any edge $e = (v_+, v_-)$,

$$t_{v_+} t_{v_-}^{-1} \in T_{\mu(e),\mathbb{C}}.$$

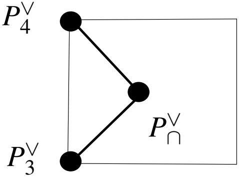
- The relative action across an edge e can only be in the direction of the trivial cylinder asymtotically close to the node.

Tropical symmetry group

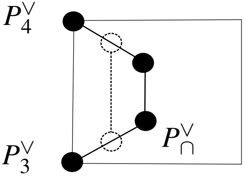
- Symmetries of the tropical graph are the infinitesimal generators of the tropical symmetry group.

Tropical symmetry group

- Symmetries of the tropical graph are the infinitesimal generators of the tropical symmetry group.
- Symmetries of the tropical graph are the ways of moving the vertices in Γ without changing the edge slope.



A rigid tropical graph: $T_{\text{trop}}(\Gamma)$ is finite



$T_{\text{trop}}(\Gamma) = \mathbb{C}^\times$

Tropical symmetry group

- A broken map of index 0 or 1 has a rigid tropical graph. Indeed, since the action of $T_{\mathrm{trop}}(\Gamma)$ does not have infinitesimal stabilizers

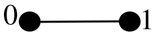
$$\mathrm{ind}(u) \geq \dim(T_{\mathrm{trop}}(\Gamma)).$$

Tropical symmetry group

- A broken map of index 0 or 1 has a rigid tropical graph. Indeed, since the action of $T_{\mathrm{trop}}(\Gamma)$ does not have infinitesimal stabilizers

$$\mathrm{ind}(u) \geq \dim(T_{\mathrm{trop}}(\Gamma)).$$

- For a single cut, broken maps of index 0 and 1 do not have componets in the neck piece. This is because the only rigid tropical graph for a single cut is

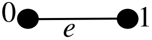


- In case of multiple cuts, components in neck pieces occur generically.

Discrete tropical symmetry

Maps with a rigid tropical graph may have a non-empty symmetry group $T_{\mathrm{trop}}(\Gamma)$. For example,

- In the single cut case the rigid tropical graph

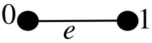
  $$\mathcal{T}(e) = 2$$

  has a symmetry group of $\mathbb{Z}_2$. Indeed, the broken map has two choices of *framing* at the node and thus, there are 2 ways of gluing the node.

Discrete tropical symmetry

Maps with a rigid tropical graph may have a non-empty symmetry group $T_{\text{trop}}(\Gamma)$. For example,

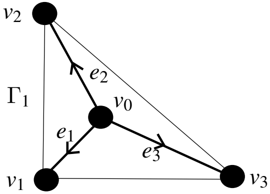
- In the single cut case the rigid tropical graph



$$\mathcal{T}(e) = 2$$

  has a symmetry group of $\mathbb{Z}_2$. Indeed, the broken map has two choices of *framing* at the node and thus, there are 2 ways of gluing the node.

- The tropical graph



$$\mathcal{T}(e_1) = (-1, -1)$$
$$\mathcal{T}(e_2) = (-1, 2)$$
$$\mathcal{T}(e_3) = (2, -1)$$

  has 3 choices of framings, and $T_{\text{trop}}(\Gamma) = \mathbb{Z}_3$.

Main result

- Suppose $L \subset X$ is a Lagrangian submanifold in the complement of separating hypersurfaces.
- After the multiple cut $L \subset \mathcal{X}$ is in one of the cut spaces in the complement of relative divisors.

Result

Given $E \geq 0$, for a large enough neck length, the moduli space of holomorphic disks of index zero with area $\leq E$

$$\mathcal{M}(X, L)_0^{\leq E} := \{u : (C, \partial C) \to ((X, J_\nu), L), \operatorname{ind}(u) = 0\}$$

is bijective to the the moduli space holomorphic broken disks of index zero with area $\leq E$

$$\mathcal{M}_{\text{brok}}(\mathcal{X}, L)_0^{\leq E} := \{u : (C, \partial C) \to (\mathcal{X}, L), \operatorname{ind}(u) = 0\}.$$

Main Result

The composition maps of the **broken Fukaya algebra**

$$CF_{\mathrm{brok}}(\mathcal{X}, L)$$

are defined by counts of broken disks of index zero.

