MTH931 Riemannian Geometry II
Problem Set #1

Thomas Walpuski

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This assignment is **due**: January 29, 2019.

Do five of the seven problems below.

Exercise 1.1. Consider $\mathbf{R}^{n+1}$ with the Lorentzian metric

$$g_L = -\mathrm{d}x^0 \otimes \mathrm{d}x^0 + \sum_{a=1}^{n} \mathrm{d}x^a \otimes \mathrm{d}x^a.$$

Set

$$H^n := \left\{x \in \mathbf{R}^{n+1} : g_L(x,x) = 1 \text{ and } x_0 > 0\right\}.$$

Prove that:

1. The symmetric bilinear form g_{-1} obtained by restricting g_L to H^n is positive definite; that is: a Riemannian metric.
2. The Riemannian curvature tensor of (H^n, g_{-1}) is given by

$$R(u,v)w = -\langle v, w\rangle u + \langle u, w\rangle v; \quad \text{that is:} \quad \sec = -1.$$

Exercise 1.2. Let $n \in \{2,3,\ldots\}$ and $\kappa \in \mathbf{R}$. Denote by vol_κ^n the Riemannian volume form of (S_κ^n, g_κ). Prove that, in geodesic polar coordinates,

$$g_\kappa = \mathrm{d}r \otimes \mathrm{d}r + \sin_\kappa(r)^2 g_{S^{n-1}} \quad \text{and}$$
$$\mathrm{vol}_\kappa^n = \sin_\kappa(r)^{n-1}\mathrm{d}r \wedge \mathrm{vol}_{S^{n-1}}.$$

Exercise 1.3. Prove that, in normal coordinates,

$$\frac{\mathrm{vol}}{\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n} = 1 - \frac{1}{6}\sum_{a,b=1}^{n} \mathrm{Ric}_{ab}x^a x^b + O(|x|^3).$$

Exercise 1.4. Let (M, g) be a closed Riemannian manifold. Let $v \in \mathrm{Vect}(M)$ be Killing field and let $\alpha \in \Omega^1(M)$ be a harmonic 1–form. Prove that the function $\alpha(v)$ is constant.

Exercise 1.5. Prove the following theorem.

Theorem. *Let $\kappa > 0$. Let (M, g) be a closed Riemannian manifold of dimension n with $\mathrm{Ric}_g \geqslant (n-1)\kappa g$. If λ is a non-zero eigenvalue of the Laplacian, then*

$$\lambda \geqslant n\kappa.$$

Equality is achieved in **??** *if and only if (M, g) is isometric to (S^n_κ, g_κ).*

Exercise 1.6. Prove the following theorem.

Theorem (Relative Volume Comparison Theorem for annular sectors)**.** *Let (M, g) be a complete Riemannian manifold of dimension n and let $x \in M$. Let $\kappa \in \mathbf{R}$. Suppose that*

$$\mathrm{Ric}_g \geqslant (n-1)\kappa g.$$

If $0 \leqslant r \leqslant R$ and $0 \leqslant s \leqslant S$ with $r \leqslant s$ and $R \leqslant S$, then

$$\frac{\mathrm{vol}(A^\Gamma_{s,S}(x))}{\mathrm{vol}(A^\Gamma_{r,R}(x))} \leqslant \frac{V^n_\kappa(\Gamma, s, S)}{V^n_\kappa(\Gamma, r, R)}.$$

Exercise 1.7. Let (M, g) be a complete Riemannian manifold with $\mathrm{Ric}_g \geqslant 0$. Prove that if

$$\lim_{r\to\infty} \frac{\mathrm{vol}(B_r(x))}{r^n} \geqslant V^n_0(1) = \frac{\pi^{n/2}}{\Gamma(n/2+1)},$$

then (M, g) is isometric to $\mathbf{R}^n$.