Differential Geometry IV
Lecture Notes

Prof. Dr. Thomas Walpuski

2022-05-12

Contents

Lecture 1

Why should you care about spin geometry? There is a plethora of reasons, but here are three:

(1) Modern physics requires spinors, Dirac operators, etc.

(2) The topology of manifolds (or at least certain aspects of it) is deeply intertwined with differential operators, and certain questions can only be answered using Dirac operators.

(3) Seiberg–Witten theory requires spin^c–structure and Dirac operators.

In the following, I will discuss the first point briefly and elaborate on an instance of the second point. I will discuss Seiberg–Witten theory towards the end of the semester.

1 Dirac's problem: a square root of the Laplace operator?

Dirac [Dir28, §2] came across the following question when trying to find a relativistic theory of the electron. Consider a free particle of energy E, momentum $\mathbf{p}$, and mass m. According to Special Relativity,

$$E = \sqrt{\mathbf{p}^2 + m^2}.$$

The **rules of quantisation** dictate that E and $\mathbf{p}$ are to be replaced by the differential operators $i\partial_t$ and $-i\nabla$. Therefore,

$$i\partial_t = \sqrt{\Delta + m^2}$$

with $\Delta \coloneqq -\sum_{i=1}^{3} \partial_{x_i}^2$.

Question 1.1. Is there a differential operator $D\!\!\!/$ satisfying $\Delta = D\!\!\!/^2$?

The *ansatz*

$$D\!\!\!/ = \sum_{i=1}^{3} \gamma_i \partial_{x_i}$$

with γ_i constant for the **Dirac operator** leads to system of algebraic equations

$$\gamma_i^2 = -1 \quad \text{and} \quad \gamma_i\gamma_j + \gamma_j\gamma_i = 0.$$

This does not have any solutions in $\mathbf{R}$ or $\mathbf{C}$; but it does have solutions in $\mathbf{H}^{\oplus 2}$ and $\mathrm{M}_2(\mathbf{C})^{\oplus 2}$ (which can be found by hand). However, it is important to observe that $D\!\!\!/$ does not act on functions but rather more complicated objects: **spinors**; cf. Cartan [Car13]. Brauer and Weyl [BW35] realised that the construction of the Dirac operator and spinors is closely related with the Clifford algebra.

2 The signature of a manifold

Situation 2.1. Let X be an closed equidimensional oriented smooth manifold. ×

Definition 2.2. The **intersection form** of X is the bilinear form $b_X\colon \mathrm{H}_{\mathrm{dR}}(X) \to \mathbf{R}$ defined by

$$b_X([\alpha],[\beta]) := \int_X \alpha \wedge \beta.$$ •

Proposition 2.3.

(1) *If* $\dim X = 4k$*, then* $b_X|_{\mathrm{H}^{2k}_{\mathrm{dR}}(X)}$ *is symmetric.*

(2) *If* $\dim X = 4k+2$*, then* $b_X|_{\mathrm{H}^{2k+1}_{\mathrm{dR}}(X)}$ *is alternating.* ■

Definition 2.4. If $\dim X = 4k$, then **intersection form** is the quadratic form $q_X\colon \mathrm{H}^{2k}_{\mathrm{dR}}(X) \to \mathbf{R}$ defined by

$$q_X([\alpha]) := b_X([\alpha],[\alpha]).$$

The **signature** of X is

$$\sigma(X) := \begin{cases} \sigma(q_X) & \text{if } \dim X = 0 \bmod 4 \\ 0 & \text{otherwise} \end{cases}$$

with $\sigma(q_X)$ denoting the signature of q_X. •

Remark 2.5. By Poincaré duality, q_X is non-degenerate. Therefore, q_X is determined by its signature up to isometry according to Sylvester's Law of Inertia. ♣

Remark 2.6. Hirzebruch [Hir72] says that the concept was introduced by Weyl [Wey24] (but I cannot get a hold of the latter). ♣

3 Relative de Rham cohomology

Situation 3.1. Let Y be a smooth manifold. Let $\iota\colon X \hookrightarrow Y$ be a closed submanifold. ×

Definition 3.2. The **relative de Rham complex** of $\iota\colon X \hookrightarrow Y$ is

$$(\Omega(Y,X) := \ker(\iota^*\colon \Omega(X) \to \Omega(Y)), \mathrm{d}).$$ •

The **relative de Rham cohomology** of $\iota\colon X \hookrightarrow Y$ is

$$\mathrm{H}_{\mathrm{dR}}(Y,X) := \mathrm{H}(\Omega(Y,X),\mathrm{d}).$$

Proposition 3.3. *The sequence of differential graded algebras*

$$0 \to (\Omega^\bullet(Y,X),\mathrm{d}) \hookrightarrow (\Omega^\bullet(Y),\mathrm{d}) \xrightarrow{\iota^*} (\Omega^\bullet(X),\mathrm{d}) \to 0$$

is exact. ■

Proposition 3.4. *There is a exact sequence*

$$\cdots \to \mathrm{H}^k_{\mathrm{dR}}(Y,X) \to \mathrm{H}^k_{\mathrm{dR}}(Y) \xrightarrow{i^*} \mathrm{H}^k_{\mathrm{dR}}(X) \xrightarrow{\delta} \mathrm{H}^{k+1}_{\mathrm{dR}}(Y,X) \to \cdots;$$

moreover, the connecting homomorphism δ satisfies $\delta[i^*\alpha] = [\mathrm{d}\alpha]$.

Proof. This is an immediate consequence of Proposition 3.3 and the Snake Lemma. ■

Remark 3.5. It is possible to construct a relative de Rham cohomology $\mathrm{H}_{\mathrm{dR}}(f)$ for every smooth map $f\colon X \to Y$ such that the analogue of Proposition 3.4 holds; cf. [BT82, pp. 78–79]. ♣

4 Poincaré–Lefschetz duality

Situation 4.1. Let Y be a compact connected smooth manifold of dimension n. Let $\iota\colon X \hookrightarrow Y$ be a closed submanifold. ×

Theorem 4.2 (Poincaré–Lefschetz duality). *Let $k \in \{0,\dots,n\}$. The bilinear map $\mathrm{H}^k_{\mathrm{dR}}(Y,X) \otimes \mathrm{H}^{n-k}_{\mathrm{dR}}(Y) \to \mathbf{R}$, $[\alpha]\otimes[\beta] \mapsto \int_Y \alpha\wedge\beta$ is a perfect pairing; that is: it induces an isomorphism*

$$\mathrm{H}^k_{\mathrm{dR}}(Y,X) \to \mathrm{H}^{n-k}_{\mathrm{dR}}(Y)^*.$$

Remark 4.3. See [Sch95, Corollary 2.6.2] for a proof. (In my opinion the proof in [Sch95] is needlessly complicated and circuitous, but at least it does have most of the details.) ♣

5 Bordism invariance of the signature

Situation 5.1. Let X_1, X_2 be closed equidimensional oriented smooth manifolds. ×

Proposition 5.2 (Thom [Tho52, Corollaire V.11]). *If X_1 and X_2 are bordant, then $\sigma(X_1) = \sigma(X_2)$.*

Proof. It suffices to show that if $X = \partial Y$ with Y compact and $\dim Y = 4k+1$, then $\sigma(X) = 0$. Denote by $\mathrm{res}\colon \mathrm{H}^{2k}_{\mathrm{dR}}(Y) \to \mathrm{H}^{2k}_{\mathrm{dR}}(X)$ the restriction map. The upcoming argument shows that $\mathrm{im}\,\mathrm{res} \subset \mathrm{H}^{2k}_{\mathrm{dR}}(X)$ is totally isotropic and $\dim \mathrm{im}\,\mathrm{res} = \frac{1}{2}b^{2k}(X)$. Therefore, by Proposition 13.3, q_X has vanishing signature.

Since

$$q_x \circ \mathrm{res}([\alpha]) = \int_X \alpha\wedge\alpha = \int_Y \mathrm{d}(\alpha\wedge\alpha) = 2\int_Y \mathrm{d}\alpha\wedge\alpha = 0,$$

$q_X \circ \mathrm{res} = 0$. Therefore, $\mathrm{im}\,\mathrm{res}$ is isotropic. Moreover, res fits into the commutative diagram

$$\begin{array}{ccccc} \mathrm{H}^{2k}_{\mathrm{dR}}(Y) & \xrightarrow{\mathrm{res}} & \mathrm{H}^{2k}_{\mathrm{dR}}(X) & \xrightarrow{\delta} & \mathrm{H}^{2k+1}_{\mathrm{dR}}(Y,X) \\ & & \downarrow\cong & & \downarrow\cong \\ & & \mathrm{H}^{2k}_{\mathrm{dR}}(X)^* & \xrightarrow{\mathrm{res}^*} & \mathrm{H}^{2k}_{\mathrm{dR}}(Y)^*. \end{array}$$

Here the vertical isomorphism are Poincaré duality and Poincaré–Lefschetz duality. The diagram commutes because of Proposition 3.4 and Stokes' theorem. Therefore,

$$\dim \operatorname{res} = \dim \ker \operatorname{res}^* = b^{2k}(X) - \dim \operatorname{im} \operatorname{res}^* = b^{2k}(X) - \dim \operatorname{im} \operatorname{res}.$$

Hence, $\dim \operatorname{res} = \frac{1}{2} b^{2k}(X)$. ■

Proposition 5.3.

$$\sigma(X_1 \times X_2) = \sigma(X_1)\sigma(X_2).$$

Proof. Without loss of generality $\dim X_1 + \dim X_2 = 4k$. The Künneth theorem identifies

$$\mathrm{H}_{\mathrm{dR}}(X_1 \times X_2) = \mathrm{H}_{\mathrm{dR}}(X_1) \otimes \mathrm{H}_{\mathrm{dR}}(X_2) \quad \text{and} \quad b_{X_1 \times X_2} = b_{X_1} \otimes b_{X_2}.$$

(Here $\otimes$ denotes the graded tensor product.)

Set $n \coloneqq \dim X_1$. Set

$$V_0 \coloneqq \begin{cases} \mathrm{H}^{n/2}_{\mathrm{dR}}(X_1) \otimes \mathrm{H}^{2k-n/2}_{\mathrm{dR}}(X_2) & \text{if } n \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

For $\ell = 0, \ldots, \lfloor n/2 \rfloor$ set

$$V_\ell \coloneqq I_\ell \oplus I_{n-\ell} \quad \text{with} \quad I_\ell \coloneqq \mathrm{H}^{\ell}_{\mathrm{dR}}(X_1) \otimes \mathrm{H}^{2k-\ell}_{\mathrm{dR}}(X_2).$$

The decomposition

$$\mathrm{H}^{2k}_{\mathrm{dR}}(X_1 \times X_2) = V_0 \oplus \bigoplus_{\ell=0}^{\lfloor n/2 \rfloor} V_\ell$$

is perpendicular with respect to $q_{X_1 \times X_2}$; that is:

$$q_{X_1 \times X_2} = q_0 \perp \cdots \perp q_{\lfloor n/2 \rfloor} \quad \text{with} \quad q_\ell \coloneqq q_{X_1 \times X_2}|_{V_\ell}.$$

Therefore, it remains to determine $\sigma(q_\ell)$:

(1) Evidently, If $n = 0 \bmod 4$, then $\sigma(q_0) = \sigma(X_1) q(X_2)$.

(2) If $n = 2 \bmod 4$, then $b_{X_1}|_{\mathrm{H}^{n/2}_{\mathrm{dR}}(X_1)}$ and $b_{X_2}|_{\mathrm{H}^{2k-n/2}_{\mathrm{dR}}(X_2)}$ are alternating. Therefore, there is a totally isotropic $I \subset \mathrm{H}^{n/2}_{\mathrm{dR}}(X_1)$ with $\dim I = \frac{1}{2}\mathrm{H}^{n/2}_{\mathrm{dR}}(X_1)$. Since $I \otimes \mathrm{H}^{2k-n/2}_{\mathrm{dR}}(X_2)$ is totally isotropic, by Proposition 13.3, $\sigma(q_0) = 0$.

(3) For $\ell = 0, \ldots, \lfloor n/2 \rfloor$, $I_\ell \subset V_\ell$ is totally isotropic and $\dim I_\ell = \frac{1}{2} V_\ell$. Therefore, by Proposition 13.3, $\sigma(q_\ell) = 0$. ■

Corollary 5.4 (Thom [Tho54, paragraph after Théorème IV.1]). *The signature induces a ring homomorphism* $\sigma \colon \Omega^{\mathrm{SO}} \to \mathbf{Z}$. ■

6 Hirzebruch signature theorem

Situation 6.1. Let X be a closed oriented smooth manifold. ×

Definition 6.2. Let $r \in \mathbf{N}_0$. Let V be a vector bundle of rank r over X. Denote by $\phi_{V\otimes\mathbf{C}}\colon \operatorname{Hom}(S^\bullet\mathfrak{gl}_r(\mathbf{C}),\mathbf{C}) \to \mathrm{H}_{\mathrm{dR}}(X)$ denoting the Chern–Weil homomorphism associated with $V \otimes \mathbf{C}$. The L **genus** of a vector bundle V over X is

$$L(V) \coloneqq \phi_{V\otimes\mathbf{C}}(\det \ell) \in \mathrm{H}_{\mathrm{dR}}(X) \quad \text{with} \quad \ell(x) \coloneqq \frac{\sqrt{\frac{2\pi x}{i}}}{\tanh\sqrt{\frac{2\pi x}{i}}}. \qquad \bullet$$

Remark 6.3. Since $\tanh(x)$ is odd, $x/\tanh(x)$ is a power series in x^2 and $\ell(x)$ is a power series in x. Indeed,

$$\ell(x) = \sum_{k=0}^{\infty} \frac{2^{2k}B_{2k}}{(2k)!}\left(\frac{2\pi x}{i}\right)^k$$

with B_{2k} denoting the Bernoulli numbers. ♣

Remark 6.4. The L genus is a **multiplicative characteristic class**; that is:

$$L(V_1 \oplus V_2) = L(V_1)L(V_2). \qquad ♣$$

Remark 6.5. $L(V)$ can be expressed in terms of the Pontrjagin classes of V and the L **polynomials**:

$$L(V) = \sum_{k=1}^{\infty} L_k(p_1,\ldots,p_k).$$

A computation shows that

$$L_1 = \frac{1}{3}p_1, \quad L_2 = \frac{1}{45}(7p_2 - p_1^2), \quad \ldots.$$

see OEIS: A237111. The L polynomials are **multiplicative sequence**; cf. [Hir95, §1; LM89, Chapter III §11]. ♣

Theorem 6.6 (Hirzebruch [Hir53, Theorem 3.1; Hir95, Chapter II Theorem 8.2.2]).

$$\sigma(X) = \int_X L(TX). \tag{6.7}$$

Remark 6.8. Hirzebruch [Hir72, §2] remarks on the proof of his signature theorem:

> How to prove it? After conjecturing it I went to the library of the Institute of Advanced Studies (June 2, 1953). Thom's *Comptes Rendus* note [Tho53] had just arrived. This finished the proof.

[Tho53] introduced the concept of bordism and announced that $\Omega^{\mathrm{SO}} \otimes_{\mathbf{Z}} \mathbf{R}$ is a polynomial algebra generated by $[\mathbf{C}P^{2n}]$; cf. [Tho54, Théorème IV.17]. Since both sides of (6.7), define ring homomorphisms $\Omega^{\mathrm{SO}} \otimes \mathbf{R} \to \mathbf{R}$ it suffices to verify (6.7) on $\mathbf{C}P^{2n}$. Surely, Hirzebruch did this before going to the library. ♣

Remark 6.9. The Hirzebruch signature theorem implies a sequence of dazzling integrality theorems; e.g.: if $\dim X = 4$, then $\frac{1}{3}\langle p_1(TX), [X]\rangle \in \mathbf{Z}$. The same does not hold for arbitrary real vector bundles. ♣

7 Rohklin's theorem

Theorem 7.1 (Rokhlin [Rok52]). *Let X be a closed oriented 4–manifold. If $w_2(X) = 0$, then $\sigma(X)$ is divisible by 16.*

8 The Euler characteristic operator

Situation 8.1. Let (X, g) be an closed oriented Riemannian manifold. ×

Definition 8.2. The **Euler characteristic operator** associated with (X, g) is

$$\delta := \mathrm{d} + \mathrm{d}^* \colon \Omega^{\mathrm{even}}(X) \to \Omega^{\mathrm{odd}}(X). \qquad \bullet$$

Proposition 8.3.

$$\operatorname{index} \delta = \chi(X).$$

Proof. This is an immediate consequence of Hodge theory. ■

Remark 8.4. [LM89, Chapter II Example 6.1] exhibits δ as a Dirac operator. ♣

9 The signature operator

Situation 9.1. Let (X, g) be an closed oriented Riemannian manifold with $\dim X = 2n$. ×

Definition 9.2. Define $\varepsilon \in \mathrm{End}(\Lambda T^*X \otimes \mathbf{C})$ by

$$\varepsilon\alpha := i^{k(k-1)+n} * \alpha \quad \text{with} \quad k := \deg \alpha. \qquad \bullet$$

Proposition 9.3.

(1) $\varepsilon^2 = \mathbf{1}$.

(2) $\mathrm{d} + \mathrm{d}^*$ *and* ε *anti-commute.*

Proof. To prove (1), observe that for every $\alpha \in \Lambda^k T^*X \otimes \mathbf{C}$

$$\varepsilon^2\alpha = (-1)^{k(2n-k)} i^{k(k-1)+n} i^{(2n-k)(2n-k-1)+n} \alpha = \alpha$$

because

$$2k(2n-k) + k(k-1) + n + (2n-k)(2n-k-1) + n = 4n^2 = 0 \bmod 4.$$

To prove (2) observe that for every $\alpha \in \Omega^k(X, \mathbf{C})$

$$\begin{aligned} \mathrm{d}^*\alpha &= (-1)^{2n(k-1)+1} * \mathrm{d} * \alpha = - * \mathrm{d} * \alpha \\ &= -(-i)^{(2n-k+1)(2n-k)+k(k-1)+2n} \varepsilon \mathrm{d} \varepsilon \alpha = -\varepsilon \mathrm{d} \varepsilon \alpha \end{aligned}$$

because

$$(2n-k+1)(2n-k) + k(k-1) + 2n = 2k(k-1) + 4(n - kn + n^2) = 0 \bmod 4. \qquad ■$$

Definition 9.4. Set
$$\Omega_\pm(X,\mathbf{C}) \coloneqq \{\alpha \in \Omega(X,\mathbf{C}) : \varepsilon\alpha = \pm\alpha\}.$$ •

The **signature operator** associated with (X,g) is
$$D \coloneqq \mathrm{d} + \mathrm{d}^*\colon\ \Omega_+(X,\mathbf{C}) \to \Omega_-(X).$$

Proposition 9.5.
$$\mathrm{index}_{\mathbf{C}}\, D = \sigma(X).$$

Proof. Since $(\mathrm{d}+\mathrm{d}^*)^2 = \Delta$,
$$\ker D = \mathscr{H}_+(X,\mathbf{C}) \coloneqq \mathscr{H}(X,\mathbf{C}) \cap \Omega_+(X,\mathbf{C}) \quad \text{and}$$
$$\operatorname{coker} D \cong \mathscr{H}_-(X,\mathbf{C}) \coloneqq \mathscr{H}(X,\mathbf{C}) \cap \Omega_-(X,\mathbf{C}).$$

The inclusions $\mathscr{H}^k(X,\mathbf{C}) \hookrightarrow \mathscr{H}_\pm(X,C), \alpha \mapsto \alpha \pm \varepsilon\alpha$ $(k = 0,\dots,n-1)$ and $\mathscr{H}^n_\pm(X,\mathbf{C}) \hookrightarrow \mathscr{H}_\pm(X,C)$ assemble into an isomorphism
$$\mathscr{H}^0(X,\mathbf{C}) \oplus \cdots \oplus \mathscr{H}^{n-1}(X,\mathbf{C}) \oplus \mathscr{H}^n_\pm(X,\mathbf{C}) \cong \mathscr{H}_\pm(X,\mathbf{C}).$$

Therefore,
$$\mathrm{index}_{\mathbf{C}}\, D = \dim_{\mathbf{C}} \mathscr{H}^n_+(X,\mathbf{C}) - \dim_{\mathbf{C}} \mathscr{H}^n_-(X,\mathbf{C}).$$

If n is even, then
$$\mathscr{H}^n_\pm(X,\mathbf{C}) = \mathscr{H}^n_\pm(X) \otimes \mathbf{C} \quad \text{with} \quad \mathscr{H}^n_\pm(X) \coloneqq \{\alpha \in \mathscr{H}^n(X) : *\alpha = \pm\alpha\}.$$

Therefore,
$$\mathrm{index}_{\mathbf{C}}\, D = \dim \mathscr{H}^n_+(X) - \dim \mathscr{H}^n_-(X) = \sigma(X)$$
by Hodge theory.

If n is odd and $\alpha \in \Lambda^n T^*X \otimes \mathbf{C}$, then
$$\varepsilon\alpha = i * \alpha.$$

Therefore,
$$\overline{\mathscr{H}^n_+}(X,\mathbf{C}) = \mathscr{H}^n_-(X,\mathbf{C})$$
and
$$\mathrm{index}_{\mathbf{C}}\, D = 0 = \sigma(X).$$ ■

Remark 9.6. Hirzebruch signature theorem predates the Atiyah–Singer index theorem by about decade—but, of course, it can be derived from it because of the above. ♣

Remark 9.7. Is there a differential operator $\not{D}$ whose index explains Rokhlin's theorem? Yes! If X is a closed oriented 4–manifold with $w_2(X) = 0$, then the **Atiyah–Singer operator** $\not{D}$ satisfies
$$\mathrm{index}\, \not{D} = \frac{1}{8}\sigma(X) \quad \text{and} \quad \mathrm{index}\, \not{D} = 0 \bmod 2.$$ ♣

Remark 9.8. [LM89, Chapter II Example 6.2] exhibits D as a Dirac operator. ♣

Lecture 2

A problem at the root of spin geometry is to find solutions to the algebraic equation

$$\gamma_i^2 = -1 \quad \text{and} \quad \gamma_i\gamma_j + \gamma_j\gamma_i = 0.$$

The systematic answer to this question is the theory of Clifford algebras. Although it is possible to quite directly attack the problem of determining the Clifford algebras over **R** and **C**, I would find it uncultured to so. There is a Clifford algebra associated with every quadratic form and Clifford algebras play an important role in the theory of quadratic forms. The main purpose of this lecture is to show that every quadratic form can be decomposed (possibly non-uniquely) into simple elementary pieces. At the end, I will mention the Cartan–Dieudonné Theorem, which plays an important role in many developments of the spin group in the literature (but can be replaced by the super Skolem–Noether theorem).

10 Quadratic forms

Situation 10.1. Let k be a field. ×

Definition 10.2. Let V be a k–vector space. A **quadratic form** on V is a map $q\colon V \to k$ such that:

(1) it is homogenous of degree 2; i.e.: for every $\lambda \in k$ and $v \in V$

$$q(\lambda v) = \lambda^2 q(v),$$

and

(2) the map

$$(v, w) \mapsto q(v + w) - q(v) - q(w)$$

defines a symmetric bilinear form $p \in \mathrm{Hom}(S^2V, k)$—the **polarisation** of q. •

Example 10.3. Let $a_1, \dots, a_n \in k$. The **diagonal quadratic form** $\langle a_1, \dots, a_n\rangle\colon k^{\oplus n} \to k$ is defined by

$$\langle a_1, \dots, a_n\rangle(x_1, \dots, x_n) \coloneqq a_1x_1^2 + \cdots + a_nx_n^2.$$ ♠

Example 10.4. Let $a, b \in k$. The quadratic form $[a, b]\colon k \oplus k \to k$ is defined by

$$[a, b](x, y) \coloneqq ax^2 + xy + by^2.$$ ♠

Example 10.5. The **hyperbolic plane** is $[0, 0]$. ♠

Definition 10.6. Let $q_i\colon V_i \to k$ $(i = 1, 2)$ be quadratic forms. The **perpendicular sum** of q_1 and q_2 is the quadratic form $q_1 \perp q_2\colon V_1 \oplus V_2 \to k$ defined by

$$q_1 \perp q_2 \coloneqq q_1 + q_2.$$ •

Remark 10.7. The structure theory of quadratic forms shows that every quadratic form can be decomposed into the above pieces. ♣

Proposition 10.8. *Let V be a k–vector space. Define $Q\colon \operatorname{Hom}(V\otimes V, k) \to \{\textit{quadratic forms on } V\}$ by*

$$Q(b)(v) := b(v \otimes v)$$

Q is surjective and $\ker Q = \operatorname{Hom}(\Lambda^2 V, k)$; that is: the sequence

$$\operatorname{Hom}(\Lambda^2 V, k) \hookrightarrow \operatorname{Hom}(V \otimes V, k) \overset{Q}{\twoheadrightarrow} \{\textit{quadratic forms on } V\}$$

is exact.

Proof. Evidently, $\ker Q = \operatorname{Hom}(\Lambda^2 V, k)$. To see that Q is surjective, let $q\colon V \to k$ be a quadratic form on V and denote its polarisation by p. Choose a basis $\{e_i : i \in I\}$ of V and choose an order $\prec$ on the index set I. Define the $b \in \operatorname{Hom}(V \otimes V, k)$ by

$$b(e_i \otimes e_j) := \begin{cases} 0 & \text{if } i < j, \\ q(e_i) & \text{if } i = j, \\ p(e_i, e_j) & \text{if } i > j. \end{cases}$$

Evidently, $q = Q(b)$. ■

Remark 10.9. If $2 \neq 0 \in k$, then $\frac{1}{2}$ of the polarisation defines a right inverse of Q; therefore, the theory of quadratic form is equivalent to the theory of symmetric bilinear forms. If $2 = 0 \in k$, then the polarisation is alternating. ♣

Definition 10.10. Let $q_i\colon V_i \to k$ $(i = 1, 2)$ be quadratic forms. A **quadratic morphism** $f\colon q_1 \to q_2$ is a linear map such that

$$q_2 \circ f = q_1.$$ •

Remark 10.11. Quadratic forms and quadratic morphisms form a category. ♣

Definition 10.12. Let q be a quadratic form. The **orthogonal group** associated with q is the group

$$\mathrm{O}(q) := \{f\colon q \to q : f \text{ is a quadratic isomorphism}\}.$$ •

[Che54, §1.2; Bou07, §3.4; MH73; Lam73; Lam05; KS80; Knu91; EKM08, §7].