Theorem (VW)

The broken and unbroken Fukaya algebras are homotopy equivalent :

$$CF(X, L) \simeq CF_{\mathrm{brok}}(\mathcal{X}, L).$$

Gromov topology on broken maps

We introduce some notation.

- $\widetilde{\mathcal{M}}_\Gamma(\mathcal{X})$:= moduli space of broken holomorphic disks with tropical graph Γ.

Gromov topology on broken maps

We introduce some notation.

- $\widetilde{\mathcal{M}}_\Gamma(\mathcal{X}) :=$ moduli space of broken holomorphic disks with tropical graph Γ.
- $\mathcal{M}_\Gamma(\mathcal{X}) := \widetilde{\mathcal{M}}_\Gamma / T_{\mathrm{trop}}(\Gamma)$.

Gromov topology on broken maps

We introduce some notation.

- $\widetilde{\mathcal{M}}_\Gamma(\mathcal{X}) :=$ moduli space of broken holomorphic disks with tropical graph Γ.
- $\mathcal{M}_\Gamma(\mathcal{X}) := \widetilde{\mathcal{M}}_\Gamma / T_{\text{trop}}(\Gamma)$.
- The moduli space of broken holomorphic disks with d boundary markings is
$$\mathcal{M}_d(\mathcal{X}) = \cup_\Gamma \mathcal{M}_\Gamma(\mathcal{X})$$
where the union is over all tropical graphs Γ with d boundary markings.

Gromov topology on broken maps

We introduce some notation.

- $\widetilde{\mathcal{M}}_\Gamma(\mathcal{X}) :=$ moduli space of broken holomorphic disks with tropical graph Γ.
- $\mathcal{M}_\Gamma(\mathcal{X}) := \widetilde{\mathcal{M}}_\Gamma / T_{\mathrm{trop}}(\Gamma)$.
- The moduli space of broken holomorphic disks with d boundary markings is

$$\mathcal{M}_d(\mathcal{X}) = \cup_\Gamma \mathcal{M}_\Gamma(\mathcal{X})$$

  where the union is over all tropical graphs Γ with d boundary markings.
- Let $\mathcal{M}_d(\mathcal{X})^{\leq E} \subset \mathcal{M}_d(\mathcal{X})$ be the subset of broken holomorphic disks with area $\leq E$.

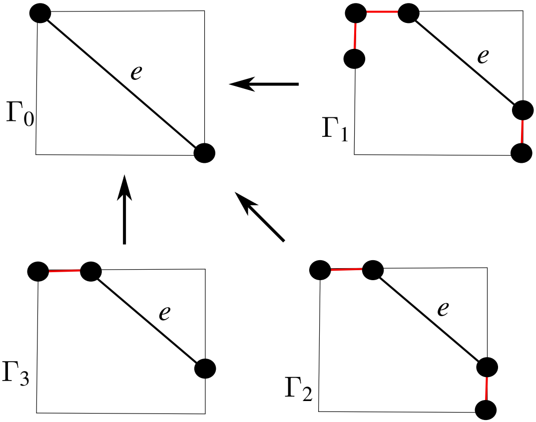
Gromov topology on broken maps

Theorem (Gromov convergence)

Given a sequence $u_\nu \in \widetilde{\mathcal{M}}_\Gamma(\mathcal{X})^{\leq E}$ *there is a subsequence that converges to* $u \in \widetilde{\mathcal{M}}_{\Gamma'}(\mathcal{X})^{\leq E}$ *where* Γ' *is a tropical graph with an* edge collapse *morphism* $\Gamma' \to \Gamma$. *Further* $\mathrm{ind}(u) = \mathrm{ind}(u_\nu)$.

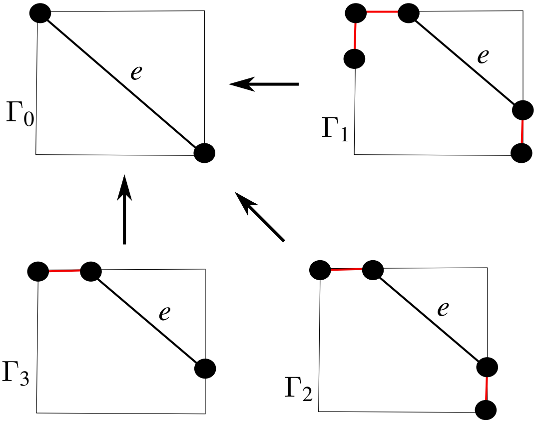
Gromov topology on broken maps: edge collapse

$\Gamma_1 \to \Gamma_0$, $\Gamma_2 \to \Gamma_0$, $\Gamma_3 \to \Gamma_0$ are edge collapse morphisms. In all graphs $\mathcal{T}(e) = (1, 1)$.



