Differential Geometry IV
Problem Set 9

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(1) Consider a smooth quartic Q in $\mathbf{C}P^3$; that is: Q is the zero locus of a transverse $s \in \mathrm{H}^0(\mathscr{O}_{\mathbf{C}P^3}(4))$. Q is a $K3$ surface (or "the" $K3$ surface if you consider them as smooth 4–manifolds only.)

Determine the canonical bundle $\mathscr{K}_Q$ and prove that Q admits a spin structure.

Prove that index $D^+ = 2$ (and, therefore, $\sigma(Q) = -16$.

(2) Let X be a closed spin 4–manifold. Let E be a Hermitian vector bundle with a unitary connection. Compute $\operatorname{index}(D^+ \colon \Gamma(S^+ \otimes E) \to \Gamma(S^- \otimes E))$ in terms of $\sigma(X)$, $c_1(E)$ and $c_2(E)$.

(3) Let X be a closed $2n$–manifold with a $\mathrm{spin}^{\mathrm{U}(1)}$ structure. Determine index $D^+ \colon \Gamma(S^+) \to \Gamma(S^-)$ in terms of $\hat{A}(TX)$ and the characteristic classes of the characteristic line bundle L (the complex line bundle determined by $\mathrm{Spin}^{\mathrm{U}(1)}_{2n} \to \mathrm{U}(1)$).

(4) Let X be a closed oriented 4–manifold. Let V be a oriented rank r Euclidean vector bundle equipped with a connection A. Consider $\delta_A \colon \Omega^1(X, V) \to \Omega^0(X, V) \to \Omega^+(X, V)$ defined by

$$\delta_A \alpha \coloneqq (\mathrm{d}_A^* \alpha, \mathrm{d}_A^+ \alpha).$$

Figure out how to regard δ_A as a Dirac operator.

Determine index δ_A [in terms of $p_1(V)$ and the (refined) Betti numbers of X].