MTH931 Riemannian Geometry II

in progress

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Contents

1 Riemannian metrics

Definition 1.1. Let M be a manifold. A **Riemannian metric** on M is a bilinear form $g \in \Gamma(S^2T^*M)$ on TM which is positive definite. A **Riemannian manifold** is a pair (M, g) consisting of a manifold M and a Riemannian metric g on M.

Notation 1.2. If (M, g) is a Riemannian manifold, $x \in M$, and $v, w \in T_xM$, then we set

$$\langle v, w\rangle_g := g(v, w) \quad \text{and} \quad |v|_g := \sqrt{g(v, v)}. \tag{1.3}$$

Notation 1.4. If $x^1, \dots, x^n \colon M \supset U \to \mathbf{R}$ are local coordinates, for $a, b \in \{1, \dots, n\}$, we set

$$\partial_a := \frac{\partial}{\partial x^a} \quad \text{and} \quad g_{ab} := g(\partial_a, \partial_b).$$

Definition 1.5. The **musical isomorphisms** $\cdot^\flat \colon TM \to T^*M$ and $\cdot^\sharp \colon T^*M \to TM$ are defined by

$$v^\flat := \langle v, \cdot\rangle \quad \text{and} \quad \langle \alpha^\sharp, \cdot\rangle := \alpha(\cdot).$$

Definition 1.6. Let (M, g) be Riemannian manifold. Let $f \in C^\infty(M)$. The **gradient** of f is the vector field ∇f defined by

$$\langle \nabla f, \cdot\rangle := \mathrm{d}f.$$

The **Hessian** of f is the bilinear form $\operatorname{Hess} f \in \Gamma(S^2T^*M)$ defined by

$$\operatorname{Hess} f := \frac{1}{2}\mathscr{L}_{\nabla f}\, g.$$

2 The Riemannian distance

Definition 2.1. The **length** of a curve $\gamma \colon [t_0, t_1] \to M$ is defined by

$$\ell(\gamma) := \int_{t_0}^{t_1} |\dot\gamma(t)|\,\mathrm{d}t. \tag{2.2}$$

Remark 2.3. The length functional ℓ is invariant under reparametrizations of γ.

Definition 2.4. A curve $\gamma \colon [t_0, t_1] \to M$ is **parametrized by arc-length** or has **unit speed** if

$$\ell(\gamma|_{[t_0,t]}) = t - t_0 \quad \text{or, equivalently,} \quad |\dot\gamma| = 1. \tag{2.5}$$

Definition 2.6. The **Riemannian distance** associated with (M, g) is the function $d \colon M \times M \to [0, \infty]$ defined by

$$d(x, y) := \inf\bigl\{\ell(\gamma) : \gamma \in C^\infty([t_0, t_1], M) \text{ with } \gamma(t_0) = x \text{ and } \gamma(t_1) = y\bigr\}. \tag{2.7}$$

Proposition 2.8. *(M, d) is a metric space.*

3 The Riemanian volume form

Definition 3.1. Let (M, g) be an oriented Riemannian manifold. The **Riemannian volume form** is the unique positive volume form

$$\mathrm{vol}_g \quad \text{satisfying} \quad |\mathrm{vol}_g| = 1. \tag{3.2}$$

Proposition 3.3. *In local coordinates $x^1, \ldots, x^n$,*

$$\mathrm{vol}_g = \sqrt{\det g}\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n. \tag{3.4}$$

Definition 3.5. Let (M, g) be an oriented Riemannian manifold of dimension n. The **Hodge star operator** is the linear map $\star\colon \Lambda^\bullet T^*M \to \Lambda^{\bullet - n}T^*M$ defined by

$$\alpha \wedge \star\beta = \langle\alpha, \beta\rangle \mathrm{vol}_g.$$

Definition 3.6. Let (M, g) be an Riemannian manifold. The **divergence** of $v \in \mathrm{Vect}(M)$ is the function $\mathrm{div}\, v \in C^\infty(M)$ defined by

$$\mathrm{div}(v)\mathrm{vol}_g = \mathscr{L}_v \mathrm{vol}_g.$$

(Here vol_g need only be locally defined.)

Definition 3.7. Let (M, g) be Riemannian manifold. The **Laplacian** of $f \in C^\infty(M)$ is the function $\Delta f \in C^\infty(M)$ defined by

$$\Delta f = -\,\mathrm{div}\,\nabla f.$$

4 The Levi-Civita connection

Definition 4.1. Let M be a manifold. An **affine connection** is a connection on TM. An affine connection ∇ is called **torsion-free** if for all $v, w \in \mathrm{Vect}(M)$,

$$\nabla_v w - \nabla_w v = [v, w].$$

Definition 4.2. Let (M, g) be Riemannian manifold. An affine connection ∇ is called **metric** if

$$\nabla g = 0;$$

that is: for all $v, w \in \mathrm{Vect}(M)$,

$$dg(v, w) = g(\nabla v, w) + g(v, \nabla w).$$

Theorem 4.3 (Fundamental Theorem of Riemannian Geometry). *Let (M, g) be a Riemannian manifold.*

1. *There exists a unique affine connection ∇^{LC} which is torsion-free and metric.*

2. *The affine connection ∇^{LC} satisfies **Koszul's formula**:*

$$\begin{aligned} 2\langle \nabla^{\mathrm{LC}}_u v, w\rangle &= \mathscr{L}_u\langle v, w\rangle + \mathscr{L}_v\langle w, u\rangle - \mathscr{L}_w\langle u, v\rangle \\ &\quad + \langle [u, v], w\rangle - \langle [u, w], v\rangle - \langle [v, w], u\rangle. \end{aligned} \tag{4.4}$$

3. *Suppose $x^1, \dots, x^n$ are local coordinates on M. The **Christoffel symbols** Γ^c_{ab} defined by*

$$\Gamma^k_{ab} \coloneqq \langle \nabla^{\mathrm{LC}}_{\partial_a} \partial_b, \partial_c\rangle \tag{4.5}$$

satisfy

$$\Gamma^c_{ab} = \frac{1}{2} g^{cd}(\partial_a g_{bd} - \partial_d g_{ab} + \partial_b g_{ad}). \tag{4.6}$$

Definition 4.7. We call ∇^{LC} the **Levi-Civita connection** associated with (M, g).

Remark 4.8. It is customary to drop the super-script LC.

5 The Riemann curvature tensor

Theorem 5.1. *Let (M, g) be a Riemannian manifold.*

1. *There exists a unique tensor field $R_g \in \Omega^2(M, \mathfrak{o}(TM))$ satisfying*

$$R_g(u, v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w. \tag{5.2}$$

2. *The tensor field R_g satisfies*

$$\langle R_g(u, v)w, z\rangle = \langle R_g(w, z)u, v\rangle. \tag{5.3}$$

3. *The tensor field R_g satisfies the **algebraic Bianchi identity**:*

$$R_g(u, v)w + R_g(v, w)u + R_g(w, u)v = 0. \tag{5.4}$$

4. *The tensor field R_g satisfies the **differential Bianchi identity**:*

$$\mathrm{d}^\nabla R_g = 0. \tag{5.5}$$

Definition 5.6. We call R_g the **Riemann curvature tensor** of (M, g).

Remark 5.7. Let V be a Euclidean space of dimension n. The space of **algebraic curvature tensors** on V is

$$\mathscr{R}(V) := \ker\left(S^2\Lambda^2 V \xrightarrow{\wedge} \Lambda^4 V\right) \subset \Lambda^2 V \otimes \Lambda^2 V.$$

Since

$$\dim \mathscr{R}(V) = \frac{n^4 - n^2}{12},$$

at each point $x \in M$, the Riemann curvature tensor R_g has $(n^4 - n^2)/12$ components.

Definition 5.8. The **sectional curvature** of (M, g) is the map $\sec_g\colon \Lambda^2 TM \to \mathbf{R}$ defined by

$$\sec_g(v \wedge w) := \frac{\langle R_g(v, w)w, v\rangle}{|v \wedge w|^2}. \tag{5.9}$$

Remark 5.10. The Riemann curvature tensor R_g can be recovered from the sectional curvature $\sec_g$ *algebraically*.

Remark 5.11. The sectional curvature *really* is a map $\mathrm{Gr}_2(TM) \to \mathbf{R}$.

Definition 5.12. The **curvature operator** is the self-adjoint map $\mathfrak{R}_g \in \Gamma(\mathrm{Sym}^2(\Lambda^2 TM))$ defined by

$$\langle \mathfrak{R}_g(u \wedge v), w \wedge z\rangle := \langle R_g(u, v)z, w\rangle.$$

6 Model spaces

Example 6.1 ($\mathbf{R}^n$). $\mathbf{R}^n$ with the Riemannian metric

$$g_0 := \sum_{a=1}^{n} \mathrm{d}x^a \otimes \mathrm{d}x^a$$

has vanishing Riemann curvature tensor: $R = 0$.

Exercise 6.2 (S^n). Consider the n–dimensional unit sphere

$$S^n := \left\{x \in \mathbf{R}^{n+1} : |x| = 1\right\}$$

with the Riemannian metric g_1 induced by g_0 on $\mathbf{R}^{n+1}$. Prove that:

1. If $u, v, w \in \mathrm{Vect}(S^n) \subset C^\infty(S^n, \mathbf{R}^{n+1})$, then at every point $x \in S^n$

$$\nabla_v w = \partial_v w + \langle v, w\rangle x.$$

2. The Riemannian curvature tensor of (S^n, g_1) is given by

$$R(u,v)w = \langle v, w\rangle u - \langle u, w\rangle v; \quad \text{that is:} \quad \sec = 1.$$

Exercise 6.3 (H^n). Consider $\mathbf{R}^{n+1}$ with the Lorentzian metric

$$g_L = -\mathrm{d}x^0 \otimes \mathrm{d}x^0 + \sum_{a=1}^{n} \mathrm{d}x^a \otimes \mathrm{d}x^a.$$

Set

$$H^n := \left\{x \in \mathbf{R}^{n+1} : g_L(x,x) = 1 \text{ and } x_0 > 0\right\}.$$

Prove that:

1. The symmetric bilinear form g_{-1} obtained by restricting g_L to H^n is positive definite; that is: a Riemannian metric.
2. The Riemannian curvature tensor of (H^n, g_{-1}) is given by

$$R(u,v)w = -\langle v, w\rangle u + \langle u, w\rangle v; \quad \text{that is:} \quad \sec = -1.$$

Definition 6.4. Let $n \in \{2, 3, \ldots\}$ and $\kappa \in \mathbf{R}$. The n–dimensional **model space** of constant sectional curvature κ is

$$(S^n_\kappa, g_\kappa) := \begin{cases} \left(S^n, \kappa^{-1/2} g_1\right) & \text{if } \kappa > 0, \\ (\mathbf{R}^n, g_0) & \text{if } \kappa = 0, \\ \left(H^n, (-\kappa)^{-1/2} g_1\right) & \text{if } \kappa < 0. \end{cases}$$

Theorem 6.5 (Riemann [Rie68], Killing [Kil91], and Hopf [Hop25]). *If (M,g) is a simply-connected Riemannian manifold of constant sectional curvature $\kappa \in \mathbf{R}$, then it is isometric to an open subset of (S^n_κ, g_κ).*

Definition 6.6. Let $n \in \{2, 3, \ldots\}$ and $\kappa \in \mathbf{R}$. The function $V^n_\kappa \colon [0,\infty) \to [0,\infty)$ is defined by

$$V^n_\kappa(r) := \mathrm{vol}(B_r(x)) \tag{6.7}$$

for $B_r(x) \subset S^n_\kappa$.

Remark 6.8. The functions V^n_κ satisfy the scaling relation

$$V^n_\kappa(r) = V^n_{r^2\kappa}(1).$$

Definition 6.9. For $\kappa \in \mathbf{R}$, set

$$\sin_\kappa(r) := \begin{cases} \sin(\sqrt{\kappa} r) & \text{if } \kappa > 0, \\ r & \text{if } \kappa = 0, \\ \sinh(\sqrt{-\kappa} r) & \text{if } \kappa < 0. \end{cases}$$

Exercise 6.10. Let $n \in \{2, 3, \dots\}$ and $\kappa \in \mathbf{R}$. Denote by

$$\mathrm{vol}_\kappa^n$$

the Riemannian volume form of (S_κ^n, g_κ). Prove that, in geodesic polar coordinates,

$$g_\kappa = \mathrm{d}r \otimes \mathrm{d}r + \sin_\kappa(r)^2 g_{S^{n-1}} \quad \text{and} \tag{6.11}$$

$$\mathrm{vol}_\kappa^n = \sin_\kappa(r)^{n-1} \mathrm{d}r \wedge \mathrm{vol}_{S^{n-1}}. \tag{6.12}$$

Remark 6.13. It is exercise to compute that

$$\mathrm{vol}(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}; \quad \text{and thus:} \quad V_{n,0}(r) = \frac{\pi^{n/2}}{\Gamma(n/2+1)} r^n. \tag{6.14}$$

V_κ^n for $\kappa \neq 0$ can be expressed in terms of trigonometric/hyperbolic functions and Gauß' hypergeometric function ${}_2F_1$; but these formulae are unwieldy.

If $\kappa > 0$, then V_κ^n is constant equal to $\kappa^{n/2}\mathrm{vol}(S^n)$ on $[\pi/\sqrt{\kappa}, \infty)$.

For $\kappa < 0$ and $r \gg 1$,

$$\sinh(\sqrt{-\kappa} r) \sim \frac{e^{(n-1)\sqrt{-\kappa} x}}{2^{n-1}}.$$

Therefore,

$$V_\kappa^n(r) \sim \frac{\pi^{n/2}}{(n-1)2^{n-2}\Gamma(n/2)\sqrt{-\kappa}} e^{(n-1)\sqrt{-\kappa} r}. \tag{6.15}$$

7 Geodesics

Definition 7.1. Let $I \subset \mathbf{R}$ be an interval. A curve $\gamma \colon I \to M$ is called a **geodesic** if

$$\nabla_t \dot\gamma = 0. \tag{7.2}$$

Here ∇_t is the pull-back of the Levi-Civita connection to $\gamma^* TM$.

Remark 7.3. Suppose $x^1, \dots, x^n$ are local coordinates on M. Setting $\gamma^i := x^i \circ \gamma$, (7.2) becomes

$$\ddot\gamma^k + \Gamma_{ij}^k \dot\gamma^i \dot\gamma^j = 0. \tag{7.4}$$

Theorem 7.5 (Existence and Uniqueness of Geodesics).

1. *Given $t_\star \in \mathbf{R}$, $x \in M$, and $v \in T_xM$, there exists an open interval I containing $t_\star$ and a geodesic $\gamma\colon I \to M$ satisfying $\gamma(t_\star) = x$ and $\dot\gamma(t_\star) = v$.*

2. *Let $\gamma\colon I \to M$ and $\delta\colon J \to M$ be two geodesics. If there is and $t_\star \in I \cap J$ such that $\gamma(t_\star) = \delta(t_\star)$ and $\dot\gamma(t_\star) = \dot\delta(t_\star)$, then γ and δ agree on $I \cap J$.*

3. *Let $\gamma\colon I \to M$ be a geodesic with $I = (t_0, t_1)$ maximal. If $t_0 \neq -\infty$, then for every compact subset $K \subset M$, there exists a $t_0^K \in (t_0, t_1)$ such that if $t \in (t_0, t_0^K)$, then $\gamma(t) \notin K$. An analogous statement holds if $t_1 \neq +\infty$.*

   In particular, if M is compact, then $I = \mathbf{R}$.

Definition 7.6. We say that (M, g) is **geodesically complete** (at $x \in M$) if every geodesic (passing through x) can be extended to $\mathbf{R}$.

8 The exponential map

Definition 8.1. Given $x \in M$ and $v \in T_xM$, denote by $\gamma_v^x\colon I_v^x \to M$ the maximal geodesic with $\gamma_v^x(0) = x$ and $\dot\gamma_v^x(0) = v$. Set

$$\mathscr{O}_x \coloneqq \left\{v \in T_xM : 1 \in I_v^x\right\} \quad \text{and} \quad \mathscr{O} \coloneqq \bigcup_{x\in M} \mathscr{O}_x \subset TM \tag{8.2}$$

The **exponential map** at x is the map $\exp_x\colon \mathscr{O}_x \to M$ defined by

$$\exp_x(v) \coloneqq \gamma_v(1). \tag{8.3}$$

The **exponential map** $\exp\colon \mathscr{O} \to M$ is defined by $\exp|_{\mathscr{O}_x} \coloneqq \exp_x$.

Proposition 8.4.

1. *$\mathscr{O}$ is open and* $\exp$ *is smooth.*

2. *The derivative at* $0 \in \mathscr{O}_x$ *of the exponential map* $\exp_x$*,*

$$\mathrm{d}_0 \exp_x \colon T_0 T_x M \to T_x M, \tag{8.5}$$

   is invertible; in fact, it is the inverse of the canonical isomorphism $T_x M \cong T_0 T_x M$*.*

   In particular, $\exp_x$ *induces a diffeomorphism between a neighborhood of the origin in* $T_x M$ *and a neighborhood of* x *in* M*.*

3. *The derivative of the map* $(\pi, \exp) \colon \mathscr{O} \to M \times M$ *along the zero section is invertible.*

   In particular, this map induces a diffeomorphism between a neighborhood of the zero section of TM *and a neighborhood if the diagonal in* $M \times M$*.*

Lemma 8.6 (Gauß' Lemma). *Let* $x \in M$*. Denote by* $\partial_r \in \mathrm{Vect}(T_x M)$ *the radial vector field. Suppose* $B_\varepsilon(0) \subset \mathscr{O}_x$ *is such that* $\exp|_{B_\varepsilon(0)}$ *is a diffeomorphism onto its image. On* $B_\varepsilon(0)$*,*

$$\langle (\exp_x)_* \partial_r, (\exp_x)_* w \rangle = \langle \partial_r, w \rangle. \tag{8.7}$$

Theorem 8.8 (Short geodesics are minimal). *Let* $r > 0$ *and* $x \in M$*. If* $\exp_x \colon B_r(0) \to M$ *is a diffeomorphism onto its image, then, for every* $v \in B_r(0)$*,* $\gamma \colon [0,1] \to M$ *defined by* $\gamma(t) := \exp_x(tv)$ *is the unique minimal geodesic from* x *to* $\exp_x(v)$*; in particular:* $\exp_x(B_r(0)) = B_r(x)$*.*

Proof. We have $\ell(\gamma) = |v|$. Let $\delta \colon [0,1] \to M$ be a curve from x to $y = \exp_x(v)$. We will show that

$$\ell(\delta) \geqslant |v|. \tag{8.9}$$

We can assume that the image δ is contained $\exp_x(\bar{B}_{|v|}(0))$; otherwise, the upcoming argument shows that part of δ already has length at least $|v|$. Set

$$w(t) := \exp_x^{-1}(\delta(t)). \tag{8.10}$$

By the Cauchy–Schwarz inequality and Lemma 8.6,

$$\begin{aligned}
\ell(\delta) &= \int_0^1 |\dot{\delta}(t)| \, \mathrm{d}t \\
&\geqslant \int_0^1 \langle \dot{\delta}(t), (\exp_p)_* \partial_r \rangle \, \mathrm{d}t \\
&= \int_0^1 \langle \dot{w}(t), \partial_r \rangle \, \mathrm{d}t = \int_0^1 \frac{\langle \dot{w}(t), w(t) \rangle}{|w(t)|} \, \mathrm{d}t = \int_0^1 \partial_t |w(t)| \, \mathrm{d}t = |v|.
\end{aligned}$$

Equality holds if and only if $\dot{w}(t)$ and ∂_r are parallel. ∎

Definition 8.11. Let $x \in M$. **Normal coordinates** of M at x are coordinates obtained by composing $\exp_x^{-1}$ with an isometry $T_xM \cong \mathbf{R}^n$.