11 Structure theory of quadratic forms

Situation 11.1. Let k be a field. Let $q\colon V \to k$ be a quadratic form with $\dim V < \infty$. Denote by $p \in \operatorname{Hom}(S^2 V, k)$ the polarisation of q. ×

Remark 11.2. An isomorphism $f\colon k^{\oplus n} \cong V$ (that is: a choice of basis) identifies q with the polynomial

$$\sum_{i=1}^{n} a_i x_i^2 + \sum_{i<j} a_{ij} x_i x_j \coloneqq q \circ f(x_1, \ldots, x_n).$$

A primary objective of the structure theory of quadratic forms is to simplify this expression as much as possible though a particularly clever choice of f. ♣

Example 11.3. If $\operatorname{char} k \neq 2$ and $a \in k^\times$, then $\langle a, a\rangle \cong [0, 0]$. ♠

Example 11.4. For every $a \in k$, $[0, a] \cong [0, 0]$. ♠

Example 11.5. If $\operatorname{char} k = 2$, then $[a, b] \perp [a, c] \cong [a, b + c] \perp [0, 0]$. ♠

Definition 11.6. The quadratic form q is **non-degenerate** if the map $\flat\colon V \to V^*$ defined by

$$v^\flat(w) \coloneqq p(v, w).$$

is an isomorphism. •

Example 11.7. $\langle a_1, \ldots, a_n\rangle$ is non-degenerate if and only if $2^n a_1 \cdots a_n \neq 0$ (in particular: $\operatorname{char} k \neq 2$). ♠

Example 11.8. $[a, b]$ is non-degenerate if and only if $4ab - 1 \neq 0$; e.g. if $\operatorname{char} k = 2$ or $a = 0$. ♠

Definition 11.9. The **radical** of p and the **radical** of q are

$$\operatorname{rad} p \coloneqq \ker \flat \quad \text{and} \quad \operatorname{rad} q \coloneqq \operatorname{rad} p \cap q^{-1}(0)$$

respectively. The **nullity** and **defect** of q are

$$n(q) \coloneqq \dim \operatorname{rad} q \quad \text{and} \quad d(q) \coloneqq \dim \operatorname{rad} p - \dim \operatorname{rad} q$$

respectively. •

Remark 11.10. As an immediate consequence of Definition 10.2 (2), $\operatorname{rad} q \subset \operatorname{rad} p$ is a linear subspace. ♣

Proposition 11.11. *If $V = \operatorname{rad} p \oplus W$, then $q = q|_{\operatorname{rad} p} \perp q|_W$ and $q|_W$ is non-degenerate.*

Proof. By definition of $\operatorname{rad} p$, $q = q|_{\operatorname{rad} p} \perp q|_W$. Since $\ker \flat|_W = 0$, $q|_W$ is non-degenerate. ■

Proposition 11.12. *If $\operatorname{char} k \neq 2$, then $\operatorname{rad} q = \operatorname{rad} p$.*

Proof. This is a consequence of $q = Q(\frac{1}{2}p)$. ■

Definition 11.13. The quadratic form q is **anisotropic** if $q^{-1}(0) = \{0\}$. •

Proposition 11.14. *If $\operatorname{rad} p = \operatorname{rad} q \oplus D$, then $q|_D$ is anisotropic and diagonal with respect to every basis.*

Proof. By definition of $\operatorname{rad} q$, $q|_D$ is anisotropic. If $e_1, \ldots, e_n$ is a basis of D, then

$$q|_D(x_1e_1 + \cdots + x_ne_n) = \sum_{i=1}^{n} a_i x_i^2 + \sum_{i<j} \underbrace{p(e_i, e_j)}_{=0} x_i x_j. \qquad \blacksquare$$

Definition 11.15. Let $W \subset V$. The **perpendicular** of W is

$$W^\perp := \{v \in V : p(v, w) = 0 \text{ for every } w \in W\}. \qquad \bullet$$

Proposition 11.16 (Splitting off non-degenerate summands). *Let $W \subset V$. If q and $q|_W$ are non-degenerate, then $q = q|_W \perp q|_{W^\perp}$, and $q|_{W^\perp}$ is non-degenerate.*

Proof. Since $q|_W$ is non-degenerate, $W \cap W^\perp = 0$. Since $W^\perp = \ker(V \xrightarrow{\flat} V^* \to W^*)$ and q is non-degenerate, $\dim W^\perp = \dim V - \dim W$. Therefore, $W + W^\perp = V$. Evidently, $q = q|_W + q|_{W^\perp}$, and $q|_{W^\perp}$ is non-degenerate. $\blacksquare$

Proposition 11.17 (Diagonalisation of non-degenerate quadratic forms if $\operatorname{char} k \neq 2$). *If $\operatorname{char} k \neq 2$ and q is non-degenerate, then q can be diagonalised; that is: there are $a_1, \ldots, a_n \in k$ with*

$$q \cong \langle a_1, \ldots, a_n \rangle.$$

Proof. The proof is by induction on $\dim V$. If $V = 0$, then the assertion is trivial. If $V \neq 0$, then $q \neq 0$ because it is non-degenerate. Choose $v \in V$ with $q(v) \neq 0$. Since $\operatorname{char} k \neq 2$, $q|_{\langle v \rangle}$ is non-degenerate. Therefore, by Proposition 11.16, $q = q|_{\langle v \rangle} \perp q|_{\langle v \rangle^\perp}$, and $q|_{\langle v \rangle^\perp}$ is non-degenerate. Since $\dim \langle v \rangle^\perp = \dim V - 1$, the assertion follows. $\blacksquare$

Remark 11.18. The proof of Proposition 11.17 can be turned into an algorithm—a variant of the Gram–Schmidt process. ♣

Proposition 11.19. *Suppose that* $\dim V \geqslant 2$. *Let* $v \in V \setminus \operatorname{rad} p$.

(1) *If $q(v) = 0$, then there is a $W \subset V$ with $v \in W$ and* $\dim W = 2$ *such that*

$$q|_W \cong [0, 0].$$

(2) *If* $\operatorname{char} k = 2$, *then there are $W \subset V$ with $v \in W$ and* $\dim W = 2$, *and $a, b \in k$ such that*

$$q|_W \cong [a, b].$$

Proof. Since $v \notin \operatorname{rad} p$, there is a $w' \in V$ with $p(v, w') = 1$. Since $q(v) = 0$ or $\operatorname{char} k = 2$, $w' \notin \langle v \rangle$. This proves (2) with $a := q(v)$ and $b := q(w')$. To prove (1), set $w := w' - q(w')v$ and observe that $q(w) = 0$. $\blacksquare$

Definition 11.20. $S \subset V$ is **totally isotropic** if $q|_S = 0$. The **Witt index** of q is

$$i(q) := \max\{\dim S : S \subset W \subset V \text{ is totally isotropic and } \operatorname{rad} p \cap W = 0\}. \qquad \bullet$$

Theorem 11.21 (Structure theorem for quadratic forms). *Set $n := n(q)$, $d := d(q)$, and $i := i(q)$.*

(1) *If* char $k \neq 2$, *then there are $a_1, \ldots, a_r \in k^\times$ such that*

$$q \cong \langle 0\rangle^{\perp n} \perp [0,0]^{\perp i} \perp \langle a_1, \ldots, a_r\rangle.$$

Moreover, $\langle a_1, \ldots, a_r\rangle$ is anisotropic.

(2) *If* char $k = 2$, *then there are $a_1, \ldots, a_d, b_1, c_1, \ldots, b_r, c_r \in k^\times$ such that*

$$q \cong \langle 0\rangle^{\perp n} \perp \langle a_1, \ldots, a_d\rangle \perp [0,0]^{\perp i} \perp [b_1, c_1] \perp \ldots \perp [b_r, c_r].$$

Moreover, $\langle a_1, \ldots, a_d\rangle$ and $[b_1, c_1] \perp \ldots \perp [b_r, c_r]$ are anisotropic.

Remark 11.22. At first glance, Theorem 11.21 seems very satisfactory. However, it does not answer the question to what extend $a_1, \ldots, a_r \in k$ (resp. $a_1, \ldots, a_d, b_1, c_1, \ldots, b_r, c_r \in k$) are uniquely determined by q. If k is quadratically closed (e.g., $k = \mathbf{C}$), then there is a simple answer. If k is a Euclidean field (e.g., $k = \mathbf{R}$), then Sylvester's Law of Inertia answers this question. For general k one has to delve into Witt theory; cf. [EKM08, §8]. ♣

Proof of Theorem 11.21. Choose $W \subset V$ with $V = \operatorname{rad} p \oplus W$. By Proposition 11.11, it suffices to analyse $q|_{\operatorname{rad} p}$ and $q|_W$, and the latter is non-degenerate.

If char $k \neq 2$, then by Proposition 11.12, $q_{\operatorname{rad} p} = \langle 0\rangle^{\perp n}$. If char $k = 2$, then, by Proposition 11.14, $q|_{\operatorname{rad} p} \cong \langle 0\rangle^{\perp n} \perp \langle a_1, \ldots, a_d\rangle$.

Let $S \subset W$ totally isotropic with $\dim S = i$. Repeated application of Proposition 11.19 and Proposition 11.16 constructs a totally isotropic $S' \subset W$ with $S \cap S' = 0$ and $q|_{S\oplus S'} \cong [0,0]^{\perp i}$. Set $R := (S \oplus S')^\perp$. By Proposition 11.16, $q|_W = q|_{S\oplus S'} \perp q|_R$, $q|_R$ is non-degenerate and anisotropic. If char $k \neq 2$, then, by Proposition 11.17, $q|_R \cong \langle a_1, \ldots, a_r\rangle$. If char $k = 2$, then, by repeated application of Proposition 11.19 and Proposition 11.16, $q|_R \cong [b_1, c_1] \perp \ldots \perp [b_r, c_r]$. ■

[Che54, §1.3; EKM08, §7].

12 Quadratic forms over quadratically closed fields

Situation 12.1. Let k be a quadratically closed field. ×

Theorem 12.2. *Let $q\colon V \to k$ be a quadratic form with* $\dim V < \infty$. *Set $n := n(q)$, $i := i(q)$, and $r := \dim V - n - 2i$. In this situation, $r \in \{0,1\}$ and*

$$q \cong \langle 0\rangle^{\perp n} \perp [0,0]^{\perp i} \perp \langle 1\rangle^r.$$

Proof. Since k is quadratically closed, for every $a \in k^\times$, $\langle a\rangle \cong \langle 1\rangle$. Moreover, $\langle 1\rangle^{\perp r}$ is anisotropic if and only if $r \in \{0,1\}$. Since k is quadratically closed, if $a, b \in k^\times$, then

$$[a,b](x,y) = ax^2 + xy + by^2$$

has a non-trivial zero; therefore: $[a,b]$ fails to be anisotropic. Therefore, Theorem 11.21 finishes the proof. ■

Remark 12.3. If char $k \neq 2$, then $[0,0] \cong \langle 1,1\rangle$ because $x^2+y^2 = (x+iy)(x-iy)$ with $i^2 = -1$. ♣

Exercise 12.4. Prove that every quadratic form $q\colon V \to \mathbf{C}$ with $\dim V < \infty$ is isomorphic to $\langle 0\rangle^n \perp \langle 1\rangle^{r+2i}$ by repeatedly completing the square.

13 Sylvester's Law of Inertia

Situation 13.1. Let k be a ordered field. In particular, $\operatorname{char} k = 0$. Let $q\colon V \to k$ be a quadratic form with $\dim V < \infty$. $\times$

Definition 13.2. The quadratic form q is **positive definite** (**negative definite**) if $q(v) > 0$ ($q(v) < 0$) for every $0 \neq v \in V$. The **signature** of q is

$$\sigma(q) := r_+(q) - r_-(q)$$

with

$$r_\pm(q) := \max\{\dim W : \pm q|_W \text{ is positive definite}\}$$ •

Proposition 13.3. *If q is non-degenerate and $i(q) \geqslant \frac{1}{2}\dim V$, then $i(q) = \frac{1}{2}\dim V$ and $\sigma(q) = 0$.*

Proof. By Theorem 11.21 (1), $q \cong [0,0]^{i(q)}$. In particular, $i(q) = \frac{1}{2}\dim V$. Since, $x^2 - y^2 = (x+y)(x-y)$, $[0,0] \cong \langle 1,-1\rangle$. Therefore, $\sigma(q) = 0$. ■

Definition 13.4. A **Euclidean field** is an ordered field k such that every $x \in k$ with $x \geqslant 0$ admits a square root. •

Theorem 13.5 (Sylvester's Law of Inertia). *Set $r := r(q)$, $s := s(q)$, and $n := n(q)$. There are $a_1^+, \ldots, a_r^+ > 0$ and $a_1^-, \ldots, a_s^- < 0$ such that*

$$q \cong \langle a_1^+, \ldots, a_r^+\rangle \perp \langle a_1^-, \ldots, a_s^-\rangle \perp \langle 0\rangle^{\perp n};$$

moreover, if k is Euclidean, then $a_1^+ = \ldots = a_r^+ = 1$ and $a_1^- = \ldots = a_s^- = -1$.

Proof. By Proposition 11.11 and Proposition 11.12, without loss of generality, q is non-degenerate; that is: $n = 0$. Let $V_+ \subset V$ with $q|_{V_+}$ positive definite and $\dim V_+ = r$. Set $V_- := V_+^\perp$. The restriction $q|_{V_-}$ is negative definite; indeed: if $0 \neq v \in V_-$ and $q(v) > 0$, then $q_{V_+ \oplus \langle v\rangle}$ is positive definite—a contradiction. By Proposition 11.17, $q|_{V_\pm}$ can be diagonalised. If k is Euclidean, then, for every $a \in k^\times$, $\langle a\rangle \cong \langle \pm 1\rangle$. ■

[MH73, (2.5)]

14 Cartan–Dieudonné Theorem

Situation 14.1. Let k be a field. Let $q\colon V \to k$ be a quadratic form with $\dim V < \infty$. Denote by $p \in \operatorname{Hom}(S^2V, k)$ the polarisation of q. $\times$

Definition 14.2. A vector $v \in V$ is **isotropic** if $q(v) = 0$ and **anisotropic** if $q(v) \neq 0$. •

Definition 14.3. Let $v \in V$ be anisotropic. The **reflection** $r_v \in \operatorname{End}(V)$ along v is defined by

$$r_v(w) := w - \frac{p(v,w)}{q(v)}v.$$ •

Proposition 14.4. *For every anisotropic $v \in V$, $r_v^2 = \mathrm{id}_V$ and $r_v \in \mathrm{O}(q)$.*

Proof. By direct computation,

$$\begin{aligned} r_v \circ r_v(w) &= r_v(w) - \frac{p(v, r_v(w))}{q(v)} \\ &= w + \frac{2p(v,w)}{q(v)} v - \frac{p(v,w)p(v,v)}{q(v)^2} v = w \end{aligned}$$

and

$$q(r_v(w)) = q(w) - \frac{p(v,w)^2}{q(v)^2} q(v) + \frac{p(v,w)^2}{q(v)} = q(w). \qquad \blacksquare$$

Theorem 14.5 (Cartan–Dieudonné). *Denote by $R(q) < \mathrm{O}(q)$ the subgroup generated by reflections along anisotropic vectors. If q is non-degenerate and $\dim V < \infty$, then $\mathrm{O}(q) = R(q)$*— unless *$k = \mathbf{F}_2$, $\dim V = 4$, and q is of Witt index $i(q) = 2$; in which case: $R(q) < \mathrm{O}(q)$ has index two.*

Proof. See [Che54, §1.5; Lam05, Chapter I, Theorem 7.1]. ∎

Remark 14.6 (The exceptional case in the Cartan–Dieudonné Theorem). By Theorem 11.21, if $k = \mathbf{F}_2$, $\dim V = 4$, and $i(q) = 2$, then $q \cong [0,0] \perp [0,0] \cong [1,1] \perp [1,1]$. The latter model is more convenient to understand why $\mathrm{O}(q) \neq R(q)$.

For every $0 \neq v \in \mathbf{F}_2^{\oplus 2}$, $[1,1](v) = 1$. Therefore, $\mathrm{O}([1,1]) = \mathrm{GL}_2(\mathbf{F}_2)$. Since $\mathbf{F}_2^2$ has three non-zero elements, $\mathrm{GL}_2(\mathbf{F}_2) \subset S_3$. Indeed, for every $0 \neq v \in \mathbf{F}_2^{\oplus 2}$, r_v transposes the remaining two non-zero elements. Therefore, $\mathrm{GL}_2(\mathbf{F}_2) = S_3$.

Let $(v,w) \in \left(\mathbf{F}_2^{\oplus 2}\right)^{\oplus 2}$. Evidently, $q(v,w) = ([1,1] \perp [1,1])(v,w) \neq 0$ if and only if either ($v = 0$ and $w \neq 0$) or ($v \neq 0$ and $w = 0$). Therefore, reflections in anisotropic vectors generate $S_3 \times S_3 \subset \mathrm{O}(q)$. However, $\sigma \in \mathrm{O}(q)$ defined by $\sigma(v,w) \coloneqq (w,v)$ fails to be of this form. By Theorem 14.5,

$$\mathrm{O}(q) \cong C_2 \ltimes (S_3 \times S_3). \qquad ♣$$

Lecture 3

This lectures constructs the Clifford algebra as a functor from quadratic forms to super algebras. At the end of the lecture, I will explain how to (in principle) compute the Clifford algebra of any non-degenerate finite-dimensional quadratic form.

15 The Clifford algebra of a quadratic form

Situation 15.1. Let k be a field. Let $q\colon V \to k$ be quadratic form. Denote by $p \in \mathrm{Hom}(S^2V, k)$ the polarisation of q. ×

Definition 15.2. A **Clifford algebra** of q is pair $(\mathrm{C}\ell(q), \gamma)$ consisting of a k–algebra $\mathrm{C}\ell(q)$ and a linear map $\gamma\colon V \to \mathrm{C}\ell(q)$ such that:

(1) for every $v \in V$

$$\gamma(v)^2 = q(v) \cdot \mathbf{1},$$

and

(2) if A is a k–algebra together with a linear map $\delta\colon V \to A$ such that for every $v \in V$

$$\delta(v)^2 = q(v) \cdot \mathbf{1},$$

then there is a unique algebra homomorphism $f\colon \mathrm{C}\ell(q) \to A$ such that

$$f \circ \gamma = \delta.$$

•

Remark 15.3. By polarisation, $\gamma(v)^2 = q(v) \cdot \mathbf{1}$ implies

$$\gamma(v)\gamma(w) + \gamma(w)\gamma(v) = p(v, w) \cdot \mathbf{1},$$

but these are not equivalent unless $\mathrm{char}\, k = 2$. ♣

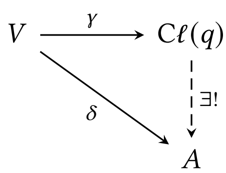


Figure 1: Universal property of the Clifford algebra.

Proposition 15.4 (Construction of the Clifford algebra). *Denote by (TV, i) the tensor algebra of V. Denote by $I_q \subset TV$ the ideal generated by elements of the form*

$$i(v) \otimes i(v) - q(v).$$

Set

$$\mathrm{C}\ell(q) := TV/I_q \quad \textit{and} \quad \gamma := \pi \circ i\colon V \to \mathrm{C}\ell(q)$$

with $\pi\colon TV \to \mathrm{C}\ell(q)$ denoting the canonical projection. $(\mathrm{C}\ell(q), \gamma)$ is a Clifford algebra of q.

Proof. By construction, $\gamma(v)^2 = q(v) \cdot \mathbf{1}$. By the universal property of the tensor algebra, there exists a unique $\tilde{f} \colon TV \longrightarrow A$ such that $\tilde{f} \circ \delta = i$. Since $\delta(v)^2 = q(v) \cdot \mathbf{1}$, $\tilde{f}$ factors through $f \colon \mathrm{C}\ell(q) \longrightarrow A$. Evidently, $f \circ \delta = \gamma$. By the universal property of the quotient, f is unique. ■

Remark 15.5. The proof of Proposition 15.4 is a good exercise to practise the use of universal properties. ♣

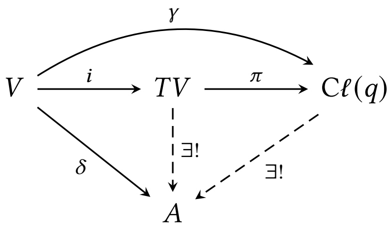


Figure 2: The proof of Proposition 15.4.

Proposition 15.6. *The Clifford algebra* $(\mathrm{C}\ell(q), \gamma)$ *of* q *is unique up to unique isomorphism.* ■

Proposition 15.7. $(\mathrm{C}\ell(0), \gamma) = (\Lambda V, i)$. ■

Proposition 15.8. *Let* q_1, q_2 *be quadratic forms. Denote by* $(\mathrm{C}\ell(q_i), \gamma_i)$ *the Clifford algebra of* q_i *(*$i = 1, 2$*). If* $f \colon q_1 \longrightarrow q_2$ *is a quadratic morphism, then there is a unique algebra homomorphism* $\mathrm{C}\ell(f) \colon \mathrm{C}\ell(q_1) \longrightarrow \mathrm{C}\ell(q_2)$ *such that*

$$\mathrm{C}\ell(f) \circ \gamma_1 = \gamma_2 \circ f.$$ ■

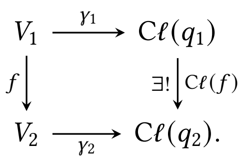


Figure 3: Construction of $\mathrm{C}\ell(f)$.

Remark 15.9. The Clifford algebra defines a functor from the category of quadratic forms to the category of algebras. It is an important invariant of a quadratic form and crucial for understanding spin geometry. ♣

16 The Clifford algebras of $\langle a \rangle$, $\langle a, b \rangle$, and $[a, b]$

Situation 16.1. Let k be a field. ×

Example 16.2. For every $a \in k^\times$

$$\mathrm{C}\ell(\langle a \rangle) = k[i]/(i^2 - a) \quad \text{with} \quad \gamma(x) := xi.$$

More concretely:

(1) Suppose that $\operatorname{char} k \neq 2$. If a has a square root $\sqrt{a} \in k^\times$, then $k[i]/(i^2 - a) \cong k \oplus k$ via

$$x + yi \mapsto (x + \sqrt{a}y, x - \sqrt{a}y).$$

(2) If a does not have a square root in $k^\times$, then $k[i]/(i^2 - a)$ is the quadratic field extension $k(\sqrt{a})$. In particular, for $k = \mathbf{R}$, $\mathrm{C}\ell(\langle -1 \rangle) \cong \mathbf{C}$. ♠

Example 16.3. For every $a, b \in k$

$$\mathrm{C}\ell([a,b]) = \left[\begin{smallmatrix} a,b \\ k \end{smallmatrix}\right] := k\langle i, j\rangle/(i^2 - a, j^2 - b, ij + ji - 1) \quad \text{with} \quad \gamma(x,y) := xi + yj.$$

In particular, $\left[\begin{smallmatrix} 0,0 \\ k \end{smallmatrix}\right] \cong \mathrm{M}_2(k)$ via

$$i \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad j \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$ ♠

Example 16.4. For every $a, b \in k$

$$\mathrm{C}\ell(\langle a,b \rangle) = \left(\begin{smallmatrix} a,b \\ k \end{smallmatrix}\right) := k\langle i, j\rangle/(i^2 - a, j^2 - b, ij + ji) \quad \text{with} \quad \gamma(x,y) := xi + yj.$$

$\left(\begin{smallmatrix} a,b \\ k \end{smallmatrix}\right)$ is the **quaternion algebra**; cf. WP: Quaternion algebra. Here are some concrete instances:

(1) $\left(\begin{smallmatrix} -1,-1 \\ \mathbf{R} \end{smallmatrix}\right) = \mathbf{H}$: **Hamilton's quaterions**; cf. WP: History of the quaternions.

(2) $\left(\begin{smallmatrix} a^2,\pm b^2 \\ k \end{smallmatrix}\right) \cong \mathrm{M}_2(k)$ via

$$i \mapsto \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \quad \text{and} \quad j \mapsto \begin{pmatrix} 0 & b \\ \pm b & 0 \end{pmatrix}.$$ ♠

17 The Clifford algebra of hyperbolic quadratic forms

Let I be a finite-dimensional k–vector space. Set $V := I^* \oplus I$. Define $q \colon V \to k$ by

$$q(\ell, v) := \ell(v).$$

Define $\delta \colon V \to \mathrm{End}(\Lambda I)$ by

$$\delta(\ell, v) = v \wedge \cdot + i_\ell.$$

By direct computation,

$$\delta(\ell, v)^2 = \ell(v)\mathbf{1} = q(\ell, v)\mathbf{1}.$$

This induces an algebra homomorphism $f \colon \mathrm{C}\ell(q) \to \mathrm{End}(\Lambda I)$. Since $\mathrm{End}(\Lambda I)$ is a simple algebra and $\dim \mathrm{C}\ell(q) = \dim \mathrm{End}(\Lambda I)$, f is an isomorphism.

[Can any of this be done more canonical?]

18 Injectivity of $\gamma\colon V \to C\ell(q)$

Proposition 18.1. *The map $\gamma\colon V \to C\ell(q)$ is injective.*

Notation 18.2. In the light of Proposition 18.1, it appears excessive not to drop γ from the notation and tacitly identify $v \in V$ and $\gamma(v) \in C\ell(q)$. ○

Remark 18.3. In principle, it should be possible to prove that $\operatorname{im} i \cap I_q = \{0\}$ by a combinatorial argument. It might even appear to be obvious [ABS64, (1.1); Frioo, §1.2]; however, it is difficult not to mess this up in some way; cf. the notorious argument in [LM89, p.8]. ("Let anyone among you who is without sin be the first to throw a stone [..].") *I would be genuinely interested to see a correct proof along those lines.* The following proof constructs a representation of $C\ell(q)$ in which $\gamma(v)$ obviously acts non-trivially. ♣

Proof of Proposition 18.1. Let $b \in \operatorname{Hom}(V \otimes V, k)$ such that $q = Q(b)$; that is: $q(v) = b(v \otimes v)$; cf. Proposition 10.8. Every $v \in V$ defines an endomorphism $v \wedge \cdot \in \operatorname{End}(\Lambda V)$ of degree 1. Every $\lambda \in V^*$ defines a unique derivation $i_\lambda \in \operatorname{Der}_{-1}(\Lambda V)$ of degree -1 satisfying $i_\lambda(v) = \lambda(v)$; indeed:

$$i_\lambda(v_1 \wedge \cdots \wedge v_k) = \sum_{i=1}^{k} (-1)^{i+1} \lambda(v_i) \cdot v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_k.$$

(This is the differential of the **Koszul complex** associated with λ.) Define $\cdot^\flat\colon V \to V^*$ by

$$v^\flat(w) \coloneqq b(v \otimes w).$$

Define $\delta\colon V \to \operatorname{End}(\Lambda V)$ by

$$\delta(v)\alpha \coloneqq v \wedge \alpha + i_{v^\flat}\alpha$$

Evidently, δ is injective. By direct computation,

$$\delta(v)^2 = q(v) \cdot \mathbf{1}.$$

By the universal property of the Clifford algebra, there is a unique $f\colon C\ell(q) \to \operatorname{End}(\Lambda V)$ such that $f \circ \gamma = \delta$. Therefore, γ is injective. ■

19 Orthogonal maps as automorphisms of the Clifford algebra

Situation 19.1. Let k be a field. Let $q\colon V \to k$ be quadratic form. ×

Proposition 19.2. *The homomorphism $C\ell\colon O(q) \to \operatorname{Aut}(C\ell(q))$ defined by Proposition 15.8 is injective; moreover: $\phi \in \operatorname{im} C\ell$ if and only if $\phi(V) \subset V$.*

Notation 19.3. In the light of Proposition 19.2, it is convenient to identify $O(q)$ as a subgroup of $\operatorname{Aut}(C\ell(q))$. ○

Proof. Only the sufficiency in the second part requires a proof. Suppose $\phi \in \operatorname{Aut}(C\ell(q))$ satisfies $\phi(V) \subset V$. Define $\Phi \in \operatorname{End}(V)$ by $\Phi(v) \coloneqq \phi(v)$. Since ϕ is invertible, $\Phi \in GL(V)$. Moreover, for every $v \in V$

$$q(v)\mathbf{1} = v^2 = \phi(v^2) = \Phi(v)^2 = q(\Phi(v))^2 \cdot 1.$$

Therefore, $\Phi \in O(q)$. Evidently, $\phi = C\ell(\Phi)$. ■

20 Graded algebras

Situation 20.1. Let k be a field. Let G, H be monoids. ×

Definition 20.2. A G–**graded algebra** is an k–algebra A together with a G–**grading**; that is: a direct sum decomposition

$$A = \bigoplus_{g \in G} A_g$$

such that

$$A_g \cdot A_h \subset A_{gh}.$$

An element $a \in A$ is **homogeneous of degree** g if $a \in A_g$. For every $a \in A$ the **homogeneous component of degree** g is the projection a_g of a onto A_g. •

Definition 20.3. A **super algebra** is a $\mathbf{Z}/2\mathbf{Z}$–graded algebra. •

Example 20.4. The tensor algebra TV is $\mathbf{N}_0$–graded; indeed:

$$TV = \bigoplus_{k \in \mathbf{N}_0} TV^k \quad \text{with} \quad TV^k := V^{\otimes k}.$$ ♠

Remark 20.5. Let A be a G–graded algebra. Every homomorphism $f\colon G \to H$ induces a H–grading via

$$A = \bigoplus_{h \in H} A_h \quad \text{with} \quad A_h := \bigoplus_{g \in f^{-1}(h)} A_g.$$ ♣

Example 20.6. Since $\mathbf{N}_0 \to \mathbf{Z} \to \mathbf{Z}/2$, the $\mathbf{N}$–grading on the tensor algebra TV induces $\mathbf{Z}$– and $\mathbf{Z}/2\mathbf{Z}$–gradings. ♠

Definition 20.7. Let A be a G–graded k–algebra. An ideal $I \subset A$ is **homogeneous** if, for every $a \in I$ and $g \in G$, $a_g \in I$. •

Proposition 20.8. *Let A be a G–graded k–algebra. Let A/I be a homogenous ideal. A/I has a unique G–grading such that $\pi\colon A_g \to (A/I)_g$ is surjective.* ■

Remark 20.9. If A, B are G–graded algebras, then $A \otimes B$ is G–graded:

$$(A \otimes B)_g := \bigoplus_{g = hk} A_h \otimes B_k.$$ ♣

In the presence of a G–grading with $G \to \mathbf{Z}/2\mathbf{Z}$, the usual multiplication of the tensor product of algebras often is inappropriate because it violates the Koszul sign convention.

