Differential Geometry IV
Problem Set 2

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(1) Let I be a finite-dimensional vector space. Set $V \coloneqq I^* \oplus I$ and define the quadratic form $q\colon V \to k$ by

$$q(\alpha, v) \coloneqq \alpha(v).$$

Prove that

$$C\ell(q) \cong \mathrm{End}(\Lambda I).$$

(2) Let $q\colon V \to k$ be a quadratic form. Let $b \in \mathrm{Hom}(V \otimes V, k)$. Set $v^\flat(w) \coloneqq b(v \otimes w)$. Define the algebra homomorphism $\Psi_b\colon TV \to \mathrm{End}(TV)$ by

$$\Psi_b(v)x \coloneqq v \otimes x + i_{v^\flat}(x).$$

Define $\Theta_b \in \mathrm{End}(TV)$ by

$$\Theta_b(x) \coloneqq \Psi_b(x)\mathbf{1}.$$

Prove the following result from the lectures.

Lemma 0.1. *Let $b, b_1, b_2 \in \mathrm{Hom}(V \otimes V, k)$.*

(a) *$\Theta_b\colon TV \to TV$ is uniquely characterised by $\Theta_b(\mathbf{1}) = \mathbf{1}$ and*

$$\Theta_b(v \otimes x) = v \otimes \Theta_b(x) + i_{v^\flat}\Theta_b(x)$$

for every $v \in V$ and $x \in TV$.

(b) *$\Theta_0 = \mathrm{id}_{TV}$ and $\Theta_{b_1} \circ \Theta_{b_2} = \Theta_{b_1 + b_2}$; in particular: Θ_b is an isomorphism.*

(c) *$\Theta_b I_q \subset I_{q - Q(b)}$; in particular, Θ_b descends to a linear isomorphism*

$$\theta_b\colon C\ell(V, q) \to C\ell(V, q - Q(b)).$$

The above is at the heart of the elegant construction of the symbol and quantisation maps due to Bourbaki.

(3) Prove Schur's Lemma.

(4) Prove Frobenius' theorem on real division algebras.

(5) Let A be an $\mathbf{R}$–algebra. Let V be an A–module.

   (a) Suppose that an isomorphism $\mathrm{End}_A(V) \cong \mathbf{C}$ exists. Does V have a "canonical" complex structure?
   (b) Suppose that an isomorphism $\mathrm{End}_A(V) \cong \mathbf{H}$ exists. Does V have a "canonical" quaternionic structure?

   Hint: Of course, you first have to make precise what you want "canonical" to mean. The answers are not the same for $\mathbf{C}$ and $\mathbf{H}$.