Proposition 8.12. *Suppose $x^1, \dots, x^n$ are normal coordinates.*

1. *The Christoffel symbols Γ_{ij}^k vanish at the origin.*
2. *We have $g_{ij}x^j = \delta_{ij}x^j$.*
3. *Setting*

(8.13)
$$R_{ijk\ell} := \langle R(\partial_i, \partial_j)\partial_k, \partial_\ell \rangle,$$

we have

(8.14)
$$g_{ij} = \delta_{ij} + \frac{1}{3}\sum_{k,\ell=1}^{n} R_{ik\ell j}x^k x^\ell + O(|x|^3).$$

9 The energy functional

Definition 9.1. A **variation** of a curve $\gamma\colon [t_0, t_1] \to M$ is a smooth map $\boldsymbol{\gamma}\colon (-\varepsilon, \varepsilon) \times [t_0, t_1] \to M$ such that $\boldsymbol{\gamma}(0, \cdot) = \gamma$. The variation $\boldsymbol{\gamma}$ is called **proper** if $\boldsymbol{\gamma}(\cdot, t_0)$ and $\boldsymbol{\gamma}(\cdot, t_1)$ are constant. We set $\gamma_s(t) := \boldsymbol{\gamma}(s, t)$.

Proposition 9.2 (First Variation Formula). *Given a variation $\boldsymbol{\gamma}$ of a curve $\gamma\colon [t_0, t_1] \to M$,*

(9.3)
$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} E(\gamma_s) = &-\int_{t_0}^{t_1} \langle \nabla_t \dot\gamma_0(t), \partial_s\boldsymbol{\gamma}(0, t)\rangle\,\mathrm{d}t \\ &+ \langle \dot\gamma_0(t_0), \partial_s\boldsymbol{\gamma}(0, t_0)\rangle - \langle \dot\gamma_0(t_1), \partial_s\boldsymbol{\gamma}(0, t_1)\rangle.\end{aligned}$$

Corollary 9.4. *A curve $\gamma\colon [t_0, t_1] \to M$ is a geodesic if and only if, for every proper variation $\boldsymbol{\gamma}$, 0 is a critical point of $s \mapsto E(\gamma_s)$; that is:*

(9.5)
$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} E(\gamma_s) = 0.$$

Proposition 9.6. *For every curve $\gamma\colon [t_0, t_1] \to M$,*

(9.7)
$$\ell(\gamma) \leqslant \sqrt{2E(\gamma)} \cdot \sqrt{t_1 - t_0}$$

with equality if and only if $|\dot\gamma|$ is constant.

Corollary 9.8. *Let $t_0 < t_1$ and $x, y \in M$. Set*

$$\gamma \in P_{x,y} := \{\delta \in C^\infty([t_0, t_1], M) : \delta(t_0) = x \text{ and } \delta(t_1) = y\}. \tag{9.9}$$

If $\gamma \in P_{x,y}$ satisfies $\partial_t |\dot\gamma| = 0$ and minimizes ℓ in $P_{x,y}$, then γ also minimizes E in $P_{x,y}$; in particular: it is a geodesic.

10 The second variation formula

Lemma 10.1 (The second variation formula). *Let $\gamma\colon [t_0, t_1] \to M$ be a geodesic. If $\boldsymbol{\gamma}$ is a variation of γ, then*

$$\begin{aligned} \left.\frac{\mathrm{d}^2}{\mathrm{d}s^2}\right|_{s=0} E(\gamma_s) = \int_{t_0}^{t_1} &\left|\nabla_t \partial_s \boldsymbol{\gamma}(0,t)\right|^2 - \langle R(\partial_s \boldsymbol{\gamma}(0,t), \dot\gamma(t))\dot\gamma(t), \partial_s \boldsymbol{\gamma}(0,t)\rangle \,\mathrm{d}t \\ &+ \langle \nabla_s \partial_s \boldsymbol{\gamma}(0,t), \partial_t \gamma(t_1)\rangle - \langle \nabla_s \partial_s \boldsymbol{\gamma}(0,t), \partial_t \gamma(t_0)\rangle. \end{aligned} \tag{10.2}$$

Remark 10.3. If $\boldsymbol{\gamma}$ is a proper variation, then (10.2) depends only on $V := \partial_s \boldsymbol{\gamma}(0, \cdot)$.

Definition 10.4. Let $\gamma\colon [t_0, t_1] \to M$ be a geodesic. The **index form** of γ is the bilinear map $I\colon S^2\Gamma(\gamma^* TM) \to \mathbf{R}$ is defined by

$$I(v, w) = \int_{t_0}^{t_1} \langle \nabla_t v, \nabla_t w\rangle - \langle R(v, \dot\gamma(t))\dot\gamma(t), w\rangle \,\mathrm{d}t$$

Definition 10.5. Let $\gamma\colon [t_0, t_1] \to M$ be a geodesic passing through x and y. We say that x and y are **conjugate along** γ if there is a non-zero Jacobi field J along γ with $J(t_0) = 0$ and $J(t_1) = 0$.

Definition 10.6. Let $x \in M$. The **conjugate locus** of x in $T_x M$ is the set of points $v \in \mathcal{O}_x$ such that x and $\exp_x(v)$ are conjugate along $t \mapsto \exp_x(tv)$.

Remark 10.7. The **conjugate locus** of x is the set of points $v \in \mathcal{O}_x$ such that $\mathrm{d}_v \exp_x$ is not injective.

Theorem 10.8 (Jacobi). *Let $\gamma\colon [t_0, t_1] \to M$ with $\gamma(t_0) = x$ and $y = \gamma(t_\star)$ for $t_\star \in (t_0, t_1)$. If x and y are are conjugate along $\gamma|_{[t_0, t_\star]}$, then γ is not minimal; that is: there is a proper variation $\boldsymbol{\gamma}$ of γ with*

$$\ell(\boldsymbol{\gamma}(s, \cdot)) < \ell(\gamma) \tag{10.9}$$

for all $s \neq 0$.

Sketch proof. Since x and y are conjugate along $\gamma|_{[t_0, t_\star]}$, there is a non-trivial Jacobi field J along $\gamma|_{[t_0, t_\star]}$ vanishing at t_0 and $t_\star$. Extend J to a piecewise smooth Jacobi field along γ by declaring it to vanish on $[t_\star, t_1]$.

$\nabla_t J(t_\star) \neq 0$ for otherwise J would be trivial. Choose $V \in \Gamma(\gamma^* TM)$ with

(10.10) $$V(t_0) = 0, \quad V(t_1) = 0, \quad \text{and} \quad \langle \nabla_t J(t_\star), V(t_\star) \rangle < 0.$$

For $0 < \varepsilon \ll 1$, set

$$J_\varepsilon := J + \varepsilon V.$$

Define the pricewise smooth proper variation $\boldsymbol{\gamma}_\varepsilon$ of γ by

(10.11) $$\boldsymbol{\gamma}_\varepsilon(s,t) := \exp(sJ_\varepsilon(t)).$$

By Lemma 10.1,

(10.12) $$\left.\frac{\mathrm{d}^2}{\mathrm{d}s^2}\right|_{s=0} E(\gamma_s) = I(J_\varepsilon, J_\varepsilon) = 2\varepsilon I(J,V) + \varepsilon^2 I(V,V).$$

An integration by parts, shows that

(10.13) $$I(J,V) = -\int_{t_0}^{t_1} \langle \nabla_t^2 J + R(J, \dot\gamma(t))\dot\gamma(t), V \rangle \,\mathrm{d}t + \langle \nabla_t J(t_\star), V(t_\star) \rangle.$$

The first term vanishes since J is a Jacobi field and the second term is negative. Consequently, $I(J_\varepsilon, J_\varepsilon) < 0$ provided $0 < \varepsilon \ll 1$. □

Definition 10.14. Let $x \in M$. The **cut locus** of x is the subset of those $v \in \mathcal{O}_x$ such that $\gamma(t) := \exp_x(tv)$ is minimizing for $t \in [0,1]$ but fails to be minimizing for $t \in [0, 1+\varepsilon)$ for every $\varepsilon > 0$.

Proposition 10.15. *If v is in the cut locus if x, then*

1. *v is in the conjugate locus of x or*
2. *there is more than one minimal geodesic from x to $\exp_x(v)$.*

11 Jacobi fields

Proposition 11.1. *Let $\boldsymbol{\gamma}$ be a variation of a geodesic $\gamma\colon [t_0, t_1] \to M$. If every γ_s is a geodesic, then the vector field $J \in \Gamma(\gamma^* TM)$ defined by*

(11.2) $$J := \partial_t \boldsymbol{\gamma}(0, \cdot)$$

*satisfies the **Jacobi equation**:*

(11.3) $$\nabla_t^2 J + R(J, \dot\gamma)\dot\gamma = 0.$$

Definition 11.4. Let γ be a geodesic. A vector field $J \in \Gamma(\gamma^* TM)$ is called a **Jacobi field** along γ if (11.3) holds.

Proposition 11.5. *Let $\gamma\colon [0,1] \to M$ be a geodesic. Given $J_0, \dot{J}_0 \in T_{\gamma(0)}M$, there exists a unique Jacobi field along γ with $J(0) = J_0$ and $\nabla_t J(0) = \dot{J}_0$.*

Proposition 11.6. *Let $x \in M$, $v \in \mathcal{O}_x$, and $w \in T_xM = T_vT_xM$. Denote by J the Jacobi field along $t \mapsto \exp_x(tv)$ with*

$$J(0) = 0 \quad \textit{and} \quad \nabla_t J(0) = w. \tag{11.7}$$

Then

$$\mathrm{d}_v \exp_x(w) = J(1).$$

Theorem 11.8 (Hadamard). *If (M, g) is complete and $\sec \leqslant 0$, then $\exp_x\colon T_xM \to M$ is a covering map. In particular, the universal cover of M is diffeomorphic to $\mathbf{R}^n$.*

Sketch proof. Suppose that J is a Jacobi field along $\gamma(t) = \exp(tv)$ with $J(0) = 0$ and $\nabla_t J(0) = w$. Since

$$\partial_t |J|^2 = 2\langle \nabla_t J, J\rangle \quad \text{and} \quad \partial_t^2 |J|^2 = -2\langle R(J, \dot{\gamma})\dot{\gamma}, J\rangle + 2|\nabla_t J|^2, \tag{11.9}$$

the function $t \mapsto |J(t)|^2$ vanishes at 0 and is strictly convex. Consequently, $J(1) \neq 0$. □

12 Ricci curvature

Definition 12.1. Let (M, g) be a Riemannian manifold. The **Ricci curvature** of (M, g) is the tensor field $\mathrm{Ric}_g \in \Gamma(S^2T^*M)$ defined by

$$\mathrm{Ric}_g(v, w) := \mathrm{tr}(R_g(\cdot, v)w) = \sum_{a=1}^n \langle R(e_a, v)w, e_a\rangle. \tag{12.2}$$

Here $(e_1, \ldots, e_n)$ is a orthonormal basis of T_xM.

Remark 12.3. If $\sec_g = \kappa$, then

$$\mathrm{Ric}_g = (n-1)\kappa g.$$

Remark 12.4. The map

$$\mathrm{tr}\colon \mathscr{R}(V) \to S^2V$$

is injective if $n < 3$, bijective for $n = 3$, and surjective for $n > 3$. Therefore, for $n \geqslant 3$, the Ricci curvature Ric_g has $\binom{n+1}{2}$ components. For a Riemannian 3–manifold (M, g), Ric_g determines all of R_g.

Proposition 12.5. *In normal coordinates,*

$$\frac{\mathrm{vol}_g}{\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n} = 1 - \frac{1}{6}\sum_{a,b=1}^n \mathrm{Ric}_{ab}x^a x^b + O(|x|^3). \tag{12.6}$$

Exercise 12.7. Prove Proposition 12.5. *Extended hint:* Let $x \in M$. Set

$$\theta := \frac{\mathrm{vol}_g}{\mathrm{vol}_{T_xM}}.$$

Let $v \in T_xM$ with $|v| = 1$. The task at hand is to find the first few terms in the Taylor expansion of $t \mapsto \theta(tv)$. To find these proceed as follows. Set $\gamma(t) = \exp_t(v)$. Let $e_1 = v, \dots, e_n$ be a positive orthonormal basis of T_xM. For $a = 1, \dots, n$, let $J_a(t)$ be the Jacobi field along γ with $J_a(0) = 0$ and $\nabla_t J_a(0) = e_a$. By Proposition 11.6,

$$d_{tv} \exp_x(e_a) = \frac{J_a(t)}{t}.$$

Therefore,

$$\theta(tv) = t^{-n+1}\sqrt{\det G(t)} \quad \text{with} \quad G_{ab}(t) = \langle J_a(t), J_b(t)\rangle.$$

Now use the Jacobi equation to find the first few terms of the Taylor expansion of $t \mapsto J_a(t)$.

13 Scalar curvature

Definition 13.1. Let (M, g) be a Riemannian manifold. The **scalar curvature** of (M, g) is the function $\mathrm{scal}_g \in C^\infty(M)$ defined by

$$\mathrm{scal}_g := \mathrm{tr}(\mathrm{Ric}_g) = \sum_{a,b=1}^{n} \langle R_g(e_a, e_b)e_b, e_a\rangle. \tag{13.2}$$

Here $(e_1, \dots, e_n)$ is a orthonormal basis of T_xM.

Proposition 13.3. *If (M, g) is a Riemannian manifold and $x \in M$, then, for $0 < r \ll 1$,*

$$\frac{\mathrm{vol}(B_x(r))}{V_0^n(r)} = 1 - \frac{\mathrm{scal}_g(x)}{6(n+2)} r^2 + O(r^3).$$

14 Einstein Metrics

Definition 14.1. A Riemannian metric g is called a **Einstein metric** if there is a constant $\lambda \in \mathbf{R}$ such that

$$\mathrm{Ric}_g = \lambda g. \tag{14.2}$$

If g is a Einstein metric on M, then (M, g) is called a **Einstein manifold**.

Lemma 14.3 (Schur's Lemma). *Let (M, g) be a Riemannian manifold of dimension $n \geqslant 3$. If $\lambda \in C^\infty(M)$ and*

$$\mathrm{Ric}_g = \lambda g,$$

then λ is locally constant.

Corollary 14.4. *Let (M, g) be a connected Riemannian manifold of dimension $n \geqslant 3$. If*

$$\mathrm{Ric}_g^\circ := \mathrm{Ric}_g - \frac{\mathrm{scal}_g}{n} g = 0,$$

then g is an Einstein metric.

Proposition 14.5 (Contracted Bianchi identity). *If (M, g) is a Riemannian manifold, then*

$$\mathrm{dscal}_g = -2\nabla^*\mathrm{Ric}_g. \tag{14.6}$$

Proof. Let $x \in M$ and let $e_1, \dots, e_n$ be a local orthonormal frame such that $(\nabla_{e_a} e_b)(x) = 0$. At the point x, by the differential Bianchi identity (5.5),

$$\begin{aligned}
e_c \cdot \mathrm{scal} &= \sum_{a,b=1}^n \langle (\nabla_{e_c} R)(e_a, e_b)e_b, e_a \rangle \\
&= -\sum_{a,b=1}^n \langle (\nabla_{e_a} R)(e_b, e_c)e_b, e_a \rangle + \langle (\nabla_{e_b} R)(e_c, e_a)e_b, e_a \rangle \\
&= 2\sum_{a,b=1}^n \langle (\nabla_{e_a} R)(e_b, e_a)e_c, e_b \rangle \\
&= -(\nabla^*\mathrm{Ric})(e_c).
\end{aligned}$$

□

Proof of Lemma 14.3. By hypothesis, $\mathrm{dscal}_g = n\mathrm{d}\lambda$. However, by the contracted Bianchi identity (14.6), $\mathrm{dscal}_g = 2\mathrm{d}\lambda$. Therefore, $\mathrm{d}\lambda = 0$. □

15 Bochner's vanishing theorem for harmonic 1–forms

Theorem 15.1 (Bochner's vanishing theorem for harmonic 1–forms [Boc46, Theorem 1]). *Let (M, g) be a closed, connected Riemannian manifold of dimension n. If $\mathrm{Ric}_g \geqslant 0$, then the following hold:*

1. *Every harmonic 1–form α is parallel and satisfies $\mathrm{Ric}_g(\alpha^\sharp, \alpha^\sharp) = 0$. In particular, $b_1(M) \leqslant n$.*

2. *If there exists some $x \in M$ with $\mathrm{Ric}_g(x) > 0$, then every harmonic 1–form vanishes. In particular, $b_1(M) = 0$.*

Proposition 15.2 (Bochner–Weitzenböck formula for 1–forms [Boc46, Lemma 2]). *Let (M, g) be a Riemannian manifold. For every $\alpha \in \Omega^1(M)$,*

$$(\mathrm{dd}^* + \mathrm{d}^*\mathrm{d})\alpha = \nabla^*\nabla\alpha + \mathrm{Ric}_g(\alpha^\sharp, \cdot). \tag{15.3}$$

in particular,

$$\frac{1}{2}\Delta|\alpha|^2 = \langle(\mathrm{dd}^* + \mathrm{d}^*\mathrm{d})\alpha, \alpha\rangle - |\nabla\alpha|^2 - \mathrm{Ric}_g(\alpha^\sharp, \alpha^\sharp). \tag{15.4}$$

Proof. Let $x \in M$ and let $e_1, \dots, e_n$ be a local orthonormal frame such that $(\nabla_{e_a} e_b)(x) = 0$. At the point x,

$$\begin{aligned}
(\mathrm{dd}^* + \mathrm{d}^*\mathrm{d})\alpha &= -\sum_{a,b=1}^n e^a \wedge i(e_b)\nabla_{e_a}\nabla_{e_b}\alpha + i(e_a)e^b \wedge \nabla_{e_a}\nabla_{e_b}\alpha \\
&= -\sum_{a=1}^n \nabla_{e_a}\nabla_{e_a}\alpha - \sum_{a,b=1}^n e^a i(e_b)[\nabla_{e_a}, \nabla_{e_b}]\alpha \\
&= \nabla^*\nabla\alpha + \sum_{a,b=1}^n \alpha(R(e_a, e_b)e_b)e^a \\
&= \nabla^*\nabla\alpha + \sum_{a,b,c=1}^n \alpha(e_c)\langle R(e_a, e_b)e_b, e_c\rangle e^a.
\end{aligned}$$

This proves (15.3). The identity (15.4) follows from

$$\Delta|\alpha|^2 = 2\langle\nabla^*\nabla\alpha, \alpha\rangle - 2|\nabla\alpha|^2. \qquad \square$$

Proof of Theorem 15.1. Let α be a harmonic 1–form. By the Bochner–Weitzenböck formula (15.4),

$$\begin{aligned}
0 &= -\frac{1}{2}\int_M \Delta|\alpha|^2 \\
&= \int_M |\nabla\alpha|^2 + \mathrm{Ric}(\alpha^\sharp, \alpha^\sharp).
\end{aligned}$$

Both terms under the integral are non-negative. Therefore, they vanish. This proves (1). If $\mathrm{Ric}_g(x) > 0$, then α must vanish in order for the second term to vanish. This proves (2). □

Remark 15.5. If $\mathrm{Ric}_g \geqslant 0$ and M is non-compact, then there often (always?) are many non-parallel harmonic 1–forms.