Definition 20.10. Let A, B be $\mathbf{Z}/2\mathbf{Z}$–graded algebras. The **super tensor product** $A \hat{\otimes} B$ is tensor product as a vector space and with the above grading but with the multiplication rule

$$(a_1 \hat{\otimes} b_1)(a_2 \hat{\otimes} b_2) := (-1)^{\deg b_1 \deg a_2} a_1 a_2 \hat{\otimes} b_1 b_2.$$ •

Remark 20.11. If G is a finite group and A, B are G–graded algebras, then $\mathrm{Hom}(A, B)$ is a G–graded vector space:

$$\mathrm{Hom}(A, B) = \bigoplus_{g \in G} \left(\bigoplus_{h \in G} \mathrm{Hom}(A_h, B_{gh}) \right). \qquad \clubsuit$$

If G is not finite or not a group, then the right-hand side might be a proper subspace of $\mathrm{Hom}(A, B)$.

21 The Z/2Z–grading of the Clifford algebra

Situation 21.1. Let k be a field. Let $q\colon V \to k$ be quadratic form. $\times$

Definition 21.2. Denote by TV the tensor algebra of V. Denote by $\pi\colon TV \to \mathrm{C}\ell(q)$ the projection map. Since I_q is homogeneous with respect to the $\mathbf{Z}/2\mathbf{Z}$ grading of TV, $\mathrm{C}\ell(q)$ inherits a $\mathbf{Z}/2\mathbf{Z}$ grading:

$$\mathrm{C}\ell(q)^i := \pi(TV^i) \quad \text{with} \quad TV^i := \bigoplus_{k=0}^{\infty} V^{\otimes 2k+i}. \qquad \bullet$$

Remark 21.3. If $\operatorname{char} k \neq 2$, then

$$\mathrm{C}\ell(q)^0 = \{x \in \mathrm{C}\ell(q) : \mathrm{C}\ell(-\mathbf{1})x = x\} \quad \text{and} \quad \mathrm{C}\ell(q)^0 = \{x \in \mathrm{C}\ell(q) : \mathrm{C}\ell(-\mathbf{1})x = -x\}. \qquad \clubsuit$$

Remark 21.4. If $f\colon q_1 \to q_2$ is a quadratic morphism, then $\mathrm{C}\ell(f)$ is a super algebra homomorphism of degree 0. Therefore, the Clifford algebra defines a functor from the category of quadratic forms to the category of super algebras. $\clubsuit$

Remark 21.5. As rule of thumb, most things involving Clifford algebras should be done in the super way. $\clubsuit$

Remark 21.6. The isomorphism $\mathrm{C}\ell(\langle a^2, \pm b^2 \rangle) \cong \mathrm{M}_2(k) \cong \mathrm{C}\ell([0,0])$ typically are not homomorphisms of super algebras; that is: they disrespect the $\mathbf{Z}/2\mathbf{Z}$ grading. $\clubsuit$

22 The Clifford algebra of a perpendicular sum

Situation 22.1. Let k be a field. $\times$

Proposition 22.2 (Clifford algebra of a perpendicular sum). *Let $q_i\colon V_i \to k$ be quadratic forms ($i = 1, 2$). Denote by $(\mathrm{C}\ell(q_i), \gamma_i)$ the Clifford algebra of q_i ($i = 1, 2$). Set*

$$\gamma := \gamma_1 \hat{\otimes} \mathbf{1} + \mathbf{1} \hat{\otimes} \gamma_2 \colon V_1 \oplus V_2 \to \mathrm{C}\ell(q_1) \hat{\otimes} \mathrm{C}\ell(q_2).$$

$(\mathrm{C}\ell(q_1) \hat{\otimes} \mathrm{C}\ell(q_2), \gamma)$ *is the Clifford algebra of* $q_1 \perp q_2$.

Proof. By direct computation,

$$\begin{aligned}\gamma(v_1 \oplus v_2) &= \big(\gamma_1(v_1) \hat{\otimes} \mathbf{1} + \mathbf{1} \hat{\otimes} \gamma_2(v_2)\big)^2 \\ &= \gamma_1(v_1)^2 \hat{\otimes} \mathbf{1} + \gamma_1(v_1) \hat{\otimes} \gamma_2(v_2) - \gamma_1(v_1) \hat{\otimes} \gamma_2(v_2) + \mathbf{1} \hat{\otimes} \gamma_2(v_2)^2 \\ &= (q_1(v_1) + q_2(v_2))\mathbf{1}.\end{aligned}$$

Let A be an algebra together with a linear map $\delta\colon V_1 \oplus V_2 \to A$ satisfying

$$\delta(v_1 \oplus v_2)^2 = (q_1(v_1) + q_2(v_2))\mathbf{1}.$$

Set $\delta_i \coloneqq \delta|_{V_i}$. By the universal property of the Clifford algebra, there are unique homomorphisms $f_i\colon \mathrm{C}\ell(V_i, q_i) \to A_i$ such that

$$\delta_i = f_i \circ \gamma_i.$$

Define the linear map $f\colon \mathrm{C}\ell(q_1) \hat{\otimes} \mathrm{C}\ell(q_2) \to A$ by

$$f(x_1 \hat{\otimes} x_2) \coloneqq f_1(x_1) f_2(x_2).$$

By construction,

$$\delta = f \circ \gamma;$$

moreover, f is uniquely determined by this condition. It remains to verify that f is an algebra homomorphism. From

$$\delta_1(v_1)\delta_2(v_2) + \delta_2(v_2)\delta_1(v_1) = 0$$

it follows that

$$f_1(x_1) f_2(x_2) = (-1)^{\deg(x_1)\deg(x_2)} f_2(x_2) f_1(x_1).$$

Therefore, f is an algebra homomorphism. ■

Remark 22.3. Proposition 22.2 together with Theorem 11.21, Example 16.2, and Example 16.3 allows for the computation of $\mathrm{C}\ell(q)$ for every quadratic form q. See, e.g., the Computation of the real Clifford algebras and Computation of the complex Clifford algebras. ♣

Remark 22.4. The literature is littered with proofs of Proposition 22.2 that: (1) suppose that $\dim V_i < \infty$, and (2) use the dimension formula for $\mathrm{C}\ell(q_i)$. The advantage of the above Proposition 22.2 proof is that it can be used to establish $\dim \mathrm{C}\ell(q) = 2^{\dim V}$—provided $\dim V < \infty$ by appealing to Proposition 22.2, Theorem 11.21, Example 16.2, and Example 16.3. According to [Knu91], this observation is due to Kneser (but I could not locate the lecture notes that Knus refers to). ♣

Lecture 4

In this lecture, I introduce the filtration on the Clifford algebra. This will be used to prove the dimension formula. A more elegant approach is via the symbol and quantisation maps. The last part of the lecture prepares the computation of the real and complex Clifford algebras.

23 Filtered algebras

Situation 23.1. Let k be a field. ×

Definition 23.2. Let V be a vector space. A **filtration** on V is a subspace $F^r V \subset V$ for every $r \in \mathbf{N}_0$ such that

$$F^r V \subset F^{r+1} V$$

for all $r \in \mathbf{N}_0$ and

$$V = \bigcup_{r \in \mathbf{N}_0} F^r V.$$

A vector space together with a filtration is called a **filtered vector space**. •

Every graded vector space V has a canonical filtration given by

$$F^r V = V^{\leqslant r} := \bigoplus_{s \leqslant r} V^s.$$

Definition 23.3. Given a filtered vector space V, the **associated graded vector space** $\operatorname{Gr} V$ is

$$\operatorname{Gr} V := \bigoplus_{r=0}^{\infty} \operatorname{Gr}^r V \quad \text{with} \quad \operatorname{Gr}^r V := F^r V / F^{r-1} V.$$

Here we use the convention $F^{-1} V = \{0\}$. •

Remark 23.4. If V is a graded vector space, then the associated graded vector space $\operatorname{Gr} V$ of V with the canonical filtration is isomorphic to V. ♣

Remark 23.5. If V is a filtered vector space, then there is a canonical linear map $i\colon V \to \operatorname{Gr} V$. The map i is injective and, hence, an isomorphism if V is finite-dimensional. ♣

Definition 23.6. Let A be a k–algebra. A **filtration** in A is a filtration on the underlying vector space such that

$$F^r A \cdot F^s A \subset F^{r+s} A.$$

•

Remark 23.7. If A is a filtered algebra, then $\operatorname{Gr} A$ inherits the structure of graded algebra. ♣

Definition 23.8. If A is a filtered algebra, then $\operatorname{Gr} A$ is called the **associated graded algebra**. •

Remark 23.9. Let A be a filtered algebra and let I be an ideal in A. Given $r \in \mathbf{N}_0$, define

$$F^r(A/I) := (F^r A)/(I \cap F^r A).$$

This defines a filtration on A/I. ♣

24 The filtration of the Clifford algebra

Definition 24.1. Denote by $\pi\colon TV \to C\ell(q)$ the projection map. The **filtration** of $C\ell(q)$ is defined by

$$F^r C\ell(q) := \pi(F^r TV) \quad \text{with} \quad F^r TV := \bigoplus_{i=0}^{r} V^{\otimes i}. \qquad \bullet$$

Proposition 24.2. *The homomorphism* $\operatorname{Gr}\pi\colon TV = \operatorname{Gr} TV \to \operatorname{Gr} C\ell(V,q)$ *factors through an isomorphism*

$$\kappa\colon \Lambda V \cong \operatorname{Gr} C\ell(V,q).$$

Proof. The map $\operatorname{Gr}\pi\colon TV = \operatorname{Gr} TV \to \operatorname{Gr} C\ell(V,q)$ is surjective. Since

$$\ker \operatorname{Gr}\pi = \bigoplus_{r=0}^{\infty} TV^r \cap (I_q + F^{r-1}TV) = I_0,$$

$\ker \operatorname{Gr}\pi$ factors through κ, and κ is an isomorphism. ■

Since there always is a (non-canonical) vector space isomorphism $\operatorname{Gr} A \cong A$, this implies the following.

Theorem 24.3 (Poincaré–Birkhoff–Witt Theorem for Clifford algebras). $\dim C\ell(q) = 2^{\dim V}$. *Moreover, if* $\{e_i : i \in \{1, \ldots, \dim V\}\}$ *is a basis of* V*, then*

$$\{e_{i_1} \cdots e_{i_r} : 1 \leqslant i_1 < \cdots i_r \leqslant \dim V\}$$

is a basis of $C\ell(q)$.

Remark 24.4. The first section of Theorem 24.3 holds whether $\dim V < \infty$ or not. ♣

Remark 24.5. The original(?) Poincaré–Birkhoff–Witt Theorem is about Lie algebras and their universal enveloping algebras. The terminology seems to have spread to many related situations. ♣

25 The symbol map and the quantisation map

Situation 25.1. Let k be a field. Let $q\colon V \to k$ be quadratic form. ×

Remark 25.2. Proposition 24.2 implies that there exists an vector space isomorphism $C\ell(q) \to \Lambda V$. Indeed, the algebra isomorphism $\operatorname{Gr}(q) \to \Lambda V$ is canonical, but the vector space isomorphism $C\ell(q) \to \operatorname{Gr}(q)$ might not be. Chevalley [Che54, Proof of II.1.2] already observed that a lift of q to a blinear form $b \in \operatorname{Hom}(V \otimes V, k)$ induces a vector space isomorphism $C\ell(q) \to \Lambda V$. The following elegant construction is due to Bourbaki [Bou07, §9]. ♣

Definition 25.3. For every $\lambda \in V^*$ denote by $i_\lambda \in \operatorname{Der}_{-1}(TV)$ the unique derivation of degree -1 such that

$$i_\lambda(v) = \lambda(v).$$

•

Remark 25.4. If $\lambda, \mu \in V^*$, then i_λ and i_μ anti-commute. Indeed, $[i_\lambda, i_\mu] = i_\lambda i_\mu + i_\mu i_\lambda$ is a graded derivation of degree -2 and, therefore, vanishes. ♣

Definition 25.5. Let $b \in \mathrm{Hom}(V \otimes V, k)$. Set $v^\flat(w) := b(v \otimes w)$. Define the algebra homomorphism $\Psi_b \colon TV \to \mathrm{End}(TV)$ by

$$\Psi_b(v)x := v \otimes x + i_{v^\flat}(x).$$

Define $\Theta_b \in \mathrm{End}(TV)$ by

$$\Theta_b(x) := \Psi_b(x)\mathbf{1}.$$

•

Lemma 25.6. *Let $b, b_1, b_2 \in \mathrm{Hom}(V \otimes V, k)$.*

(1) $\Theta_b \colon TV \to TV$ *is uniquely characterised by* $\Theta_b(\mathbf{1}) = \mathbf{1}$ *and*

$$\Theta_b(v \otimes x) = v \otimes \Theta_b(x) + i_{v^\flat}\Theta_b(x)$$

for every $v \in V$ and $x \in TV$.

(2) $\Theta_0 = \mathrm{id}_{TV}$ *and* $\Theta_{b_1} \circ \Theta_{b_2} = \Theta_{b_1+b_2}$*; in particular:* Θ_b *is an isomorphism.*

(3) $\Theta_b I_q \subset I_{q-Q(b)}$*; in particular,* Θ_b *descends to a linear isomorphism*

$$\theta_b \colon \mathrm{C}\ell(V, q) \to \mathrm{C}\ell(V, q - Q(b)).$$

Proof. (1) is obvious.

Evidently, $\Theta_0 = \mathrm{id}_{TV}$. The proof of the second assertion in (2) requires the identity

$$\Theta_b i_\lambda = i_\lambda \Theta_b.$$

The proof of this identity is by induction on the degree. Evidently, it holds on $TV^0 = k$. Moreover, if the identity is know to hold on x, then

$$\begin{aligned}
\Theta_b(i_\lambda(v \otimes x)) &= \Theta_b(\lambda(v)x - v \otimes i_\lambda x) \\
&= \lambda(v)\Theta_b(x) - v \otimes \Theta_b(i_\lambda x) - i_{v^\flat}\Theta_b(i_\lambda x) \\
&= \lambda(v)\Theta_b(x) - v \otimes i_\lambda\Theta_b(x) - i_{v^\flat} i_\lambda \Theta_b(x) \\
&= \lambda(v)\Theta_b(x) - v \otimes i_\lambda\Theta_b(x) + i_\lambda i_{v^\flat} \Theta_b(x) \\
&= i_\lambda(v \otimes \Theta_b(x) + i_{v^\flat}\Theta_b(x)) \\
&= i_\lambda \Theta_b(v \otimes x).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Theta_{b_1}(\Theta_{b_2}(v \otimes x))) &= \Theta_{b_1}(v \otimes \Theta_{b_2}(x) + i_{v^{\flat_2}}\Theta_{b_2}(x)) \\
&= v \otimes \Theta_{b_1}(\Theta_{b_2}(x)) + i_{v^{\flat_1}}\Theta_{b_1}(\Theta_{b_2}(x)) + \Theta_{b_1} i_{v^{\flat_2}}\Theta_{b_2}(x)
\end{aligned}$$

This proves (2).

To prove (3), observe the following. Since

$$i_{v^\flat}(w \otimes w - q(w)) = 0,$$

if $\Theta_b(x) \in I_{q-Q(b)}$, then (by induction) $\Theta_b(v \otimes x) = v \otimes \Theta_b(x) + i_{v^\flat}\Theta_b(x) \in I_{q-Q(b)}$. Finally,

$$\begin{aligned}\Theta_b((v \otimes v - q(v)) \otimes x) &= v \otimes \Theta_b(v \otimes x) + i_{v^\flat}\Theta_b(v \otimes x) - q(v) \otimes \Theta_b(x) \\ &= (v \otimes v - q(v) + b(v \otimes v)) \otimes \Theta_b(x).\end{aligned}$$

■

Corollary 25.7. *If $q = Q(b)$, then the **symbol map***

$$\sigma_b \coloneqq \theta_{-b} \colon \mathrm{C}\ell(V, q) \to \Lambda V$$

*and the **quantisation map***

$$\kappa_b \coloneqq \theta_b \colon \Lambda V \to \mathrm{C}\ell(V, q)$$

are vector space isomorphisms (and inverses of each other). ■

Remark 25.8. See `https://empg.maths.ed.ac.uk/Activities/Spin/SpinNotes.pdf` for some background on on the terminology. ♣

This immediately implies Theorem 24.3.

Remark 25.9. If char $k \neq 2$, then $b = \frac{1}{2}p$ is the canonical choice to define $\sigma \coloneqq \sigma_\beta$ and $\kappa \coloneqq \kappa_b$. ♣

26 Artin–Wedderburn Theorem

Remark 26.1. *What does it mean to determine an algebra?* A key reason to care about algebras are their modules. The Artin–Wedderburn theorem says that the structure of finite-dimensional semi-simple algebras is determined by their simple modules. ♣

Situation 26.2. Let k be a field. Let A be a k–algebra. ×

Definition 26.3.

(1) An A**–module** is a k–vector space V together with an algebra homomorphism $A \to \mathrm{End}(V)$.

(2) An A**–submodule** W of V is an is a k–vector subspace such that $xW \subset W$ for every $x \in A$.

(3) An A–module V is **simple** if 0 and V are its only submodule.

(4) If V, W are A–modules, then

$$\mathrm{Hom}_A(V, W) \coloneqq \{f \colon V \to W : f \text{ is } A\text{–linear}\}.$$

(5) The **commuting algebra** of a module V is

$$\mathrm{End}_A(V) \coloneqq \mathrm{Hom}_A(V, V).$$

•

Lemma 26.4 (Schur's Lemma). *Let V, W be simple A–modules. If $f \in \mathrm{Hom}_A(V, W)$, then $f = 0$ or f is invertible.* ■

Corollary 26.5. *If V is an simple A–module, then $\mathrm{End}_A(V)$ is a division algebra over k; that is, every non-zero $x \in \mathrm{End}_A(V)$ is invertible.* ■

Theorem 26.6 (Frobenius). *If D is a finite-dimensional division algebra over $\mathbf{R}$, then D is isomorphic to either $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$.*

Proposition 26.7. *If k is an algebraically closed field, e.g., $k = \mathbf{C}$, then any division algebra over k is isomorphic to k.*

Definition 26.8. The **Jacobson radical** of A is

$$J(A) \coloneqq \{x \in A : xV = 0 \text{ for every simple } A\text{–module } V\}.$$

A is **semi-simple** if $J(A) = 0$. •

Remark 26.9. $J(A)$ is an ideal of A. ♣

Example 26.10. Let V be a vector space. The Jacobson radical of ΛV is

$$J(\Lambda V) = \bigoplus_{k \geqslant 1} \Lambda^k V;$$

that is: the ideal generated by $V \subset \Lambda V$. To see this, let $x \in V$ and let W be an simple module of ΛV. Set $\ker x \coloneqq \{w \in W : xw = 0\}$ and $\mathrm{im}\, x \coloneqq xW$. Since ΛV is graded commutative, $\ker x$ and $\mathrm{im}\, x$ are submodule of W. Since W is simple, $\ker x = 0$ or $\ker x = W$. Since $x^2 = 0$, $\mathrm{im}\, x \subset \ker x$. This forces, $\ker x = W$. ♠

Theorem 26.11 (Artin–Wedderburn Theorem). *Suppose that* $\dim A < \infty$.

(1) *A has only finitely many simple modules $V_1, \ldots, V_r$ (up to isomorphism) and each V_i is finite-dimensional.*

(2) *If D_i denotes the commuting algebra of V_i, then*

$$A/J(A) \cong \prod_{i=1}^{r} \mathrm{End}_{D_i}(V_i).$$ ■

See Representation theory of finite groups.

27 Frobenius' theorem on real division algebras

Definition 27.1. Let k be a field. A k–algebra D is a **division algebra** if it has no zero divisors. •

Theorem 27.2 (Frobenius [Fro77, last paragraph]). *If D is a finite-dimensional real division algebra, then D is isomorphic to either $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$.*

Proof. Since D is a division algebra, left multiplication defines an inclusion

$$D \hookrightarrow \mathrm{End}(D).$$

Set $d := \dim D$ and

$$\mathrm{Im}\, D := \{x \in D : x^2 \in \mathbf{R} \text{ and } x^2 \leqslant 0\}.$$

Lemma 27.3. $\mathrm{Im}\, D = \ker(\mathrm{tr}\colon\ D \to \mathbf{R})$. *In particular:* $D = \mathbf{R} \oplus \mathrm{Im}\, D$.

Proof. Let $x \in D \hookrightarrow \mathrm{End}(D)$. Denote the minimal and the characteristic polynomial of x by $\mu \in \mathbf{R}[\lambda]$ and $\chi \in \mathbf{R}[\lambda]$ respectively. Decompose χ into irreducible factors as follows

$$\chi(\lambda) = \prod_{i=1}^{a}(\lambda - r_i) \cdot \prod_{j=1}^{b}(\lambda - s_i)(\lambda - \bar{s}_i)$$

with

$$r_1, \ldots, r_a \in \mathbf{R} \quad \text{and} \quad s_1, \bar{s}_1, \ldots, s_b, \bar{s}_b \in \mathbf{C}\backslash\mathbf{R}.$$

By the Cayley–Hamilton theorem,

$$\chi(x) = 0 \in \mathrm{End}(D).$$

Since D is a division algebra, the minimal polynomial μ is one of the irreducible factors of χ. Since μ and χ have the same roots, either:

(1) $\chi = \mu^d$ and $\mu(\lambda) = \lambda - r$ with $r \in \mathbf{R}$, or

(2) $\chi = \mu^{d/2}$ and $\mu(\lambda) = (\lambda - s)(\lambda - \bar{s}) = \lambda^2 - 2\,\mathrm{Re}\, s \cdot \lambda + |s|^2$ with $s \in \mathbf{C}\backslash\mathbf{R}$.

In the former case, for $x \in \mathbf{R}$ the assertion is obvious because $\mathbf{R} \cap \mathrm{Im}\, D = 0 = \ker(\mathrm{tr}\colon\ \mathbf{R} \to \mathbf{R})$. Since

$$\chi(\lambda) = \lambda^d - \mathrm{tr}(x)\lambda^{d-1} + \cdots,$$

in the latter case,

$$\mathrm{tr}(x) = -d\,\mathrm{Re}(s).$$

Therefore, if $\mathrm{tr}(x) = 0$, then $\mathrm{Re}\, s = 0$; hence: $x^2 = -|s|^2 \leqslant 0$. Conversely, if $x^2 = -t^2$ with $t \in \mathbf{R}$, then $\mu(\lambda) = \lambda^2 + t^2$; hence: $\mathrm{tr}(\lambda) = 0$. ■

Define the quadratic form $q\colon\ \mathrm{Im}\, D \to \mathbf{R}$ by

$$q(x) := -x^2.$$

Denote the polarisation of q by p. Set $b := \frac{1}{2}p$. By construction, b is a Euclidean inner product on $\mathrm{Im}\, D$. Let $S \subset \mathrm{Im}\, D$ be a minimal subspace which generates D as an $\mathbf{R}$–algebra. Let $e_1, \ldots, e_s$ be an orthonormal basis of S. By construction,

$$e_i^2 = -1 \quad \text{and} \quad e_i e_j + e_j e_i = 0 \quad \text{and} \quad (i \neq j \in \{1, \ldots, s\}).$$

Evidently:

(1) If $n = 0$, then $D \cong \mathbf{R}$.

(2) If $n = 1$, then $D \cong \mathbf{C}$.

(3) If $n = 2$, then $D \cong \mathbf{H}$.

Finally, if $n \geqslant 3$, then $x := e_1 e_2 e_3 \neq \pm 1$ satisfies

$$0 = x^2 - 1 = (x+1)(x-1).$$

This contradicts D being a division algebra. ■

Remark 27.4. Palais [Pal68] has another short elementary proof. ♣

Remark 27.5. Frobenius' theorem on real division algebras implies that the Brauer group $B(\mathbf{R})$ agrees with C_2 and is generated by $\mathbf{H}$: $\mathbf{H} \otimes \mathbf{H} = \mathrm{M}_4(\mathbf{R}) \sim \mathbf{R}$ with $\sim$ denoting Morita equivalence. ♣

28 Representation theory of finite groups

Situation 28.1. Let k be a field. Let G be a finite group. ×

Theorem 28.2 (Maschke's Theorem). *If* $\operatorname{char} k$ *does not divide* $|G|$*, then* $k[G]$ *is semi-simple.*

Proof. Let V be a $k[G]$–module. Let $W \subset V$ be submodule. Let $\pi \in \operatorname{End}(V)$ be a k–linear projection onto W. By averaging over G, π can be assumed to be G–invariant; that is: $k[G]$–linear. Therefore, $W' := \ker \pi$ is a complementary $k[G]$–submodule.

By the above, $k[G]$ decomposes into simple modules $k[G] = \bigoplus_{i=1}^r V_i$. Every non-zero $x \in k[G]$ acts non-trivially on $k[G]$ and thus on at least one of the simple modules V_i. Consequently, $J(k[G]) = \{0\}$. ■

Corollary 28.3. *If* $\operatorname{char} k$ *does not divide* $|G|$*, then* G *has only finitely many irreducible representations* $V_1, \ldots, V_r$*, and*

$$k[G] \cong \prod_{i=1}^r \operatorname{End}_{D_i}(V_i).$$ ■

For comparison, Serre [Ser77] is a classical reference on the representation theory of finite groups.

Lecture 5

The goal of this lecture is to compute the Clifford algebras of all non-degenerate quadratic forms over $\mathbf{R}$ and $\mathbf{C}$. Our computation more or less follows [ABS64] and [LM89, Chapter I §4], which is quite straight-forward and a lot of fun. There are many alternative methods to arrive at the same result; see, e.g., Roe [Roe98, p. 59] for an approach using the representation theory of finite groups due to to J.F. Adams [Roe98, p. 68]. (An important consequence of the computation is a mod 8 (resp. mod 2) periodicity—related to Bott periodicity).

As a result of our computation one can read off the simple modules of the Clifford algebra. (These govern the coarse theory of spinors over pseudo-Riemannian manifolds.) Spelling out what they are is a simple exercise whose solution is written up in the lecture notes. You should look at this and make sure you understand why this works. In the lecture, I will only briefly discuss the role of the volume element.

29 Computation of the real Clifford algebras

[LM89, Chapter I §4]

Definition 29.1. For $r, s \in \mathbf{N}_0$ set

$$\mathrm{C}\ell_{r,s} \coloneqq \mathrm{C}\ell(q_{r,s}) \quad \text{with} \quad q_{r,s} \coloneqq \langle 1\rangle^{\perp r} \perp \langle -1\rangle^{\perp s} \quad \text{defined over} \quad \mathbf{R}. \qquad \bullet$$

Theorem 29.2. *For every $r, s \in \{0, \dots, 7\}$, $\mathrm{C}\ell_{r,s}$ is as in Table 1. Moreover, for every $r, s \in \mathbf{N}_0$*

$$\mathrm{C}\ell_{r+8,s} \cong \mathrm{C}\ell_{r,s} \otimes \mathrm{M}_{16}(\mathbf{R}) \quad \text{and} \quad \mathrm{C}\ell_{r,s+8} \cong \mathrm{C}\ell_{r,s} \otimes \mathrm{M}_{16}(\mathbf{R}).$$

Lemma 29.3. *Let $q\colon V \to k$ be a quadratic form with $\dim V < \infty$. For every $a, b \in k$*

$$\mathrm{C}\ell(q \perp \langle a, b\rangle) \cong \mathrm{C}\ell(-(ab)^{-1} \cdot q) \otimes \mathrm{C}\ell(\langle a, b\rangle).$$

Proof. Denote by (e_1, e_2) the standard basis of $k^{\oplus 2}$. Define $\delta\colon V \oplus k^{\oplus 2} \to \mathrm{C}\ell(-(ab)^{-1} \cdot q) \otimes \mathrm{C}\ell(\langle \pm 1\rangle^{\perp 2})$ by

$$\delta(v, x_1, x_2) \coloneqq v \otimes e_1 e_2 + 1 \otimes x_1 e_1 + 1 \otimes x_2 e_2.$$

Since

$$\delta(v, x_1, x_2)^2 = q(v) + ax^2 + by^2,$$

there is a unique algebra homomorphism $f\colon \mathrm{C}\ell(q \perp \langle a, b\rangle) \to \mathrm{C}\ell(-(ab)^{-1} \cdot q) \otimes \mathrm{C}\ell(\langle a, b\rangle)$. Since f maps onto a set of generators, it is surjective. For dimension reasons it also injective and, hence, an algebra isomorphism. ∎

Proposition 29.4.

