Differential Geometry IV
Problem Set 4

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(1) Let $D \in \{\mathbf{R}, \mathbf{C}, \mathbf{H}\}$ Let P be a $D \otimes \mathrm{C}\ell_{0,n}$–module. Prove that there is a Euclidean inner product $\langle \cdot, \cdot \rangle$ on P such that

$$\langle v\phi, v\psi \rangle = -|v|^2 \langle \phi, \psi \rangle$$

and, moreover, i is orthogonal if $D = \mathbf{C}$ and i, j, k are orthogonal if $D = \mathbf{H}$.

(2) Establish the exceptional isomorphisms

(a) $\mathrm{Spin}_{0,3} \cong \mathrm{Sp}(1) \cong \mathrm{SU}(2)$.

(b) $\mathrm{Spin}_{1,2} \cong \mathrm{SL}_2(\mathbf{R})$.

(c) $\mathrm{Spin}_{0,4} \cong \mathrm{Sp}(1) \times \mathrm{Sp}(1) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$.

(d) $\mathrm{Spin}_{1,3} \cong \mathrm{SL}_2(\mathbf{C})$.

(e) $\mathrm{Spin}_{0,5} \cong \mathrm{Sp}(2)$.

(f) $\mathrm{Spin}_{0,6} \cong \mathrm{SU}(4)$.

(3) Since $\mathrm{Spin}_{r,s} \subset (\mathrm{C}\ell^0_{r,s})^\times$ is a Lie subgroup, $\mathfrak{spin}_{r,s} \subset \mathrm{C}\ell^0_{r,s}$ with the Lie bracket agreeing with the commutator. Set $b = b_{r,s} := \frac{1}{2} p_{r,s}$, $\frac{1}{2}$ of the polarisaton of $q_{r,s}$. Denote by $\kappa \colon \Lambda^2\mathbf{R}^{r+s} \to \mathrm{C}\ell^0_{r,s}$ the map induced by the quantisation map. Identify $\Lambda^2\mathbf{R}^{r+s} = \mathfrak{so}_{r,s}$ via

$$(u \wedge v)w := ub_{r,s}(v, x) - vb_{r,s}(u, x).$$

Prove the following:

(a) $\mathfrak{spin}_{r,s}$ agrees with the image of the quantisation map $\kappa \colon \Lambda^2\mathbf{R}^{r+s} \to \mathrm{C}\ell^0_{r,s}$.

(b) $\mathrm{Lie}(\mathrm{Ad}) \circ \kappa(\alpha) = 2\alpha$ for every $\alpha \in \Lambda^2\mathbf{R}^{r+s} = \mathfrak{so}_{r,s}$.

(4) What double covers $\mathrm{Pin}^{\star}_{r,s} \to \mathrm{O}_{r,s}$ are there such that the following diagram commutes

$$\begin{array}{ccc}
\{\pm 1\} & = & \{\pm 1\} \\
\downarrow & & \downarrow \\
\mathrm{Spin}^{+}_{r,s} & \longrightarrow & \mathrm{Pin}^{\star}_{r,s} \\
\downarrow & & \downarrow \\
\mathrm{SO}^{+}_{r,s} & \longrightarrow & \mathrm{O}_{r,s}
\end{array}?$$