16 Bochner's vanshing theorem for Killing fields

Definition 16.1. Let (M, g) be a Riemannian manifold. A vector field $v \in \mathrm{Vect}(M)$ is called a **Killing field** if

$$\mathscr{L}_v g = 0. \tag{16.2}$$

The space of Killing fields is denoted by

$$\mathfrak{iso}(M, g) := \{v \in \mathrm{Vect}(M) : \mathscr{L}_v g = 0\}.$$

Remark 16.3. If $u, v, w \in \mathrm{Vect}(M)$, then

$$(\mathscr{L}_u g)(v, w) = \langle \nabla_v u, w \rangle + \langle \nabla_w u, v \rangle \tag{16.4}$$

Exercise 16.5 (Bochner). Let v be Killing field and let $\alpha \in \Omega^1(M)$ be a harmonic 1–form. Prove that the function $\alpha(v)$ is constant.

Theorem 16.6 (Bochner's vanishing theorem for Killing fields [Boc46, Theorem 2]). *Let (M, g) be a closed, connected Riemannian manifold of dimension n. If $\mathrm{Ric}_g \leqslant 0$, then the following hold:*

1. *Every Killing field v is parallel and satisfies $\mathrm{Ric}_g(v, v) = 0$. In particular, $\dim \mathfrak{iso}(M, g) \leqslant n$.*
2. *If there exists some $x \in M$ with $\mathrm{Ric}_g(x) > 0$, then every Killing field vanishes. In particular, $\mathrm{Iso}(M, g)$ is finite.*

This follows from the following proposition and the argument from the proof of Theorem 15.1.

Proposition 16.7 (Bochner–Weitzenböck formula for vector fields [Boc46, Lemma 2]). *Let (M, g) be a Riemannian manifold. For every $v \in \mathrm{Vect}(M)$,*

$$\nabla^* \mathscr{L}_v g - \mathrm{d}\,\mathrm{div}\, v = \langle \nabla^* \nabla v, \cdot \rangle - \mathrm{Ric}_g(v, \cdot) \tag{16.8}$$

in particular,

$$\frac{1}{2}\Delta |v|^2 = (\nabla^* \mathscr{L}_v g)(v) - \mathscr{L}_v\, \mathrm{div}\, v + \mathrm{Ric}_g(v, v) - |\nabla v|^2. \tag{16.9}$$

Proof. Let $x \in M$ and let $e_1, \ldots, e_n$ be a local orthonormal frame such that $(\nabla_{e_a} e_b)(x) = 0$. At the point x,

$$\begin{aligned}(\nabla^*\mathscr{L}_v g)(w) &= -\sum_{a=1}^n (\nabla_{e_a}\mathscr{L}_v g)(e_a, w)\\ &= -\sum_{a=1}^n \nabla_{e_a}[(\mathscr{L}_v g)(e_a, w)] - (\mathscr{L}_v g)(e_a, \nabla_{e_a} w)\\ &= -\sum_{a=1}^n \nabla_{e_a}(\langle \nabla_{e_a} v, w\rangle + \langle \nabla_w v, e_a\rangle) - \langle \nabla_{e_a} v, \nabla_{e_a} w\rangle - \langle e_a, \nabla_{\nabla_{e_a} w} v\rangle\\ &= \langle \nabla^*\nabla v, w\rangle - \sum_{a=1}^n \nabla_{e_a}\langle \nabla_w v, e_a\rangle - \langle e_a, \nabla_{\nabla_{e_a} w} v\rangle;\end{aligned}$$

and, furthermore,

$$\begin{aligned}-\sum_{a=1}^n \nabla_{e_a}\langle \nabla_w v, e_a\rangle - \langle \nabla_{\nabla_{e_a} v} v, e_a\rangle &= -\sum_{a=1}^n \langle \nabla_{e_a}\nabla_w v, e_a\rangle - \langle \nabla_{\nabla_{e_a} w} v, e_a\rangle\\ &= -\mathrm{Ric}(v, w) + \sum_{a=1}^n \langle \nabla_w \nabla_{e_a} v, e_a\rangle\\ &= -\mathrm{Ric}(v, w) + \mathscr{L}_w \operatorname{div} w.\end{aligned}$$

□

Corollary 16.10. *If (M, g) is a compact Riemannian manifold, then $v \in \mathrm{Vect}(M)$ is a Killing field if and only if*

$$\nabla^*\nabla v - \mathrm{Ric}_g(v, \cdot)^\flat = 0.$$

Application to Riemann surfaces Theorem 16.6 can be used to proof Hurwitz' automorphism theorem.

Theorem 16.11 (Uniformization Theorem). *Let (Σ, j) be a closed Riemann surface.*

1. *If $\chi(\Sigma) = 2$, then $(\Sigma, j) \cong \mathbf{C}P^1$.*
2. *If $\chi(\Sigma) = 0$, then $(\Sigma, j) \cong \mathbf{C}/\Lambda$ with $\Lambda \subset \mathbf{C}$ a co-compact lattice.*
3. *If $\chi(\Sigma) < 0$, then $(\Sigma, j) \cong H^2/\Gamma$ with $\Gamma \subset \mathrm{PSL}_2(\mathbf{R})$ a discrete subgroup acting freely on $\mathbf{H}^2$.*

Theorem 16.12 (Metric Uniformization Theorem). *Let (Σ, j) be a closed Riemann surface. In the conformal class determined by j there is a unique Riemannian metric g satisfying*

$$\mathrm{Ric}_g = \lambda g \quad \textit{with} \quad \lambda = \begin{cases} 1 & \textit{if } \chi(\Sigma) = 2,\\ 0 & \textit{if } \chi(\Sigma) = 0,\\ -1 & \textit{if } \chi(\Sigma) < 0.\end{cases}$$

Theorem 16.13 (Hurwitz' Automorphism Theorem [Hur93, p. 424]). *If (Σ, j) is a closed Riemann surface with $\chi(\Sigma) < 0$, then*

$$\# \operatorname{Aut}(\Sigma, j) \leqslant -42\chi(\Sigma). \tag{16.14}$$

Equality holds in (16.14) if and only if Σ is a branched cover of $\mathbf{C}P^1$ with ramification indicies 2, 3, and 7.

Remark 16.15. See de Saint-Gervais [dSai16] for an account of the history of the Uniformization Theorem.

Proof of Theorem 16.13. By Theorem 16.12, for the Einstein metric g,

$$\operatorname{Aut}(\Sigma, j) = \operatorname{Iso}(\Sigma, g).$$

Therefore, by Theorem 16.6, $\operatorname{Aut}(\Sigma, j)$ is finite.

The inequality is proved by a somewhat tedious—but nevertheless enlightening—case distinction. Set $\Gamma := \operatorname{Aut}(\Sigma, j)$. Consider the quotient map

$$\pi \colon \Sigma \to S := \Sigma/\Gamma.$$

Since $\operatorname{Aut}(\Sigma, j)$ acts holomorphically, π is locally given by $z \mapsto z^n$. Therefore, S is a Riemann surface and π is a branched covering map.

A point $z \in \Sigma$ is called a **ramification point** of the stabilizer Γ_z is non-trivial. In this case, $\Gamma_z = \mathbf{Z}/n\mathbf{Z}$ and we call $e_z := \#G_z$ the **ramification index** of z. A point $w \in S$ is called a **branched point** if it is the image of a ramification point. Since Γ acts transitively on $\pi^{-1}(w)$, every $z \in \pi^{-1}(w)$ is a ramification point and they all have the same ramification index. The **ramification index** w is the ramification index of any of its preimages and denoted by e_w. By the preceding discussion,

$$\#\pi^{-1}(w) = \frac{\#\Gamma}{e_w}.$$

Denote by $z_1, \ldots, z_m$ the ramification points and by $w_1, \ldots, w_k$ the branch points of π. By the Riemann–Hurwitz formula,

$$\begin{aligned} -\chi(\Sigma) &= -\#\Gamma \cdot \chi(S) + \sum_{a=1}^{m} (e_{z_a} - 1) \\ &= \#\Gamma \cdot \underbrace{\left[-\chi(S) + \sum_{a=1}^{k} \left(1 - \frac{1}{e_{w_a}} \right) \right]}_{=:A}. \end{aligned}$$

Therefore,

$$\#\Gamma = -\frac{\chi(\Sigma)}{A}.$$

Since $\chi(\Sigma) < 0$, $A > 0$. The following case distinction shows that $A \geqslant \frac{1}{42}$:

- If $\chi(S) < 0$, then $A \geqslant 2$.
- If $\chi(S) = 0$, then $k \geqslant 1$; therefore: $A \geqslant \frac{1}{2}$.
- It remains to analyze the case $\chi(S) = 2$. In this case $k \geqslant 3$.
  - If $k \geqslant 5$, then $A \geqslant \frac{1}{2}$.
  - If $k = 4$, then at least one of the ramification indices is bigger than two; therefore: $A \geqslant 1/6$.
  - A further case distinction in the case $k = 3$ shows that $A \geqslant 1/42$ with equality achieved if the ramification indices are 2, 3, and 7.

This finishes the proof. □

Remark 16.16. It should be stressed that they crucial part of the above proof is establishing that $\mathrm{Aut}(\Sigma, j)$ is finite. The remainder, although much longer, is really just bookkeeping.

17 Myers' Theorem

Theorem 17.1 (Myers [Mye41]). *Let (M, g) be a connected, complete Riemannian manifold. Let $\kappa > 0$. If*

$$\mathrm{Ric}_g \geqslant (n-1)\kappa g,$$

then

$$\mathrm{diam}(M, g) \leqslant \pi/\sqrt{\kappa}.$$

In particular, $\pi_1(M)$ is finite.

Proof. If $\pi_1(M)$ is not finite, then the universal cover $\tilde{M}$ has infinite diameter but also satisfies the lower Ricci bound: a contradiction to the asserted diameter bound.

The diameter bound follows once we prove that if $\gamma\colon [0, T] \to M$ is a minimal geodesic parametrized by arc-length, then

$$T \leqslant \pi/\sqrt{\kappa}.$$

To see this, let $e_1 = \dot{\gamma}, e_2, \ldots, e_n$ be a parallel orthonormal frame along γ. For $a = 2, \ldots, n$, set

$$V_a := \sin\left(\frac{\pi}{T} t\right) e_a \tag{17.2}$$

and let $\boldsymbol{\gamma}_a$ be a proper variation of γ with $\partial_s \boldsymbol{\gamma}_a(0, \cdot) = V_a$. By the second variation formula for the energy functional,

$$\begin{aligned}\sum_{a=2}^{n} \frac{\mathrm{d}^2}{\mathrm{d}s^2}\bigg|_{s=0} E(\gamma_{a,s}) &= \sum_{a=2}^{n} \int_0^T |\nabla_t V_a(t)|^2 - \langle R(V_a(t), \dot\gamma(t))\dot\gamma(t), V_a(t)\rangle \,\mathrm{d}t \\ &= (n-1)\left(\frac{\pi}{T}\right)^2 \int_0^T \cos\left(\frac{\pi}{T}t\right)^2 \mathrm{d}t - \int_0^T \sin\left(\frac{\pi}{T}t\right)^2 \mathrm{Ric}(e_1(t), e_1(t))\,\mathrm{d}t \\ &\leqslant (n-1)\left(\frac{\pi}{T}\right)^2 \int_0^T \cos\left(\frac{\pi}{T}t\right)^2 \mathrm{d}t - (n-1)\kappa \int_0^T \sin\left(\frac{\pi}{T}t\right)^2 \mathrm{d}t \\ &= \left[\left(\frac{\pi}{T}\right)^2 - \kappa\right] \frac{(n-1)T}{2}.\end{aligned}$$

If $T > \pi/\sqrt{\kappa}$, then this is negative; hence, one of the $\boldsymbol{\gamma}_a$ is an energy decreasing (hence: length decreasing) variation; therefore, γ is cannot be minimal. □

Remark 17.3. The diameter bound in Theorem 17.1 is sharp since S^n has $\mathrm{Ric} = g$.

Remark 17.4. The conclusion of Theorem 17.1 is much stronger than that of Theorem 15.1; however, so is its hypothesis. In fact, $\mathrm{Ric}_g \geqslant (n-1)\kappa g$ with $\kappa > 0$ is a much stronger condition than $\mathrm{Ric}_g > 0$ which in turn is much stronger than $\mathrm{Ric}_g \geqslant 0$.

18 Laplacian comparison

Definition 18.1. Let (M, g) be a Riemannian manifold of dimension n and $x \in M$. The **distance function** associated with x is the function $r\colon M \to [0, \infty)$ defined by

$$r(y) := d(x, y).$$

Remark 18.2. Within of the cut-locus of x in M, the distance function r associated with x is smooth and, by Gauß' lemma, satisfies

$$|\nabla r| = 1.$$

Theorem 18.3 (Laplacian Comparison Theorem). *Let (M, g) be a Riemannian manifold and let $x \in M$. Let $\kappa \in \mathbf{R}$. If*

$$\mathrm{Ric}_g \geqslant (n-1)\kappa g,$$

then, within the cut-locus of x in M,

$$-\Delta r \leqslant (n-1)\frac{\sin_\kappa'(r)}{\sin_\kappa(r)}. \tag{18.4}$$

Moreover, at a point y within the cut-locus of x in M, equality holds in (18.4) if and only if all radial sectional curvatures are equal to κ; that is: for all $v \in \partial_r^\perp \subset T_yM$,

$$\sec_g(\partial_r, v) = \kappa.$$

This might seems to be a "technical" result, but we will see that it has far-reaching consequences. As a first indication, we give a second proof of Myers' theorem.

Proof of Theorem 17.1 using Theorem 18.3. Suppose $\kappa > 0$ and $\mathrm{Ric}_g \geqslant (n-1)\kappa g$. If $\mathrm{diam}(M, g) > \pi/\sqrt{k}$, then there is a minimal geodesic $\gamma\colon [0, T] \to M$ parametrized by arc-length with $T > \pi/\sqrt{k}$. Set $x := \gamma(0)$. The geodesic is contained in within of the cut-locus of x in M and $r \circ \gamma(t) = t$. This contradicts (18.4) because the function $\sin_\kappa'(t)/\sin_\kappa(t) = \sqrt{\kappa}\cot(\sqrt{\kappa}t)$ diverges to $-\infty$ as t tends $\pi/\sqrt{k}$. □

The following propositions prepare the proof of Theorem 18.3.

Proposition 18.5 (Bochner–Weitzenböck formula for gradients). *Let (M, g) be a Riemannian manifold. For $f \in C^\infty(M)$,*

$$\frac{1}{2}\Delta|\nabla f|^2 = \langle \nabla\Delta f, \nabla f\rangle - |\mathrm{Hess}\, f|^2 - \mathrm{Ric}(\nabla f, \nabla f). \tag{18.6}$$

Proof. By the Bochner–Weitzenböck formula (15.4) for $\alpha = \mathrm{d}f$,

$$\begin{aligned}\frac{1}{2}\Delta|\nabla f|^2 &= \langle \mathrm{d}\mathrm{d}^*\mathrm{d}f, \nabla f\rangle - |\nabla \mathrm{d}f|^2 - \mathrm{Ric}(\mathrm{d}f^\sharp, \mathrm{d}f^\sharp)\\ &= \langle \nabla\Delta f, \nabla f\rangle - |\mathrm{Hess}\, f|^2 - \mathrm{Ric}(\nabla f, \nabla f).\end{aligned}$$ □

Proposition 18.7. *In the situation of Theorem 18.3,*

$$-\partial_r\Delta r + \frac{(\Delta r)^2}{n-1} + (n-1)\kappa \leqslant 0. \tag{18.8}$$

Equality holds in (18.8) if and only if

$$\mathrm{Hess}\, r = -\frac{\Delta r}{n-1}(g - \mathrm{d}r \otimes \mathrm{d}r) \quad \textit{and} \quad \mathrm{Ric}_g = (n-1)\kappa g. \tag{18.9}$$

Proof. By the Cauchy–Schwarz inequality, if $A \in \mathbf{R}^{m\times m}$ is symmetric, then

$$\frac{(\mathrm{tr}\, A)^2}{m} \leqslant |A|^2$$

with equality if and only if $A = \frac{\mathrm{tr}\, A}{m}\mathbf{1}$. Therefore and since $\mathrm{Hess}\, r(\partial_r, \cdot) = 0$,

$$\frac{(\Delta r)^2}{n-1} \leqslant |\mathrm{Hess}\, r|^2$$

with equality if and only if the first part of (18.9) holds.