(1) $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C} \cong \mathbf{C} \oplus \mathbf{C}$,

(2) $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{H} \cong \mathrm{M}_2(\mathbf{C})$, *and*

$C\ell_{r,s}$	$r = 0$	1	2	3	4	5	6	7
$s = 0$	$\mathbf{R}$	$\mathbf{R}^{\oplus 2}$	$M_2(\mathbf{R})$	$M_2(\mathbf{C})$	$M_2(\mathbf{H})$	$M_2(\mathbf{H})^{\oplus 2}$	$M_4(\mathbf{H})$	$M_8(\mathbf{C})$
1	$\mathbf{C}$	$M_2(\mathbf{R})$	$M_2(\mathbf{R})^{\oplus 2}$	$M_4(\mathbf{R})$	$M_4(\mathbf{C})$	$M_4(\mathbf{H})$	$M_4(\mathbf{H})^{\oplus 2}$	$M_8(\mathbf{H})$
2	$\mathbf{H}$	$M_2(\mathbf{C})$	$M_4(\mathbf{R})$	$M_4(\mathbf{R})^{\oplus 2}$	$M_8(\mathbf{R})$	$M_8(\mathbf{C})$	$M_8(\mathbf{H})$	$M_8(\mathbf{H})^{\oplus 2}$
3	$\mathbf{H}^{\oplus 2}$	$M_2(\mathbf{H})$	$M_4(\mathbf{C})$	$M_8(\mathbf{R})$	$M_8(\mathbf{R})^{\oplus 2}$	$M_{16}(\mathbf{R})$	$M_{16}(\mathbf{C})$	$M_{16}(\mathbf{H})$
4	$M_2(\mathbf{H})$	$M_2(\mathbf{H})^{\oplus 2}$	$M_4(\mathbf{H})$	$M_4(\mathbf{C})$	$M_{16}(\mathbf{R})$	$M_{16}(\mathbf{R})^{\oplus 2}$	$M_{32}(\mathbf{R})$	$M_{32}(\mathbf{C})$
5	$M_4(\mathbf{C})$	$M_4(\mathbf{H})$	$M_4(\mathbf{H})^{\oplus 2}$	$M_8(\mathbf{H})$	$M_8(\mathbf{C})$	$M_{32}(\mathbf{R})$	$M_{32}(\mathbf{R})^{\oplus 2}$	$M_{64}(\mathbf{R})$
6	$M_8(\mathbf{R})$	$M_8(\mathbf{C})$	$M_8(\mathbf{H})$	$M_8(\mathbf{H})^{\oplus 2}$	$M_{16}(\mathbf{H})$	$M_{16}(\mathbf{C})$	$M_{64}(\mathbf{R})$	$M_{64}(\mathbf{R})^{\oplus 2}$
7	$M_8(\mathbf{R})^{\oplus 2}$	$M_{16}(\mathbf{R})$	$M_{16}(\mathbf{C})$	$M_{16}(\mathbf{H})$	$M_{16}(\mathbf{H})^{\oplus 2}$	$M_{32}(\mathbf{H})$	$M_{32}(\mathbf{C})$	$M_{128}(\mathbf{R})$

Table 1: The periodic table of Clifford algebras.

(3) $\mathbf{H} \otimes_{\mathbf{R}} \mathbf{H} \cong \mathrm{M}_4(\mathbf{R})$.

Proof. The isomorphism $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C} \to \mathbf{C} \oplus \mathbf{C}$ is given by

$$z \otimes w \mapsto zw \oplus z\bar{w}.$$

Identifying $\mathbf{H} = \mathbf{C} \oplus \mathbf{C}j = \mathbf{C}^2$, $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{H}$ acts on $\mathbf{C}^2$ via

$$(z \otimes q) \cdot v = zv\bar{q}.$$

This action is $\mathbf{C}$–linear. A computation shows that the resulting map $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{H} \to \mathrm{End}_{\mathbf{C}}(\mathbf{C}^2) \cong \mathrm{M}_2(\mathbf{C})$ is an isomorphism.

Identifying $\mathbf{H} = \mathbf{R}^4$, $\mathbf{H} \otimes_{\mathbf{R}} \mathbf{H}$ acts on $\mathbf{R}^4$ via

$$(p \otimes q) \cdot v = pv\bar{q}.$$

This action is $\mathbf{C}$–linear. A computation shows that the resulting map $\mathbf{H} \otimes_{\mathbf{R}} \mathbf{H} \to \mathrm{End}_{\mathbf{R}}(\mathbf{R}^4) \cong \mathrm{M}_4(\mathbf{R})$ is an isomorphism. ■

Corollary 29.5. *For every $r, s \in \mathbf{N}_0$:*

(1) $C\ell_{r+1,s+1} \cong C\ell_{r,s} \otimes \mathrm{M}_2(\mathbf{R})$.

(2) $C\ell_{r+2,s} \cong C\ell_{s,r} \otimes \mathrm{M}_2(\mathbf{R})$.

(3) $C\ell_{r,s+2} \cong C\ell_{s,r} \otimes \mathbf{H}$.

(4) $C\ell_{r+4,s} \cong C\ell_{r+4,s} \cong C\ell_{r,s} \otimes \mathrm{M}_2(\mathbf{H})$.

(5) $C\ell_{r+8,s} \cong C\ell_{r+8,s} \cong C\ell_{r,s} \otimes \mathrm{M}_{16}(\mathbf{R})$. ■

Proof of Theorem 29.2. It remains to determine the entries of Table 1 by the following procedure:

(1) Example 16.2 and Example 16.4 determine

$$C\ell_{0,0} \cong \mathbf{R}, \quad C\ell_{1,0} \cong \mathbf{R}^{\oplus 2}, \quad C\ell_{0,1} \cong \mathbf{C}, \quad C\ell_{2,0} \cong C\ell_{1,1} \cong \mathrm{M}_2(\mathbf{R}), \quad \text{and} \quad C\ell_{0,2} \cong \mathbf{H}.$$

(2) $C\ell_{r+2,0} \cong C\ell_{0,r} \otimes \mathrm{M}_2(\mathbf{R}) \cong C\ell_{r+2,0}$ determines $C\ell_{3,0}$.

(3) $C\ell_{0,s+2} \cong C\ell_{s,0} \otimes \mathbf{H}$ determines $C\ell_{0,3}$.

(4) $C\ell_{r+4,0} \cong C\ell_{0,r} \otimes \mathrm{M}_2(\mathbf{H})$ determines $C\ell_{r,0}$ for every $r \in \mathbf{N}_0$.

(5) $C\ell_{0,s+4} \cong C\ell_{0,s} \otimes \mathrm{M}_2(\mathbf{H})$ determines $C\ell_{0,s}$ for every $s \in \mathbf{N}_0$.

(6) $C\ell_{r+1,s+1} \cong C\ell_{r,s} \otimes \mathrm{M}_2(\mathbf{R})$ determines the $C\ell_{r,s}$ for every $r, s \in \mathbf{N}_0$. ■

Remark 29.6. Theorem 29.2 immediately determines the commuting algebra of $C\ell_{r,s}$ up to isomorphism (of $\mathbf{R}$–algebras):

$$D_{r,s} := \mathrm{End}_{C\ell_{r,s}}(C\ell_{r,s}) \cong \begin{cases} \mathbf{R} & \text{if } r-s = 0,2 \bmod 8 \\ \mathbf{C} & \text{if } r-s = 3,7 \bmod 8 \\ \mathbf{H} & \text{if } r-s = 4,6 \bmod 8 \\ \mathbf{R}\oplus\mathbf{R} & \text{if } r-s = 1 \bmod 8 \\ \mathbf{H}\oplus\mathbf{H} & \text{if } r-s = 5 \bmod 8. \end{cases}$$

In particular:

(1) If $r-s = 3,6 \bmod 8$, then every irreducible representation P of $C\ell_{r,s}$ admits complex structure; that is: an homomorphism $\rho\colon \mathbf{C} \to \mathrm{End}_{C\ell_{r,s}}(P)$ of $\mathbf{R}$–algebras. Since $\mathrm{Aut}(\mathbf{C}) = \mathrm{O}(2) = C_2$—generated by complex conjugation $\bar{\cdot}$, there is no distinguished complex structure on P. In fact, ρ and $\rho' := \rho \circ \bar{\cdot}$ are not equivalent; that is: there is no $C \in \mathrm{End}_{C\ell_{r,s}}(P)^\times \cong \mathbf{C}^\times$ satisfying $\rho' = C \circ \rho \circ C^{-1}$.

(2) If $r-s = 4,5,6 \bmod 8$, then every irreducible representation P of $C\ell_{r,s}$ admits quaternionic structure $\rho\colon \mathbf{H} \to \mathrm{End}_{C\ell_{r,s}}(P)$. Since $\mathrm{Aut}(\mathbf{H}) = \mathrm{SO}(3)$, there is no distinguished quaternionic structure either. However, for every $\phi \in \mathrm{Aut}(\mathbf{H})$, there is a $y \in \mathbf{H}^\times$ such that $\phi(\cdot) = y \cdot y^{-1}$. Therefore, ρ and $\rho' := \rho \circ \phi$ are equivalent via $\Phi := \rho(y)$.

(3) If $r-s = 1,5 \bmod 8$, then $C\ell_{r,s}$ has two nonequivalent irreducible representations P, P'. Again, among those two none is distinguished.

It turns out that a choice of orientation of $\mathbf{R}^{r+s}$ resolves the above ambiguities. ♣

Remark 29.7. Here is a slightly more abstract perspective of the salient point of Remark 29.6.

Let A be an k–algebra. Let V be an A–module. Denote by $D := \mathrm{End}_A(V) \subset \mathrm{End}_k(V)$ the commuting algebra of V. Evidently, V can be regarded as $D \otimes_k A$–module.

By Schur's lemma, D is a division algebra if V is simple. If $k = \mathbf{R}$, then by Frobenius' Theorem on real division algebras D is *isomorphic* to $K \in \{\mathbf{R}, \mathbf{C}, \mathbf{H}\}$. Therefore, V can be regarded as a $K \otimes_k A$–module after a choice of isomorphism $D \cong K$ has been made: $K \otimes_k A \cong D \otimes_k A \to \mathrm{End}_k(V)$. If $\phi \in \mathrm{Aut}(K)$, then $K \otimes_k A \overset{\phi}{\cong} K \otimes_k A \cong D \otimes_k A \to \mathrm{End}_k(V)$ yields another structure of a $K \otimes_k A$–module on V. If ϕ is an inner automorphism, then these structures are isomorphic. In general, there are (up to) $\mathrm{Out}(K) = \mathrm{Aut}(K)/\mathrm{Inn}(K)$ many non-equivalent choices. ♣

Remark 29.8. The above discussion is somewhat unsatisfactory in that it treats $C\ell_{r,s}$ as an algebra an not a superalgebra. Section 33 determines $C\ell^0_{r,s}$. This ameliorates the situation to an extend. This approach is common in the literature on spin geometry, but it would probably be better to carry out the computation with the supermindeset. Lam [Lam05, Chapter V §4] shows how it should be done. *Oh well!* ♣

30 Computation of the complex Clifford algebras

Definition 30.1. For $r \in \mathbf{N}_0$ set

$$\mathbf{C}\ell_r := C\ell(\langle 1\rangle^{\perp r}) \quad \text{defined over} \quad \mathbf{C}.$$ •

Remark 30.2. By Proposition 11.17 every non-degenerate quadratic form $q\colon V \to \mathbf{C}$ with $r := \dim V < \infty$ is isomorphic to $\langle 1\rangle^{\perp r}$. ♣

Theorem 30.3. *For every $r \in \mathbf{N}_0$*

$$\mathbf{C}\boldsymbol{\ell}_r \cong \begin{cases} \mathrm{M}_{2^{r/2}}(\mathbf{C}) & \text{if } r \text{ is even} \\ \mathrm{M}_{2^{(r-1)/2}}(\mathbf{C})^{\oplus 2} & \text{if } r \text{ is odd.} \end{cases}$$

Lemma 29.3 and Example 16.4 immediately imply the following.

Corollary 30.4. *For every $r \in \mathbf{N}_0$*

$$\mathbf{C}\boldsymbol{\ell}_{r+1} \cong \mathbf{C}\boldsymbol{\ell}_r \otimes_{\mathbf{C}} \mathrm{M}_2(\mathbf{C}). \qquad ■$$

Proof of Theorem 30.3. $\mathbf{C}\boldsymbol{\ell}_0 \cong \mathbf{C}$ and Example 16.2 determines $\mathbf{C}\boldsymbol{\ell}_1 \cong \mathbf{C} \oplus \mathbf{C}$. Therefore, Corollary 30.4 finishes the proof. ■

Remark 30.5. Of course, for $\mathbf{C}\boldsymbol{\ell}_r$ only the ambiguity discussed in Remark 29.6 (3) remains (if r is odd). ♣

31 Pinor modules

Situation 31.1. Let $r, s \in \mathbf{N}_0$. Let $\mathrm{C}\ell_{r,s}$ be as in Definition 29.1. Suppose that an orientation on $\mathbf{R}^{\oplus(r+s)}$ has been chosen. ×

Definition 31.2. The **volume element**

$$\omega := e_1 \cdots e_{r+s} \in \mathrm{C}\ell_{r,s}$$

with $e_1, \dots, e_{r+s}$ denoting a positive orthonormal basis for $q_{r,s}$. (A moment's thought reveals that ω does not depend on the choice.) •

Proposition 31.3. *For every homogeneous $x \in \mathrm{C}\ell_{r,s}$*

$$\omega x = (-1)^{(\deg \omega + 1)\deg x} x\omega;$$

in particular: if $r + s$ is odd, then

$$\omega \in Z(\mathrm{C}\ell_{r,s}).$$

Moreover:

$$\omega^2 = \begin{cases} \mathbf{1} & \text{if } r - s = 0, 1 \bmod 4 \\ -\mathbf{1} & \text{if } r - s = 2, 3 \bmod 4. \end{cases}$$

Proof. By direct computation, for every $v \in V$

$$v\omega = (-1)^{r+s-1}\omega v.$$

This implies the first assertion. Moreover,

$$\begin{aligned} e_1 \cdots e_{r+s} \cdot e_1 \cdots e_{r+s} &= (-1)^{r+s-1} e_1^2 e_2 \cdots e_{r+s} \cdot e_2 \cdots e_{r+s} \\ &= (-1)^{\frac{(r+s)(r+s-1)}{2}} e_1^2 e_2^2 \cdots e_{r+s}^2 \\ &= (-1)^{\frac{(r+s)(r+s-1)}{2}+s} \end{aligned}$$

Finally, observe that

$$(r+s)(r+s-1) + 2s = (r-s)(r-s-1) \bmod 4. \qquad \blacksquare$$

Remark 31.4. This resolves the ambiguities in Remark 29.6 (1) and (3). ♣

Definition 31.5.

(1) If $r - s = 0, 2 \bmod 8$, then the **pinor module** P is the irreducible $C\ell_{r,s}$–module.

(2) If $r - s = 1 \bmod 8$, then the **positive pinor module** P^+ and **negative pinor module** P^- are the irreducible $C\ell_{r,s}$–module on which ω acts as $\mathbf{1}$ and $-\mathbf{1}$ respectively.

(3) If $r - s = 3, 7 \bmod 8$, then the **pinor module** P and the **conjugate pinor module** $\bar{P}$ are the irreducible $\mathbf{C} \otimes C\ell_{r,s}$–modules on which ω acts as i and $-i$ respectively.

(4) If $r - s = 4, 6 \bmod 8$, then the **pinor module** P is the irreducible $\mathbf{H} \otimes C\ell_{r,s}$–module.

(5) If $r - s = 5 \bmod 8$, then the **positive pinor module** P^+ and **negative pinor module** P^- are the irreducible $\mathbf{H} \otimes C\ell_{r,s}$–module on which ω acts as $\mathbf{1}$ and $-\mathbf{1}$ respectively. •

Remark 31.6. By Theorem 29.2, the pinor modules exist and are unique up to isomorphism. ♣

32 Complex pinor modules

Situation 32.1. Let $r \in \mathbf{N}_0$. Let $\mathbf{C\ell}_r$ be as in Definition 30.1. Suppose that an orientation on $\mathbf{R}^{\oplus r}$ has been chosen. ×

Definition 32.2. The **complex volume element**

$$\omega^{\mathbf{C}} := i^{\lfloor \frac{r+1}{2} \rfloor} e_1 \cdots e_{r+s} \in \mathbf{C\ell}_r$$

with $e_1, \ldots, e_{r+s}$ denoting a positive orthonormal basis. (A moment's though reveals that $\omega^{\mathbf{C}}$ does not depend on the choice.) •

Proposition 32.3. *If r is odd, then $\omega^{\mathbf{C}} \in Z(\mathbf{C\ell}_r)$; moreover, for every r, $(\omega^{\mathbf{C}})^2 = \mathbf{1}$.* ■

Definition 32.4.

(1) If r is even, then the **complex pinor module** P is the irreducible $\mathbf{C\ell}_r$–modules.

(2) If r is odd, then the **positive complex pinor module** P^+ and **negative complex pinor module** P^- are the irreducible $\mathbf{C\ell}_r$–modules on which $\omega^{\mathbf{C}}$ acts as $\mathbf{1}$ and $-\mathbf{1}$ respectively. •

Remark 32.5. By Theorem 30.3, the complex pinor modules exist and are unique up to isomorphism. ♣

Lecture 6

The pinor modules over $C\ell_{r,s}$ from the last lecture can also be considered as modules of the even subalgebra $C\ell^0_{r,s} \subset C\ell_{r,s}$. The simple modules of $C\ell^0_{r,s}$ are the spinor modules. It is important to understand how the pinor and spinor modules are related. Again, this is largely an exercise given the classification derived in the last lecture. The lecture notes contain the answer to this exercise, but in the lecture I will only dicuss a few examples to illustrate how this is done.

The second part of this lecture works towards the construction of spin group $\mathrm{Spin}(q)$. First I will explain that if q is non-degenerate, then $C\ell(q)$ is a CSSA, a central simple superalgebra. Then I will discuss the Clifford group $\Gamma(q)$ and the special Clifford group $S\Gamma(q)$, a precursor of $\mathrm{Spin}(q)$.

33 The even subalgebra of the Clifford algebra

Situation 33.1. Let k be a field. Let $q\colon V \to k$ be a quadratic form with $\dim V < \infty$. ×

Proposition 33.2. *For every $a \in k^\times$*

$$C\ell(q \perp \langle a\rangle)^0 \cong C\ell(V, -a^{-1} \cdot q).$$

Proof. Set $e := (0,1) \in V \oplus k$. Define $\delta\colon V \to C\ell(q \perp \langle a\rangle)^0$ by

$$\delta(v) := ev.$$

Since

$$\delta(v) = evev = -e^2v^2 = q(v),$$

there is a unique algebra homomorphism $f\colon C\ell(q) \to C\ell(q \perp \langle a\rangle)^0$ with $\delta = f \circ \gamma$. Evidently, f is surjective (it maps to a set of generators). Multiplication with e induces a vector space isomorphism $C\ell(q \perp \langle a\rangle)^0 \cong C\ell(q \perp \langle a\rangle)^1$. Consequently, $\dim C\ell(q \perp \langle a\rangle)^0 = 2^{\dim V+1}/2 = \dim C\ell(q)$. Therefore, f is an isomorphism. ■

Proposition 33.3. *For every $r, s \in \mathbf{N}_0$*

$$C\ell^0_{r+1,s} \cong C\ell_{s,r} \quad \textit{and} \quad C\ell^0_{r,s+1} \cong C\ell_{r,s}.$$ ■

Remark 33.4. Proposition 33.3 explains the symmetry $C\ell_{r+1,s} \cong C\ell_{s+1,r}$ apparent from Table 1. ♣

Proposition 33.5. *For every $r, s \in \mathbf{N}_0$*

$$C\ell^0_{r,s} \cong C\ell_{s,r-1} \cong C\ell^0_{s,r}.$$ ■

34 Spinor modules

Situation 34.1. Let $r, s \in \mathbf{N}_0$. Let $C\ell_{r,s}$ be as in Definition 29.1. Suppose that an orientation on $\mathbf{R}^{\oplus(r+s)}$ has been chosen. Denote the volume element by $\omega \in C\ell_{r,s}$. ×

Definition 34.2.

(1) If $r - s = 0 \bmod 8$, then the **positive spinor module** S^+ and the **negative spinor module** S^- are the irreducible $C\ell^0_{r,s}$–module on which ω acts as $\mathbf{1}$ and $-\mathbf{1}$ respectively.

(2) If $r - s = 1, 7 \bmod 8$, then the **spinor module** S is the irreducible $C\ell^0_{r,s}$–module.

(3) If $r - s = 2, 6 \bmod 8$, then the **spinor module** S and the **conjugate spinor module** $\bar{S}$ are the irreducible $\mathbf{C} \otimes C\ell^0_{r,s}$–modules on which ω acts as i and $-i$ respectively.

(4) If $r - s = 3, 5 \bmod 8$, then the **spinor module** S is the irreducible $\mathbf{H} \otimes C\ell^0_{r,s}$–module.

(5) If $r - s = 4 \bmod 8$, then the **positive spinor module** S^+ and the **negative spinor module** S^- are the irreducible $\mathbf{H} \otimes C\ell^0_{r,s}$–module on which ω acts as $\mathbf{1}$ and $-\mathbf{1}$ respectively. •

Remark 34.3. By Theorem 29.2, the spinor modules exist and are unique up to isomorphism. ♣

Remark 34.4. The symmetry of Definition 34.2 under exchanging r and s is a consequence of the isomorphism $C\ell^0_{r,s} \cong C\ell^0_{s,r}$. ♣

35 Decomposition of pinor into spinor modules

Proposition 35.1.

(0) If $r - s = 0 \bmod 8$, *then*

(a) $P = S^+ \oplus S^-$ *as* $C\ell^0_{r,s}$*–modules, and*

(b) $\gamma\colon\ \mathbf{R}^{\oplus(r+s)} \longrightarrow \mathrm{End}(S^+, S^-) \oplus \mathrm{End}(S^-, S^+)$.

(1) If $r - s = 1 \bmod 8$, *then* $P^\pm = S$ *as* $C\ell^0_{r,s}$*–modules.*

(2) If $r - s = 2 \bmod 8$, *then*

(a) $\mathbf{C} \otimes_{\mathbf{R}} P = S \oplus \bar{S}$ *as* $\mathbf{C} \otimes C\ell^0_{r,s}$*–modules and*

(b) $\mathbf{1}_{\mathbf{C}} \otimes_{\mathbf{R}} \gamma\colon\ \mathbf{R}^{\oplus(r+s)} \longrightarrow \mathrm{End}_{\mathbf{C}}(S, \bar{S}) \oplus \mathrm{End}_{\mathbf{C}}(\bar{S}, S)$.

(3) If $r - s = 3 \bmod 8$, *then*

(a) $P = \bar{P} = S$ *as* $\mathbf{C} \otimes C\ell^0_{r,s}$*–modules and*

(b) $\gamma\colon\ \mathbf{R}^{\oplus(r+s)} \longrightarrow \mathrm{End}_{\mathbf{C}}(S)$.

(4) If $r - s = 4 \bmod 8$, *then*

(a) $P = S^+ \oplus S^-$ *as* $\mathbf{H} \otimes C\ell^0_{r,s}$*–modules, and*

(b) $\gamma\colon \mathbf{R}^{\oplus(r+s)} \to \mathrm{End}_{\mathbf{H}}(S^+, S^-) \oplus \mathrm{End}_{\mathbf{H}}(S^-, S^+)$.

(5) *If* $r - s = 5 \bmod 8$, *then*

(a) $P^\pm = S$ *as* $\mathbf{H} \otimes \mathrm{C}\ell^0_{r,s}$*–modules, and*

(b) $\gamma\colon \mathbf{R}^{\oplus(r+s)} \to \mathrm{End}_{\mathbf{H}}(S)$.

(6) *If* $r - s = 6 \bmod 8$, *then*

(a) $P = \mathbf{H} \otimes_{\mathbf{C}} S = S \oplus \bar{S}$ *as* $\mathbf{H} \otimes \mathrm{C}\ell^0_{r,s}$*–modules, and*

(b) $\gamma\colon \mathbf{R}^{\oplus(r+s)} \to \mathrm{Hom}_{\mathbf{C}}(S, \bar{S}) \oplus \mathrm{Hom}_{\mathbf{C}}(\bar{S}, S)$

(7) *If* $r - s = 7 \bmod 8$, *then*

(a) $P = \mathbf{C} \otimes_{\mathbf{R}} S$ *as* $\mathbf{C} \otimes \mathrm{C}\ell^0_{r,s}$*–modules, and*

(b) $\gamma\colon \mathbf{R}^{\oplus(r+s)} \to \mathbf{1}_{\mathbf{C}} \otimes \mathrm{End}(S)$.

36 Complex spinor modules

Situation 36.1. Let $r \in \mathbf{N}_0$. Let $\mathbf{C}\boldsymbol{\ell}_r$ be as in Definition 30.1. Suppose that an orientation on $\mathbf{R}^{\oplus r}$ has been chosen. Denote the complex volume element by $\omega^{\mathbf{C}} \in \mathbf{C}\boldsymbol{\ell}_r$. ×

Definition 36.2.

(1) If r is even, then the **positive complex pinor module** S^+ and **negative complex pinor module** S^- are the irreducible $\mathbf{C}\boldsymbol{\ell}^0_r$–modules on which $\omega^{\mathbf{C}}$ acts as $\mathbf{1}$ and $-\mathbf{1}$ respectively.

(2) If r is odd, then the **complex spinor module** S is the irreducible $\mathbf{C}\boldsymbol{\ell}^0_r$–modules. •

Remark 36.3. By Theorem 30.3, the complex spinor modules exist and are unique up to isomorphism. ♣

37 Decomposition of complex pinor into complex spinor modules

Situation 37.1. Let $r \in \mathbf{N}_0$. Let $\mathbf{C}\boldsymbol{\ell}_r$ be as in Definition 30.1. Suppose that an orientation on $\mathbf{R}^{\oplus r}$ has been chosen. Denote the complex volume element by $\omega^{\mathbf{C}} \in \mathbf{C}\boldsymbol{\ell}_r$. Consider the complex pinor and complex spinor modules; see Definition 32.4 and Definition 36.2. ×

Proposition 37.2.

(1) *If* r *is even, then*

(a) $P = S^+ \oplus S^-$ *as* $\mathbf{C}\boldsymbol{\ell}^0_r$*–modules, and*

(b) $\gamma\colon \mathbf{C}^{\oplus r} \to \mathrm{Hom}_{\mathbf{C}}(S^+, S^-) \oplus \mathrm{Hom}_{\mathbf{C}}(S^-, S^+)$.

(2) *If* r *is odd, then* $P^+ = P^- = S$ *as* $\mathbf{C}\boldsymbol{\ell}^0_r$*–modules.*

38 The Lipschitz group

Situation 38.1. Let k be a field. Let $q\colon V \to k$ be a quadratic form. Let S be a $\mathrm{C}\ell(q)$–module. ×

Definition 38.2. The **Lipschitz group** of S is

$$L(S) := \{x \in \mathrm{GL}(S) : x\gamma(V)x^{-1} \subset \gamma(V) \subset \mathrm{End}(S)\}.$$ •

Remark 38.3. For $k = \mathbf{C}$ the Lipschitz group was introduced by Friedrich and Trautman [FT00, Definition 4]. It mediates the Clifford module perspective and the principal bundle perspective on spin geometry. Trautman [Tra08] contains a very clear and concise discussion of the role of the (complex) Lipschitz group (even for real spinors). Lazaroiu and Shahbazi [LS19] develop the real theory in exuberant detail—indeed; the theory turns out to be suprisingly intricate. ♣

Definition 38.4 (Lazaroiu and Shahbazi [LS19, Definition 3.9]). S is **weakly faithful** if the map $\gamma\colon V \to \mathrm{C}\ell(q) \to \mathrm{End}(S)$ is injective. •

Proposition 38.5. *If S is weakly faithful, then there is a unique homomorphism* $\mathrm{Ad}\colon L(S) \to \mathrm{O}(q)$ *such that*

$$\gamma(\mathrm{Ad}(x)v) = x\gamma(v)x^{-1};$$

that is: $\gamma\colon V \to \mathrm{End}(S)$ is $L(S)$–equivariant.

Proof. Evidently, the above condition defines a homomorphism $L(S) \to \mathrm{GL}(V)$. Since

$$q(\mathrm{Ad}(x)v) = \gamma(\mathrm{Ad}(x)v)^2 = \left(x\gamma(v)x^{-1}\right)^2 = x\gamma(v)^2x^{-1} = q(v),$$

it factors through $\mathrm{O}(q)$. ■

Proposition 38.6. *If S is weakly faithful, then*

$$D^\times = \ker(\mathrm{Ad}\colon L(S) \to \mathrm{O}(q)).$$

with $D := \mathrm{End}_{\mathrm{C}\ell(q)}(S)$ denoting the commuting algebra of S. ■

Remark 38.7. Determining $\mathrm{im}(\mathrm{Ad}\colon L(S) \to \mathrm{O}(q))$ is a bit more involved; cf. [LS19, Theorem 4.9]. ♣

39 The supercentre of $\mathrm{C}\ell(q)$

Situation 39.1. Let k be a field. ×

Definition 39.2. Let A be a k–superalgebra. The **supercommutator** is the bilinear map $[\cdot,\cdot]\colon A \otimes A \to A$ defined by

$$[x, y] := xy - (-1)^{\deg x \cdot \deg y} yx$$

for homogeneous $x, y \in A$. The **supercenter** of A is

$$\hat{Z}(A) := \{x \in A : [x, y] = 0 \text{ for every } y \in A\}.$$

A is **supercentral** if $\hat{Z}(A) = k$. •

Lemma 39.3. *If A, B are k–superalgebras, then*

$$\hat{Z}(A \hat{\otimes} B) = \hat{Z}(A) \hat{\otimes} \hat{Z}(B).$$

Proof. Evidently, $\hat{Z}(A) \hat{\otimes} \hat{Z}(B) \subset \hat{Z}(A \hat{\otimes} B)$.

