Differential Geometry IV

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Prerequisites Any participant in this lecture course should have a firm understanding of the theory of smooth manifolds, vector bundles, covariant derivatives, etc. Some familiarity with Riemannian Geometry and the theory of principal fibre bundles is assumed, but the course will not rely heavily on this.

Topics The lecture course has three parts: (1) an introduction to spinors, Dirac operators, etc., (2) a treatment (almost from scratch) of the elementary theory of elliptic differential operators on manifolds, (3) an introduction to Seiberg–Witten theory and its applications. Here is a tentative plan of what I plan to cover in the lectures week by week.

Part I: Spin geometry

(1) Three motivations for spin geometry: (a) Dirac's problem: how to make sense of $\sqrt{\Delta}$? (b) Rokhlin's theorem and the Hirzebruch signature theorem. (c) An extremely vague outline of Seiberg–Witten theory.

The role of Clifford algebra's in spin geometry.

The basic theory of quadratic forms.

(2) Construction of the Clifford algebra.

Examples of Clifford algebras.

The grading and filtration of the Clifford algebra. The symbol and quantisation maps.

(3) Computation of the real and complex Clifford algebras.

The pinor and spinor modules.

(4) Construction of the spin groups $\mathrm{Spin}(q)$, $\mathrm{Spin}_{r,s}$, $\mathrm{Spin}^c_{r,s}$, etc.

(5) Clifford algebra bundles, Clifford module bundles, and Dirac operators on smooth manifolds.

The Weitzenböck formula and some applications.

(6) Spin structure, spinor bundles, and the Atiyah–Singer operator.

(7) Dirac operators on homogeneous spaces and spheres.

Part II: elliptic theory on manifolds

(8) The Fourier transform, Schwartz space, Fourier inversion, Plancherel's theorem.

(9) The Sobolev spaces $W^{s,2}$. Sobolev multiplication theorem. Sobolev embedding theorem. Rellich's theorem. Schwartz representation theorem.

(10) Elliptic differential operators on $\mathbf{R}^n$: interior elliptic estimate and interior elliptic regularity.

(11) Elliptic differential operators on manifolds.

The Atiyah–Singer index theorem.

Possibly: a very rough sketch of the heat kernel proof.

Some applications of the index theorem.

Part III: Seiberg–Witten theory

(12) The Seiberg–Witten equation.

Construction of the Seiberg–Witten invariant of a 4–manifold.

(13) Computation of the Seiberg–Witten invariant for complex surfaces.

Some applications of the Seiberg–Witten invariant.

(14) Maybe: the wall-crossing formula for the Seiberg–Witten invariant.

Maybe: a sketch of the Bauer–Furuta construction.