By the Bochner–Weitzenböck formula for gradients (18.6),

$$0 = -\partial_r\Delta r + |\mathrm{Hess}\, r|^2 + \mathrm{Ric}(\partial_r, \partial_r).$$

Consequently, (18.8) holds with equality if and only if (18.9). □

Proposition 18.10 (Riccati Comparison Principle). *Let $\kappa \in \mathbf{R}$. If $f\colon (0,T) \to \mathbf{R}$ satisfies*

$$f' + f^2 + \kappa \leqslant 0 \quad \textit{and} \quad f(t) = \frac{1}{t} + O(1),$$

then

$$f(t) \leqslant \frac{\sin_\kappa'(t)}{\sin_\kappa(t)}.$$

Proof. The function $f_\kappa = \sin_\kappa' / \sin_\kappa$ satisfies the **Riccati equation**

$$f_\kappa' + f_\kappa^2 + \kappa = 0 \quad \text{and} \quad f_\kappa(t) = \frac{1}{t} + O(1),$$

Choose a smooth function $G\colon (0,T) \to \mathbf{R}$ such that

$$G' = f + f_\kappa \quad \text{and} \quad G(t) = 2\log t + O(1).$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^G(f - f_\kappa)\right] = e^G\left(f' - f_\kappa' + f^2 - f_\kappa^2\right) \leqslant 0,$$

the function $e^G(f - f_\kappa)$ is decreasing. This implies the assertion because

$$\lim_{t\to 0} e^{G(t)}(f(t) - f_\kappa(t)) = 0. \qquad \square$$

Proof of Theorem 18.3. Let $\gamma\colon [0,T] \to M$ be a geodesic emerging from x and parametrized by arc-length. Set

$$f(t) := -\frac{\Delta r}{n-1} \circ \gamma(t).$$

By (18.8),

$$f' + f^2 + \kappa \leqslant 0.$$

Since

$$\Delta r = \frac{n-1}{r} + O(1),$$

(18.4) follows from Proposition 18.10.

If equality holds in (18.4) at y, then, by Proposition 18.7,

$$\operatorname{Hess} r = \frac{\sin_\kappa'(r)}{\sin_\kappa(r)}(g - \mathrm{d}r \otimes \mathrm{d}r).$$

Let $e_1 = \partial_r, e_2, \ldots, e_n$ be a local orthonormal frame defined near y such that at y, for all $a, b = 2, \ldots, n$,

$$\operatorname{Hess} r(e_a, e_b) = \frac{\sin_\kappa'(r)}{\sin_\kappa(r)}\delta_{ab} \quad \text{and} \quad [\partial_r, e_a] = 0.$$

Since

$$\operatorname{Hess} r(e_a, e_b) = \langle \nabla_{e_a} \partial_r, e_b \rangle,$$

the former means that

$$\nabla_{e_a} \partial_r = \frac{\sin'_\kappa(r)}{\sin_\kappa(r)} e_a = f_\kappa e_a.$$

Therefore,

$$\begin{aligned}
\sec_g(\partial_r \wedge e_a) &= -\langle \nabla_{\partial_r} \nabla_{e_a} \partial_r, e_a \rangle \\
&= -\langle \nabla_{\partial_r} f_\kappa e_a, e_a \rangle \\
&= -\langle (f'_\kappa + f_\kappa^2) e_a, e_a \rangle \\
&= \kappa.
\end{aligned}$$

□

Remark 18.11. There is also is a proof of Theorem 18.3 using Jacobi fields.

Theorem 18.12 (Lichnerowicz [Lic58] and Obata [Oba62, Theorems 1 and 2]). *Let $\kappa > 0$. Let (M, g) be a closed Riemannian manifold of dimension n with $\operatorname{Ric}_g \geqslant (n-1)\kappa g$. If λ is a non-zero eigenvalue of the Laplacian, then*

$$\lambda \geqslant n\kappa. \tag{18.13}$$

Equality is achieved in (18.13) if and only if (M, g) is isometric to (S^n_κ, g_κ).

Exercise 18.14. Prove Theorem 18.12.

19 Bishop–Gromov volume comparison

Theorem 19.1 (Bishop–Gromov's Relative Volume Comparison Theorem [BC64, Section 11.10 Corollary 3; Gro81a, Section 2.1]). *Let (M, g) be a complete Riemannian manifold of dimension n and let $x \in M$. Let $\kappa \in \mathbf{R}$ and $0 < r \leqslant R$. If*

$$\operatorname{Ric}_g \geqslant (n-1)\kappa g$$

on $B_R(x)$, then

$$\frac{\operatorname{vol}(B_R(x))}{\operatorname{vol}(B_r(x))} \leqslant \frac{V^n_\kappa(R)}{V^n_\kappa(r)}. \tag{19.2}$$

Moreover, equality holds in (19.2) if and only if all radial sectional curvatures are equal to κ on $B_R(x)$.

Theorem 19.3 (Bishop's Absolute Volume Comparison Theorem [Bis63; BC64, Section 11.10 Corollary 4]). *Let (M, g) be a complete Riemannian manifold of dimension n. Let $\kappa \in \mathbf{R}$. If*

$$\mathrm{Ric}_g \geqslant (n-1)\kappa g,$$

then

$$\mathrm{vol}(B_r(x)) \leqslant V_\kappa^n(r).$$

Proof of Theorem 19.3 assuming Theorem 19.1. This is a consequence of

$$\lim_{r\to 0} \frac{\mathrm{vol}(B_r(x))}{V_\kappa^n(r)} = 1. \qquad \square$$

The following proposition prepares the proof of Theorem 19.1.

Definition 19.4. For $n \in \{2, 3, \ldots\}$ and $\kappa \in \mathbf{R}$, set

$$v_\kappa^n(r) := \begin{cases} 0 & \text{if } \kappa > 0 \text{ and } r \geqslant \pi/\sqrt{\kappa} \text{ and} \\ \sin_\kappa(r)^{n-1} & \text{otherwise.} \end{cases}$$

Proposition 19.5. *Let (M, g) be a Riemannian manifold of dimension n and let $x \in M$. Within of the cut-locus of x in T_xM, define v by*

$$\exp_x^* \mathrm{vol}_g = v\mathrm{d}r \wedge \mathrm{vol}_{S^{n-1}}.$$

Let $\kappa \in \mathbf{R}$. If

$$\mathrm{Ric}_g \geqslant (n-1)\kappa g,$$

then

$$\partial_r \left(\frac{v}{v_\kappa^n}\right) \leqslant 0. \tag{19.6}$$

Moreover, equality holds in (19.6) if and only if all radial sectional curvatures are equal to κ within of the cut-locus of x in M.

Proof. For $f \in C^\infty(M)$,

$$\mathscr{L}_{\nabla f}\mathrm{vol}_g = \mathrm{d}i(\nabla f)\mathrm{vol}_g = (\mathrm{div}\,\nabla f)\mathrm{vol}_g = -\Delta f \mathrm{vol}_g.$$

Therefore,

$$\partial_r v = -v\Delta r.$$

Hence, by Theorem 18.3,

$$\partial_r v \leqslant (n-1)\frac{\sin_\kappa'(r)}{\sin_\kappa(r)} v.$$

Since

$$\partial_r v_\kappa^n = (n-1)\frac{\sin_\kappa'(r)}{\sin_\kappa(r)} v_\kappa^n,$$

it follows that

$$\partial_r\left(\frac{v}{v_\kappa^n}\right) = \frac{\partial_r v}{v_\kappa^n} - \frac{v}{v_\kappa^n}\frac{\partial_r v_\kappa^n}{v_\kappa^n} \leqslant 0.$$

Equality holds in (19.6) if and only if equality holds in (18.4). By Theorem 18.3, the latter holds if and only if all radial sectional curvatures are equal to κ within the cut-locus of x in M. □

Proof of Theorem 19.1. The inequality (19.2) is equivalent to $\theta\colon (0,R] \to (0,\infty)$ defined by

$$\theta(r) := \frac{\mathrm{vol}(B_R(r))}{V_\kappa^n(r)}$$

being non-increasing. To prove this we proceed as follows.

Let $v\colon T_xM \to [0,\infty)$ be such that, within the cut-locus of x in T_xM,

$$\exp_x^* \mathrm{vol}_g = v\mathrm{d}r \wedge \mathrm{vol}_{S^n};$$

and $v = 0$ on and beyond the cut-locus of x in T_xM. For $r > 0$,

$$\mathrm{vol}(B_r(x)) = \int_{B_r(0)} v\mathrm{d}r \wedge \mathrm{vol}_{S^{n-1}} \quad \text{and} \quad V_\kappa^n(r) = \int_{B_r(0)} v_\kappa^r \mathrm{d}r \wedge \mathrm{vol}_{S^{n-1}}.$$

Therefore,

$$\begin{aligned}
&V_\kappa^n(r)^2\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{vol}(B_r(x))}{V_\kappa^n(r)}\right)\\
&= \mathrm{vol}(\partial B_r(x)) \cdot V_\kappa^n(r) - \mathrm{vol}(B_r(x)) \cdot (V_\kappa^n)'rR)\\
&= \int_0^r \left(\int_{S^{n-1}} v(r\hat{x}) \cdot \int_{S^{n-1}} v_\kappa^n(s) - \int_{S^{n-1}} v(s\hat{x}) \cdot \int_{S^{n-1}} v_\kappa^n(r)\right) \mathrm{d}s\\
&= \mathrm{vol}(S^{n-1}) \int_0^r \left(\int_{S^{n-1}} v(r\hat{x})v_\kappa^n(s) - v(\cdot))v_\kappa^n(r)\right) \mathrm{d}s.
\end{aligned}$$

The integrand is non-positive if and only if, for $0 < s \leqslant r$,

$$\frac{v(r\hat{x})}{v_\kappa^n(r)} \leqslant \frac{v(s\hat{x})}{v_\kappa^n(s)}$$

which follows from (19.6). This proves that θ is non-increasing.

Equality holds in (19.2) if and only if equality holds in (19.6) on $B_R(x)$. By Proposition 19.5, the latter holds if and only if and only if all radial sectional curvatures are equal to κ on $B_R(x)$. □

A minimal modification of the proof of Theorem 19.1 establishes the following variant.

Definition 19.7. Let $\Gamma \subset S^{n-1}$ be measurable and $0 \leqslant r \leqslant R$. Set

$$A_{r,R}^{\Gamma} := \left\{\rho\hat{x} \in \mathbf{R}^n : \hat{x} \in \Gamma \text{ and } \rho \in [r,R]\right\}$$

Let (M,g) a Riemannian manifold, $x \in M$, and $0 \leqslant r \leqslant R$. The **annular sector** associated with Γ centered at x and with radii r and R is

$$A_{r,R}^{\Gamma}(x) = A_{r,R}^{\Gamma,M}(x) := \exp_x(A_{r,R}^{\Gamma}).$$

For $n \in \{2,3,\ldots\}$ and $\kappa \in \mathbf{R}$, set

$$V_\kappa^n(\Gamma,r,R) := \mathrm{vol}(A_{r,R}^{\Gamma,S_\kappa^n}(x)).$$

Theorem 19.8 (Relative Volume Comparison Theorem for annular sectors [Zhu97, Theorem 3.1]). *Let (M,g) be a complete Riemannian manifold of dimension n and let $x \in M$. Let $\kappa \in \mathbf{R}$. Suppose that*

$$\mathrm{Ric}_g \geqslant (n-1)\kappa g.$$

If $0 \leqslant r \leqslant R$ and $0 \leqslant s \leqslant S$ with $r \leqslant s$ and $R \leqslant S$, then

$$\frac{\mathrm{vol}(A_{s,S}^{\Gamma}(x))}{\mathrm{vol}(A_{r,R}^{\Gamma}(x))} \leqslant \frac{V_\kappa^n(\Gamma,s,S)}{V_\kappa^n(\Gamma,r,R)}. \tag{19.9}$$

Moreover, equality holds in (19.9) if and only if all all radial sectional curvatures are equal to κ on $A_{r,S}^{\Gamma}(x)$.

20 Volume growth

Exercise 20.1. Let (M,g) be a complete Riemannian manifold with $\mathrm{Ric}_g \geqslant 0$. Prove that, for $r \geqslant 1$,

$$\mathrm{vol}(B_r(x)) \leqslant \mathrm{vol}(B_1(x))r^n.$$

Exercise 20.2. Let (M,g) be a complete Riemannian manifold with $\mathrm{Ric}_g \geqslant 0$. Prove that if

$$\lim_{r\to\infty} \frac{\mathrm{vol}(B_r(x))}{r^n} \geqslant V_0^n(1) = \frac{\pi^{n/2}}{\Gamma(n/2+1)},$$

then (M,g) is isometric to $\mathbf{R}^n$.

Remark 20.3. The preceding exercise shows, in particular, that asymptotically Euclidean manifold cannot be Ricci flat without being flat. There are, however, Ricci flat manifolds which are asymptotic to $\mathbf{R}^n/\Gamma$ for Γ acting freely outside the origin.

Theorem 20.4 (Yau [Yau76, Theorem 7]). *If (M,g) is a non-compact, complete Riemannian manifold of dimension n and with $\mathrm{Ric}_g \geqslant 0$, then, for $r \geqslant 1$,*

$$\mathrm{vol}(B_r(x)) \gtrsim_n \mathrm{vol}(B_1(x))r.$$

Definition 20.5. Let (M,g) be a Riemannian manifold. A **geodesic ray** is a geodesic $\gamma\colon [0,\infty) \to M$ satisfying

$$d(\gamma(s),\gamma(t)) = |t-s| \tag{20.6}$$

for all $s,t \in [0,\infty)$. A **geodesic line** is a geodesic $\gamma\colon \mathbf{R} \to M$ satisfying (20.6) for all $s,t \in \mathbf{R}$.

Proof of Theorem 20.4. Since M is non-compact, there is a geodesic ray $\gamma\colon [0,\infty) \to M$ parametrized by arc-length with $\gamma(0) = x$. By Theorem 19.8, for $t \geqslant 2$,

$$\begin{aligned}\frac{\mathrm{vol}(B_{\gamma(t)}(t+1)\backslash B_{\gamma(t)}(t-1))}{\mathrm{vol}(B_{\gamma(t)}(t-1))} &\leqslant \frac{(t+1)^n-(t-1)^n}{(t+1)^n}\\ &\leqslant 1-\left(1-\frac{2}{t+1}\right)^n \lesssim_n \frac{1}{t}.\end{aligned}$$

Therefore, for $t \geqslant 2$,

$$\begin{aligned}t\mathrm{vol}(B_x(1)) &\leqslant t\mathrm{vol}(B_{\gamma(t)}(t+1)\backslash B_{\gamma(t)}(t-1))\\ &\lesssim_n \mathrm{vol}(B_{\gamma(t)}(t-1))\\ &\lesssim \mathrm{vol}(B_x(2t)).\end{aligned}$$

This proves the assertion for $r \geqslant 4$. Since the assertion holds trivially for $r \in [1,4]$, the proof is complete. ☐

Remark 20.7. For examples of complete, Ricci-flat manifolds with linear volume growth see Hein [Hei12], Biquard and Minerbe [BM11], and Haskins, Hein, and Nordström [HHN15]

21 S.Y. Cheng's maximal diameter sphere theorem

Theorem 21.1 (S.Y. Cheng's maximal diameter sphere theorem [Che75]). *Let (M,g) be a connected, complete Riemannian manifold. Let $\kappa > 0$. If*

$$\mathrm{Ric}_g \geqslant (n-1)\kappa g \quad \textit{and} \quad \mathrm{diam}(M,g) = \pi/\sqrt{\kappa},$$

then (M,g) is isometric to (S^n_κ, g_κ).

Remark 21.2. The analogous result for $\sec_g$ is due to Topogonov.

Proof of Theorem 21.1. Without loss of generality $\kappa = 1$. Given $x \in M$, let $y \in M$ with $d(x, y) = \pi$. The balls $B_{\pi/2}(x)$ and $B_{\pi/2}(y)$ do not intersect. By hypothesis,

$$\mathrm{vol}(B_\pi(x)) = \mathrm{vol}(\bar{B}_\pi(y)) = \mathrm{vol}(M).$$

Therefore and by Theorem 19.1,

$$\begin{aligned} 2\mathrm{vol}(M) &= \mathrm{vol}(B_\pi(x)) + \mathrm{vol}(B_\pi(y)) \\ &\leqslant \frac{V_1^n(\pi)}{V_1^n(\pi/2)} \Big(\mathrm{vol}(B_{\pi/2}(x)) + \mathrm{vol}(B_{\pi/2}(y))\Big) \\ &\leqslant 2\mathrm{vol}(M). \end{aligned}$$

Therefore,

$$\frac{B_\pi(x)}{B_{\pi/2}(x)} = \frac{V_1^n(\pi)}{V_1^n(\pi/2)}.$$

By Theorem 19.1, all radial sectional curvatures are equal to 1 in $B_\pi(x)$. Since $x \in M$ is arbitrary,

$$\sec_g = 1.$$

The assertion thus follows from Theorem 6.5. □

22 The growth of groups

Definition 22.1. Let G be a group. A generating set $S \subset G$ is called **symmetric** if S^{-1}. Suppose G is finitely generated and S is a finite, symmetric generating set. The **word length** with respect to S is the map $\ell_S\colon G \to [0, \infty)$ defined by

$$\ell_S(g) := \min\{m : g = g_1 \cdots g_m \text{ with } g_a \in S\}.$$

For $r \in \mathbf{N}$, set

$$B_S(r) := \{g \in G : \ell_s(g) \leqslant r\} \quad \text{and} \quad V_S(r) := \#B_r^S.$$

Exercise 22.2. Try to work out the growth rates for a few of your favorite groups. (If you have no favorite group, try: the free abelian group $\mathbf{Z}^n$, the free group F_n, surfaces groups $\pi_1(\Sigma)$, the lamp-lighter group, ...)

Definition 22.3. Let G be a finitely generated group. G is said to have **polynomial growth** of rate at most $\nu \geqslant 0$ if, for some (hence: every) finite, symmetric generating set S,

$$V_S(r) \lesssim r^\nu.$$

G is said to have **exponential growth** if, for some (hence: every) finite, symmetric generating set S,

$$\lim_{r\to\infty} V_S(r)^{1/r} > 1.$$

The above can also be phrased in terms of Cayley graph equipped with the counting measure and the obvious metric.

Definition 22.4. Let G be a group and let S be a symmetric generating set. The **Cayley graph** associated with S is the graph whose vertices are the elements of G with $\{g, h\}$ an edge if and only if $gh^{-1} \in S$.

Proposition 22.5. *Let G be a finitely generated group and let $S, T \subset G$ both be finite, symmetric generating sets. With*

$$c := \max\left(\{\ell_S(g) : g \in T\} \cup \{\ell_T(g) : g \in S\}\right)$$

the inequalities

$$\frac{1}{c}\ell_T \leqslant \ell_S \leqslant c\ell_T.$$

hold. □

Finally, we have to mention the celebrated theorem of Gromov.

Definition 22.6. A group G is called **nilpotent** if its lower central series, defined by $G_0 = G$ and $G_{a+1} = [G, G_a]$, terminates in the trivial group after finitely many steps. A group G is called **virtually nilpotent** if it has a finite index subgroup which is nilpotent.

Theorem 22.7 (Gromov's theorem on groups of polynomial growth [Gro81b]). *A finitely generated group has polynomial growth if and only if it is virtually nilpotent.*

Remark 22.8. An elementary, but long, proof can be found on Terry Tao's blog.