If $z \in \hat{Z}(A \hat{\otimes} B)$ is homogeneous, then

$$z = \sum_{i=1}^{r} a_i \hat{\otimes} b_i.$$

with $a_1, \ldots, a_r \in A$ and $b_1, \ldots, b_r \in B$ linearly independent and homogeneous. By direct computation, for every $a \in A$ and $b \in B$

$$0 = [z, a \hat{\otimes} \mathbf{1}] = \sum_{i=1}^{r} (-1)^{\deg b_i} [a_i, a] \otimes b_i \quad \text{and} \quad 0 = [z, \mathbf{1} \hat{\otimes} b] = \sum_{i=1}^{r} (-1)^{\deg b_i} a_i \otimes [b_i, b].$$

Therefore, $a_1, \ldots, a_r \in \hat{Z}(A)$ and $b_1, \ldots, b_r \in \hat{Z}(B)$. ■

(The above argument is from [Lam05, Chapter IV Theorem 2.3(1)].)

Theorem 39.4. *If q is non-degenerate and* $\dim V < \infty$*, then*

$$\hat{Z}(\mathrm{C}\ell(q)) = k.$$

Proof. In the light of Lemma 39.3, Proposition 11.17 and Theorem 11.21 (2) it consider the following two cases:

(1) Consider $\langle a \rangle \colon V := k \to k$ with $a \in k^\times$ and $\operatorname{char} k \neq 2$. Set $e := 1 \in V$. By direct computation, for every $x = x_0 + x_1 e \in \mathrm{C}\ell(\langle a \rangle)$ $(x_0, x_1 \in k)$

$$[x, e] = 2x_1 a.$$

Therefore, $\hat{Z}(\mathrm{C}\ell(\langle a \rangle)) = k$.

(2) Consider $[a, b] \colon V := k^{\oplus 2} \to k$ with $a, b \in k$ and $\operatorname{char} k = 2$. Set $e_1 := (1, 0), e_2 := (0, 1) \in V$. By direct computation, for every $x = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_1 e_2 \in \mathrm{C}\ell([a, b])$ $(x_0, x_1, x_2, x_3 \in k)$

$$[x, e_1] = x_2 + x_3 e_1 \quad \text{and} \quad [x, e_2] = x_1 + x_3 e_2.$$

Therefore, $\hat{Z}(\mathrm{C}\ell([a, b]) = k$. ■

Remark 39.5. Although the above proof is quite straight-forward, I would be interested in a more conceptual proof. It might not be reasonable to expect one to exists, however. ♣

40 Tensor products of supercentral supersimple superalgebras

Situation 40.1. Let k be a field. ×

Definition 40.2. Let A be a k–superalgebra. A is **supersimple** if it has no non-trivial proper homogeneous ideals. •

Lemma 40.3. *Let A, B be k–superalgebras. If A is a supercentral and supersimple and B is supercentral, then $A \hat{\otimes} B$ is supersimple.*

Proof. Let $I \subset A \hat{\otimes} B$ be a non-trivial proper homogenous ideal. If $z \in I$ is homogeneous, then

$$z = \sum_{i=1}^{r} a_i \hat{\otimes} b_i.$$

with $a_1, \ldots, a_r \in A$ and $b_1, \ldots, b_r \in B$ linearly independent, homogeneous, and $\deg a_i + \deg b_i = \deg z$. Choose a non-zero homogeneous $z \in I$ of with minimal r.

Since A is supersimple, the homogeneous ideal (a_1) generated by a_1 is A. Therefore, there are homogeneous $c_1, d_1, \cdots, c_s, d_s \in A$ with

$$\mathbf{1} = \sum_{j=1}^{s} c_j a_1 d_j$$

and $\deg c_i + \deg c_i = \deg a_1$. By construction,

$$z' := \sum_{j=1}^{s} (c_j \hat{\otimes} \mathbf{1}) z (d_j \hat{\otimes} \mathbf{1}) = \pm \mathbf{1} \hat{\otimes} b_1 + \sum_{i=2}^{r} a_i' \otimes b_i \in I.$$

Since $\deg c_i + \deg c_i = \deg a_1$, a_i' is homogeneous. Since $b_1, \ldots, b_r$ are linearly independent, $z' \neq 0$. By the minimality of r, $\pm\mathbf{1}, a_2', \ldots, a_r'$ are linearly independent.

An analogous construction with b_1 instead of a_1 construct a non-zero homogeneous

$$z'' = \mathbf{1} \hat{\otimes} \mathbf{1} + \sum_{i=2}^{r} a_i' \otimes b_i' \in I.$$

with $\mathbf{1}, b_2', \ldots, b_r'$ linearly independent and homogeneous. Evidently, $\deg a_i = \deg b_i$.

If $r = 1$, then $I \supset (\mathbf{1} \hat{\otimes} \mathbf{1}) = A \hat{\otimes} B$—contradicting that I is proper. Therefore, $r \geqslant 2$.

For every $b \in B$

$$[z', \mathbf{1} \hat{\otimes} b] = \sum_{i=2}^{r} a_i' \otimes [b_i', b] \in I.$$

By the minimality of r, $[z', \mathbf{1} \hat{\otimes} b] = 0$. Since $a_2', \ldots, a_r'$ are linearly independent, $[b_i', b] = 0$. Therefore and since B is supercentral, $b_i' \in k$—contradicting that $\mathbf{1}, b_2', \ldots, b_r'$ are linearly independent. ■

(The above argument is from [Lam05, Chapter IV Theorem 2.3(2)].)

Theorem 40.4. *If q is non-degenerate, then $\mathrm{C}\ell(q)$ is supersimple.*

Proof. In the light of Lemma 40.3, Proposition 11.17 and Theorem 11.21 (2) it consider the following two cases:

(1) Consider $\langle a\rangle\colon V := k \to k$ with $a \in k^\times$. Set $e := 1 \in V$. Let $I \subset \mathrm{C}\ell(\langle a\rangle)$ be a proper homogeneous ideal. Let $x = x_0 + x_1 e \in I$ ($x_0, x_1 \in k$). Since I is homogeneous,

$$I \supset (x_0) \quad \text{and} \quad I \supset (x_1 e) \supset (x_1 e^2) = (x_1).$$

Therefore and since I is proper, $x = 0$. Hence, $I = 0$.

(2) Consider $[a, b]\colon V := k^{\oplus 2} \to k$ with $a, b \in k$ and $\operatorname{char} k = 2$. Set $e_1 := (1, 0), e_2 := (0, 1) \in V$. Let $I \subset \mathrm{C}\ell([a, b])$ be a proper homogeneous ideal. Let $x = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_1 e_2 \in I$ ($x_0, x_1, x_2, x_3 \in k$). Since I is homogeneous,

$$I \supset (x_0), \quad I \supset (x_1 e_1 + x_2 e_3), \quad \text{and} \quad I \supset (x_3 e_1 e_2).$$

Indeed, since

$$[x, e_1] = x_2 + x_3 e_1, \quad [x, e_2] = x_1 + x_3 e_2, \quad \text{and} \quad [[x, e_1], e_2] = x_3,$$

and I is homogeneous, $I \supset (x_0)$, $I \supset (x_1)$, $I \supset (x_2)$, and $I \supset (x_3)$. Therefore and since I is proper, $x = 0$. Hence, $I = 0$. ■

Remark 40.5. Although the above proof is quite straight-forward, I would be interested in a more conceptual proof. It might not be reasonable to expect one to exists, however. ♣

41 The Clifford group

Situation 41.1. Let k be a field. Let $q\colon V \to k$ be a quadratic form. ×

Definition 41.2. Let A be a k–superalgebra. The supergroup of **homogenous units** of A is

$$A^{h\times} := (A^0 \cup A^1) \cap A^\times.$$ •

Definition 41.3. The **twisted adjoint representation** is the homomorphism $\widetilde{\mathrm{Ad}}\colon \mathrm{C}\ell^0(q)^{h\times} \to \mathrm{Aut}(\mathrm{C}\ell(q))$ defined by

$$\widetilde{\mathrm{Ad}}(x)y := (-1)^{\deg x \cdot \deg y} x y x^{-1}.$$

The **Clifford group** $\Gamma(q)$ is

$$\Gamma(q) := \left\{x \in \mathrm{C}\ell(q)^{h\times} : \widetilde{\mathrm{Ad}}(x) \in \mathrm{O}(q)\right\}.$$ •

Proposition 41.4. *The diagram*

$$\begin{array}{ccc} \Gamma(q) & \xrightarrow{\widetilde{\mathrm{Ad}}} & \mathrm{O}(q) \\ \big\downarrow & & \big\downarrow \\ \mathrm{C}\ell(q)^{h\times} & \xrightarrow{\widetilde{\mathrm{Ad}}} & \mathrm{Aut}(\mathrm{C}\ell(q)). \end{array}$$

is a pullback diagram. ■

Proposition 41.5. *If q is non-degenerate and* $\dim V < \infty$*, then*

$$0 \to k^\times \to \Gamma(q) \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{O}(q) \to 0$$

is exact.

Proof. By Theorem 39.4, $\ker \widetilde{\mathrm{Ad}} = k^\times \subset \Gamma(q)$. If $v \in V$ is anisotopic, then $v \in \Gamma(q)$ and, by direct computation,

$$\widetilde{\mathrm{Ad}}(v) = r_v \in \mathrm{O}(q).$$

Therefore, the Cartan–Dieudonné Theorem implies that $\widetilde{\mathrm{Ad}}(\Gamma(q)) = \mathrm{O}(q)$—except possibly in the case $q \cong [1,1] \perp [1,1]$ with $k = \mathbf{F}_2$. According to Remark 14.6, it remains to find $x \in \Gamma(q)$ such that $\widetilde{\mathrm{Ad}}(x)$ interchanges the summands $[1,1]$. A direct computation reveals that

$$x := \mathbf{1} + e_1e_2 + e_1e_4 + e_2e_3 + e_3e_4$$

does just that.[1] ■

Remark 41.6. As a consequence of the proof, $\Gamma(q)$ is generated by anisotopic vectors (and x in the exceptional case $q \cong [1,1] \perp [1,1]$ with $k = \mathbf{F}_2$). ♣

Remark 41.7. The use of the Cartan–Dieudonné Theorem in the proof of Proposition 41.5 can be (and probably should be) replaced with an application of a the Skolem–Noether theorem for supercentral supersimple superalgebras; cf. Elduque and Villa [EV08, Proposition 8]. ♣

Remark 41.8.

(1) Chevalley [Che54, §2.3] considered the **adjoint representation** $\mathrm{Ad}\colon \tilde{\Gamma}(q) \to \mathrm{O}(q)$ defined by

$$\gamma(\mathrm{Ad}(x)v) := x\gamma(v)x^{-1}$$

with $\tilde{\Gamma}(q) := Z(\mathrm{C}\ell(q))^\times\Gamma(q)$; cf. [LS19, §1.18]. Of course, if $\deg\colon \Gamma(q) \to \mathbf{Z}/2\mathbf{Z}$ denotes the grading on $\Gamma(q)$, then

$$\widetilde{\mathrm{Ad}} = (-1)^{\deg} \mathrm{Ad}|_{\Gamma(q)}.$$

Therefore, Proposition 41.5 does not (quite) hold with Ad instead of $\widetilde{\mathrm{Ad}}$; cf. [Che54, p. II.3.1].

(2) Atiyah, Bott, and Shapiro [ABS64, Part I §3] observed that signs might be in order because $\mathrm{C}\ell(q)$ is a superalgebra, and introduced $\widetilde{\mathrm{Ad}}\colon \Gamma(q) \to \mathrm{O}(q)$—although with an a priori different definition of $\Gamma(q)$.

(3) Atiyah, Bott, and Shapiro only defined $\widetilde{\mathrm{Ad}}\colon \Gamma(q) \to \mathrm{O}(q)$ and hid the degree by using the involution $\alpha := \mathrm{C}\ell(-\mathbf{1})$. In doing so they have *laid a trap!* Most geometers (e.g.: [LM89, Chapter II (2.11); Har90, (10.9); Roe98, p.57]) define $\widetilde{\mathrm{Ad}}\colon \mathrm{C}\ell(q)^\times \to \mathrm{Aut}(\mathrm{C}\ell(q))$ with two defects: (a) the domain is too large, and (b) $\deg y$ is not taken into account. This is ultimately almost inconsequential, but it fails to take $\mathrm{C}\ell(q)$ seriously as a superalgebra. (I only realised this after reading [Knu91, Chapter IV (6.1)].) ♣

[1] I found this by a brute-force search using Sage.

42 The special Clifford group

Situation 42.1. Let k be a field. Let $q\colon V \to k$ be a quadratic form. ×

Definition 42.2. The **special Clifford group** is

$$S\Gamma(q) := \Gamma(q) \cap C\ell(q)^0.$$

The **special orthogonal group** is

$$SO(q) := \widetilde{Ad}(S\Gamma(q)) < O(q).$$ •

Remark 42.3. If char $k \neq 2$, then by Theorem 14.5 $SO(q) = \ker(\det\colon O(q) \to \{\pm 1\})$. If char $k = 2$, then the latter is clearly unsatisfactoy and the above is the correct definition. It turns out that even in char $k = 2$, there is a homomorphism $D\colon O(q) \to \mathbf{Z}/2\mathbf{Z}$, the Dickson invariant, such that $SO(q) = \ker D$; cf. [Knu91, Chapter IV §5]. ♣

Remark 42.4. On $S\Gamma(q)$, the twisted adjoint representation $\widetilde{Ad}$ agress with the adjoint representation Ad. ♣

Proposition 42.5. *The diagram*

$$\begin{array}{ccc} S\Gamma(q) & \xrightarrow{Ad} & SO(q) \\ \downarrow & & \downarrow \\ C\ell(q)^{0\times} & \xrightarrow{Ad} & \mathrm{Aut}(C\ell(q)^0) \end{array}$$

is a pullback diagram. ■

Lecture 7

The goal of this lecture is complete the construction of $\mathrm{Spin}(q)$ and investigate it in a bit more detail for non-degenerate real quadratic forms.

43 The spinor norm

Situation 43.1. Let k be a field. Let $q\colon V \to k$ be a non-degenerate quadratic form. ×

Definition 43.2. The **transposition** $\cdot^t\colon C\ell(q) \to C\ell(q)$ the unique anti-involution that extends id_V. •

Proposition 43.3. *For every $x \in \Gamma(q)$, $x^t x \in k^\times$.*

Proof. Since $\cdot^t$ extends id_V, for every $x \in \Gamma(q)$ and $v \in V$

$$xvx^{-1} = (xvx^{-1})^t = (x^t)^{-1}vx^t.$$

Therefore, $x^t x \in \ker \widetilde{\mathrm{Ad}} = k^\times$. ■

Definition 43.4. The **norm** is the homomorphism $N\colon \Gamma(q) \to k^\times$ defined by

$$N(x) := x^t x.$$ •

Remark 43.5. Since $N(x) \in k^\times$, it evidently is a homomorphism; moreover, $N(x) = xx^t$. For some reason the latter is more commonly used as the initial definition. In the literature, one also finds a version of the norm with $\cdot^t$ replaced by the anti-involution $\bar{\cdot}$ that extends $-\mathrm{id}_V$. I prefer the above, because $N(v) = q(v)$ for every anisotropic $v \in V$. ♣

Definition 43.6. The **spinor norm** $N\colon \mathrm{O}(q) \to k^\times/(k^\times)^2$ is the homomorphism induced by the norm. Set

$$\Omega(q) := \ker N \subset \mathrm{O}(q) \quad \text{and} \quad S\Omega(q) := \mathrm{SO}(q) \cap \Omega(q).$$ •

The norm and the spinor norm fit in the following commutative diagrams with exact rows:

$$\begin{array}{ccccc} k^\times & \hookrightarrow & \Gamma(q) & \twoheadrightarrow & \mathrm{O}(q) \\ \downarrow{\scriptstyle \cdot^2} & & \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle N} \\ (k^\times)^2 & \hookrightarrow & k^\times & \twoheadrightarrow & k^\times/(k^\times)^2 \end{array} \quad \text{and} \quad \begin{array}{ccccc} k^\times & \hookrightarrow & S\Gamma(q) & \twoheadrightarrow & \mathrm{SO}(q) \\ \downarrow{\scriptstyle \cdot^2} & & \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle N} \\ (k^\times)^2 & \hookrightarrow & k^\times & \twoheadrightarrow & k^\times/(k^\times)^2. \end{array} \tag{43.7}$$

Example 43.8. Let $r, s \in \mathbf{N}_0$. Consider $q_{r,s} := \langle 1\rangle^{\perp r} \perp \langle -1\rangle^{\perp s}$ defined over $\mathbf{R}$. Set

$$\mathrm{O}_{r,s} := \mathrm{O}(q_{r,s}).$$

$\mathrm{O}_{r,s}$ has $2^{c(r,s)}$ connected components with

$$c(r,s) := \begin{cases} 0 & \text{if } r = s = 0 \\ 1 & \text{if } (r \geqslant 1 \text{ and } s = 0) \text{ or } (r = 0 \text{ and } s \geqslant 1) \\ 2 & \text{if } r, s \geqslant 1. \end{cases}$$

By the Cartan–Dieudonné Theorem, for every $T \in O_{r,s}$ there are $v_i^{\pm}$ with $\pm q(v_i^{\pm}) > 0$ ($i \in \{1, \dots, a_{\pm}\}$) such that

$$T = r_{v_1^+} \dots r_{v_{a_+}^+} r_{v_1^-} \dots r_{v_{a_-}^-}.$$

Identifying $\mathbf{R}^\times/(\mathbf{R}^\times)^2 = \{\pm 1\}$,

$$\det T = (-1)^{a_+ + a_-} \quad \text{and} \quad N(T) = (-1)^{a_-}.$$

Therefore, if $r = 0$, then $N = \det$, if $s = 0$, then $N = \mathbf{1}$. A moment's thought shows that

$$\det \times N \colon \pi_0(O_{r,s}) \hookrightarrow \{\pm 1\}^2.$$

In particular, $S\Omega(q_{r,s})$ is the identity component of $O_{r,s}$. ♠

Example 43.9. Consider the above in the Lorentzian signature $(r, s) = (1, 3)$. A **Lorentz transformation** is a $T \in O_{1,3}$. T is **proper** if $\det T = 0$; that is: $T \in SO_{1,3}$.

A vector $v \in \mathbf{R}^4$ is **time-like**, **space-like**, and **light-like** if $q(v) > 0$, $q(v) < 0$, and $q(v) = 0$ respectively. The **light-cone** $q^{-1}(0)$ separates $\mathbf{R}^4$ into three connected components:

(1) the **future light-cone** $\{v \in \mathbf{R}^4 : q(v) > 0, v_0 > 0\}$,

(2) the **past light-cone** $\{v \in \mathbf{R}^4 : q(v) > 0, v_0 < 0\}$, and

(3) the **elsewhere** $\{v \in \mathbf{R}^4 : q(v) < 0\}$.

A $T \in O_{r,s}$ might swap the future and past light-cones. If it does not, it is **orthochronous**.

A moment's thought shows that $S\Omega(q_{r,s}) = SO^+_{1,3}$, the group of proper, orthochronous Lorentz transformations.

The quotient $O_{1,3}/SO^+_{1,3}$ is isomorphic to the subgroup of $O_{1,3}$ generated by

$$T = \operatorname{diag}(-1, 1, 1, 1) \quad \text{and} \quad P = \operatorname{diag}(1, -1, -1, -1).$$ ♠

44 Pin(q) and Spin(q)

Situation 44.1. Let k be a field. Let $q \colon V \to k$ be a non-degenerate quadratic form. ×

Definition 44.2. The **pin group** associated with q is the group

$$\operatorname{Pin}(q) := \ker(N \colon \Gamma(q) \to k^\times)$$

The **spin group** associated with q is the group

$$\operatorname{Spin}(q) := \operatorname{Pin}(q) \cap S\Gamma(q).$$ •

Proposition 44.3. *Consider $\{\pm 1\} \subset k^\times$. The sequences*

$$0 \to \{\pm 1\} \to \operatorname{Pin}(q) \xrightarrow{\widetilde{\operatorname{Ad}}} \Omega(q) \to 0$$

and

$$0 \to \{\pm 1\} \to \operatorname{Spin}(q) \xrightarrow{\operatorname{Ad}} S\Omega(q) \to 0$$

are exact.

Remark 44.4 (Pin(q) and Spin(q) if char $k = 2$). If char $k = 2$, then $\{\pm 1\} \subset k^\times$ is trivial; hence, $\mathrm{Pin}(q) = \Omega(q)$ and $\mathrm{Spin}(q) = S\Omega(q)$. ♣

Proof of Proposition 44.3. Since $\{\pm 1\} = \ker(\,\cdot^2\colon\; k^\times \to (k^\times)^2)$, the commutative diagrams with exact rows (43.7) extend to the commutative diagrams with exact rows:

$$\begin{array}{ccccc} \{\pm 1\} & \hookrightarrow & \mathrm{Pin}(q) & \twoheadrightarrow & \Omega(q) \\ \downarrow & & \downarrow & & \downarrow \\ k^\times & \hookrightarrow & \Gamma(q) & \twoheadrightarrow & \mathrm{O}(q) \\ \downarrow{\scriptstyle (\cdot)^2} & & \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle N} \\ (k^\times)^2 & \hookrightarrow & k^\times & \twoheadrightarrow & k^\times/(k^\times)^2 \end{array} \quad \text{and} \quad \begin{array}{ccccc} \{\pm 1\} & \hookrightarrow & \mathrm{Spin}(q) & \twoheadrightarrow & S\Omega(q) \\ \downarrow & & \downarrow & & \downarrow \\ k^\times & \hookrightarrow & S\Gamma(q) & \twoheadrightarrow & \mathrm{SO}(q) \\ \downarrow{\scriptstyle (\cdot)^2} & & \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle N} \\ (k^\times)^2 & \hookrightarrow & k^\times & \twoheadrightarrow & k^\times/(k^\times)^2. \end{array} \quad ■$$

Remark 44.5.

(1) There are numerous and sometimes inequivalent definitions of Pin(q) and Spin(q) in the literature. The above agrees with that found in most references on algebraic groups. *I believe that this is the correct definition, and I also believe that the other definitions are wrong and misguided.*

(2) Unless $k = \mathbf{F}_2$, $\dim V = 4$, and $i(q) = 2$,

$$\mathrm{Pin}(q) = \left\{ \lambda v_1 \cdots v_a \in \mathrm{C}\ell(q)^\times : \lambda^2 \prod_{i=1}^{a} q(v_i) = 1 \right\} \quad \text{and}$$
$$\mathrm{Spin}(q) = \left\{ \lambda v_1 \cdots v_{2a} \in \mathrm{C}\ell(q)^\times : \lambda^2 \prod_{i=1}^{2a} q(v_i) = 1 \right\}.$$

This is sometimes used to define Pin(q) and Spin(q). Indeed, in light of the Cartan–Dieudonné Theorem this is a rather sensible approach. But, of course, it does give the wrong answer in the exceptional case of the Cartan–Dieudonné Theorem.

(3) References on spin geometry (cf. [LM89, (2.25)]) often use

$$\widetilde{\mathrm{Pin}}(q) := \ker(N^2\colon\; \Gamma(q) \to (k^\times)^2) \quad \text{and} \quad \widetilde{\mathrm{Spin}}(q) := \widetilde{\mathrm{Pin}}(q) \cap S\Gamma(q)$$

instead of Definition 44.2, sometimes in the guise of a variation of (2). The apparent advantage is that $\widetilde{\mathrm{Ad}}$ maps $\widetilde{\mathrm{Pin}}(q)$ and $\widetilde{\mathrm{Spin}}(q)$ onto O(q) and SO(q) respectively. This, however, can also be seen as a disadvantage because it prematurely gives up the freedom the twist over the non-identity component of $\mathrm{SO}_{r,s}$; see Dąbrowski [Dąb88] and **??**.

(4) Fortunately, in the setting relevant to Riemannian geometry (that is: $k = \mathbf{R}$ and negative definite q), none of the above makes any difference for Spin(q). However, neither $\widetilde{\mathrm{Pin}}(q)$ nor Pin(q) might be appropriate for physics; cf. Janssens [Jan20]. ♣

Proposition 44.6. *Let S be weakly faithful $\mathrm{C}\ell(q)$–module. Denote by $L(S)$ the Lipschitz group of S. The action of $\mathrm{C}\ell(q)^{0\times}$ on S induces a group homomorphism $\mathrm{Spin}(q) \to L(S)$ such that*

$$\begin{array}{ccc} \mathrm{Spin}(q) & \longrightarrow & L(s) \\ \downarrow{\scriptstyle \mathrm{Ad}} & & \downarrow{\scriptstyle \mathrm{Ad}} \\ S\Omega(q) & \hookrightarrow & \mathrm{O}(q) \end{array}$$

commutes. ■

45 The spin group of a perpendicular sum

Situation 45.1. Let k be a field. Let $q_i\colon V_i \to k$ $(i = 1, 2)$ be non-degenerate quadratic forms. ×

The following is important to establish the "2 out of 3 principle" for spin structures.

Proposition 45.2.

(1) *The isomorphism $\mathrm{C}\ell(q_1) \hat{\otimes} \mathrm{C}\ell(q_2) \cong \mathrm{C}\ell(q_1 \perp q_2)$ (constructed in Proposition 22.2) induces an inclusion $\iota\colon \mathrm{Spin}(q_1) \times_{\{\pm 1\}} \mathrm{Spin}(q_2) \hookrightarrow \mathrm{Spin}(q_1 \perp q_2)$.*

(2) *The canonical inclusion $\bar{\jmath}\colon \mathrm{O}(q_1)\times\mathrm{O}(q_2) \hookrightarrow \mathrm{O}(q_1 \perp q_2)$ restricts to an inclusion $\jmath\colon S\Omega(q_1)\times S\Omega(q_q) \hookrightarrow S\Omega(q_1 \perp q_2)$.*

(3) *The diagram*

$$\begin{array}{ccc} \mathrm{Spin}(q_1) \times_{\{\pm 1\}} \mathrm{Spin}(q_2) & \overset{\iota}{\hookrightarrow} & \mathrm{Spin}(q_1 \perp q_2) \\ \downarrow{\scriptstyle \mathrm{Ad}} & & \downarrow{\scriptstyle \mathrm{Ad}} \\ S\Omega(q_1) \times S\Omega(q_2) & \overset{\jmath}{\hookrightarrow} & S\Omega(q_1 \perp q_2). \end{array}$$

is a pullback diagram.

46 $\mathrm{Spin}_{r,s}$

Situation 46.1. Let $r, s \in \mathbf{N}_0$. Set $q_{r,s} := \langle 1\rangle^{\perp r} \perp \langle -1\rangle^{\perp s}$ defined over $\mathbf{R}$. Set $\mathrm{SO}^+_{r,s} := S\Omega(q_{r,s}) = \ker N \cap \mathrm{SO}(q_{r,s})$. ×

Definition 46.2. Set

$$\mathrm{Spin}_{r,s} := \mathrm{Spin}(q_{r,s}).$$ •

Notation 46.3. Abbreviate

$$\mathrm{Spin}_n = \mathrm{Spin}(n) = \mathrm{Spin}_{0,n}.$$ ∘

Proposition 46.4.

(1) $\mathrm{Spin}_{r,s} \subset (\mathrm{C}\ell^0_{r,s})^\times$ *is a closed Lie subgroup.*

(2) *The group homomorphism $\mathrm{Ad}\colon \mathrm{Spin}_{r,s} \to \mathrm{SO}^+_{r,s}$ is a smooth $\{\pm 1\}$–principal covering map.*

(3) $\mathrm{Spin}_{r,s}$ *is connected if and only if* $(r,s) \notin \{(1,0),(0,1),(1,1)\}$.

