MTH 993 Spring 2018: Spin Geometry

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Course Description

In 1928 physical considerations lead Dirac [Dir28] to construct a square root of the Laplace operator $\Delta = \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$ on Minkowski space-time, that is, an operator $\not{D}$ such that $\not{D}^* \not{D} = \Delta$. The operator he discovered is now called the Dirac operator. The aim of Spin Geometry is to find the correct framework to construct Dirac operators on Riemannian manifolds, study the properties of Dirac operators, and explore the consequences for the geometry and topology of the underlying manifold. The course will have four parts:

I. **Algebraic underpinnings of Spin Geometry** We will construct the Clifford algebra of a quadratic form, discuss the classification of Clifford algebras and their representation theory. Based on this we will introduce the Spin groups and explore their representation theory.

II. **Spin structures, spinor bundles, and the Dirac operator** After a review of the theory of principal bundles and vector bundles on manifolds, we will answer the questions: What is a spin structure? and Which manifolds do admit spin structures? On manifolds with spin structures, we will construct spinor bundles, spin connections, and the Dirac operator. We will discuss in detail what it means for a Kähler manifold (a Riemannian manifold with a compatible complex structure) to admit a spin structure, and what the spinor bundles and the Dirac operator look like in this case.

III. **Basic analytic properties of Dirac operators** We will begin with a brief survey of the theory of elliptic operators on compact manifold and then apply this theory to the Dirac operator. We will then explain the Atiyah–Singer Index Theorem for Dirac operators and derive a number of applications. We will discuss the spectral flow of a family of Dirac operators and its relation to the Index Theorem.

IV. **Applications of Spin Geometry** The last part of this course is concerned with applications of Spin Geometry. Depending on the students' interest we might discuss the Seiberg–Witten equation and some of its consequences for the topology of 4–manifolds or Witten's proof of the Positive Mass Conjecture.

Textbook. The textbook for this class will be Lawson and Michelsohn [LM89]. Other useful references include [Mor96; Sal99; Fri00].

Prerequisites. A student participating in this topics course needs to have have a solid understanding of the material in *MTH 868: Geometry & Topology I* (smooth manifolds, vector fields, tangent bundles, tensor bundles, differential forms) as well as a basic understanding of *MTH 930: Riemannian Geometry* (Riemannian metrics, Levi-Civita connection, Riemannian curvature). Moreover, a rudimentary understanding of elliptic theory and algebraic topology (homology, cohomology) would be helpful, but it is not considered a requirement.

References

[Dir28] P. A. M. Dirac. The Quantum Theory of the Electron. In: *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences* 117.778 (1928), pp. 610–624. URL: http://rspa.royalsocietypublishing.org/content/117/778/610 (cit. on p. 1).

[Fri00] T. Friedrich. Dirac operators in Riemannian geometry. Vol. 25. Graduate Studies in Mathematics. Translated from the 1997 German original by Andreas Nestke. American Mathematical Society, Providence, RI, 2000, pp. xvi+195. URL: http://dx.doi.org/10.1090/gsm/025 (cit. on p. 2).

[LM89] H. B. Lawson Jr. and M.-L. Michelsohn. Spin geometry. Vol. 38. Princeton Mathematical Series. Princeton, NJ: Princeton University Press, 1989, pp. xii+427 (cit. on p. 2).

[Mor96] J. W. Morgan. The Seiberg-Witten equations and applications to the topology of smooth four-manifolds. Vol. 44. Mathematical Notes. Princeton University Press, Princeton, NJ, 1996, pp. viii+128 (cit. on p. 2).

[Sal99] D.A. Salamon. Spin Geometry and Seiberg–Witten Invariants. Unfinished manuscript. 1999. URL: https://people.math.ethz.ch/~salamon/PREPRINTS/witsei.pdf (cit. on p. 2).