23 Applications to $\pi_1(M)$

Theorem 23.1 (Milnor [Mil68, Theorem 1]). *If (M, g) is a complete Riemannian manifold of dimension n with $\mathrm{Ric}_g \geqslant 0$, then every finitely generated subgroup $G \subset \pi_1(M)$ has polynomial growth of rate at most n.*

Conjecture 23.2 (Milnor's finite generation conjecture [Mil68]). *If (M, g) is a complete Riemannian manifold of dimension n with $\mathrm{Ric}_g \geqslant 0$, then $\pi_1(M)$ is finitely generated.*

Remark 23.3. Li [Li86, Theorem 2] and Anderson [And90a, Corollary 1.5] proved Conjecture 23.2 assuming M has maximal volume growth. Sormani [Sor00, Theorem 1] proved Conjecture 23.2 assuming small linear diameter growth. Liu [Liu13, Corollary 1] proved Conjecture 23.2 in dimension three using minimal surface techniques; see also, Pan [Pan18, Theorem 1.1] for a proof using Cheeger–Colding theory.

Proof of Theorem 23.1. Denote by $\pi\colon \tilde{M} \to M$ the universal cover. Let $\tilde{x}_0 \in \pi^{-1}(x_0)$. The fundamental group $\pi_1(M, x_0)$ acts on $\tilde{M}$ as the deck transformation group $\mathrm{Deck}(\tilde{M})$. Let $S \subset \pi_1(M, x_0) \cong \mathrm{Deck}(M)$ be finite subset and denote by G the group generated by S. Set

$$D := \max\{d(\tilde{x}_0, g\tilde{x}_0) : g \in S\}.$$

For $r \in \mathbf{N}$, $\bar{B}_{Dr}(\tilde{x}_0)$ contains at least $V_S(r)$ distinct points of the form $g\tilde{x}_0$ with $g \in \mathrm{Deck}(\tilde{M})$.

Set

$$\delta := \inf\{d(\tilde{x}, g\tilde{x}) : g \in \mathrm{Deck}(\tilde{M})\}.$$

Since $\mathrm{Deck}(M)$ acts discretely, $\delta > 0$. The ball $B_{Dr+\delta}(\tilde{x}_0)$ contains at least $V_S(r)$ disjoint subsets of the form $B_\delta(g\tilde{x}_0)$. Therefore and by Theorem 19.1,

$$V_S(r) \leqslant \frac{\mathrm{vol}(B_{Dr+\delta}(\tilde{x}_0))}{\mathrm{vol}(B_\delta(\tilde{x}_0))} \leqslant (Dr+\delta)^n/\delta^n \leqslant (D/\delta^n + 1)r^n$$ □

Theorem 23.4 (Milnor [Mil68, Theorem 2]). *If (M, g) is a complete Riemannian manifold of dimension n with $\sec_g < 0$, then $\pi_1(M)$ has exponential growth.*

Theorem 23.5 (Anderson [And90b, Theorem 2.3]). *Given $n \in \mathbf{R}$, $\kappa \in \mathbf{R}$, $D, V > 0$, there are only finitely many isomorphism types of groups which appear as $\pi_1(M)$ for connected, closed Riemannian manifolds (M, g) of dimension n with*

$$\mathrm{Ric}_g \geqslant (n-1)\kappa, \quad \mathrm{vol}(M) \geqslant V \quad \textit{and} \quad \mathrm{diam}(M) \leqslant D.$$

The proof relies on the following lemma.

Lemma 23.6 (Gromov [Gro81c; Gro07, Proposition 3.22]). *Let (M, g) be a closed Riemannian manifold. Let $x_0 \in M$. There are loops $\gamma_1, \ldots, \gamma_m$ based at x_0 with*

$$\ell(\gamma_a) \leqslant 3\,\mathrm{diam}(M)$$

and a presentation

$$\pi_1(M, x_0) = \langle [\gamma_1], \ldots, [\gamma_m] | R \rangle$$

with every relation in R of the form

$$[\gamma_a][\gamma_b] = [\gamma_c]. \tag{23.7}$$

Remark 23.8. The lemma makes no assertion about the number of generators.

Proof. Choose a triangulation K of M such that x_0 is one of the vertices and every edge e_{ab} has length at most one-half of the injectivity radius of M. For every vertex v_a in K, denote by δ_a the minimal geodesic from x_0 to v_a. The loop

$$\gamma_{ab} := \delta_a e_{ab} \delta_b^{-1}$$

has length at most $3\operatorname{diam} M$.

Every loop based x_0 in the 1–skeleton of K is homotopic to a product of the γ_{ab}. Since every loop based at x_0 is homotopic to a loop in the 1–skeleton of K, the γ_{ab} generate $\pi(M, x_0)$.

If v_a, v_b, v_c form a 2–simplex in K, then $\gamma_{ab}\gamma_{bc} = \gamma_{ac}$. Every homotopy between two loops based at x_0 in the 1–skeleton of K is homotopic to a homotopy lying in the 2–skeleton. Since homotopies in the 2–skeleton correspond to a collection of 2–simplices, every relation is generated by relations of the form $\gamma_{ab}\gamma_{bc} = \gamma_{ac}$. ◻

Proof of Theorem 23.5. It suffices to estimate the number of loops γ_a in Lemma 23.6, because, if there are m generators, then there can be at most 2^{m^3} relations of the form (23.7).

Let (M, g) be a connected, closed Riemannian manifold of dimension n with $\operatorname{Ric}_g \geqslant (n-1)\kappa$, $\operatorname{vol}(M) \geqslant V$, and $\operatorname{diam}(M) \leqslant D$. Denote by $\pi\colon \tilde{M} \to M$ the universal cover of M. Let $\tilde{x}_0 \in \tilde{M}$ and set $x_0 \coloneqq \pi(\tilde{x}_0)$. The $[\gamma_a]$ acts as deck transformations: $\pi_1(M, x_0) \cong \operatorname{Deck}(\tilde{M})$. Set

$$K \coloneqq \left\{x \in \tilde{M} : d(x, \tilde{x}_0) \leqslant d(x, \gamma \cdot \tilde{x}_0) \text{ for all } \gamma \in \pi_1(M, x_0)\right\}$$

The sets $\gamma_a \cdot K$ all have volume equal to $\operatorname{vol}(M)$ and are all contained in $B_{6D}(\tilde{x}_0)$. Furthermore, they intersect in sets of measure zero. Therefore,

$$m \leqslant \frac{\operatorname{vol}(B_{6D}(\tilde{x}_0))}{\operatorname{vol}(K)} \leqslant \frac{V_\kappa^n(6D)}{V}.$$ ◻

24 The Maximum Principle

Definition 24.1. Let (M, g) a Riemannian manifold. A function $f \in C^\infty(M)$ is called **subharmonic** if $\Delta f \leqslant 0$ and **superharmonic** if $\Delta f \geqslant 0$.

Theorem 24.2 (E.Hopf's Maximum Principle [Hop27]). *Let (M, g) a Riemannian manifold. Let $f \in C^\infty(M)$ be subharmonic. If f has a local maximum at $x \in M$, then f is constant on a neighborhood of x. In particular, f has a global maximum if and only if it is constant.*

Definition 24.3 (Calabi [Cal58]). Let (M, g) is a Riemannian manifold. Let $f \in C^0(M)$ and $g \in C^0(M)$. A **lower barrier** (resp. **upper barrier**) for f at $x \in M$ is a smooth function f_x defined in a neighborhood of x satisfying

$$f_x(x) = f(x) \quad \text{and} \quad f_x \leqslant f \quad (\text{resp. } f_x \geqslant f).$$

We say that

$$\Delta f \leqslant g \quad (\text{resp. } \Delta f \geqslant g)$$

in the barrier sense if, for every $x \in M$ and every $\varepsilon > 0$, there exists a lower barrier (resp. upper barrier) $f_{x,\varepsilon}$ for f at x such that

$$\Delta f_{x,\varepsilon} \leqslant g + \varepsilon \quad (\text{resp. } \Delta f_{x,\varepsilon} \geqslant g - \varepsilon).$$

The function f is called **subharmonic in the barrier sense** (resp.**superharmonic in the barrier sense**) if $\Delta f \leqslant 0$ (resp. $\Delta f \geqslant 0$).

Remark 24.4. Calabi [Cal58, Theorem 3] proved that Theorem 18.3 holds in the barrier sense on all of M.

Theorem 24.5 (Calabi's Maximum Principle Calabi [Cal58, Theorem 1]). *Let (M, g) a Riemannian manifold. Let $f \in C^0(M)$ be subharmonic in the barrier sense. If f has a local maximum at $x \in M$, then f is constant on a neighborhood of x. In particular, f has a global maximum if and only if it is constant.*

Proof. Suppose that f achieves a local maximum at x. If there were a smooth function $f_{x,0}$ defined on a neighborhood of x such that

$$f_{x,0}(x) = f(x), \quad f_{x,0} \leqslant f, \quad \text{and} \quad \Delta f_{x,0} \leqslant 0,$$

then x is also a local maximum for $f_{x,0}$; hence, by the maximum principle for harmonic functions, $f_{x,0}$ is constant in a neighborhood of x. But then f is also constant in this neighborhood.

Suppose f is not constant; then there is a $0 < r \ll 1$ such that

$$f(x) = \sup_{y \in B_r(x)} f(y) \quad \text{and} \quad \Gamma := \{y \in \partial B_r(x) : f(y) = f(x)\} \neq \partial B_r(x).$$

As we will see shortly, there is a smooth function $g \colon \bar{B}_r(x) \to \mathbf{R}$ satisfying

$$g(x) = 0, \quad g|_\Gamma \leqslant -1/2, \quad \text{and} \quad \Delta g \leqslant -1.$$

For $0 < \delta \ll 1$, the function $f + \delta g$ has a strict local maximum at x. This, however, contradicts the observation in the first paragraph.

To construct g, we proceed as follows. Since $\Gamma \neq \partial B_r(x)$, we can choose a function χ satisfying

$$\chi(x) = 0, \quad \chi|_\Gamma < 0, \quad \text{and} \quad |\nabla \chi| > 0.$$

For $\Lambda \gg 1$, the function

$$g := e^{\Lambda\chi} - 1,$$

satisfies the required properties, because

$$\Delta g = \Lambda e^{\Lambda\chi}\left(\Delta\chi - \Lambda|\nabla\chi|^2\right). \qquad \square$$

Theorem 24.6. *Let (M, g) be a Riemannian manifold. If $f \in C^0(M)$ is harmonic in the barrier sense, then it is smooth and harmonic.*

Proof. For $x \in M$ and $0 \leqslant r \ll 1$, standard elliptic theory constructs a continuous function $h\colon \bar{B}_r(x) \to \mathbf{R}$ which is smooth and harmonic on $B_r(x)$ and satisfies the Dirichlet boundary condition $h|_{\partial B_r(x)} = f|_{\partial B_r(x)}$. By the maximum principle applied to $f - h$ and $h - f$, $f = h$. $\square$

25 Busemann functions

Proposition 25.1. *Let (M, g) be a connected, complete, non-compact Riemannian manifold. If M contains a compact subset K with $M\backslash K$ disconnected, then there is a geodesic ray passing through M.*

Exercise 25.2. Prove Proposition 25.1.

Proposition 25.3. *Let (M, g) be a Riemannian manifold. Given a geodesic ray γ in M, there exists a function $b_\gamma\colon M \to \mathbf{R}$ such that, for all $x \in M$,*

$$b_\gamma(x) := \lim_{t\to\infty} d(x, \gamma(t)) - t. \tag{25.4}$$

The function b_γ is Lipschitz with $\mathrm{Lip}(b_\gamma) = 1$.

Definition 25.5. In the situation of Proposition 25.3, b_γ is called the **Busemann function** associated with (M, g).

Remark 25.6. Morally, a Busemann function is a renormalized distance function associated to ∞.

Exercise 25.7. Compute the Busemann functions on $\mathbf{R}^n$.

Exercise 25.8. Compute the Busemann functions on $\mathbf{H}^2$. What are the level sets of these functions?

Proof of Proposition 25.3. Define $b_\gamma^t\colon M \to \mathbf{R}$ by

$$b_\gamma^t(x) := d(x, \gamma(t)) - t.$$

By the triangle inequality:

1. $b_\gamma^t(x)$ is non-increasing in t,

2. $|b_\gamma^t(x)| \leqslant d(x, \gamma(0))$, and

3. $|b_\gamma^t(x) - b_\gamma^t(y)| \leqslant d(x, y)$.

The first two show that the limit in (25.4) exists. The last implies $\mathrm{Lip}(b_\gamma) = 1$. □

Proposition 25.9. *If (M, g) is a Riemannian manifold with $\mathrm{Ric}_g \geqslant 0$ and γ is a geodesic ray, then*

$$\Delta b_\gamma \geqslant 0.$$

Proof. Let $x \in M$. For $s > 0$, denote by $\delta_s \colon [0, T_s] \to M$ the geodesic parametrized by arc-length from x to $\gamma(s)$. There is a sequence $(s_n)_{n\in\mathbf{N}}$ converging to infinity, such that $(\delta_{s_n})_{n\in\mathbf{N}}$ converges to a geodesic ray δ with $\delta(0) = x$.

The function $b_\gamma(x) + b_\delta^t$ is smooth in a neighborhood of x and agrees with $b_\gamma(x)$ at x. By the triangle inequality, for $s > 0$ and $0 \leqslant t \leqslant T_s$,

$$\begin{aligned} b_\gamma^s(y) - b_\gamma^s(x) &= d(y, \gamma(s)) - d(\gamma(s), x) \\ &= d(y, \gamma(s)) - d(\gamma(s), \delta_s(t)) - d(\delta_s(t), x) \\ &\leqslant d(y, \delta_s(t)) - d(\delta(t), x) + d(\delta_s(t), \delta(t)) \\ &= d(y, \delta_s(t)) - t + d(\delta_s(t), \delta(t)). \end{aligned}$$

Setting $s = s_n$ and taking the limit $n \to \infty$,

$$b_\gamma(y) \leqslant b_\gamma(x) + b_\delta^t(y). \tag{25.10}$$

By Theorem 18.3,

$$\Delta b_\delta^t(x) \geqslant -\frac{n-1}{d(x, \delta(t))}.$$

Since the right-hand side goes to zero as t tends to infinity, b_γ is superharmonic. □

26 Cheeger–Gromoll Splitting Theorem

Theorem 26.1 (Cheeger–Gromoll Splitting Theorem [CG71])**.** *Let (M, g) be a complete Riemannian manifold with $\mathrm{Ric}_g \geqslant 0$. If (M, g) contains a geodesic line, then there is a complete Riemannian manifold (N, h) and an isometry*

$$(M, g) \cong (\mathbf{R} \times N, \mathrm{d}t \otimes \mathrm{d}t + h).$$

The following propositions prepare the proof. This argument is due to Eschenburg and Heintze [EH84].

Proposition 26.2. *Let (M, g) be a complete Riemannian manifold with $\mathrm{Ric}_g \geqslant 0$. If (M, g) contains a geodesic line, then there exists a harmonic function $\ell \colon M \to \mathbf{R}$ with*

$$|\nabla \ell| = 1.$$

Proof. Denote the geodesic line by γ. Define $\gamma_\pm \colon [0, \infty) \to M$ by

$$\gamma_\pm(t) \coloneqq \gamma(\pm t).$$

Denote by $b_{\gamma_\pm}$ the Busemann function associated with $\gamma_\pm$. Both $\ell = b_{\gamma_\pm}$ have the asserted properties. To prove this we proceed as follows

Set

$$b \coloneqq b_{\gamma_+} + b_{\gamma_-}.$$

By construction,

$$b(\gamma(t)) = 0.$$

By the triangle inequality,

$$b(x) = \lim_{t\to\infty} \left(d(\gamma(-t), x) + d(x, \gamma(t)) - 2t\right) \geqslant 0.$$

By Proposition 25.9,

$$\Delta b \geqslant 0.$$

Therefore and by the maximum principle,

$$b = 0; \quad \text{that is:} \quad b_{\gamma_+} = -b_{\gamma_-}.$$

Therefore, $b_{\gamma_\pm}$ is harmonic.

Let $x \in M$. Let $\delta_\pm$ be geodesic rays emanating from x, constructed as in the proof of Proposition 25.3. By (25.10),

$$b^t_{\delta_+}(y) \geqslant b_{\gamma_+}(y) - b_{\gamma_+}(x) = b_{\gamma_-}(x) - b_{\gamma_-}(y) \geqslant -b^t_{\delta_-}(y).$$

Therefore and since $b^t_{\delta_\pm}(x) = 0$, for all $v \in T_xM$,

$$\langle \nabla b^t_{\delta_+}(x) + \nabla b^+_{\delta_-}(x), v\rangle = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left[b^t_{\delta_+}(\exp_x(\varepsilon v)) + b^t_{\delta_-}(\exp_x(\varepsilon v))\right] \geqslant 0.$$

This implies

$$\nabla b^t_{\delta_+} = -\nabla b^+_{\delta_-} \quad \text{and thus} \quad b^t_{\delta_+} = -b^t_{\delta_-}.$$

Plugging this back into the above series of inequalities shows that

$$b_{\gamma_\pm} = b^t_{\delta_\pm} + b_{\gamma_\pm}(x).$$

This completes the proof since

$$|\nabla b_{\gamma_\pm}| = |\nabla b^t_{\delta_\pm}| = 1.$$

□

Proposition 26.3. *Let (M, g) be a complete Riemannian manifold. Suppose $\ell \in C^\infty(M)$ satisfies*

$$|\nabla \ell| = 1 \quad \textit{and} \quad \operatorname{Hess} \ell = 0.$$

Set

$$N := \ell^{-1}(0) \quad \textit{and} \quad h = g|_N.$$

The map $(\mathbf{R} \times N, \mathrm{d}t \otimes \mathrm{d}t) \to (M, g)$ defined by

$$(t, x) \mapsto \exp_x(t\nabla \ell)$$

is an isometry.

Exercise 26.4. Prove Proposition 26.3

Proof of Theorem 26.1. Proposition 26.2 provides us with a harmonic function $\ell\colon M \to \mathbf{R}$ with $|\nabla \ell| = 1$. By Proposition 18.5,

$$0 = |\operatorname{Hess} \ell|^2 + \operatorname{Ric}(\nabla \ell, \nabla \ell).$$

Therefore and since $\operatorname{Ric} \geqslant 0$,

$$\nabla\nabla \ell = 0.$$

The assertion thus follows from Proposition 26.3. □

Exercise 26.5 (Gallot). Prove that if (M, g) is closed Riemannian manifold with $\operatorname{Ric}_g \geqslant 0$ and $b_1(M) = \dim M$, then it is isometric to a flat torus.