Proof. (1) and (2) are obvious. To prove (3) observe the following:

(1) If $(r,s) \notin \{(1,0),(0,1),(1,1)\}$, then there are e_1, e_2 with $q(e_1) = q(e_2) = \pm 1$ and $p(e_1,e_2) = 0$. The path $\gamma\colon [0,\pi/2] \to \mathrm{Spin}_{r,s}$ defined by

$$\gamma(t) = (e_1 \cos(t) + e_2 \sin(t))(e_1 \cos(t) - e_2 \sin(t))$$

satisfies

$$\gamma(\pi/2) = e_1^2 = \pm 1 \quad \text{and} \quad \gamma(\pi/2) = -e_2^2 = \mp 1,$$

Therefore and since $\mathrm{SO}^+_{r,s}$ is connected, $\mathrm{Spin}_{r,s}$ is connected.

(2) $\mathrm{SO}^+_{1,0} = \mathrm{SO}^+_{0,1} = \{\mathbf{1}\}$ and

$$\mathrm{SO}^+_{1,1} = \left\{ \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} : t \in \mathbf{R} \right\} \cong \mathbf{R}.$$

Therefore, the covering map Ad must be trivial in these cases; hence: $\pi_0(\mathrm{Spin}_{r,s}) = \{\pm 1\}$. ■

Remark 46.5. The $\{\pm 1\}$–principal covering maps $\mathrm{Ad}\colon \mathrm{Spin}_{r,s} \to \mathrm{SO}^+_{r,s}$ and $\mathrm{Ad}\colon \mathrm{Spin}_{s,r} \to \mathrm{SO}^+_{s,r} = \mathrm{SO}^+_{r,s}$ is isomorphic. ♣

Remark 46.6. If r, s are quite small, then $\mathrm{Spin}_{r,s}$ can often be identified with more familiar groups. For example:

(1) $\mathrm{Spin}_{0,3} \cong \mathrm{Sp}(1) \cong \mathrm{SU}(2)$.

(2) $\mathrm{Spin}_{1,2} \cong \mathrm{SL}_2(\mathbf{R})$.

(3) $\mathrm{Spin}_{0,4} \cong \mathrm{Sp}(1) \times \mathrm{Sp}(1) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$.

(4) $\mathrm{Spin}_{1,3} \cong \mathrm{SL}_2(\mathbf{C})$.

(5) $\mathrm{Spin}_{0,5} \cong \mathrm{Sp}(2)$.

(6) $\mathrm{Spin}_{0,6} \cong \mathrm{SU}(4)$.

Finding the above isomorphisms and the corresponding adjoint representations is an exercise. (It can be solved by reading [Har90, pp. 298–307]; see also `https://en.wikipedia.org/wiki/Exceptional_isomorphism`.) ♣

Remark 46.7. It also is a fun exercise to determine the

$$\mathrm{Ad}^{-1} \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}. \qquad ♣$$

47 $\mathfrak{spin}_{r,s}$

Definition 47.1. Set

$$\mathfrak{spin}_{r,s} := \mathrm{Lie}(\mathrm{Spin}_{r,s}). \qquad \bullet$$

Since $\mathrm{Spin}_{r,s} \subset (\mathrm{C}\ell^0_{r,s})^\times$ is a Lie subgroup, $\mathfrak{spin}_{r,s} \subset \mathrm{C}\ell^0_{r,s}$ with the Lie bracket agreeing with the commutator. Set $b = b_{r,s} := \frac{1}{2}p_{r,s}$. Denote by $\kappa\colon \Lambda^2\mathbf{R}^{r+s} \to \mathrm{C}\ell^0_{r,s}$ the map induced by the quantisation map. Of course, with respect to the standard basis $e_1, \ldots, e_{r+s}$,

$$\kappa(e_i \wedge e_j) = e_i e_j.$$

Identify $\Lambda^2\mathbf{R}^{r+s} = \mathfrak{so}_{r,s}$ via

$$(u \wedge v)w := ub_{r,s}(v, x) - vb_{r,s}(u, x).$$

Proposition 47.2.

(1) $\mathfrak{spin}_{r,s}$ *agrees with the image of the quantisation map* $\kappa\colon \Lambda^2\mathbf{R}^{r+s} \to \mathrm{C}\ell^0_{r,s}$.

(2) $\mathrm{Lie}(\mathrm{Ad}) \circ \kappa(\alpha) = 2\alpha$ *for every* $\alpha \in \Lambda^2\mathbf{R}^{r+s} = \mathfrak{so}_{r,s}$.

Proof. Denote the standard basis of $\mathbf{R}^{r+s}$ by $e_1, \ldots, e_{r+s}$. Define $x_{ij}\colon \mathbf{R} \to \mathrm{Spin}_{r,s}$ by

$$\begin{aligned} x_{ij}(t) &:= \begin{cases} (e_i\cos(t) - e_j\sin(t) \cdot (e_i\cos(t) + e_j\sin(t)) & \text{if } i < j \leqslant r \\ (e_i\cos(t) + e_j\sin(t)) \cdot (-e_i\cos(t) + e_j\sin(t)) & \text{if } r+1 \leqslant i < j \\ (\sinh(t)e_i - \cosh(t)e_2)(\sinh(t)e_i + \cosh(t)e_2) & \text{if } i \leqslant r < j \end{cases} \\ &= \begin{cases} \cos(2t) + \sin(2t)e_ie_j & \text{if } i < j \leqslant r \text{ or } r+1 \leqslant i < j \\ \cosh(2t) + \sinh(2t)e_ie_j & \text{if } i \leqslant r < j. \end{cases} \end{aligned}$$

Since $\dot{x}_{ij}(0) = e_ie_j = \kappa(e_i \wedge e_j)$, this proves (1).

To prove (2), observe that

$$[e_ie_j, v] = e_ie_jv - ve_ie_j = 2e_ib(e_j, v) - 2e_jb(e_j, v) = 2(e_i \wedge e_j)v. \qquad \blacksquare$$

Lecture 8

The first part of this lectures completes the algebraic discussion. The second part introduces the “axiomatic” approach to Dirac operators.

48 $\mathrm{Spin}^G_{r,s}$

Definition 48.1. Let $r, s \in \mathbf{N}_0$. Let G be a Lie group. Let $\varepsilon\colon \{\pm 1\} \to G$ be a group homomorphism. Set

$$\mathrm{Spin}^G_{r,s} := \mathrm{Spin}_{r,s} \times_{\{\pm 1\}} G. \qquad \bullet$$

Proposition 48.2. *The sequences*

$$1 \to G \to \mathrm{Spin}^G_{r,s} \xrightarrow{\mathrm{Ad}} \mathrm{SO}^+_{r,s} \to 0$$

and

$$0 \to \mathrm{Spin}_{r,s} \to \mathrm{Spin}^G_{r,s} \to (G/\{\pm 1\}) \to 0$$

are exact. ■

Remark 48.3. Of course, there is also a homomorphism $\mathrm{Spin}^G_{r,s} \to G/\{\pm 1\}$. ♣

49 $\mathrm{Spin}^{\mathrm{U}(1)}_{r,s}$

Remark 49.1. Possibly the most important instance is $G = \mathrm{U}(1)$ with $\varepsilon(-1) = -\mathbf{1}$. This group is often denoted by $\mathrm{Spin}^{\mathbf{C}}_{r,s}$ (or similarly). *A warning is in order:* $\mathrm{Spin}^{\mathbf{C}}_{r,s}$ is not the spin group for $q_{r,s}$ over $\mathbf{C}$, but it can be obtained from it using the real structure on $\mathbf{C}^{r+s}$; cf. [ABS64, p. 9]. ♣

The significance of $\mathrm{Spin}^{\mathrm{U}(1)}_{r,s}$ is that for $\mathbf{C} \otimes \mathrm{C}\ell_{r,s}$–modules it can take the role of the spin group.

Proposition 49.2. *Let S be a $\mathbf{C} \otimes \mathrm{C}\ell_{r,s}$–module which is weakly faithful as a $\mathrm{C}\ell_{r,s}$–module. Denote by $L(S)$ the Lipschitz group of S. The action of $\mathbf{C} \times \mathrm{C}\ell^{0\times}_{r,s}$ on S induces a group homomorphism $\mathrm{Spin}^{\mathrm{U}(1)}_{r,s} \to L(S)$ such that*

$$\begin{array}{ccc} \mathrm{Spin}^{\mathrm{U}(1)}_{r,s} & \longrightarrow & L(s) \\ \big\downarrow{\scriptstyle \mathrm{Ad}} & & \big\downarrow{\scriptstyle \mathrm{Ad}} \\ \mathrm{SO}^+_{r,s} & \hookrightarrow & \mathrm{O}_{r,s} \end{array}$$

commutes. ■

$\mathrm{Spin}^{\mathrm{U}(1)}_n$ also interacts well with $\mathrm{U}(n)$.

Proposition 49.3 (Atiyah, Bott, and Shapiro [ABS64, pp. 10, 13, 14]). *Let $n \in \mathbf{N}_0$.*

(1) *The map $\rho\colon \mathrm{U}(n) \to \mathrm{SO}(2n)$ does not lift to Spin_{2n}.*

(2) *The map* $\rho \times \det\colon \mathrm{U}(n) \to \mathrm{SO}(2n) \times \mathrm{U}(1)$ *lifts to* $\mathrm{Spin}_{2n}^{\mathrm{U}(1)}$*; that is,*

$$\begin{array}{ccc} & & \mathrm{Spin}_{2n}^{\mathrm{U}(1)} \\ & \nearrow & \big\downarrow \widetilde{\mathrm{Ad}} \times (-)^2 \\ \mathrm{U}(n) & \xrightarrow[\rho\times\det]{} & \mathrm{SO}(2n) \times \mathrm{U}(1). \end{array}$$

(3) *The complex pinor module* P *be can be identified with* $\Lambda_{\mathbf{C}}(\mathbf{C}^n)^*$ *such that the lift* $\mathrm{U}(n) \to \mathrm{Spin}^c(n)$ *makes the following diagram commutative:*

$$\begin{array}{ccc} \mathrm{U}(n) & \longrightarrow & \mathrm{Spin}_n^c \\ \big\downarrow & & \big\downarrow \\ \mathrm{End}_{\mathbf{C}}(\mathbf{C}^n) & \xrightarrow{(-)^*\circ\Lambda} & \mathrm{End}_{\mathbf{C}}(\Lambda_{\mathbf{C}}(\mathbf{C}^n)^*). \end{array}$$

The Clifford multiplication on $\Lambda_{\mathbf{C}}(\mathbf{C}^n)^*$ *is given by*

$$\gamma(v)\alpha = v^* \wedge \alpha - i(v)\alpha.$$

Proof. (1) is a consequence of the fact that $\pi_1(\rho)\colon \pi_1(\mathrm{U}(n)) \to \pi_1(\mathrm{SO}(2n))$ is surjective, but $\pi_1(\widetilde{\mathrm{Ad}})\colon \pi_1(\mathrm{Spin}(2n)) \to \pi_1(\mathrm{SO}(2n))$ is not.

(2) is proved by constructing the lift explicitly. Given $f \in \mathrm{U}(n)$, chose a unitary basis $(e_1, \dots, e_n)$ of $\mathbf{C}^n$ in which f is diagonal; that is: $f = \mathrm{diag}(e^{i\alpha_1}, \cdots, e^{i\alpha_n})$. An orthonormal basis of the $2n$–dimensional real Euclidean space $\mathbf{C}^n$ is given by $(e_1, ie_1, \dots, e_n, ie_n)$. Define $\tilde{f} \in \mathrm{Spin}^c(2n)$ by

$$\tilde{f} \coloneqq \prod_{j=1}^{n} [(\cos(\alpha_j/2) + \sin(\alpha_j/2)e_j(ie_j)), e^{\frac{i}{2}\alpha_j}].$$

Observe that $\alpha_j \in \mathbf{R}/2\pi\mathbf{Z}$, so $\alpha_j/2 \in \mathbf{R}/\pi\mathbf{Z}$. Consequently, the both factors individually are only defined up to a sign. Their product, however, is well-defined. Clearly, $\left(\prod_{j=1}^{n} e^{\frac{i}{2}\alpha_j}\right)^2 = \det(f)$. The fact that $\rho(f) = \widetilde{\mathrm{Ad}}(\tilde{f})$ follows from following observation.

Proposition 49.4. *Let* (e_1, e_2) *be an orthonormal basis of* $\mathbf{R}^2$ *and let* $\alpha \in \mathbf{R}$*. We have*

$$\widetilde{\mathrm{Ad}}(\cos(\alpha/2) + \sin(\alpha/2)e_1e_2) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}.$$

Proof. Since

$$\alpha\,(\cos(\alpha/2) + \sin(\alpha/2)e_1e_2)^{-1} = \cos(\alpha/2) - \sin(\alpha/2)e_1e_2,$$

we have

$$\begin{aligned} \widetilde{\mathrm{Ad}}\,(\cos(\alpha/2) + \sin(\alpha/2)e_1e_2)\,e_i &= (\cos(\alpha/2) + \sin(\alpha/2)e_1e_2)^2\,e_i \\ &= \big(\cos(\alpha/2)^2 - \sin(\alpha/2)^2 \\ &\qquad + 2\cos(\alpha/2)\sin(\alpha/2)e_1e_2\big)e_i \\ &= (\cos(\alpha) + \sin(\alpha)e_1e_2)\,e_i. \end{aligned}$$

From this the assertion follows directly. ∎

The formula for the Clifford multiplication defines how $\mathrm{Spin}^c(2n)$ acts on $\Lambda_{\mathbf{C}}(\mathbf{C}^n)^*$. Proving (3) is a matter of a calculation using the explicit formula for the lift constructed above. ■

50 Bilinear forms on pinor modules

`I should have probably said the following in an earlier lecture.`

The (complex) (s)pinor modules P (defined in Section 31, Section 34, Section 32, and Section 36) admit non-degenerate bilinear forms $b \in \mathrm{Hom}(P \otimes P, \mathbf{R})$ with respect to which the γv is skew-adjoint for every $v \in \mathbf{R}^{r+s}$. The construction of these b is quite formidable and discussed in [Har90, §13]. If $r = 0$ or $s = 0$, then b can be assumed to be positive definite.

51 Clifford algebra bundles

Situation 51.1. Let $r, s \in \mathbf{N}_0$. Let X be a smooth manifold. Let $\pi\colon V \to X$ be a vector bundle of rank $r+s$. Let $q \in \Gamma(\mathrm{Hom}(S^2V, \mathbf{R}))$ be a symmetric bilinear form on V of signature $r-s$. Denote the $\mathrm{O}_{r,s}$ frame bundle of (V, q) by $(\rho\colon \mathrm{Fr}(V, q) \to X, R)$. Identify, $V = \mathrm{Fr}(V, q) \times_{\mathrm{O}_{r,s}} \mathbf{R}^{r+s}$. ×

Definition 51.2. The **Clifford algebra bundle** associated with q is the algebra bundle

$$\mathrm{C}\ell(q) := \mathrm{Fr}(V, q) \times_{\mathrm{O}_{r,s}} \mathrm{C}\ell_{r,s}.$$

Denote by $\gamma\colon V \to \mathrm{C}\ell(q)$ the canonical bundle map. •

Remark 51.3. The Serre–Swan Theorem identifies vector bundles V over X (more precisely: their spaces of sections $\Gamma(V)$) with finitely-generated projective modules over the ring $R := C^\infty(X)$. The symmetric bilinear form is nothing but a quadratic form $q\colon \Gamma(V) \to R$. Clifford algebras can be defined over arbitrary rings. An the above construction can be carried out in this framework. I do not know if this is useful for anything. ♣

Example 51.4. The most important example is $V = TX$ and $g := -g$ with g a pseudo-Riemannian metric. ♠

Proposition 51.5. *For every orthogonal corvariant derivative $\nabla\colon \Gamma(V) \to \Omega^1(X, V)$ there is a unique covariant derivative $\nabla_{\mathrm{C}\ell}\colon \Gamma(\mathrm{C}\ell(V)) \to \Omega^1(X, \mathrm{C}\ell(V))$ such that*

$$\nabla_{\mathrm{C}\ell}\gamma(v) = \gamma(\nabla v) \quad \textit{and} \quad \nabla_{\mathrm{C}\ell}(xy) = (\nabla_{\mathrm{C}\ell}x)y + x(\nabla_{\mathrm{C}\ell}y).$$

■

52 Clifford module bundles

[ABS64]

Situation 52.1. Let (X, g) be a pseudo-Riemannian manifold. ×

Definition 52.2. A **Clifford module bundle** over X is a vector bundle $\pi\colon S \to X$ together with a smooth map of algebra bundles $C\ell(-g) \to \mathrm{End}(S)$. If S is a Clifford module bundle, then the induced map $\gamma\colon TX \to \mathrm{End}(S)$ is called the **Clifford multiplication**. •

Remark 52.3. *The minus sign is not a mistake*, but might appear somewhat awkward. A typical "solution" is to push the minus sign into the definition of the Clifford algebra. Another "solution" is to keep in mind that (in an not unreasonable convention) the symbol of the differential operator Δ is $-g$. ♣

Example 52.4. $C\ell(-g)$ is a Clifford module bundle. ♠

Example 52.5. The bundle of exterior algebras

$$S \coloneqq \Lambda T^*X = \bigoplus_{r=0}^{n} \Lambda^r T^*X$$

is a Clifford module bundle with

$$\gamma(v)\alpha \coloneqq v^\flat \wedge \alpha - i_v\alpha.$$ ♠

53 Dirac bundles

Situation 53.1. Let (X, g) be a pseudo-Riemannian manifold. ×

Definition 53.2. A **Dirac bundle** of (X, g) consists of:

(1) a Clifford module bundle (S, γ),

(2) a non-degenerate bilinear form $b \in \mathrm{Hom}(S \otimes S, \mathbf{R})$, and

(3) a covariant derivative $\nabla = \nabla^S\colon \Gamma(S) \to \Omega^1(X, S)$

such that:

(4) $\nabla^S(\gamma(v)\phi) = \gamma(\nabla^{\mathrm{LC}}v)\phi + \gamma(v)\nabla^S\phi$,

(5) $b(\gamma(v)\phi, \psi) + b(\phi, \gamma(v)\psi) = 0$, and

(6) $\mathrm{d}b(\phi, \psi) = b(\nabla^S\phi, \psi) + b(\phi, \nabla^S\psi)$.

A **complex Dirac bundle** is a Dirac bundle together with an almost complex structure i which commutes with γ and is ∇^S–parallel. A **quaternionic Dirac bundle** is a Dirac bundle together with an almost complex structures i, j, k which commute with γ, are ∇^S–parallel, and satisfy $ij = -ji = k$. •

Remark 53.3. This deviates from [LM89, Part II Definition 5.2] in that b is not required to be a Euclidean inner product; that is: symmetric and positive definite. This is necessary because (in indefinite signature) the pinor modules do not always have $\mathrm{Spin}_{r,s}$–invariant Euclidean inner products. ♣

Once a Dirac bundle has been found, further examples can be obtained by twisting.

Definition 53.4. Let (S, γ, b, ∇_S) be a Dirac bundle. Let (E, h, ∇_E) be a vector bundle together with a bilinear form h and a covariant derivative ∇_E with $\nabla h = 0$. The twist of (S, γ, b, ∇_S) by (E, h, ∇_E) is

$$(S \otimes E, \gamma \otimes \mathrm{id}_E, b \otimes h, \nabla_S \otimes \nabla_E).$$ •.

For index theory, the following concept is fundamental.

Definition 53.5. A **grading** of a Dirac bundle (S, γ, b, ∇_S) is an $\varepsilon \in \Gamma(\mathrm{End}(S))$ such that:

(1) $\varepsilon^2 = \mathrm{id}$,

(2) $\gamma\varepsilon + \varepsilon\gamma = 0$,

(3) $b(\varepsilon\phi, \varepsilon\psi) = 0$, and

(4) $\nabla\varepsilon = 0$. •

Example 53.6. ΛT^*X has an obvious grading (leading to the Euler characteristic), but its complexification $\Lambda T^*X \otimes \mathbf{C}$ has another grading (leading to the signature); see Section 9. ♠

54 Dirac operators

Situation 54.1. Let (X, g) be a pseudo-Riemannian manifold of signature (r, s). Let (S, γ, b, ∇_S) be a Dirac bundle over (X, g). ×

Definition 54.2. The **Dirac operator** $D\colon \Gamma(S) \to \Gamma(S)$ associated with (S, γ, ∇_S, b) is the composition

$$\Gamma(S) \xrightarrow{\nabla_S} \Gamma(T^*X \otimes S) \xrightarrow{\sharp\otimes \mathrm{id}_S} \Gamma(TX \otimes S) \xrightarrow{\gamma} \Gamma(S).$$

Here the isomorphism $\sharp\colon T^*X \to TX$ defined by $g(\sharp\alpha, v) := \alpha(v)$. •

Remark 54.3. Suppose that dim $= r + s$ and g has signature $r - s$. If $e_1, \dots, e_{r+s}$ is a (local) orthonormal frame then

$$D\phi = \sum_{i=1}^{r+s} \varepsilon_i \gamma(e_i) \nabla_{e_i} \phi \quad \text{with} \quad \varepsilon_i := g(e_i, e_i) \in \{\pm 1\}.$$

It is not difficult to see that this expression does not depend on the choice of $e_1, \dots, e_{r+s}$. The signs ε_i are crucial! ♣

Example 54.4. $S = \Lambda TM$ with its natural Euclidean metric and covariant derivative is a Dirac bundle. The corresponding Dirac operator is

$$D = \mathrm{d} + \mathrm{d}^*\colon \Lambda TM \to \Lambda TM.$$ ♠

Proposition 54.5. *For every $\phi, \psi \in \Gamma(S)$*

$$b(D\phi, \psi) - b(\phi, D\psi) = \operatorname{div} v \quad \textit{with} \quad \langle v, \cdot \rangle := b(\gamma(\cdot)\phi, \psi).$$

In particular, if X is closed, then

$$\int_X b(D\phi, \psi)\,\mathrm{vol}_g = \int_X b(\phi, D\psi)\,\mathrm{vol}_g.$$

Proof. This is proved by direct computation. ■

Remark 54.6. If ε is a grading, then S be decomposed into the ± 1–eigenbundles $S^{\pm}$ of ε and the Dirac operator decomposes accordingly as

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}.$$

♣

Lecture 9

This lecture discusses two fundamental facts about Dirac operators: the Weitzenböck formula and the conformal invariance.

55 The Weitzenböck formula

Situation 55.1. Let (X, g) be a pseudo-Riemannian manifold of signature (r, s). Let (S, γ, b, ∇) be a Dirac bundle over (X, g). Denote by D the associated Dirac operator. ×

Definition 55.2. Define $\mathscr{F}_S \in \Gamma(\mathrm{End}(S))$ by

$$\mathscr{F}_S \coloneqq \frac{1}{2}\sum_{i,j=1}^{n} \varepsilon_i\varepsilon_j\gamma(e_i)\gamma(e_j)F_S(e_i, e_j)$$

with F_S denoting the curvature of ∇_S. •

Proposition 55.3 (Weitzenböck formula for Dirac bundles). *D satisfies the **Weitzenböck formula***

$$D^2 = \nabla_S^*\nabla_S + \mathscr{F}_S.$$

Proof. Let $x \in X$. Pick a local orthonormal frame $(e_1, \ldots, e_{r+s})$ defined in a neighborhood of x with $(\nabla e_i)_x = 0$. By direct computation at $x \in X$,

$$\begin{aligned}\sum_{i,j=1}^{n} \varepsilon_i\varepsilon_j\gamma(e_i)\nabla_{S,e_i}\gamma(e_j)\nabla_{S,e_j} &= \sum_{i,j=1}^{n} \varepsilon_i\varepsilon_j\gamma(e_i)\gamma(e_j)\nabla_{S,e_i}\nabla_{S,e_j}\\ &= -\sum_{i=1}^{n}\nabla_{S,e_i}\nabla_{S,e_i} + \sum_{i<j}^{n}\varepsilon_i\varepsilon_j\gamma(e_i)\gamma(e_j)[\nabla_{S,e_i}, \nabla_{S,e_j}]\\ &= \nabla_S^*\nabla_S + \sum_{i<j}^{n}\varepsilon_i\varepsilon_j\gamma(e_i)\gamma(e_j)F_s(e_i, e_j).\end{aligned}$$ ■

Remark 55.4. According to Proposition 55.3, the Dirac operator still is almost a square root of the Laplace operator. ♣

Remark 55.5. The Weitzenböck formula is at the heart of vanishing and estimating theorems established using the Bochner technique. Most of these applications require further information on $\mathscr{F}_S$. ♣

56 The refined Weitzenböck formula

Situation 56.1. Let (X, g) be a pseudo-Riemannian manifold of signature (r, s). Let (S, γ, b, ∇) be a Dirac bundle over (X, g). Denote by D the associated Dirac operator. ×

Proposition 56.2 (refined Weitzenböck formula for Dirac Bundles). *There is a* $C\ell(-g)$*–linear* $F_S \in \Omega^2(X, \mathrm{End}(S))$ *such that with*

$$\mathscr{F}_S^{\mathrm{tw}} := \frac{1}{2}\sum_{i,j=1}^{r+s} \varepsilon_i\varepsilon_j \gamma(e_i)\gamma(e_j)F_S^{\mathrm{tw}}(e_i, e_j)$$

and scal_g *denoting the scalar curvature of* g

$$D^2 = \nabla_S^*\nabla_S + \frac{1}{4}\mathrm{scal}_g + \mathscr{F}_S^{\mathrm{tw}}.$$

Remark 56.3. Proposition 56.2 exhibits the role of scalar curvature for Dirac operators as the universal term in the Weitzenböck formula. The fact that F_S^{tw} is $C\ell(-g)$–linear often restricts it severely and facilitates the computation of $\mathscr{F}_S^{\mathrm{tw}}$. ♣

Remark 56.4. cf. [LM89, Theorem 8.17] where this is proved for $S \otimes E$. ♣

Remark 56.5. Schrödinger [Sch32, pp. 126–128] first computed D^2 and observed the appearance of $\frac{1}{4}\mathrm{scal}_g$. ♣

The proof relies on a sequence of computations.

Proposition 56.6. *For every* $x \in X$ *and* $u, v, w \in T_xX$

$$[F_S(u, v), \gamma(w)] = \gamma(R(u, v)w).$$

Proof. Pick a local orthonormal frame $(e_1, \dots, e_{r+s})$ defined in a neighborhood of x with $(\nabla e_i)_x = 0$. By direct computation at $x \in X$,

$$\begin{aligned}
[F_S(e_i, e_j), \gamma(e_k)] &= [[\nabla_{S,e_i}, \nabla_{S,e_j}], \gamma(e_k)] \\
&= [\nabla_{S,e_i}, [\nabla_{S,e_j}, \gamma(e_k)]] - [\nabla_{S,e_j}, [\nabla_{S,e_i}, \gamma(e_k)]] \\
&= [\nabla_{S,e_i}, \gamma(\nabla_{e_j}e_k)] - [\nabla_{e_j}, \gamma(\nabla_{e_i}e_k)] \\
&= \gamma(\nabla_{e_i}\nabla_{e_j}e_k) - \gamma(\nabla_{e_j}\nabla_{e_i}e_k) \\
&= \gamma(R(e_i, e_j)e_k).
\end{aligned}$$

■

Proposition 56.7. *Define* $R_S \in \Omega^2(X, \mathrm{End}(S))$ *by*

$$R_S(v, w) := \frac{1}{4}\sum_{i,j=1}^{n} \varepsilon_i\varepsilon_j\gamma(e_i)\gamma(e_j)\langle R(v, w)e_i, e_j\rangle.$$

For every $x \in X$ *and* $u, v, w \in T_xX$

$$[R_S(u, v), \gamma(w)] = \gamma(R(u, v)w).$$

Proof. Pick a local orthonormal frame $(e_1, \dots, e_{r+s})$. By direct inspection

$$[\gamma(e_i)\gamma(e_j), \gamma(e_k)] = -2\gamma(e_i)\varepsilon_k\delta_{jk} + 2\varepsilon_k\gamma(e_j)\delta_{ik}.$$

Therefore,

$$
\begin{aligned}
[R_S(e_k, e_\ell), \gamma(e_m)] &= \frac{1}{4} \sum_{i,j=1}^{n} \varepsilon_i \varepsilon_j \langle R(e_k, e_\ell)e_i, e_j \rangle [\gamma(e_i)\gamma(e_j), \gamma(e_m)] \\
&= \frac{1}{2} \sum_{j=1}^{n} \varepsilon_j \langle R(e_k, e_\ell)e_m, e_j \rangle \gamma(e_j) - \frac{1}{2} \sum_{i=1}^{n} \varepsilon_i \langle R(e_k, e_\ell)e_i, e_m \rangle \gamma(e_i) \\
&= \sum_{j=1}^{n} \varepsilon_j \langle R(e_k, e_\ell)e_m, e_j \rangle \gamma(e_j). \qquad \blacksquare
\end{aligned}
$$

Proposition 56.8. *If $e_1, \dots, e_{r+s}$ is an orthonormal basis, then*

$$
\sum_{i,j,\ell=1}^{n} \varepsilon_\ell \varepsilon_i \varepsilon_j \gamma(e_\ell)\gamma(e_i)\gamma(e_j) \langle R(e_k, e_\ell)e_i, e_j \rangle = -2 \sum_{i=1}^{n} \varepsilon_i \gamma(e_i) \mathrm{Ric}(e_k, e_i) \quad \textit{and}
$$

$$
\sum_{i,j,k,\ell=1}^{n} \varepsilon_k \varepsilon_\ell \varepsilon_i \varepsilon_j \gamma(e_k)\gamma(e_\ell)\gamma(e_i)\gamma(e_j) \langle R(e_k, e_\ell)e_i, e_j \rangle = 2\mathrm{scal}_g.
$$

Proof. The first identity implies the second directly. To prove the first identity, observe the following:

(1) If i, j, ℓ are pairwise distinct, then

$$
\gamma(e_\ell)\gamma(e_i)\gamma(e_j) = \gamma(e_i)\gamma(e_j)\gamma(e_\ell) = \gamma(e_j)\gamma(e_\ell)\gamma(e_i).
$$

By the algebraic Bianchi identity

$$
\langle R(e_k, e_\ell)e_i, e_j \rangle + \langle R(e_k, e_i)e_j, e_\ell \rangle + \langle R(e_k, e_j)e_\ell, e_i \rangle = 0.
$$

Therefore, the sum of terms with i, j, ℓ pairwise distinct appearing the the left-hand side vanishes.