Gromov topology on broken maps: edge collapse

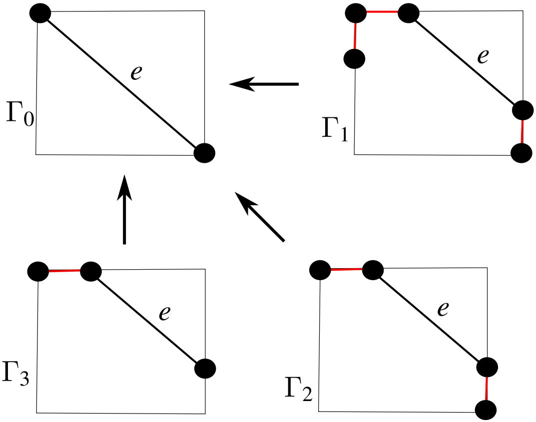
$\Gamma_1 \to \Gamma_0$, $\Gamma_2 \to \Gamma_0$, $\Gamma_3 \to \Gamma_0$ are edge collapse morphisms. In all graphs $\mathcal{T}(e) = (1,1)$.



Remark : For an edge collapse $\Gamma' \to \Gamma$, $\Gamma' \neq \Gamma \implies T_{\text{trop}}(\Gamma) \subsetneq T_{\text{trop}}(\Gamma')$.

Gromov topology on broken maps: edge collapse

$\Gamma_1 \to \Gamma_0$, $\Gamma_2 \to \Gamma_0$, $\Gamma_3 \to \Gamma_0$ are edge collapse morphisms. In all graphs $\mathcal{T}(e) = (1,1)$.



Remark : For an edge collapse $\Gamma' \to \Gamma$, $\Gamma' \neq \Gamma \implies T_{\text{trop}}(\Gamma) \subsetneq T_{\text{trop}}(\Gamma')$.

Corollary : In Gromov convergence, $\operatorname{ind}(u_\nu) \leq 1$ implies that the limit has the same tropical type as u_ν.

Gromov topology on broken maps

Conjecture

For generic perturbations, the space $\mathcal{M}_d(\mathcal{X})^{<E}$

- *is compact,*
- *and is the coarse moduli space of a smooth Deligne-Mumford stack.*
- *In any connected component, generic points have a finite tropical symmetry group.*
- *For any tropical graph* Γ*, the moduli space* $\mathcal{M}_\Gamma(\mathcal{X})^{<E}$ *is a stratum of codimension* $\dim(T_{\mathrm{trop},\mathbb{C}}(\Gamma))$ *in* $\mathcal{M}_d(\mathcal{X})^{<E}$.

Broken maps vs holomorphic buildings

In the single cut case, there are differences between broken maps and holomorphic buildings of SFT.

Broken maps vs holomorphic buildings

In the single cut case, there are differences between broken maps and holomorphic buildings of SFT.

- A holomorphic buildings is a *continuous* map from a nodal curve to

$$\mathcal{X}[k] := X_+ \cup_Y Z(\mathbb{P}^1) \cup_Y \cdots \cup_Y Z(\mathbb{P}^1) \cup_Y X_-,$$

  for some $k \geq 1$.
- A broken map is not continuous and does not remember the ordering of neck piece components.

Broken maps vs holomorphic buildings

In the single cut case, there are differences between broken maps and holomorphic buildings of SFT.

- A holomorphic buildings is a *continuous* map from a nodal curve to

$$\mathcal{X}[k] := X_+ \cup_Y Z(\mathbb{P}^1) \cup_Y \cdots \cup_Y Z(\mathbb{P}^1) \cup_Y X_-,$$

  for some $k \geq 1$.
- A broken map is not continuous and does not remember the ordering of neck piece components.
- We expect that the glued family corresponding to buildings with different orderings are part of the same connected component of the moduli space of unbroken maps. Thus the ordering is not a combinatorial invariant.

Table of Contents

1. Introducing the problem
2. Single cut : Unbroken to broken
3. Multiple cut : Unbroken to broken
4. Degenerating matching conditions

Degenerating matching conditions

NEXT WEEK