Exercise 26.6 (Gallot). Prove that if (M, g) is closed Riemannian manifold with $\operatorname{Ric}_g \leqslant 0$ and $\dim \cong (M) = \dim M$, then it is isometric to a flat torus.

Definition 26.7. A subgroup $B_n \subset \mathrm{O}(n) \ltimes \mathbf{R}^n$ is called a **Bieberbach group** if it acts freely on $\mathbf{R}^n$ and $\mathbf{R}^n / B_n$ is compact.

Remark 26.8. $\mathbf{Z}^n$ obviously is a Bieberbach group. The group B_2 generated by

$$(x, y) \mapsto (x + 1/2, -y) \quad \text{and} \quad (x, y) \mapsto (x, y + 1)$$

also is a Bieberbach group. What is $\mathbf{R}^2 / B_2$?

Theorem 26.9 (Bieberbach). *Every Bieberbach group B_k contains $\mathbf{Z}_k$ as finite index subgroup.*

Theorem 26.10 (Structure Theorem for Nonnegative Ricci Curvature [CG71]). *If (M, g) is a closed Riemannian manifold with $\operatorname{Ric}_g \geqslant 0$, then the following hold:*

1. *The universal cover $\tilde{M}$ is isometric to $\mathbf{R}^k \times N$ with N compact.*
2. *There is a finite group G, a Bieberbach group B_k, and an exact sequence*

$$0 \to G \to \pi_1(M) \to B_k \to 0.$$

Proof. By Theorem 26.1, $\tilde{M} \cong \mathbf{R}^k \times N$ with N containing no geodesic lines. Every geodesic line in $\tilde{M}$ must of the form $t \mapsto (\gamma(t), x)$ If $g \in \mathrm{Deck}(M)$, then it is an isometry of $\tilde{M}$ and thus maps geodesic lines to geodesic lines. Therefore, $\mathrm{Deck}(M) \subset \mathrm{Iso}(\mathbf{R}^k) \times \mathrm{Iso}(N)$. Since M is compact, there is a compact subset $K \subset \tilde{M}$ with

$$\mathrm{Deck}(M) \cdot K = \tilde{M} \quad \text{and thus} \quad \mathrm{Deck}(M) \cdot \pi_{\mathbf{R}^k}(K) = \mathbf{R}^k \quad \text{and} \quad \mathrm{Deck}(M) \cdot \pi_N(K) = N.$$

To prove (1), observe that if N were not compact, then it would contain a geodesic ray γ. By the above, there is a sequence $g_n \in \mathrm{Deck}(M)$ such that $g_n(\gamma(n)) \in \pi_N(K)$. Since K and S^{n-1} are compact, after passing to a subsequence, $g_n(\gamma(n))$ converges to a limit $x \in \pi_N(K)$ and $(g_n)_*(\dot{\gamma}(n))$ converges to a limit $v \in T_x M$. This shows that the geodesics $\gamma_n \colon [-n, \infty) \to N$ defined by

$$\gamma_n(t) = g_n(\gamma(n+t))$$

converges to the geodesic $\gamma_\infty \colon \mathbf{R} \to N$ defined by $\gamma_\infty(t) \coloneqq \exp_x(tv)$. By construction, γ_∞ is a geodesic line in N: a contradiction.

It remains to prove (2). Set

$$B_k \coloneqq \mathrm{im}(\mathrm{Deck}(M) \to \mathrm{Iso}(\mathbf{R}^k)) \quad \text{and} \quad G \coloneqq \ker(\mathrm{Deck}(M) \to \mathrm{Iso}(\mathbf{R}^k)).$$

By construction, B_k acts freely. Since $B_k \cdot K = \mathbf{R}^k$, B_k is a Bieberbach group. G acts discretely on N; hence, is finite because N is compact. □

27 Poincaré and Sobolev inequalities

See Petersen [Pet16] for proofs.

Theorem 27.1 (Neumann–Poincaré–Sobolev inequality). *Let $\kappa \leqslant 0$ and $D \geqslant 0$. Let (M, g) be a Riemannian manifold of dimension n with*

$$\mathrm{Ric}_g \geqslant (n-1)\kappa g.$$

For all $f \in C^\infty(M)$, $x \in M$, and $0 < r \leqslant D$,

$$\|f - \bar{f}_{x,r}\|_{L^{\frac{n}{n-1}}(B_x(r))} \leqslant c(n, \kappa D^2)\|\nabla f\|_{L^1(B_x(r))}. \tag{27.2}$$

Theorem 27.3 (Sobolev inequality). *Let $\kappa \leqslant 0$ and $D \geqslant 0$. Let (M, g) be a Riemannian manifold of dimension n with*

$$\mathrm{Ric}_g \geqslant (n-1)\kappa g.$$

For all $f \in C^\infty(M)$, $x \in M$, $0 < r \leqslant D$, and $p \in [1, n)$,

$$\|f\|_{L^{\frac{pn}{n-p}}(B_x(r))} \leqslant \frac{p(n-1)}{n-p} c(n, \kappa D^2)\|\nabla f\|_{L^p(B_x(r))} + \|f\|_{L^p(B_x(r))}. \tag{27.4}$$

Remark 27.5. The above shows that uniform lower Ricci bounds do give uniform upper bounds on Poincaré constants, Sobolev constants, etc. However, there are situation in which uniform lower Ricci bounds are not available, but Poincaré upper bounds can be established, for example, using discretization techniques [GS05].

28 Betti number bounds

Theorem 28.1 (Gromov, Gallot). *Let $\kappa \leqslant 0$ and $D \geqslant 0$. If (M, g) is a closed Riemannian manifold of dimension $n \geqslant 3$ with*

$$\mathrm{Ric}_g \geqslant (n-1)\kappa g \quad \textit{and} \quad \operatorname{diam} M \leqslant D,$$

then

$$b_1(M) \leqslant c(n, \kappa D^2). \tag{28.2}$$

Remark 28.3. This result complements Theorem 15.1.

The following propositions prepare the proof of Theorem 28.1.

Proposition 28.4 (P. Li). *Let (M, g) be a closed Riemannian manifold and let E be a Euclidean vector bundle over M. If $V \subset \Gamma(E)$ is a linear subspace, then*

$$\dim V \leqslant \operatorname{rk} E \cdot \sup\{\|s\|_{L^\infty} : s \in V \textit{ with } \|s\|_{L^2} = 1\}^2.$$

Proof. Without loss of generality, V is finite dimensional, Set $m := \dim V$ and $r := \operatorname{rk} E$. Let $s_1, \ldots, s_m$ be a L^2–orthonormal basis of V. Set

$$f(x) := \sum_{a=1}^{m} |s_a(x)|^2.$$

This function only depends on V and not on the choice of $s_1, \ldots, s_m$. By construction,

$$m = \fint_M f \leqslant \max_{x \in M} f(x).$$

Let $x_\star \in M$ be a point at which f achieves its maximum. Without loss of generality, the image of the restriction map $V \to E_{x_\star}$ is spanned by $s_1, \ldots, s_r$. Therefore,

$$m \leqslant \sum_{a=1}^{r} |s_a(x_\star)|^2 \leqslant r \cdot \sup\{\|s\|_{L^\infty} : s \in V \text{ with } \|s\|_{L^2} = 1\}^2. \qquad \square$$

Proposition 28.5. *Let (M, g) be a closed Riemannian manifold. Let $\nu > 1$. Suppose that, for all $f \in C^\infty(M)$,*

$$\|f\|_{L^\nu} \leqslant c\|\nabla f\|_{L^2} + \|f\|_{L^2}.$$

If $\lambda \geqslant 0$ and $f \in C^\infty(M)$ satisfies

$$\Delta f \leqslant \lambda f,$$

then

$$\|f\|_{L^\infty} \leqslant \exp\left(\frac{c\sqrt{\lambda\nu}}{\sqrt{\nu} - 1}\right)\|f\|_{L^2}.$$

Proof. Moser iteration. □

Proof of Theorem 28.1. By Proposition 28.4 applied to

$$\mathscr{H}^1(M, g) := \{\alpha \in \Omega^1(M) : \Delta\alpha = 0\},$$

it suffices derive an L^∞ bound for harmonic 1–form. By Proposition 15.2 and hypothesis, for $\alpha \in \mathscr{H}^1(M, g)$,

$$\Delta|\alpha|^2 \leqslant 2(n-1)|\kappa||\alpha|^2.$$

The desired L^∞ bound thus follows from Proposition 28.5. □

Exercise 28.6. What is the generalization of Theorem 28.1 to b_k? What replaces the Ricci lower bound?

29 Metric spaces

Definition 29.1. A metric space (X, d) is called **separable** if it contains a countable, dense subset.

Definition 29.2. A metric space (X, d) is called **totally bounded** if for every $\varepsilon > 0$ there is a finite collection of balls of radius ε covering X.

Theorem 29.3. *A metric space (X, d) is compact if and only if it is complete and totally bounded.*

Exercise 29.4. Prove Theorem 29.3.

Theorem 29.5. *Every totally bounded metric space is separable.*

Exercise 29.6. Prove Theorem 29.5.

Theorem 29.7 (Kuratowski). *Let (X, d) be a metric space. For every $x_\star \in X$, the $\delta\colon X \to L^\infty(X)$ defined by*

$$\delta(x) := d(x, \cdot) - d(x_\star, \cdot)$$

is an isometric embedding.

Proof. This immediately follows from the fact that, for $x, y \in X$,

$$\|\delta(x) - \delta(y)\|_{L^\infty} = \sup_{z \in X} d(x,z) - d(z,y) = d(x,y).$$ □

Theorem 29.8. *Every separable metric space (X,d) admits an isometric embedding in $\ell^\infty(\mathbf{N})$.*

Proof. By Theorem 29.7, every countable subset of X admits an isometric embedding into $\ell^\infty(\mathbf{N})$. This extends extends to an isometric embedding of X if the subset is dense. □

30 Hausdorff distance

Definition 30.1. Let X be a set. The **power set** of X is the set of all subsets of X and denoted by $\mathfrak{P}(X)$.

Definition 30.2. Let (X,d) be a metric space. The **Hausdorff distance** is the map $d_H = d_H^X \colon \mathfrak{P}(X) \times \mathfrak{P}(X) \to [0,\infty]$ defined by

$$d_H(A,B) := \inf\{\varepsilon > 0 : A \subset B_\varepsilon(B) \text{ and } B \subset B_\varepsilon(A)\}.$$

Proposition 30.3. *Let (X,d) be a metric space. Denote by $\mathfrak{C}(X) \subset \mathfrak{P}(X)$ the set of all closed subsets of X. Then $(\mathfrak{C}(X), d_H)$ is a metric space.*

Exercise 30.4. Prove Proposition 30.3.

Theorem 30.5. *If (X,d) is a complete metric space, then $(\mathfrak{C}(X), d_H)$ is complete.*

Proof. Let $(A_n)_{n\in\mathbf{N}}$ be a Cauchy sequence in $(\mathfrak{C}(X), d_H)$. Set

$$A_\infty := \bigcap_{n=1}^\infty \overline{\bigcup_{m=n}^\infty A_m}.$$

Let $\varepsilon > 0$. Let $N \in \mathbf{N}$ such that, for $n,m \geqslant N$, $d_H(A_n, A_m) \leqslant \varepsilon$. By definition, for every $x \in A_\infty$, there is an $m \geqslant N$ such that $x \in B_\varepsilon(A_m)$. Therefore and since $x \in A_\infty$ was arbitrary, for every $n \geqslant N$, $A_\infty \subset B_{2\varepsilon}(A_n)$

Let $x \in A_n$ with $n \geqslant N$. Choose $(n_k)_{k\in\mathbf{N}}$ with $N_1 = n$ such that, for every $k \in \mathbf{N}$, $d_H(A_{n_k}, A_{n_{k+1}}) < \varepsilon/2^k$. Furthermore, choose $(x_k)_{k\in\mathbf{N}}$ with $x_1 = x$, $x_k \in A_{n_k}$, and such that, for every $k \in \mathbf{N}$, $d(x_k, x_{k+1}) < \varepsilon/2^k$. Since X is complete, the Cauchy sequence $(x_k)_{k\in\mathbf{N}}$ converges to a limit x_∞. By definition, $x_\infty \in A_\infty$. Since

$$d(x, x_\infty) = \sum_{k=1}^\infty \varepsilon/2^k \leqslant \varepsilon$$

and $x \in A_n$ was arbitrary, for every $n \geqslant N$, $A_n \subset B_{2\varepsilon}(A_\infty)$. □

Theorem 30.6. *If (X,d) is a compact metric space, then $(\mathfrak{C}(X), d_H)$ is compact.*

Exercise 30.7. Prove Theorem 30.6.

31 The Gromov–Hausdorff distance

The theory of Gromov–Hausdorff distance was invented by Edwards [Edw75] and then invented again by Gromov [Gro81b].

Definition 31.1 ([Gro81b, Section 6; Gro07, Chapter 3.A]). Let (X, d_X) and (Y, d_Y) be two metric spaces. The **Gromov–Hausdorff distance** between (X, d_X) and (Y, d_X) is denoted by

$$d_{GH}(X, Y)$$

and defined as the infimum of the numbers

$$d_H^Z(i(X), j(Y))$$

for all metric spaces (Z, d_Z) and all isometric embeddings $i\colon X \to Z$ and $j\colon Y \to Z$.

Proposition 31.2. *Let (X, d_X) and (Y, d_Y) be metric spaces and let $\varepsilon > 0$. If $d_{GH}(X, Y) < \varepsilon$, then there are maps $f\colon X \to Y$ and $g\colon Y \to X$ such that:*

1. *For every $x_1, x_2 \in X$, $y_1, y_2 \in Y$:*

$$\begin{aligned} |d_X(x_1, x_2) - d_Y(f(x_1), f(x_2))| &\leqslant 2\varepsilon \quad \text{and} \\ |d_Y(y_1, y_2) - d_X(g(y_1), g(y_2))| &\leqslant 2\varepsilon. \end{aligned} \tag{31.3}$$

2. *For every $\varepsilon > 0$, $x \in X$, $y \in Y$:*

$$\begin{aligned} d_X(x, g(f(x))) &\leqslant 2\varepsilon \quad \text{and} \\ d_Y(y, f(g(y))) &\leqslant 2\varepsilon. \end{aligned} \tag{31.4}$$

Proof. Let (Z, d_Z) be a metric space and let $i\colon X \to Z$ and $j\colon Y \to Z$ be isometric embeddings such that

$$d_H^Z(i(X), j(Y)) \leqslant \varepsilon.$$

By definition, there are maps $f\colon X \to Y$ and $g\colon Y \to X$ such that, for every $x \in X$ and $y \in Y$,

$$d_Z(x, f(x)) \leqslant \varepsilon \quad \text{and} \quad d_Z(y, g(y)) \leqslant \varepsilon.$$

By the triangle, (31.3) and (31.4) hold. □

Proposition 31.5. *If (X, d_X) and (Y, d_Y) are separable, then $d_{GH}(X, Y)$ is equal to the infimum of the numbers*

$$d_H^{\ell^\infty(\mathbf{N})}(i(X), j(Y))$$

for all isometric embeddings $i\colon X \to \ell^\infty(\mathbf{N})$ and $j\colon Y \to \ell^\infty(\mathbf{N})$.

Proof. This is a consequence of Theorem 29.8. □

Proposition 31.6. *The Gromov–Hausdorff distance satisfies the triangle inequality: If X, Y, Z are metric spaces, then*

$$d_{GH}(X,Z) \leqslant d_{GH}(X,Y) + d_{GH}(Y,Z).$$

Exercise 31.7. Prove Proposition 31.6.

Proposition 31.8. *Two compact metric space (X, d_X) and (Y, d_Y) are isometric if and only $d_{GH}(X,Y) = 0$.*

Proof. If (X, d_X) and (Y, d_Y), then trivially $d_{GH}(X,Y) = 0$.

If $d_{GH}(X,Y) = 0$, then, for every $\varepsilon > 0$, there are $f_\varepsilon\colon X \to Y$ and $g_\varepsilon\colon Y \to X$ as in Proposition 31.2. Let $\Gamma \subset X$ and $\Delta \subset Y$ be countable, dense subsets. Since X and Y are compact and by a diagonal sequence argument, there is a null-sequence $(\varepsilon_n)_{n\in\mathbf{N}}$ such that, for every $x \in \Gamma$ and $y \in \Delta$, the limits

$$f(x) \coloneqq \lim_{n\to\infty} f_{\varepsilon_n}(x) \quad \text{and} \quad g(y) \coloneqq \lim_{n\to\infty} g_{\varepsilon_n}(y)$$

exist. By (31.3), the maps $f\colon \Gamma \to Y$ and $g\colon \Delta \to X$ are isometric embeddings, and extend to isometric embeddings $f\colon X \to Y$ and $g\colon Y \to X$. By (31.4), f and g are mutual inverses. Therefore, X and Y are isometric. □

Definition 31.9. Denote by $\mathfrak{M}$ the set of isometry classes of non-empty, compact metric spaces. The metric space $(\mathfrak{M}, d_{GH})$ is called **Gromov–Hausdorff space**.

Theorem 31.10 (Edwards [Edw75, Theorems III.3 and III.7])**.** *$(\mathfrak{M}, d_{GH})$ is separable and complete.*

Proof. The subset of finite metric spaces (X, d) with d taking only rational values is dense in $(\mathfrak{M}, d_{GH})$. Therefore, $(\mathfrak{M}, d_{GH})$ is separable.

Let $(X_n, d_{X_n})_{n\in\mathbf{N}}$ be a Cauchy sequence in $(\mathfrak{M}, d_{GH})$. By Proposition 31.5, there are isometric embeddings $i_n\colon X_n \to \ell^\infty(\mathbf{N})$ such that $A_n \coloneqq i_n(X_n)$ is a Cauchy sequence in $\mathfrak{C}(\ell^\infty(\mathbf{N}))$. By Theorem 30.5, $(A_n)_{n\in\mathbf{N}}$ converges to a limit A_∞ in $\mathfrak{C}(\ell^\infty(\mathbf{N}))$. A_∞ is complete and totally bounded; hence, compact. By Proposition 31.5, X_n converges to $X_\infty \coloneqq A_\infty$ in $(\mathfrak{M}, d_{GH})$. □

Exercise 31.11. Prove that $(\mathfrak{M}, d_{GH})$ is contractible, has infinite diameter, is a length space, and is not locally compact.

Definition 31.12. Let $\mathfrak{X}$ be a set of isometry classes of metric spaces. $\mathfrak{X}$ is **uniformly bounded** if $\sup\{\operatorname{diam}(X,d) : [X,d] \in \mathfrak{X}\} < \infty$. $\mathfrak{X}$ is **uniformly totally bounded** if, for every $\varepsilon > 0$, there is an $n \in \mathbf{N}$ such that every $[X,d] \in \mathfrak{X}$ can be covered by at most n balls of radius ε.