(2) The terms with $i = j$ vanish because $R(e_k, e_\ell)$ is skew-symmetric.

(3) If $i \neq j = \ell$, then

$$
\begin{aligned}
\varepsilon_\ell \varepsilon_i \varepsilon_j \gamma(e_\ell)\gamma(e_i)\gamma(e_j) \langle R(e_k, e_\ell)e_i, e_j \rangle &= \varepsilon_i \gamma(e_i) \langle R(e_k, e_j)e_i, e_j \rangle \\
&= -\varepsilon_i \gamma(e_i) \langle R(e_j, e_k)e_i, e_j \rangle.
\end{aligned}
$$

Therefore, the sum of these expressions contributes

$$
-\sum_{i=1}^{n} \varepsilon_i \gamma(e_i) \mathrm{Ric}(e_k, e_i).
$$

(4) If $j \neq i = \ell$, then

$$\varepsilon_\ell \varepsilon_i \varepsilon_j \gamma(e_\ell)\gamma(e_i)\gamma(e_j)\langle R(e_k, e_\ell)e_i, e_j\rangle = -\varepsilon_j \gamma(e_j)\langle R(e_i, e_k)e_j, e_i\rangle$$

Therefore, the sum of these expressions also contributes

$$-\sum_{i=1}^{n} \varepsilon_i \gamma(e_i)\mathrm{Ric}(e_k, e_i).$$

■

Proof of Proposition 56.2. By the above, $F_S = R_S + F_S^{\mathrm{tw}}$ with F_S^{tw} being linear. Moreover,

$$\frac{1}{2}\sum_{i,j=1}^{n} \varepsilon_i \varepsilon_j \gamma(e_i)\gamma(e_j) R_S(e_i, e_j) = \frac{1}{4}\mathrm{scal}_g.$$

This finishes the proof. ■

57 Conformal invariance of Dirac operators

Proposition 57.1.

$$\tilde{\nabla}_v w = \nabla_v w + (\mathscr{L}_v f)w + (\mathscr{L}_w f)v - g(v, w)\nabla f$$

Proposition 57.2 (Hitchin [Hit74, §1.4]). *Let (X, g) be a pseudo-Riemannian manifold. Let $f \in C^\infty(X)$. Set $\tilde{g} \coloneqq e^{2f} g$. Let (S, γ, b, ∇) be a Dirac bundle over (X, g).*

(1) *Set*

$$\tilde{\gamma} \coloneqq e^f \gamma. \quad \textit{and} \quad \tilde{\nabla}_v \coloneqq \nabla_v + \frac{1}{4}[\gamma(\nabla f), \gamma(v)]$$

$(S, \tilde{\gamma}, b, \tilde{\nabla})$ is a Dirac bundle over $(X, \tilde{g})$.

(2) *The Dirac operator D associated with (S, γ, b, ∇) and the Dirac operator $\tilde{D}$ associated with $(S, \tilde{\gamma}, b, \tilde{\nabla})$ satisfy*

$$\tilde{D} = e^{-\frac{n+1}{2}f} D e^{\frac{n-1}{2}f}.$$

Proof. Evidently, $\tilde{\gamma}(v)^2 = e^{2f} g(v, v)$. Moreover, $\frac{1}{4}[\gamma(\nabla f), \gamma(v)]$ is b–skew-adjoint. Therefore, b is $\tilde{\nabla}$–parallel. It remains to verify that $\tilde{\gamma}$ is $\tilde{\nabla}$–parallel; that is:

$$\tilde{\nabla}_v(\tilde{\gamma}(w)\phi) - \tilde{\gamma}(\tilde{\nabla}_v w)\phi - \tilde{\gamma}(w)\tilde{\nabla}_v \phi = 0.$$

Since the Levi–Civita connection $\tilde{\nabla}$ of $\tilde{g}$ satisfies

$$\tilde{\nabla}_v w = \nabla_v w + (\mathscr{L}_v f)w + (\mathscr{L}_w f)v - g(v, w)\nabla f,$$

this amounts to verifying that

$$e^f\left(\frac{1}{4}[[\gamma(\nabla f), \gamma(v)], \gamma(w)]\phi - g(\nabla f, w)\gamma(v) + g(v, w)\gamma(\nabla f)\right)\phi = 0.$$

The latter is a consequence of the following lemma. This proves (1)

If $e_1, \dots, e_n$ is g–orthonormal, then $\tilde{e}_1 \coloneqq e^{-f} e_1, \dots, e_n \coloneqq e^{-f} e_n$ is $\tilde{g}$–orthonormal. Since $\varepsilon_i \gamma(e_i)[\gamma(e_j), \gamma(e_j)] = 2(1-\delta_{ij})\gamma(e_j)$,

$$\begin{aligned}
\tilde{D} &= \sum_{i=1}^{n} \varepsilon_i \tilde{\gamma}(\tilde{e}_i) \tilde{\nabla}_{\tilde{e}_i} \\
&= e^{-f} \sum_{i=1}^{n} \varepsilon_i \gamma(e_i) \left(\nabla_{e_i} + \frac{1}{4} [\gamma(\nabla f), \gamma(e_i)] \right) \\
&= e^{-f} \left(D + \frac{n-1}{2} \gamma(\nabla f) \right) \\
&= e^{-\frac{n+1}{2} f} D e^{\frac{n-1}{2} f}.
\end{aligned}$$

This proves (2). ■

Lemma 57.3. $\frac{1}{4}[[\gamma(u), \gamma(v)], \gamma(w)] = \gamma(u) g(v, w) - \gamma(v) g(v, w)$.

Proof. Pick a local orthonormal frame $(e_1, \dots, e_{r+s})$. It suffices to prove the above for $\{u, v, w\} \subset \{e_1, \dots, e_{r+s}\}$. By direct inspection

$$[\gamma(e_i)\gamma(e_j), \gamma(e_k)] = -2\gamma(e_i) g(e_j, e_k) + 2\gamma(e_j) g(e_i, e_k).$$

Therefore,

$$\begin{aligned}
[[\gamma(e_i), \gamma(e_j)], \gamma(e_k)] &= [\gamma(e_i)\gamma(e_j), \gamma(e_k)] - [\gamma(e_j)\gamma(e_i), \gamma(e_k)] \\
&= -4\gamma(e_i)\varepsilon_k \delta_{jk} + 4\gamma(e_j)\varepsilon_k \delta_{ik} \\
&= -4\gamma(e_i) g(e_j, e_k) + 4\gamma(e_j) g(e_i, e_k).
\end{aligned}$$ ■

58 Reduction of the structure group

Situation 58.1. Let G, H be Lie groups. Let $\lambda\colon H \to G$ be a Lie group homomorphism. Let $(p\colon P \to B, R\colon P \circlearrowleft G)$ be a G–principal bundle. ×

Definition 58.2. A **λ–reduction** of (p, R) consists of:

(1) a smooth manifold Q,

(2) a smooth right action $S\colon Q \circlearrowleft H$, and

(3) an H–equivariant smooth map $\xi\colon Q \to P$; that is: for every $x \in Q$ and $g \in H$:

$$\xi(xg) = \xi(x)\lambda(g)$$

such that $(q \coloneqq p \circ \xi\colon Q \to B, S)$ is an H–principal fibre bundle. •

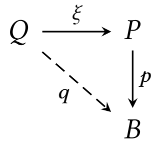


Figure 4: A λ–reduction of (p, R).

Remark 58.3. Let (Q, S, ξ) be a λ–reduction of (p, S). Define $r\colon\ Q \times_H G \longrightarrow B$ by $r([x, g]) \coloneqq p(x)$ and $T\colon\ (Q \times_H G) \times G \longrightarrow Q \times_H G$ by $T([x, g], h) \coloneqq [x, gh]$. The map $\phi\colon\ Q \times_H G \longrightarrow P$ defined by

$$\phi([x, g]) \coloneqq \xi(x)\lambda(g)$$

is an isomorphism $(r, T) \longrightarrow (p, R)$ of G–principal bundles. ♣

Definition 58.4. Let (Q_i, S_i, ξ_i) $(i = 1, 2)$ be λ–reductions of (p, R). An **isomorphism** $\phi\colon\ (Q_1, S_1, \xi_1) \longrightarrow (Q_2, S_2, \xi_2)$ is an isomorphism $\phi\colon\ (q_1, S_1) \longrightarrow (q_2, S_2)$ of H–principal bundles such that

$$\xi_2 \circ \phi = \xi_1.$$

•

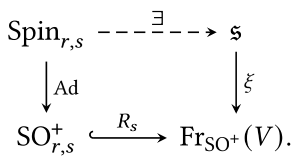
59 Spin structures on pseudo-Euclidean vector bundles

Situation 59.1. Let X be a manifold. Let $V \longrightarrow X$ be an space- and time-oriented Euclidean vector bundle of signature (r, s). Denote the frame bundle of V by $(p\colon\ \mathrm{Fr}_{\mathrm{SO}^+}(V) \longrightarrow X, R)$. Denote by $\mathrm{Ad}\colon\ \mathrm{Spin}_{r,s} \longrightarrow \mathrm{SO}^+_{r,s}$ the adjoint representation. ×

Definition 59.2. A **spin structure** on V is a Ad–reduction $(\mathfrak{s}, S, \xi)$ of $(p\colon\ \mathrm{Fr}_{\mathrm{SO}}(V) \longrightarrow X, R)$. •

Definition 59.3. A **spin manifold** is a pseudo-Riemannian manifold (X, g) together with a spin structure on TX. •

Remark 59.4 (M. Hirsch [Mil63, Alternative Definition 1]). Since $\{\pm 1\} \hookrightarrow \mathrm{Spin}^+_{r,s} \twoheadrightarrow \mathrm{SO}^+_{r,s}$, if $(\mathfrak{s}, S, \xi)$ is a spin structure on V, then $\xi\colon\ \mathfrak{s} \longrightarrow \mathrm{Fr}_{\mathrm{SO}^+}(V)$ is a $\{\pm 1\}$–principal covering map. Moreover, for every $s \in \mathfrak{s}$ the pull-back of $\mathfrak{s}$ via $R_s\colon\ \mathrm{SO}^+_{r,s} \hookrightarrow \mathrm{Fr}_{\mathrm{SO}^+}(V)$ is isomorphic to $\mathrm{Ad}\colon\ \mathrm{Spin}_{r,s} \longrightarrow \mathrm{SO}^+_{r,s}$:



Conversely, every such $\{\pm 1\}$–principal covering map arises from a spin structure on V. Moreover, $(\mathfrak{s}_i, S_i, \xi_i)$ $(i = 1, 2)$ are isomorphic if and only if ξ_i $(i = 1, 2)$ are isomorphic. ♣

Proposition 59.5. *Let $(\mathfrak{s}, s, \xi)$ be a spin structure of V. The map $\mathscr{A}(p, R) \longrightarrow \mathscr{A}(p \circ \xi, S)$ defined by*

$$\theta_A \longmapsto \mathrm{Lie}(\mathrm{Ad})^{-1} \circ \xi^* \theta_A$$

is a bijection. ■

Remark 59.6. Since $\pi_1(\mathrm{GL}^+_{r+s}(\mathbf{R})) = \mathbf{Z}/2\mathbf{Z}$, $\mathrm{GL}^+_{r+s}(\mathbf{R})$ has a unique non-trivial $\{\pm 1\}$–principal covering map $\rho\colon \widetilde{\mathrm{GL}}^+_{r+s}(\mathbf{R}) \to \mathrm{GL}^+_{r+s}(\mathbf{R})$. The composition of $\mathrm{Ad}\colon \mathrm{Spin}_{r,s} \to \mathrm{SO}^+_{r,s}$ with the inclusion $\mathrm{SO}^+_{r,s} \hookrightarrow \mathrm{GL}^+_{r+s}(\mathbf{R})$ lifts to an inclusion $\mathrm{Spin}_{r,s} \hookrightarrow \widetilde{\mathrm{GL}}^+_{r+s}(\mathbf{R})$. Indeed, these maps form a pullback diagram:

$$\begin{array}{ccc} \mathrm{Spin}_{r,s} & \hookrightarrow & \widetilde{\mathrm{GL}}^+_{r+s}(\mathbf{R}) \\ \downarrow{\scriptstyle \mathrm{Ad}} & & \downarrow{\scriptstyle \rho} \\ \mathrm{SO}^+_{r,s} & \hookrightarrow & \mathrm{GL}^+_{r+s}(\mathbf{R}). \end{array}$$

This can be understood as a very efficient construction of $\mathrm{Spin}_{r,s}$. However, it does not help with understanding the representation theory of $\mathrm{Spin}_{r,s}$. Indeed, the spinor representations do not extend to $\mathrm{GL}^+_{r+s}(\mathbf{R})$. ♣

Remark 59.7. Remark 59.6 induces a bijection between Ad–reductions of $\mathrm{Fr}_{\mathrm{SO}^+}(V)$ (up to isomorphism) and ρ–reduction of $\mathrm{Fr}_{\mathrm{GL}^+}(V)$ (up to isomorphism). This observation is sometimes useful to compare spin structure with respect to different Euclidean metrics on V. ♣

60 Stiefel–Whitney classes

See Milnor and Stasheff [MS74, §4, §8].

61 Existence and classification of spin structures

Situation 61.1. Let X be a manifold. Let $V \to X$ be an space- and time-oriented Euclidean vector bundle of signature (r, s). Denote the frame bundle of V by $(p\colon \mathrm{Fr}_{\mathrm{SO}^+}(V) \to X, R)$. Denote by $\pi\colon \mathrm{Spin}_{r,s} \to \mathrm{SO}^+_{r,s}$ the vector representation. ×

Definition 61.2. Let $(\mathfrak{s}, S, \xi)$ be a spin structure on V. Let $\lambda\colon \tilde{X} \to X$ be a $\{\pm 1\}$–principal covering map. The **twist** of $(\mathfrak{s}, S, \xi)$ by λ is the spin structure $(\mathfrak{t}, T, \eta)$ defined by

$$\mathfrak{t} := \tilde{X} \times_{\{\pm 1\}} \mathfrak{s}, \quad T([x, s], g) := [x, sg], \quad \text{and} \quad \eta([x, s]) := \xi(s). \qquad \bullet$$

Remark 61.3. $\{\pm 1\}$–principal covering maps $\tilde{X} \to X$ are classified via their monodromy by $\mathrm{H}^1(X, \mathbf{Z}/2\mathbf{Z})$. Twisting defines an action of $\mathrm{H}^1(X, \mathbf{Z}/2\mathbf{Z})$ on the set of isomorphism classes of spin structure on V. ♣

Proposition 61.4.

(1) *V admits a spin structure if and only if $w_2(V) = 0$.*

(2) *If $w_2(V) = 0$, then the set of isomorphism classes of spin structures on V is a $\mathrm{H}^1(X, \mathbf{Z}/2\mathbf{Z})$–torsor.*

(See also Haefliger [Hae56] and Greub and Petry [GP78].)

Proof. The following argument is due to Milnor [Mil63, p. 199].

Without loss of generality X is connected. By Remark 59.4, it suffices to classify $\{\pm 1\}$–principal covering maps $\xi\colon \mathfrak{s} \to \mathrm{Fr}_{\mathrm{SO}^+}(V)$ whose restriction to every fibre is non-trivial. The 5–term exact sequence associated to the Leray–Serre spectral sequence of $p\colon \mathrm{Fr}_{\mathrm{SO}^+}(V) \to X$ is

$$0 \to \mathrm{H}^1(X, \mathbf{Z}/2\mathbf{Z}) \xrightarrow{p^*} \mathrm{H}^1(\mathrm{Fr}_{\mathrm{SO}^+}(V), \mathbf{Z}/2\mathbf{Z}) \xrightarrow{\mathrm{res}} \mathrm{H}^1(\mathrm{SO}^+_{r,s}, \mathbf{Z}/2\mathbf{Z}) \xrightarrow{\delta_V} \mathrm{H}^2(X, \mathbf{Z}/2\mathbf{Z}).$$

(This can also be obtained by applying $\mathrm{Hom}(\cdot, \mathbf{Z}/2\mathbf{Z})$ to the long exact sequence of homotopy groups associated with p.) Since $\{\pm 1\}$–principal covering maps of $\mathrm{Fr}_{\mathrm{SO}^+}(V)$ are classified through their monodromy by

$$\mathrm{H}^1(\mathrm{Fr}_{\mathrm{SO}^+}(V), \mathbf{Z}/2\mathbf{Z}) = \mathrm{Hom}(\pi_1(\mathrm{Fr}_{\mathrm{SO}^+}(V)), \{\pm 1\})$$

and similarly for $\mathrm{SO}^+_{r,s}$. [See, e.g., my lecture notes on Differential Geometry III]. Therefore, the set of isomorphism classes of spin structure on V is $\mathrm{res}^{-1}([\pi])$ with $[\pi] \in \mathrm{H}^1(\mathrm{SO}^+_{r,s}, \mathbf{Z}/2\mathbf{Z})$ denoting the class of $\pi\colon \mathrm{Spin}_{r,s} \to \mathrm{SO}^+_{r,s}$.

Since the above sequence is exact, V admits a spin structure if and only if

$$\delta_V([\pi]) \in \mathrm{H}^2(X, \mathbf{Z}/2\mathbf{Z})$$

vanishes. This is nothing but $w_2(V)$. This either requires a proof (cf. [Kar68, Proposition (1.1.26)]) or can be taken as the definition of $w_2(V)$.

The above exact sequence exhibits $\mathrm{res}^{-1}([\pi])$ as a $\mathrm{H}^1(X, \mathbf{Z}/2\mathbf{Z})$–torsor. The $\mathrm{H}^1(X, \mathbf{Z}/2\mathbf{Z})$ action is given by twisting with isomorphism classes of $\{\pm 1\}$–principal covering maps. ■

Remark 61.5. Milnor's warning about spin structure vs spin PFB [LM89, Chapter II Remark 1.14]. ♣

62 Spinor bundles and the Atiyah–Singer operator

Situation 62.1. Let X be a manifold. Let $V \to X$ be an space- and time-oriented Euclidean vector bundle signature (r, s). Denote the frame bundle of V by $(p\colon \mathrm{Fr}_{\mathrm{SO}^+}(V) \to X, R)$. Denote by $\pi\colon \mathrm{Spin}_{r,s} \to \mathrm{SO}^+_{r,s}$ the vector representation. Let $(\mathfrak{s}, S, \xi)$ be a spin structure on V. ×

Definition 62.2. Denote by P an irreducible $\mathrm{C}\ell_{r,s}$–module; cf. Section 31 and b as in Section 50. The **spinor bundle** associated with P and b is the Dirac bundle (S, γ, b, ∇) defined as follows:

(1) Set
$$S := \mathfrak{s} \times_{\mathrm{Spin}_{r,s}} P \to X.$$

(2) The Clifford multiplication induces $\gamma\colon TX \to \mathrm{End}(S)$.

(3) The bilinear form b is induced by the bilinear form b above compatible with γ.

(4) The Levi-Civita connection of $\mathrm{Fr}_{\mathrm{SO}^+}(V)$ induces a connection on $\mathfrak{s}$ which induces a covariant derivative ∇_S on S. By construction, γ and b are parallel.

•

Remark 62.3. As a vector bundle S does not depend on the choice of P, but γ does.

(1) If $r - s = -3, -7 \mod 8$, then P carries a complex structure.

(2) If $r - s = -4, -5, -6 \mod 8$, then P carries a quaternionic structure.

(3) If $r - s = 0, -2 \mod 4$, then P is essentially unique.

(4) If $r - s = -1$, then either the volume element $\omega \in C\ell(-g)$ acts as $+i$ or as $-i$; but then these are unique.

(5) If $r - s = -5$, then either the volume element $\omega \in C\ell(-g)$ acts as $+1$ or as -1; but then these are unique.

The discussion in Section 35 governs the further decomposition. ♣

Definition 62.4. Assume the situation of Definition 62.2. The **Atiyah–Singer operator** is the Dirac operator D associated with the spinor bundle (S, γ, b, ∇). •

63 Weitzenböck formula for the Atiyah–Singer operator

`This should probably be called the Schrödinger formula or the Lichnerowicz formula (or both).`

Situation 63.1. Let X be spin manifold. Denote by (S, γ, b, ∇) a spinor bundle. ×

Definition 63.2. Define $R_S \in \Omega^2(X, \mathrm{End}(S))$ by

$$R_S(v, w) := \frac{1}{4} \sum_{i,j=1}^{n} \varepsilon_i \varepsilon_j \gamma(e_i)\gamma(e_j)\langle R(v, w)e_i, e_j\rangle.$$

•

Proposition 63.3. *The curvature F_S of the covariant derivative induced by the Levi-Civita connection is*

$$F_S = R_S.$$

In particular,

$$D^2 = \nabla_S^* \nabla_S + \frac{1}{4}\mathrm{scal}_g.$$

Therefore, if $\mathrm{scal}_g \geqslant 0$, then every harmonic spinor is parallel; if scal_g is positive somewhere, then harmonic spinors must vanish.

Proof. The twisting curvature F_S^{tw} is 2–form with values in skew-symmetric endomorphisms of $\$$ which commute with the Clifford multiplication. Since $\$$ arises from an irreducible representation, by Schur's Lemma an endomorphism of $\$$ commuting with the Clifford multiplication must be a scalar. A skew-symmetric scalar vanishes. This shows that $F_S^{\mathrm{tw}} = 0$. ■

Alternative proof/Exercise. One can proof directly that $F_S = R_S$. ∎

Exercise 63.4. If $S = W$ is a complex spinor bundle, associated to a spin^c−structure prove that $F_S^{\mathrm{tw}} \in \Omega^2(M, i\mathbf{R})$. Identify F_S^{tw} in terms of the curvature of the connection on the characteristic line bundle L. More precisely, prove that $F_S^{\mathrm{tw}} = \frac{1}{2}F_A$ where F_A denotes the curvature of the connection on L.

64 Bochner technique

`This is really a baby version: the vanishing version. More interesting applications use the estimating technique; see https://walpu.ski/Teaching/RiemannianGeometry.pdf.`

Proposition 64.1 (Bochner). *If M is compact and $\mathscr{F}_S$ is non-negative definite (that is: $\langle \mathscr{F}_S\Phi, \Phi\rangle \geqslant 0$), then $D\Phi = 0$ implies $\nabla_S\Phi = 0$. Moreover, $\mathscr{F}_S$ is positive definite somewhere, then $\Phi = 0$.*

Proof. If $D\Phi = 0$, then we have

$$\int_M |\nabla\Phi|^2 + \langle \mathscr{F}_S\Phi, \Phi\rangle = 0.$$

∎

Lecture 10

In this lecture we will warm-up by study spin structures on S^1. Of course, this is very simple, but it already holds a little surprise. Then I will briefly mention the significance of parallel spinors. The bulk of this lecture is concerned with $\mathrm{spin}^{\mathrm{U}(1)}$ structures and Kähler manifolds.

65 Restriction of spin structures

Recall the pullback diagram

$$\begin{array}{ccccc} \mathrm{Spin}_{r_1,s_1} & \longrightarrow & \mathrm{Spin}_{r_1,s_1} \times_{\{\pm 1\}} \mathrm{Spin}_{r_2,s_2} & \longrightarrow & \mathrm{Spin}_{r,s} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{SO}^+_{r_1,s_1} & \longrightarrow & \mathrm{SO}^+_{r_1,s_1} \times \mathrm{SO}^+_{r_2,S_2} & \longrightarrow & \mathrm{SO}^+_{r,s}. \end{array}$$

Definition 65.1. Let X be a manifold. Let $(V_1, q_1) \to X$ be space- and time-oriented Euclidean vector bundles of signature (r_1, s_1). If $(\mathfrak{s}, S, \xi)$ is spin structure on $(V \oplus \mathbf{R}^n, q = q_1 \perp q_2)$ with $q_2 \coloneqq q_{r_2,s_2}$, then the above diagram induces a spin structure $(\mathfrak{s}_1, S_1, \xi_1)$ on (V_1, q_1). This is the **restriction of the spin structure**. •

The main use of this construction is to restrict spin structures on (X, g) to immersed hypersurfaces $\iota\colon Y \to X$ such that $N\iota$ is trivial and $\iota^* g$ is pseudo-Riemannian.

66 Spin structures on S^1

`coming soon`

67 Parallel spinors and Ricci flat metrics

Proposition 67.1 (cf. Hitchin [Hit74, Theorem 1.2]). *Let X be a spin manifold. If there exists a non-zero spinor $\Phi \in \Gamma(\$)$ such that*

$$\nabla \Phi = 0,$$

then X is Ricci flat.

Remark 67.2. This is well-known among physicists, because non-zero parallel spinor are closely related to super symmetry. ♣

Proof. Since Ric is a symmetric tensor, we can chose a local orthonormal frame and functions $\lambda_1, \ldots, \lambda_n$ such that

$$\mathrm{Ric}(e_i, e_j) = \lambda_i \delta_{ij}.$$

If Φ is parallel, then in particular $R_S\Phi = 0$. By Definition 63.2 and Proposition 56.8, this means that

$$\begin{aligned}
0 &= \sum_{\ell=1}^{n} \gamma(e_\ell) R_S(e_k, e_\ell)\Phi \\
&= \frac{1}{4}\sum_{i,j,\ell=1}^{n} \gamma(e_\ell)\gamma(e_i)\gamma(e_j)\langle R(e_k, e_\ell)e_i, e_j\rangle\Phi \\
&= -\frac{1}{2}\sum_{i=1}^{n} \gamma(e_i)\mathrm{Ric}(e_k, e_i)\Phi \\
&= -\frac{1}{2}\lambda_k\gamma(e_k)\Phi.
\end{aligned}$$

It follows that $\lambda_1 = \cdots = \lambda_n = 0$ and therefore $\mathrm{Ric} = 0$. ■

All known Ricci flat manifold have special holonomy, that is, $\mathrm{Hol}(g)$ is a strict subgroup of $\mathrm{SO}(n)$. It is a famous open question whether there are any compact Ricci-flat manifolds with $\mathrm{Hol}(g) = \mathrm{SO}(n)$. If M admits a parallel spinor, then it is impossible that $\mathrm{Hol}(g) = \mathrm{SO}(n)$, because the holonomy group of the spin bundle must reduce to a subgroup $\mathrm{Spin}(n-1) \subset \mathrm{Spin}(n)$. The possible holonomy groups have been classified by Berger [Ber55]. The following theorem clarifies the relation between parallel spinors and special holonomy.

Theorem 67.3 (Wang [Wan89]). *Let X be a complete, simply connected, irreducible spin manifold of dimension n. Set $d \coloneqq \dim\ker \not{D}$. If X is not flat, then one of the following holds:*

(1) $n = 2m$, $\mathrm{Hol}(g) = \mathrm{SU}(m)$ *(that is: M is Calabi–Yau,) and* $d = 2$.

(2) $n = 4m$, $\mathrm{Hol}(g) = \mathrm{Sp}(m)$ *(that is: M is hyperkähler), and* $d = m + 1$.

(3) $n = 7$, $\mathrm{Hol}(g) = G_2$, *and* $d = 1$.

(4) $n = 8$, $\mathrm{Hol}(g) = \mathrm{Spin}(7)$, *and* $d = 1$.