Definition 31.13. Let (X,d) be a metric space and $A \subset X$. A is **relatively compact** if $\bar{A}$ is compact.

Theorem 31.14 (Gromov's compactness criterion [Gro81b, p.64]). *A subset $\mathfrak{X} \subset \mathfrak{M}$ is relatively compact if and only if it is uniformly bounded and uniformly totally bounded.*

Proof. By Theorem 31.10, it suffices to show that $\mathfrak{X}$ is totally bounded. Since $\mathfrak{X}$ is uniformly bounded,

$$D := \sup\{\operatorname{diam}(X, d) : [X, d] \in \mathfrak{X}\} < \infty$$

Let $\varepsilon > 0$. Since $\mathfrak{X}$ is uniformly totally bounded, there is an $n \in \mathbf{N}$ such that every $[X, d] \in \mathfrak{X}$ contains n points $x_1, \dots, x_n$ such that

$$X \subset B_\varepsilon(x_1) \cup \cdots \cup B_\varepsilon(x_n).$$

Therefore, (X, d) and $(\{x_1, \dots, x_n\}, d)$ have Gromov–Hausdorff distance at most ε. There is a metric $\tilde{d}$ on $\{1, \dots, n\}$ taking values in $\{0, \varepsilon, 2\varepsilon, \dots, \lceil D/\varepsilon \rceil \varepsilon\}$ and such that, for $a, b = 1, \dots, n$,

$$|d(x_a, x_b) - \tilde{d}(a, b)| \leqslant \varepsilon.$$

By construction, (X, d) and $(\{1, \dots, n\}, \tilde{d})$ have Gromov–Hausdorff distance at most 2ε. Since there are finitely many metrics $\tilde{d}$ on $\{1, \dots, n\}$ as above and $[X, d]$ was arbitrary, it follows that $\mathfrak{X}$ is totally bounded. □

Definition 31.15. A metric space (X, d) is said to have **doubling constant** at most $c > 0$ if, for every $x \in X$ and $r > 0$, $B_{2r}(x)$ can be covered by at most c balls of radius r.

Proposition 31.16. *Let $\mathfrak{X} \subset \mathfrak{M}$ be uniformly bounded. If there exists a constant $c > 0$ such that every $[X, d] \in \mathfrak{X}$ has doubling constant at most c, then $\mathfrak{X}$ is uniformly totally bounded.*

Exercise 31.17. Prove Proposition 31.16.

Theorem 31.18 ([Gro07, Theorem 5.3]). *Let $\kappa \in \mathbf{R}$, $r > 0$, and $n > 0$. There is a constant $c = c(r^2\kappa, n) > 0$ such that if (M, g) is a complete Riemannian manifold of dimension n with*

$$\operatorname{Ric}_g \geqslant (n-1)\kappa g,$$

then every ball $B_{2r}(x)$ can be covered by at most c balls of radius r.

Proof. Given $B_{4r}(x)$, let $x_1, \dots, x_m \in B_{4r}(x)$ such that

$$B_{4r}(x) \subset \bigcup_{a=1}^{m} B_{2r}(x_a)$$

and, for $a \neq b$,

$$B_r(x_a) \cap B_r(x_b) = \varnothing.$$

Denote by $x_\star$ the center of the ball $B_r(x_a)$ with smallest volume. By construction and by Theorem 19.1,

$$\begin{aligned} m &\leqslant \frac{\mathrm{vol}(B_{4r}(x))}{\mathrm{vol}(B_r(x_\star))} \\ &\leqslant \frac{\mathrm{vol}(B_{8r}(x_\star))}{\mathrm{vol}(B_r(x_\star))} \\ &\leqslant \frac{V_\kappa^n(8r)}{V_\kappa^n(r)} \\ &= \frac{V_{r^2\kappa}^n(8)}{V_{r^2\kappa}^n(1)} =: c(r^2\kappa, n). \end{aligned}$$

□

Corollary 31.19. *Let $\kappa \in \mathbf{R}$, $D > 0$, and $n > 0$. The subset of isometry classes of complete Riemannian manifolds (M, g) of dimension n with*

$$\mathrm{diam}(M) \leqslant D \quad \textit{and} \quad \mathrm{Ric}_g \geqslant (n-1)\kappa g$$

is relatively compact in $(\mathfrak{M}, d_{GH})$.

32 Pointed Gromov–Hausdorff convergence

Definition 32.1. A **pointed metric space** is a metric space (X, d) together with a point $x \in X$.

Definition 32.2. Let (X, x, d_X) and (Y, y, d_Y) be two pointed metric spaces. The **uniform pointed Gromov–Hausdorff distance** between (X, x, d_X) and (Y, y, d_X) is denoted by

$$d_{pGH}(X, Y)$$

and defined as the infimum of the numbers

$$d_H^Z(i(X), j(Y))$$

for all pointed metric spaces (Z, z, d_Z) and all pointed isometric embeddings $i\colon X \to Z$ and $j\colon Y \to Z$.

For non-compact spaces, it is too restrictive to demand convergence with respect to d_{upGH}.

Definition 32.3. A metric space is called **proper** if every closed ball is compact.

Definition 32.4. Denote by $\mathfrak{M}_\star$ the set of isometry classes of proper, pointed metric spaces. The **pointed Gromov–Hausdorff topology** is the topology on $\mathfrak{M}_\star$ generated by the subbasis consisting of the subsets

$$\left\{[Y, y, d_Y] \in \mathfrak{M}_\star : \text{there is an } s > 0 \text{ with } |r - s| < \delta \text{ such that } d_{pGH}(B_s(x), B_r(y)) < \varepsilon\right\}$$

for $[X, x, d_X] \in \mathfrak{M}_\star$ and $r, \delta, \varepsilon > 0$.

Proposition 32.5. *Let $(X_n, x_n, d_{X_n})_{n\in\mathbf{N}}$ be a sequence of proper, pointed metric spaces and let $(X_\infty, x_\infty, d_{X_\infty})$ be a proper, pointed metric space. In the pointed Gromov–Hausdorff topology,*

$$\lim_{n\to\infty} [X_n, x_n, d_{X_n}] = [X_\infty, x_\infty, d_{X_\infty}]$$

if and only if, for every $r > 0$, there is a sequence $(r_n)_{n\in\mathbf{N}}$ converging to r such that

$$\lim_{n\to\infty} d_{pGH}(B_{r_n}(x_n), B_r(x_\infty)) = 0.$$

Remark 32.6. The variation of r is needed to make sure that $((1 + 1/n) \cdot \mathbf{Z}, 0)$ converges to $(\mathbf{Z}, 0)$, etc.

Theorem 32.7 (Gromov's compactness criterion for the pointed Gromov–Hausdorff topology [Gro81b, p.64]). *A subset $\mathfrak{X} \subset \mathfrak{M}_\star$ is relatively sequentially compact if, for every $r > 0$, the set $\{B_r(x) : [X, x, d_X] \in \mathfrak{X}\}$ is uniformly totally bounded.*

33 Weyl curvature tensor

Definition 33.1. The **Kulkarni–Nomizu product** is defined as

$$(h \owedge k)(u, v, w, z) := h(u, z)k(v, w) + h(v, w)k(u, z) - h(u, w)k(v, z) - h(v, z)k(u, w).$$

Proposition 33.2.

1. *If $e_1, \ldots, e_n$ is an orthonormal basis, then*

$$\sum_{a=1}^{n} (h \owedge g)(e_a, v, w, e_a) = (n - 2)h(v, w) + \operatorname{tr} h \cdot \langle v, w\rangle.$$

2. *$(g \owedge g)(u, v, w, z) = -2\langle u \wedge v, w \wedge z\rangle$.*

Proof. If $e_1, \dots, e_n$ is an orthonormal basis, then

$$\begin{aligned}
&\sum_{a=1}^{n} (h \owedge g)(e_a, e_b, e_b, e_a) \\
&\quad = \sum_{a=1}^{n} h(e_a, e_a)g(e_b, e_b) + h(e_b, e_b)g(e_a, e_a) - h(e_a, e_b)g(e_b, e_a) - h(e_b, e_a)g(e_a, e_b) \\
&\quad = \sum_{a=1}^{n} h(e_a, e_a) + (n-2)h(e_b, e_b).
\end{aligned}$$

This implies the assertion. □

Proposition 33.3.

Definition 33.4. The **Weyl curvature tensor** $W \in \Gamma(S^2\Lambda^2 T^*M)$ of (M, g) is defined by

$$W(u, v, w, z) := \langle R(u, v)w, z\rangle - \frac{1}{n-2}(\mathrm{Ric}_g^\circ \owedge g)(u, v, w, z) - \frac{\mathrm{scal}_g}{2n(n-1)}(g \owedge g)(u, v, w, z).$$

Definition 33.5. In dimension 4, the **self-dual Weyl curvature tensor** $W^+ \in \Gamma(S^2\Lambda^+ T^*M)$ and the **anti-self-dual Weyl curvature tensor** $W^- \in \Gamma(S^2\Lambda^- T^*M)$ are the corresponding components of W.

34 Hitchin–Thorpe inequality

Theorem 34.1 (Hitchin–Thorpe inequality). *If (M, g) is a closed, oriented 4–manifold, then*

$$2\chi(M) \pm 3\sigma(M) = \frac{1}{4\pi^2}\int_M 2|W_\pm|^2 - |\mathrm{Ric}_g^\circ| + \frac{\mathrm{scal}_g^2}{24}. \tag{34.2}$$

In particular, if g is Einstein,

$$2\chi(M) \pm 3\sigma(M) = \frac{1}{4\pi^2}\int_M 2|W_\pm|^2 + \frac{\mathrm{scal}_g^2}{24} \geqslant 0.$$

Equality holds in (34.1) if and only if g is flat or the universal cover of (M, g) is a K3 surface equipped with a Ricci-flat metric.

This is a direct consequence of the following (tedious to prove) Chern–Weil formulae.

Proposition 34.3.

$$\chi(M) = \frac{1}{8\pi^2}\int_M |W|^2 - |\mathrm{Ric}_g^\circ|^2 + \frac{\mathrm{scal}_g^2}{24}.$$

Proposition 34.4.
$$\sigma(M) = \frac{1}{12\pi^2}\int_M |W_+|^2 - |W_-|^2.$$

35 Hyperkähler manifolds

Definition 35.1. A **hyperkähler manifold** is a Riemannian manifold (X, g) together with a triple of complex structures I, J, K with respect to which g is a Kähler metric and which satisfy the relations

$$IJ = K = -IJ. \tag{35.2}$$

Proposition 35.3. *If (X, g, I, J, K) is a hyperkähler manifold, then g is Ricci flat.*

Proposition 35.4 (Hitchin [Hit92, Theorem 2])**.** *Let (X, g) be a Riemannian manifold together with a triple of almost complex structures I, J, K which are compatible with g and which satisfy (35.2). If the 2–forms*

$$\omega_I \coloneqq g(I\cdot,\cdot), \quad \omega_J \coloneqq g(J\cdot,\cdot), \quad \textit{and} \quad \omega_K \coloneqq g(K\cdot,\cdot) \tag{35.5}$$

are closed, then (X, g, I, J, K) is a hyperkähler manifold.

Proof. We need to prove that I, J, and K are integrable.

Set $n \coloneqq \dim X/2$ and define

$$\theta_I \coloneqq (\omega_J + i\omega_K)^{n/2}. \tag{35.6}$$

We have

$$TM_I^{0,1} = \ker\left(i(\cdot)\theta_I \colon TM \otimes \mathbf{C} \to \Lambda_{\mathbf{C}}^{n-1} TM_I^{1,0}\right). \tag{35.7}$$

Therefore, for $v, w \in \Gamma(T_I^{0,1}M)$,

$$i([u,v])\theta_I = i(u)i(v)\mathrm{d}\theta_I = 0. \tag{35.8}$$

This shows that $[v, w] \in \Gamma(T_I^{0,1}M)$. By the Newlander–Nirenberg Theorem, I is integrable. The same argument proves that J and K are integrable as well. □

36 The Gibbons–Hawking ansatz

Let U be an open subset of $\mathbf{R}^3$. Denote by $g_{\mathbf{R}^3}$ the restriction of the standard metric on $\mathbf{R}^3$ to U. Let $\pi\colon X \to U$ be a principal $\mathrm{U}(1)$–bundle. Denote by $\partial_\alpha \in \mathrm{Vect}(X)$ the generator of the

U(1)–action. Let $i\theta \in \Omega^1(X, i\mathbf{R})$ be a U(1)–connection 1–form and let $f \in C^\infty(U, (0,\infty))$ be a positive smooth function such that

$$\mathrm{d}\theta = -*_3\,\mathrm{d}f. \tag{36.1}$$

Set

$$g := f\pi^* g_{\mathbf{R}^3} + \frac{1}{f}\theta \otimes \theta \tag{36.2}$$

and define complex structures I_1, I_2, I_3 by

$$I_i\partial_\alpha = f^{-1}\partial_{x_i} \quad \text{and} \quad I_i\partial_{x_j} = \sum_{k=1}^{3} \varepsilon_{ijk}\partial_{x_k}. \tag{36.3}$$

The corresponding Hermitian forms are

$$\omega_i := \mathrm{d}\theta \wedge \mathrm{d}x_i + \frac{1}{2}\sum_{j,k=1}^{3} \varepsilon_{ijk} f\,\mathrm{d}x_j \wedge \mathrm{d}x_k. \tag{36.4}$$

Writing (36.1) as

$$\mathrm{d}\theta = -\frac{1}{2}\sum_{\ell,j,k=1}^{3} \varepsilon_{\ell jk}\partial_{x_\ell} f\,\mathrm{d}x_j \wedge \mathrm{d}x_k,$$

we see that

$$\mathrm{d}\omega_i = \mathrm{d}\theta \wedge \mathrm{d}x_i + \frac{1}{2}\sum_{j,k=1}^{3} \varepsilon_{ijk}\mathrm{d}f \wedge \mathrm{d}x_j \wedge \mathrm{d}x_k = 0. \tag{36.5}$$

Therefore, we have proved the following.

Proposition 36.6. *(X, g, I_1, I_2, I_3) is hyperkähler manifold.*

This construction is called the **Gibbons–Hawking ansatz**.

Remark 36.7. By construction, the length of the U(1)–orbit over $x \in U$ is $f(x)^{-1/2}$.

Remark 36.8. The fact that

$$i(\partial_\alpha)\omega_i = -\mathrm{d}x_i \tag{36.9}$$

means that the map $\pi\colon X \to U \subset \mathbf{R}^3$ is a hyperkähler moment map for the action of U(1) on X (with $\mathbf{R}^3$ and $(\mathfrak{u}(1) \otimes \operatorname{Im}\mathbf{H})^*$ identified suitably.

Remark 36.10. By (36.1),

$$\Delta f = 0. \tag{36.11}$$

Conversely, suppose that $f\colon U \to \mathbf{R}$ is harmonic and the cohomology class of $*_3 \mathrm{d}f$ lies in $\mathrm{im}(H^2(U, 2\pi\mathbf{Z}) \to H^2(U, \mathbf{R}))$, then there is a U(1)–bundle X over U together a connection $i\theta$ satisfying

$$\mathrm{d}\theta = - *_3 \mathrm{d}f. \tag{36.12}$$

Example 36.13 ($\mathbf{R}^4$). Let $U = \mathbf{R}^3 \backslash \{0\}$ and define $f\colon U \to \mathbf{R}$ by

$$f(x) = \frac{1}{2|x|}. \tag{36.14}$$

This function is harmonic and satisfies

$$- *_3 \mathrm{d}f = \frac{1}{2}\mathrm{vol}_{S^2}. \tag{36.15}$$

Since $\mathrm{vol}(S^2) = 4\pi$, there is a U(1)–bundle X over U together with a connection $i\theta$ such that (36.1). Therefore, the Gibbins–Hawking ansatz yields a hyperkähler metric on X.

By Chern–Weil theory the first Chern number of the restriction of X to S^2 is

$$\int_{S^2} i\frac{i}{4\pi}\mathrm{vol}_{S^2} = -1. \tag{36.16}$$

Up to is isomorphism, there is only one principal U(1)–bundle over S^2: the Hopf bundle $\pi\colon S^3 \to S^2$ and the U(1)–action given by $e^{i\alpha} \cdot (z_0, z_1) = (e^{i\alpha} z_0, e^{i\alpha} z_1)$. If g_{S^3} denotes the standard metric on S^3, then

$$\theta = g_{S^3}(\partial_\alpha, \cdot) \tag{36.17}$$

satisfies

$$\mathrm{d}\theta = \pi^* \mathrm{vol}_{S^2}. \tag{36.18}$$

It follows that

$$X = S^3 \times (0, \infty) = \mathbf{R}^4 \backslash \{0\} \tag{36.19}$$

and the Gibbons–Hawking ansatz gives the metric

$$g = 2r\, \theta \otimes \theta + \frac{1}{2r}(\mathrm{d}r \otimes \mathrm{d}r + r^2 g_{S^2}). \tag{36.20}$$

The change of coordinates $\rho = \sqrt{2r}$ rewrites this metric as

$$g = \mathrm{d}\rho \otimes \mathrm{d}\rho + \rho^2(\theta \otimes \theta + \frac{1}{4} g_{S^2}) = \mathrm{d}\rho \otimes \mathrm{d}\rho + \rho^2 g_{S^3}.$$

This means that the Gibbons–Hawking ansatz yield the standard metric on $\mathbf{R}^4$.

Example 36.21 (Taub–NUT). Let $U = \mathbf{R}^3\backslash\{0\}$, let $c > 0$, and define $f_c\colon U \to \mathbf{R}$ by

$$f_c(x) = \frac{1}{2|x|} + c. \tag{36.22}$$

This function is harmonic and we have

$$\mathrm{d}f_c = \mathrm{d}f. \tag{36.23}$$

By the preceding discussion, $X = S^3 \times (0, \infty)$ and the Gibbons–Hawking ansatz gives the metric

$$g = \left(\frac{1}{2r} + c\right)^{-1} \theta \otimes \theta + \left(\frac{1}{2r} + c\right) (\mathrm{d}r \otimes \mathrm{d}r + r^2 g_{S^2}). \tag{36.24}$$

As r tends to zero this metric is asymptotic to

$$c^{-1}\theta \otimes \theta + g_{\mathbf{R}^3}. \tag{36.25}$$

Although, the metric appears singular at $r = 0$, the coordinate change $\rho = \sqrt{2r}$ rewrites it as

$$(1 + c\rho^2)\mathrm{d}\rho \otimes \mathrm{d}\rho + \rho^2 \left((1 + c\rho^2)^{-1}\theta \otimes \theta + (1 + c\rho^2)\frac{1}{4}g_{S^2}\right) \tag{36.26}$$

which is smooth.

This metric is called the **Taub–NUT metric**. It is non-flat hyperkähler metric on $\mathbf{R}^4$. It was first discovered by Taub [Tau51] and Newman, Tamburino, and Unti [NTU63]. The Taub–NUT space is the archetype of an **ALF space**.

Remark 36.27. It was observed by LeBrun [LeB91] that the Taub–NUT metric is in fact Kähler for the standard complex structure on $\mathbf{C}^2$. Thus it yields a non-flat Ricci-flat Kähler metric on $\mathbf{C}^2$.

Example 36.28 ($(\mathbf{R}^4\backslash\{0\})/\mathbf{Z}_k$). Let $k \in \{1, 2, 3, \ldots\}$ Let $U = \mathbf{R}^3\backslash\{0\}$ and define $f\colon U \to \mathbf{R}$ by

$$f(x) := \frac{k}{2|x|}. \tag{36.29}$$

This function is harmonic and it satisfies

$$-*_3\mathrm{d}f = k\mathrm{vol}_{S^2}. \tag{36.30}$$

Thus, the Gibbons–Hawking ansatz applies. Denote by (X_k, g_k) the Riemannian manifold obtained in this way. If $k = 1$, then this $\mathbf{R}^4$ with its standard metric. Let us understand the cases $k \geqslant 2$.

The restriction of X_k to S^2 has Chern number $-k$. This $\mathrm{U}(1)$–bundle is $S^3/\mathbf{Z}_k \to S^2$. Consequently,

$$X_k = S^3/\mathbf{Z}_k \times (0, \infty) = \mathbf{R}^4/\mathbf{Z}_k.$$

We can choose the connection 1-form $i\theta_k$ on X_k such that its pullback to X_1 is $ik\theta_1$. It follows that the pullback of g_k to X_1 can be written as

$$2kr\,\theta \otimes \theta + \frac{k}{2r}(\mathrm{d}r \otimes \mathrm{d}r + r^2 g_{S^2}). \tag{36.31}$$

Up to a coordinate change $r \mapsto kr$ this is the standard metric on $\mathbf{R}^4$. It follows that g_k is the metric induced by the standard metric on $\mathbf{R}^4$.

Example 36.32 (Eguchi–Hanson and multi-center Gibbons–Hawking). Let $x_1, \ldots, x_k$ be k distinct points in $\mathbf{R}^3$. Set $U := \mathbf{R}^3 \backslash \{x_1, \ldots, x_k\}$ and define $f\colon U \to \mathbf{R}$ by

$$f(x) = \sum_{i=1}^{k} \frac{1}{2|x - x_i|}. \tag{36.33}$$

From the in discussion Example 36.13 it is clear that the Gibbons–Hawking ansatz for f produces a Riemannian manifold whose apparent singularities over $x_1, \ldots, x_k$ can be removed. Denote the resulting manifold by (X, g).

Since

$$f(x) = \frac{k}{2|x|} + O(|x|^{-2}) \quad \text{as} \quad |x| \to \infty,$$

(X, g) is asymptotic at infinity to $\mathbf{R}^4/\mathbf{Z}_k$. These spaces are called **ALE spaces of type** A_{k-1}. For $k = 2$, this metric was discovered by Eguchi and Hanson [EH79]. The metrics for $k \geqslant 3$ were discovered by Gibbons and Hawking [GH78].

Let us understand the geometry and topology of these spaces somewhat more. Suppose γ is an arc in $\mathbf{R}^3$ from x_i to x_j avoiding all the other points x_k. The pre-image in X of any interior point of γ is an S^1 while the pre-images of the end points are points. Therefore,

$$\pi^{-1}(\gamma) \subset X \tag{36.34}$$

is diffeomorphic to S^2. Suppose γ is straight line segment in $\mathbf{R}^3$ with unit tangent vector

$$v = \sum_{i=1}^{3} a_i \partial_{x_i} \tag{36.35}$$

with $a_1^2 + a_2^2 + a_3^2 = 1$. The tangent spaces to $\pi^{-1}(\gamma)$ are spanned by ∂_α and v. In particular, they are invariant with respect to the complex structure

$$I_v := a_1 I_1 + a_2 I_2 + a_3 I_3. \tag{36.36}$$

Its volume is given by

$$\int_{\pi^{-1}(\gamma)} a_1\omega_1 + a_2\omega_2 + a_3\omega_3. \tag{36.37}$$

Therefore,

$$[\pi^{-1}(\gamma)] \neq 0 \in H_2(X, \mathbf{Z}). \tag{36.38}$$

If necessary we can reorder the points x_i so that for $i = 1, \dots, k-1$, there is a straight-line segment γ_i joining x_i and x_{i+1}. Set

$$\Sigma_i := \pi^{-1}(\gamma_i). \tag{36.39}$$

It is not difficult to see that $[\Sigma_1], \dots, [\Sigma_{k-1}]$ generate $H_2(M; \mathbf{Z})$. It is an exercise to show that

$$[\Sigma_i] \cdot [\Sigma_j] = \begin{cases} -2 & \text{if } i = j, \\ 1 & \text{if } i \neq j. \end{cases} \tag{36.40}$$

Remark 36.41. Kronheimer [Kro89b] gave an alternative construction of the ALE spaces of type A_{k-1} (in fact, all ALE spaces) as hyperkähler quotients. He also classified these spaces completely [Kro89a].

Example 36.42. Let $x_1, \dots, x_k$ be k distinct points in $\mathbf{R}^3$ and let $c > 0$. Set $U := \mathbf{R}^3 \backslash \{x_1, \dots, x_k\}$ and define $f\colon U \to \mathbf{R}$ by

$$f(x) = \sum_{i=1}^{k} \frac{1}{2|x - x_i|} + c. \tag{36.43}$$

The Gibbons–Hawking ansatz for f gives rise to the so-called **multi-center Taub–NUT** metric.

Example 36.44. The following is due to Anderson, Kronheimer, and LeBrun [AKL89]. Let $x_1, x_2, \dots$ be an infinite sequence of distinct points in $\mathbf{R}^3$ and denote by U the complement of these points. If

$$\sum_{j=2}^{\infty} \frac{1}{x_1 - x_j} < \infty, \tag{36.45}$$

then

$$f(x) := \sum_{j=1}^{\infty} \frac{1}{2|x - x_j|} \tag{36.46}$$

defines a harmonic function on U. The Gibbons–Hawking ansatz gives rise to a hyperkähler manifold X whose second homology $H_2(X, \mathbf{Z})$ is infinitely generated. Anderson, Kronheimer, and LeBrun prove that the metric g is complete.

This is not a complete list of interesting examples of hyperkähler manifold which can be produced using the Gibbons–Hawking ansatz. The most egregious omission is that of the Ooguri–Vafa metric.

References

[AKL89] M. T. Anderson, P. B. Kronheimer, and C. LeBrun. *Complete Ricci-flat Kähler manifolds of infinite topological type. Communications in Mathematical Physics* 125.4 (1989), pp. 637–642. DOI: 10.1007/BF01228345. MR: 1024931. Zbl: 0734.53051 (cit. on p. 54).

[And90a] M. T. Anderson. *On the topology of complete manifolds of non-negative Ricci curvature. Topology* 29.1 (1990), pp. 41–55. DOI: 10.1016/0040-9383(90)90024-E. MR: 1046624. Zbl: 0696.53027 (cit. on p. 31).

[And90b] M. T. Anderson. *Short geodesics and gravitational instantons. Journal of Differential Geometry* 31.1 (1990), pp. 265–275. MR: 1030673. Zbl: 0696.53029. URL: http://projecteuclid.org/euclid.jdg/1214444097 (cit. on p. 32).

[BC64] R. L. Bishop and R. J. Crittenden. *Geometry of manifolds.* Pure and Applied Mathematics. 1964. MR: 0169148. Zbl: 0132.16003 (cit. on pp. 25, 26).

[Bis63] R. L. Bishop. *A relation between volume, mean curvature, and diameter. Notices Of The American Mathematical Society* 10.4 (1963), p. 364. URL: https://www.ams.org/journals/notices/196306/196306FullIssue.pdf (cit. on p. 26).

[BM11] O. Biquard and V. Minerbe. *A Kummer construction for gravitational instantons. Communications in Mathematical Physics* 308.3 (2011), pp. 773–794. MR: 2855540. Zbl: 1234.53020 (cit. on p. 29).

[Boc46] S. Bochner. *Vector fields and Ricci curvature. Bulletin of the American Mathematical Society* 52 (1946), pp. 776–797. DOI: 10.1090/S0002-9904-1946-08647-4. MR: 0018022. Zbl: 0060.38301 (cit. on pp. 16–18).

[Cal58] E. Calabi. *An extension of E. Hopf's maximum principle with an application to Riemannian geometry. Duke Mathematical Journal* 25 (1958), pp. 45–56. MR: 0092069. Zbl: 0079.11801. URL: http://projecteuclid.org/euclid.dmj/1077467776 (cit. on p. 34).

[CG71] J. Cheeger and D. Gromoll. *The splitting theorem for manifolds of nonnegative Ricci curvature. Journal of Differential Geometry* 6 (1971), pp. 119–128. MR: 0303460. Zbl: 0223.53033. URL: http://projecteuclid.org/euclid.jdg/1214430220 (cit. on pp. 36, 38).

[Che75] S.-Y. Cheng. *Eigenvalue comparison theorems and its geometric applications. Mathematische Zeitschrift* 143 (1975), pp. 289–297. Zbl: 0329.53035 (cit. on p. 29).

[dSai16] H. P. de Saint-Gervais. *Uniformization of Riemann surfaces.* 2016. DOI: 10.4171/145. MR: 3494804. Zbl: 1332.30001 (cit. on p. 20).

[Edw75] D. A. Edwards. *The structure of superspace* (1975), pp. 121–133. MR: 0401069. Zbl: 0311.54010 (cit. on pp. 43, 44).

[EH79] T. Eguchi and An. J. Hanson. *Self-dual solutions to Euclidean gravity. Annals of Physics* 120.1 (1979), pp. 82–106. DOI: 10.1016/0003-4916(79)90282-3. MR: 540896. Zbl: 0409.53020 (cit. on p. 53).

[EH84] J.-H. Eschenburg and E. Heintze. *An elementary proof of the Cheeger-Gromoll splitting theorem. Annals of Global Analysis and Geometry* 2.2 (1984), pp. 141–151. DOI: 10.1007/BF01876506. MR: 777905. Zbl: 0548.53041 (cit. on p. 36).

[GH78] G. W. Gibbons and S. W. Hawking. *Gravitational Multi - Instantons. Phys. Lett.* 78B (1978), p. 430. DOI: 10.1016/0370-2693(78)90478-1 (cit. on p. 53).

[Gro07] M. Gromov. *Metric structures for Riemannian and non-Riemannian spaces.* English. Modern Birkhäuser Classics. 2007, pp. xx+585. MR: 2307192. Zbl: 1113.53001 (cit. on pp. 32, 43, 45).

[Gro81a] M. Gromov. *Curvature, diameter and Betti numbers. Commentarii Mathematici Helvetici* 56.2 (1981). DOI: 10.1007/BF02566208. MR: 630949. Zbl: 0467.53021 (cit. on p. 25).

[Gro81b] M. Gromov. *Groups of polynomial growth and expanding maps. Institut des Hautes Études Scientifiques. Publications Mathématiques* 53 (1981), pp. 53–73. MR: 623534. Zbl: 0474.20018. URL: http://www.numdam.org/item?id=PMIHES_1981__53__53_0 (cit. on pp. 31, 43, 45, 47).

[Gro81c] M. Gromov. *Structures métriques pour les variétés riemanniennes.* Vol. 1. Textes Mathématiques. 1981, pp. iv+152. MR: 682063. Zbl: 0509.53034 (cit. on p. 32).

[GS05] A. Grigor'yan and L. Saloff-Coste. *Stability results for Harnack inequalities. Annales de l'Institut Fourier* 55.3 (2005), pp. 825–890. MR: 2149405. URL: http://aif.cedram.org/item?id=AIF_2005__55_3_825_0 (cit. on p. 40).

[Hei12] H.-J. Hein. *Gravitational instantons from rational elliptic surfaces. Journal of the American Mathematical Society* 25.2 (2012), pp. 355–393. DOI: 10.1090/S0894-0347-2011-00723-6. MR: 2869021. Zbl: 1239.53066 (cit. on p. 29).

[HHN15] M. Haskins, H.-J. Hein, and J. Nordström. *Asymptotically cylindrical Calabi–Yau manifolds. Journal of Differential Geometry* 101.2 (2015), pp. 213–265. DOI: 10.4310/jdg/1442364651. MR: 3399097. Zbl: 1332.32028 (cit. on p. 29).

[Hit92] N. Hitchin. *Hyperkähler manifolds. Astérisque* 206 (1992). Séminaire Bourbaki, Vol. 1991/92, Exp. No. 748, 3, 137–166. MR: 1206066. Zbl: 0979.53051 (cit. on p. 49).

[Hop25] H. Hopf. *Zum Clifford-Kleinschen Raumproblem. Mathematische Annalen* 95 (1925), pp. 313–339. DOI: 10.1007/BF01206614. MR: 1512281. Zbl: 51.0439.05 (cit. on p. 7).

[Hop27] E. Hopf. *Elementare Bemerkungen über die Lösungen partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus. Sitzungsberichte der Preußischen Akademie der Wissenschaften, Physikalisch-Mathematische Klasse* (1927), pp. 147–152. Zbl: 53.0454.02 (cit. on p. 33).

[Hur93] A. Hurwitz. *Ueber algebraische Gebilde mit eindeutigen Transformationen in sich. Mathematische Annalen* 41 (1893), pp. 403–442. DOI: 10.1007/BF01443420. MR: 1510753. Zbl: 24.0380.02 (cit. on p. 20).

[Kil91] W. Killing. *Ueber die Clifford–Klein'schen Raumformen. Mathematische Annalen* 39 (1891), pp. 257–278. Zbl: 23.0529.01 (cit. on p. 7).

[Kro89a] P. B. Kronheimer. *A Torelli-type theorem for gravitational instantons. Journal of Differential Geometry* 29.3 (1989), pp. 685–697. MR: 992335. URL: http://projecteuclid.org/getRecord?id=euclid.jdg/1214443067 (cit. on p. 54).

[Kro89b] P. B. Kronheimer. *The construction of ALE spaces as hyper-Kähler quotients. Journal of Differential Geometry* 29.3 (1989), pp. 665–683. MR: 992334. URL: http://projecteuclid.org/getRecord?id=euclid.jdg/1214443066 (cit. on p. 54).

[LeB91] C. LeBrun. *Complete Ricci-flat Kähler metrics on* $\mathbf{C}^n$ *need not be flat. Several complex variables and complex geometry, Part 2 (Santa Cruz, CA, 1989).* Vol. 52. Proc. Sympos. Pure Math. 1991, pp. 297–304. MR: 1128554 (cit. on p. 52).

[Li86] P. Li. *Large time behavior of the heat equation on complete manifolds with nonnegative Ricci curvature. Annals of Mathematics* 124.1 (1986), pp. 1–21. DOI: 10.2307/1971385. MR: 847950. Zbl: 0613.58032 (cit. on p. 31).

[Lic58] A. Lichnerowicz. *Géométrie des groupes de transformations.* Travaux et Recherches Mathématiques. 1958, pp. ix+193. MR: 0124009. Zbl: 0096.16001 (cit. on p. 25).

[Liu13] G. Liu. 3*–manifolds with nonnegative Ricci curvature. Inventiones Mathematicae* 193.2 (2013), pp. 367–375. DOI: 10.1007/s00222-012-0428-x. MR: 3090181. Zbl: 1279.53035 (cit. on p. 31).

[Mil68] J. Milnor. *A note on curvature and fundamental group. Journal of Differential Geometry* 2 (1968), pp. 1–7. Zbl: 0162.25401. URL: http://projecteuclid.org/euclid.jdg/1214501132 (cit. on pp. 31, 32).

[Mye41] S. B. Myers. *Riemannian manifolds with positive mean curvature. Duke Mathematical Journal* 8 (1941), pp. 401–404. MR: 0004518. Zbl: 0025.22704. URL: http://projecteuclid.org/euclid.dmj/1077492655 (cit. on p. 21).

[NTU63] E. Newman, L. Tamburino, and T. Unti. *Empty-space generalization of the Schwarzschild metric. Journal of Mathematical Physics* 4 (1963), pp. 915–923. DOI: 10.1063/1.1704018. MR: 0152345. Zbl: 0115.43305 (cit. on p. 52).

[Oba62] M. Obata. *Certain conditions for a Riemannian manifold to be isometric with a sphere. Journal of the Mathematical Society of Japan* 14 (1962), pp. 333–340. DOI: 10.2969/jmsj/01430333. MR: 0142086. Zbl: 0115.39302 (cit. on p. 25).

[Pan18] J. Pan. *A proof of Milnor conjecture in dimension 3*. *Journal für die reine und angewandte Mathematik* (2018). DOI: 10.1515/crelle-2017-0057 (cit. on p. 31).

[Pet16] P. Petersen. *Riemannian Geometry*. 3rd edition. Vol. 171. Graduate Texts in Mathematics. 2016. DOI: 10.1007/978-3-319-26654-1. Zbl: 06520113 (cit. on p. 39).

[Rie68] B. Riemann. *Ueber die Hypothesen, welche der Geometrie zu Grunde liegen*. *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen* 13 (1868), pp. 1–20. Zbl: 01.0022.02 (cit. on p. 7).

[Sor00] C. Sormani. *Nonnegative Ricci curvature, small linear diameter growth and finite generation of fundamental groups*. *Journal of Differential Geometry* 54.3 (2000), pp. 547–559. Zbl: 1035.53045. URL: http://projecteuclid.org/euclid.jdg/1214339792 (cit. on p. 31).

[Tau51] A. H. Taub. *Empty space-times admitting a three parameter group of motions*. 53 (1951), pp. 472–490. DOI: 10.2307/1969567. MR: 0041565. Zbl: 0044.22804 (cit. on p. 52).

[Yau76] S.-T. Yau. *Some function-theoretic properties of complete Riemannian manifold and their applications to geometry*. *Indiana University Mathematics Journal* 25 (1976), pp. 659–670. DOI: 10.1512/iumj.1976.25.25051. MR: 0417452. Zbl: 0335.53041 (cit. on p. 29).

[Zhu97] S. Zhu. *The comparison geometry of Ricci curvature*. *Comparison geometry*. Vol. 30. Mathematical Sciences Research Institute Publications. 1997, pp. 221–262. MR: 1452876. Zbl: 0896.53036 (cit. on p. 28).

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