Remark 67.4 (Friedrich [Fri00, Chatper 3, Exercise 4]). For $c > 0$, the metric

$$g = \frac{x_1}{x_1 + c}(\mathrm{d}x_1)^2 + x_1^2(\mathrm{d}x_2)^2 + x_1\sin(x_2)^2(\mathrm{d}x_3)^2 + \frac{x_1 + c}{x_1}(\mathrm{d}x_4)^2$$

is Ricci flat, but does not admit a non-trivial parallel spinor. ♣

68 Spin^G structures on pseudo-Euclidean vector bundles

Situation 68.1. Let X be a manifold. Let $V \to X$ be an space- and time-oriented Euclidean vector bundle of signature (r, s). Denote the frame bundle of V by $(p\colon \mathrm{Fr}_{\mathrm{SO}^+}(V) \to X, R)$. Denote by $\pi\colon \mathrm{Spin}^G_{r,s} \to \mathrm{SO}^+_{r,s}$ the adjoint representation. ×

Definition 68.2. A **spin^G structure** on V is a π–reduction $(\mathfrak{s}, S, \xi)$ of $(p\colon \mathrm{Fr}_{\mathrm{SO}}(V) \to X, R)$. •

Proposition 68.3. *Let* $(\mathfrak{s}, S, \xi)$ *be a spin*^{G} *structure of* V*. Denote by* (q, T) *the* $G/\{\pm 1\}$*–principal bundle associated with* $\rho\colon \mathrm{Spin}^G_{r,s} \to G/\{\pm 1\}$*. Denote by* $\eta\colon \mathfrak{s} \to q$ *the induced map. The map* $\mathscr{A}(p, R) \times \mathscr{A}(q, T) \to \mathscr{A}(p \circ \xi, S)$ *defined by*

$$(\theta_A, \theta_B) \mapsto \mathrm{Lie}(\pi \times \rho)^{-1} \circ (\xi^* \theta_A, \eta^* \theta_B)$$

is a bijection. ■

Remark 68.4. Of course, every spin structure induces a spin^G structure. ♣

69 Spin$^{\mathrm{U}(1)}$ structures

Situation 69.1. Let X be a manifold. Let $V \to X$ be an space- and time-oriented Euclidean vector bundle of signature (r, s). Denote the frame bundle of V by $(p\colon \mathrm{Fr}_{\mathrm{SO}^+}(V) \to X, R)$. Denote by $\pi\colon \mathrm{Spin}_{r,s} \to \mathrm{SO}^+_{r,s}$ the vector representation. ×

Definition 69.2. Let $(\mathfrak{s}, S, \xi)$ be a spin$^{\mathrm{U}(1)}$ structure on V. The **characteristic line bundle** of a spin$^{\mathrm{U}(1)}$ structure is the Hermitian line bundle associated with the homomorphism $\mathrm{Spin}^{\mathrm{U}(1)}_{r,s} \to \mathrm{U}(1)/\{\pm 1\} \to \mathrm{U}(1)$. •

Definition 69.3. Let $(\mathfrak{s}, S, \xi)$ be a spin$^{\mathrm{U}(1)}$ structure on V with characteristic line bundle L Let A be a unitary covariant derivative on L. Denote by P an irreducible $\mathbf{C} \otimes C\ell_{r,s}$–module; cf. Section 32 and b as in Section 50. The **complex spinor bundle** associated with P and b is the Dirac bundle (S, γ, b, ∇) defined as follows:

(1) Set
$$S := \mathfrak{s} \times_{\mathrm{Spin}^{\mathrm{U}(1)}_{r,s}} P \to X.$$

(2) The Clifford multiplication induces $\gamma\colon TX \to \mathrm{End}(S)$.

(3) The bilinear form b is induced by the bilinear form b above compatible with γ.

(4) The Levi-Civita connection of $\mathrm{Fr}_{\mathrm{SO}^+}(V)$ and A induce a connection on $\mathfrak{s}$ which induces a covariant derivative ∇_S on S. By construction, γ and b are parallel.

If $\dim X$ is even, then a complex spinor bundle inherits a **canonical grading** ε from Section 37. •

Definition 69.4. Assume the situation of Definition 69.3. The **complex Atiyah–Singer operator** is the Dirac operator D associated with the spinor bundle (S, γ, b, ∇). •

Definition 69.5. Let $(\mathfrak{s}, S, \xi)$ be a spin$^{\mathrm{U}(1)}$ structure on V. Let $(\lambda\colon P \to X, \Lambda)$ be a $\mathrm{U}(1)$–principal bundle. The **twist** of $(\mathfrak{s}, S, \xi)$ by (λ, Λ) is the spin^G structure $(\mathfrak{t}, T, \eta)$ defined by

$$\mathfrak{t} := P \times_{\mathrm{U}(1)} \mathfrak{s}, \quad T([x, s], g) := [x, sg], \quad \text{and} \quad \eta([x, s]) := \xi(s).$$ •

Remark 69.6. This construction does not work for arbitrary spin^G structures. It hinges on the fact that $\mathrm{U}(1)$ is abelian $(\mathfrak{s}, \Sigma, \xi)$ ♣

Definition 69.7. Denote by $\beta_2\colon \mathrm{H}^k(X,\mathbf{Z}/2Z) \to \mathrm{H}^{k+1}(X,\mathbf{Z})$ the Bockstein homomorphism induced by the exact sequence $0 \to \mathbf{Z} \xrightarrow{2\times} \mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \to 0$. Set

$$W_{k+1}(V) := \beta_2 w_k(V).$$

•

Proposition 69.8.

(1) *V admits a spin$^{\mathrm{U}(1)}$ structure if and only if $w_2(V) \in \mathrm{im}(\mathrm{H}^2(X,\mathbf{Z}) \to \mathrm{H}^2(X,\mathbf{Z}/2\mathbf{Z})$ if and only if $W_3(V) = 0$.*

(2) *If V admits a spin$^{\mathrm{U}(1)}$ structure, then the set of spin$^{\mathrm{U}(1)}$ structures is a torsor over $\mathrm{H}^2(X,\mathbf{Z})$.*

Proof. Exercise. ■

70 Spin structures and spin$^{\mathrm{U}(1)}$ structures on Kähler manifolds

The following is based on Hitchin [Hit74, Section 2.1].

Definition 70.1. If X is a complex manifold, then its **canonical bundle** is

$$\mathscr{K}_X = \Lambda^n_{\mathbf{C}} T^{1,0}X^*$$

and the **anti-canonical bundle** is $\mathscr{K}_X^*$. •

Remark 70.2. If X is a Kähler manifold with volume form vol, then there is a pairing $(\Lambda^n T^{1,0}X^*) \otimes (\Lambda^n T^{0,1}X^*) \to \mathbf{C}$ given by

$$\alpha \otimes \beta \mapsto \frac{\alpha \wedge \beta}{\mathrm{vol}}.$$

In particular,

$$\mathscr{K}_X^* \cong \Lambda^n T^{0,1}X^* \cong \Lambda^n T^{1,0}X.$$

♣

Proposition 70.3. *Suppose X is a Kähler manifold.*

(1) *For any Hermitian line bundle L, there is a unique spin$^{\mathrm{U}(1)}$ structure $(\mathfrak{s}, S, \xi)$ on X whose complex spinor bundle is*

$$S = \bigoplus_{k=0}^{n} \Lambda^k T^{0,1}X^* \otimes L$$

whose characteristic line bundle is $L^{\otimes 2} \otimes_{\mathbf{C}} \mathscr{K}_X^*$. Moreover:

$$S^+ = \bigoplus_{k=0}^{\frac{n}{2}} \Lambda^{2k} T^{0,1}X^* \otimes L \quad \text{and} \quad S^- = \bigoplus_{k=0}^{\frac{(n-1)}{2}} \Lambda^{2k+1} T^{0,1}X^* \otimes L$$

(2) *The Clifford multiplication on S is given by*

$$\gamma(v)\alpha = \sqrt{2}(v^{0,1})^* \wedge \alpha - \sqrt{2}i(v^{0,1})\alpha.$$

(3) *If A is a Hermitian connection on L, then the corresponding connection on W induced by the Levi–Civita connection on $\Lambda^k(T^*X)^{0,1}$ is compatible with the Clifford multiplication.*

(4) *If A induces a holomorphic structure $\bar\partial_{\mathscr{L}}$ on L (that is: $F_A^{0,2} = 0$), then*

$$D = \sqrt{2}(\bar\partial_{\mathscr{L}} + \bar\partial_{\mathscr{L}}^*) \colon \Omega^{0,\bullet}(X, \mathscr{L}) \to \Omega^{0,\bullet}(X, \mathscr{L}).$$

In particular, if X is compact, then the space of positive and negative harmonic spinors can be identified with the cohomology groups

$$\bigoplus_{k=0}^{\lfloor n/2 \rfloor} H^{2k}(X, \mathscr{L}) \quad \text{and} \quad \bigoplus_{k=0}^{\lfloor (n-1)/2 \rfloor} H^{2k+1}(X, \mathscr{L}).$$

Proof. If X is a Kähler manifold, then the structure group of TX is canonically reduced from $\mathrm{SO}(2n)$ to $\mathrm{U}(n)$. It follows from Proposition 49.3, that any Kähler manifold has a canonical $\mathrm{spin}^{\mathrm{U}(1)}$ structure; moreover, the complex spinor bundle is given $\bigoplus_{k=0}^n \Lambda^k(T^*X)^{0,1}$ and the Clifford multiplication is as asserted. It is computation to verify that the characteristic line bundle of the canonical $\mathrm{spin}^{\mathrm{U}(1)}$ structure is given by $\mathscr{K}_X^*$. Taking into account that the set of $\mathrm{spin}^{\mathrm{U}(1)}$ structures is a torsor over the group of Hermitian line bundles, the above proves (1) and (2). (3) is obvious and the first half of (4) follows by a direct computation. The second half of (4) follows by Hodge theory. ■

Proposition 70.4. *A $\mathrm{spin}^{\mathrm{U}(1)}$ structure $(\mathfrak{s}, S, \xi)$ arises from a spin structure if and only if its characteristic line bundle is trivial. The set of spin structures inducing a fixed $\mathrm{spin}^{\mathrm{U}(1)}$ structure is a torsor over* $\ker(H^1(X, \mathbf{Z}_2) \to H^2(X, \mathbf{Z})$ *(that is: the group of Euclidean line bundles with trivial complexification).*

Proof. $\mathrm{Spin}^{\mathrm{U}(1)}(n) = \mathrm{Spin}(n) \times_{\mathbf{Z}_2} \mathrm{U}(1)$ and we have an exact sequence

$$0 \to \mathrm{Spin}(n) \to \mathrm{Spin}^{\mathrm{U}(1)}(n) \to \mathrm{U}(1) \to 0.$$

Since characteristic line bundle is associated to the representation $\mathrm{Spin}^{\mathrm{U}(1)}(n) \to \mathrm{U}(1)$, its triviality is precisely the obstruction to lifting a $\mathrm{spin}^{\mathrm{U}(1)}$ structure to a spin structure. This proves the first section. The second section follows by observing that any two spin structures differ by a Euclidean line bundle $\mathfrak{l}$, while any two $\mathrm{spin}^{\mathrm{U}(1)}$ structures differ by a Hermitian line bundle. ■

Remark 70.5. Serre duality asserts that for a holomorphic vector bundle $\mathscr{E}$ over a compact complex manifold,

$$H^k(X, \mathscr{E}) \cong H^{n-k}(X, \mathscr{E}^* \otimes \mathscr{K}_X)^*.$$

In terms of the Dolbeault resolution, this duality is induced on chain-level by the pairing

$$\begin{aligned}(\Lambda^k T^{0,1}X^* \otimes \mathscr{E}) \otimes (\Lambda^{n-k} T^{0,1}X^* \otimes \mathscr{E}^* \otimes \mathscr{K}_X) &\cong \mathscr{K}_X^* \otimes \mathscr{K}_X \otimes \mathscr{E} \otimes \mathscr{E}^* \\ &\to \Lambda^{2n} T^*X \otimes \mathbf{C} \\ &\to \mathbf{C}.\end{aligned}$$

This pairing induces an isomorphism

$$\Lambda^k T^{0,1}X^* \otimes \mathscr{E} \cong (\Lambda^{n-k}T^{0,1}X^* \otimes \mathscr{E}^* \otimes \mathscr{K}_X)^*.$$

Using the Hermitian inner product on $\mathscr{K}_X$, we obtain an *anti-linear* isomorphism

$$\sigma\colon \Lambda^k T^{0,1}X^* \otimes \mathscr{E} \cong \Lambda^{n-k}T^{0,1}X^* \otimes \mathscr{E}^* \otimes \mathscr{K}_X.$$

In particular, if $\mathscr{L}$ is a square root of $\mathscr{K}_X$ (that is: $\mathscr{L}^{\otimes 2} \cong \mathscr{K}_X$), then

$$\sigma\colon \Lambda^k T^{0,1}X^* \otimes \mathscr{L} \cong \Lambda^{n-k}T^{0,1}X^* \otimes \mathscr{L}.$$

♣

Proposition 70.6. *Let X be a Kähler manifold.*

(1) *X admits a spin structure if and only if there is a complex line bundle L satisfying $L^{\oplus 2} \cong \mathscr{K}_X$.*

(2) *Suppose X is compact. There is a bijective correspondence between the set of spin structures on X and the set of isomorphism classes of holomorphic line bundles $\mathscr{L}$ satisfying $\mathscr{L}^{\otimes 2} \cong \mathscr{K}_X$. (Each such $\mathscr{L}$ inherits a Hermitian metric from $\mathscr{K}_X$.)*

(3) *Suppose that $\mathscr{L}$ is a square root of $\mathscr{K}_X$ and W denotes the associated complex spinor bundle. There is a spinor bundle S such that:*

  (a) *If $\dim_{\mathbf{C}} X = 1 \bmod 4$, then*

$$\$ = W \quad \textit{and} \quad \not{D} = \sqrt{2}(\bar\partial + \bar\partial^*).$$

  The is a complex structure J on $\$$ which commutes with Clifford multiplication and anti-commutes with the complex structure i.

  (b) *If $\dim_{\mathbf{C}} X = 2 \bmod 4$, then*

$$\$^\pm = W^\pm \quad \textit{and} \quad \not{D}^\pm = \sqrt{2}(\bar\partial + \bar\partial^*);$$

  moreover, there is a complex structure J on $\$^\pm$ which commutes with Clifford multiplication and anti-commutes with the complex structure i.

  (c) *If $\dim_{\mathbf{C}} X = 3 \bmod 4$, then is a real structure on W which respect to which $\sqrt{2}(\bar\partial + \bar\partial^*)$ is real. With respect to this real structure we have*

$$\$ = \operatorname{Re} W \quad \textit{and} \quad \not{D} = \sqrt{2}(\bar\partial + \bar\partial^*).$$

  (d) *If $\dim_{\mathbf{C}} X = 4 \bmod 4$, then is a real structure on $W^\pm$ With respect to this real structure we have*

$$\$^\pm = \operatorname{Re} W^\pm \quad \textit{and} \quad \not{D} = \sqrt{2}(\bar\partial + \bar\partial^*).$$

Proof. (1) follows from Proposition 70.4.

(2) Denote $\mathscr{O}^\times$ the sheaf of nowhere vanishing holomorphic functions on X. There is a short exact sequence of sheaves

$$1 \to \mathbf{Z}_2 \to \mathscr{O}^\times \xrightarrow{x \mapsto x^2} \mathscr{O}^\times \to 1.$$

The corresponding long exact sequence in cohomology reads as follows:

$$H^0(X, \mathscr{O}^\times) \to H^0(X, \mathscr{O}^\times) \to H^1(X, \mathbf{Z}_2) \xrightarrow{\alpha} H^1(X, \mathscr{O}^\times) \to H^1(X, \mathscr{O}^\times) \xrightarrow{\beta} H^2(X, \mathbf{Z}_2)$$

The map α is injective, because the map $\mathbf{C}^\times = H^0(X, \mathscr{O}^\times) \to H^0(X, \mathscr{O}^\times) = \mathbf{C}^\times$ is surjective. Recall, that $H^1(X, \mathscr{O}^\times)$ classifies a holomorphic line bundles. A holomorphic line bundle $\mathscr{L}$ has a square root if and only if $\beta([\mathscr{L}]) = (c_1(L) \bmod 2) = 0$. If $\beta([\mathscr{L}]) = 0$, then by the above the set of square roots is a torsor over $H^1(X, \mathbf{Z}_2)$.

For the proof of (3), using 1, one first analyzes the relationship between the spinor representation S and the complex spinor representation W in dimension n and determines the following:

(1) If $n = 2 \bmod 8$, then $S = W$ and W has a complex anti-linear complex structure J. $S = \mathbf{H}$, $W = W^+ \oplus W^- = \mathbf{C} \oplus \mathbf{C}$.

(2) If $n = 4 \bmod 8$, then $S^\pm = W^\pm$ and $W^\pm$ have a complex anti-linear complex structure J.

(3) If $n = 6 \bmod 8$, then there is a real structure on W and $S = \operatorname{Re} W$. This real structure does not respect the splitting $W = W^+ \oplus W^-$. Clifford multiplication is real with respect to this real structure.

(4) If $n = 8 \bmod 8$, then there is a real structure on $W^\pm$ and $S^\pm = \operatorname{Re} W^\pm$. Clifford multiplication is real with respect to this real structure. ■

`The above linear algebra should probably have been discussed earlier.`

Index

References

[ABS64] M. F. Atiyah, R. Bott, and A. Shapiro. *Clifford modules. Topology* 3.1 (1964), pp. 3–38. DOI: 10.1016/0040-9383(64)90003-5. MR: 0167985. Zbl: 0146.19001 (cit. on pp. 22, 33, 46, 54, 56)

[Ber55] M. Berger. *Sur les groupes d'holonomie homogène des variétés à connexion affine et des variétés riemanniennes. Bulletin de la Société Mathématique de France* 83 (1955), pp. 279–330. MR: 0079806. Zbl: 0068.36002 (cit. on p. 71)

[BT82] R. Bott and L. W. Tu. *Differential forms in algebraic topology.* Graduate Texts in Mathematics 82. Springer, 1982. MR: 658304 (cit. on p. 7)

[Bou07] N. Bourbaki. *Éléments de mathématique. Algèbre. Chapitre 9.* Reprint of the 1959 original. Springer, 2007. DOI: 10.1007/978-3-540-35339-3. MR: 2325344. Zbl: 1107.13001 (cit. on pp. 13, 27)

[BW35] R. Brauer and H. Weyl. *Spinors in n Dimensions. American Journal of Mathematics* 57.2 (1935), pp. 425–449. DOI: 10.2307/2371218. Zbl: 0011.24401 (cit. on p. 5)

[Car13] E. Cartan. *Les groupes projectifs qui ne laissent invariante aucune multiplicité plane. Bulletin de la Société Mathématique de France* 41 (1913), pp. 53–96. DOI: 10.24033/bsmf.916 (cit. on p. 5)

[Che54] C. C. Chevalley. *The algebraic theory of spinors.* Columbia University Press, 1954. MR: 0060497. Zbl: 0057.25901 (cit. on pp. 13, 16, 18, 27, 46)

[Dąb88] L. Dąbrowski. *Group actions on spinors.* Monographs and Textbooks in Physical Science 9. Bibliopolis, 1988. MR: 1096953. Zbl: 0703.53001 (cit. on p. 50)

[Dir28] P. A. M. Dirac. *The Quantum Theory of the Electron. Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences* 117.778 (1928), pp. 610–624. DOI: 10.1098/rspa.1928.0023. eprint: `http://rspa.royalsocietypublishing.org/content/117/778/610.full.pdf`. (cit. on p. 5)

[EV08] A. Elduque and O. Villa. *The existence of superinvolutions. Journal of Algebra* 319.10 (2008), pp. 4338–4359. DOI: 10.1016/j.jalgebra.2007.10.044. MR: 2407903 (cit. on p. 46)

[EKM08] R. Elman, N. Karpenko, and A. Merkurjev. *The algebraic and geometric theory of quadratic forms.* American Mathematical Society Colloquium Publications 56. American Mathematical Society, 2008. DOI: 10.1090/coll/056. MR: 2427530. Zbl: 1165.11042 (cit. on pp. 13, 16)

[Fri00] T. Friedrich. *Dirac operators in Riemannian geometry.* Vol. 25. Graduate Studies in Mathematics. American Mathematical Society, 2000. DOI: 10.1090/gsm/025. MR: 1777332. Zbl: 0949.58032 (cit. on pp. 22, 71)

[FT00] T. Friedrich and A. Trautman. *Spin spaces, Lipschitz groups, and spinor bundles.* Vol. 18. 3-4. Special issue in memory of Alfred Gray (1939–1998). 2000, pp. 221–240. DOI: 10.1023/A:1006713405277. (cit. on p. 42)

[Fro77] G. Frobenius. *Ueber lineare Substitutionen und bilineare Formen. Journal für die Reine und Angewandte Mathematik* 84 (1877), pp. 1–63. DOI: 10.1515/crelle-1878-18788403 (cit. on p. 30)

[GP78] W. Greub and H.-R. Petry. *On the lifting of structure groups. Differential geometrical methods in mathematical physics, II (Proc. Conf., Univ. Bonn, Bonn, 1977).* Lecture Notes in Math. 676. Springer, 1978, pp. 217–246. MR: 519614. Zbl: 0399.55005 (cit. on p. 66)

[Hae56] A Haefliger. *Sur l'extension du groupe structural d'un espace fibré.* French. *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences* 243 (1956), pp. 558–560. Zbl: 0070.40001 (cit. on p. 66)

[Har90] F. R. Harvey. *Spinors and calibrations.* Vol. 9. Perspectives in Mathematics. Boston, MA: Academic Press Inc., 1990, pp. xiv+323 (cit. on pp. 46, 52, 56)

[Hir53] F. Hirzebruch. *On Steenrod's reduced powers, the index of inertia, and the Todd genus. Proceedings of the National Academy of Sciences of the United States of America* 39 (1953), pp. 951–956. DOI: 10.1073/pnas.39.9.951. MR: 63670. Zbl: 0051.14501 (cit. on p. 9)

[Hir72] F. Hirzebruch. *The signature theorem: reminiscences and recreation. Prospects in Mathematics.* Vol. 70. Princeton University Press, 1972, pp. 1–32. MR: 0368023. Zbl: 0252.58009 (cit. on pp. 6, 9)

[Hir95] Fr. Hirzebruch. *Topological methods in algebraic geometry.* Classics in Mathematics. Translated from the German and Appendix One by R. L. E. Schwarzenberger, With a preface to the third English edition by the author and Schwarzenberger, Appendix Two by A. Borel, Reprint of the 1978 edition. Springer-Verlag, 1995. DOI: 10.1007/978-3-642-62018-8. MR: 1335917. Zbl: 0843.14009 (cit. on p. 9)

[Hit74] N. Hitchin. *Harmonic spinors. Advances in Math.* 14 (1974), pp. 1–55. MR: 0358873 (cit. on pp. 63, 70, 73)

[Jan20] B. Janssens. *Pin groups in general relativity. Physical Review D* 101.2 (2020). DOI: 10.1103/physrevd.101.021702 (cit. on p. 50)

[Kar68] M. Karoubi. *Algèbres de Clifford et K-théorie.* French. *Annales Scientifiques de l'École Normale Supérieure. Quatrième Série* 1.2 (1968), pp. 161–270. DOI: 10.24033/asens.1163. Zbl: 0194.24101 (cit. on p. 67)

[KS80] M. Knebusch and W. Scharlau. *Algebraic theory of quadratic forms.* Generic methods and Pfister forms, Notes taken by Heisook Lee, DMV Seminar, 1. Birkhäuser, Boston, Mass., 1980. MR: 583195 (cit. on p. 13)

[Knu91] M.-A. Knus. *Quadratic and Hermitian forms over rings.* Grundlehren der mathematischen Wissenschaften 294. Springer-Verlag, 1991. DOI: 10.1007/978-3-642-75401-2. MR: 1096299 (cit. on pp. 13, 25, 46, 47)

[Lam73] T. Y. Lam. *The algebraic theory of quadratic forms.* Mathematics Lecture Note Series. W. A. Benjamin, Inc., Reading, Mass., 1973. MR: 0396410 (cit. on p. 13)

[Lam05] T. Y. Lam. *Introduction to quadratic forms over fields.* Graduate Studies in Mathematics 67. American Mathematical Society, 2005. DOI: 10.1090/gsm/067. MR: 2104929. Zbl: 1068.11023 (cit. on pp. 13, 18, 36, 43, 44)

[LM89] H. B. Lawson Jr. and M.-L. Michelsohn. *Spin geometry.* Princeton Mathematical Series 38. Princeton University Press, 1989. MR: 1031992. Zbl: 0688.57001 (cit. on pp. 9–11, 22, 33, 46, 50, 57, 61, 67)

[LS19] C. I. Lazaroiu and C. S. Shahbazi. *Real pinor bundles and real Lipschitz structures. Asian J. Math.* 23.5 (2019), pp. 749–836. DOI: 10.4310/AJM.2019.v23.n5.a3. (cit. on pp. 42, 46)

[Mil63] J. Milnor. *Spin structures on manifolds. l'enseignement Mathématique. Revue Internationale. 2e Série* 9 (1963), pp. 198–203. MR: 157388. Zbl: 0116.40403. (cit. on pp. 65, 67)

[MH73] J. Milnor and D. H. Husemoller. *Symmetric bilinear forms.* Ergebnisse der Mathematik und ihrer Grenzgebiete 73. Springer-Verlag, 1973. MR: 0506372. Zbl: 0292.10016 (cit. on pp. 13, 17)

[MS74] J. W. Milnor and J. D. Stasheff. *Characteristic classes.* Annals of Mathematics Studies, No. 76. Princeton University Press; University of Tokyo Press, 1974. MR: 0440554. Zbl: 0298.57008 (cit. on p. 66)

[Pal68] R. S. Palais. *The classification of real division algebras. American Mathematical Monthly* 75 (1968), pp. 366–368. DOI: 10.2307/2313414. MR: 228539. Zbl: 0159.04403 (cit. on p. 32)

[Roe98] J. Roe. *Elliptic operators, topology and asymptotic methods.* Second. Pitman Research Notes in Mathematics Series 395. Longman, Harlow, 1998, pp. ii+209. MR: 1670907 (cit. on pp. 33, 46)

[Rok52] V. A. Rokhlin. *New results in the theory of four-dimensional manifolds. Doklady Akademii Nauk SSSR. Novaya Seriya* 84 (1952), pp. 221–224. MR: 0052101. Zbl: 0046.40702 (cit. on p. 10)

[Sch32] E. Schrödinger. *Diracsches Elektron im Schwerefeld. I. Sitzungsberichte der Preußischen Akademie der Wissenschaften, Physikalisch-Mathematische Klasse* 1932 (1932), pp. 105–128. DOI: 10.34663/9783945561317-15. Zbl: 0004.28100 (cit. on p. 61)

[Sch95] G. Schwarz. *Hodge Decomposition. A Method for Solving Boundary Value Problems.* Lecture Notes in Mathematics 1607. Springer-Verlag, 1995. DOI: 10.1007/BFb0095978. MR: 1367287. Zbl: 0828.58002 (cit. on p. 7)

[Ser77] J.-P. Serre. *Linear representations of finite groups.* Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42. Springer, 1977, pp. x+170. MR: 0450380 (cit. on p. 32)

[Tho52] R. Thom. *Espaces fibrés en sphères et carrés de Steenrod. Annales Scientifiques de l'École Normale Supérieure. Troisième Série* 69 (1952), pp. 109–182. DOI: 10.24033/asens.998. MR: 0054960. Zbl: 0049.40001 (cit. on p. 7)

[Tho53] R. Thom. *Variétés différentiables cobordantes. Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences* 236 (1953), pp. 1733–1735. MR: 55689. Zbl: 0050.39602 (cit. on p. 9)

[Tho54] R. Thom. *Quelques propriétés globales des variétés différentiables. Commentarii Mathematici Helvetici* 28 (1954), pp. 17–86. DOI: 10.1007/BF02566923. MR: 61823. Zbl: 0057.15502 (cit. on pp. 8, 9)

[Tra08] A. Trautman. *Connections and the Dirac operator on spinor bundles. Journal of Geometry and Physics* 58.2 (2008), pp. 238–252. DOI: 10.1016/j.geomphys.2007.11.001. MR: 2384313. Zbl: 1140.53025 (cit. on p. 42)

[Wan89] M. Y. Wang. *Parallel spinors and parallel forms. Annals of Global Analysis and Geometry* 7.1 (1989), pp. 59–68. DOI: 10.1007/BF00137402. MR: 1029845 (cit. on p. 71)

[Wey24] H. Weyl. *Análysis situs combinatorio (continuacion).* Spanish. *Revista Mat. hisp.-amer.* 6 (1924), pp. 1–9, 33–41 (cit. on p. 6)