Seminar: Elliptic boundary value problems and applications in geometry

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This seminar meets every **Friday 11:15–13:00** during the summer semester 2023 in **2.006**. If you have any questions, contact me at thomas.walpuski@hu-berlin.de. Please, sign up for the Moodle at https://moodle.hu-berlin.de/course/view.php?id=118893 (key: bvp).

What is this seminar about?

If $D\colon \Gamma(E) \to \Gamma(F)$ is a linear elliptic differential operator of order k defined over a compact manifold X without boundary, then it extends to a **Fredholm operator** $D\colon H^{s+k}\Gamma(E) \to H^s\Gamma(F)$ between Sobolev spaces. Moreover, by **elliptic regularity**, if ϕ is a distributional section of E with $D\phi \in H^s\Gamma(F)$, then $\phi \in H^{s+k}\Gamma(E)$. As a consequence, the index of D is independent of s. It can be determined using the **Atiyah–Singer index theorem**. These facts are crucial for applications, e.g., in Hodge theory and the construction of moduli spaces of pseudo-holomorphic curves and in gauge theory.

If X has a boundary, then the above fail—unless suitable **boundary conditions** are imposed. The purpose of this seminar is to develop the L^2 theory of boundary value problems for Dirac operators D in the sense of Gromov and Lawson, and to discuss a few applications. The bulk of the seminar follows the outstanding exposition in [BB12].

Talks

21.4 **Introduction and overview**. (Thomas Walpuski)

28.4 **Review of L^2 theory of Dirac operators without boundary**. (TBD)

Introduce Dirac bundles and Dirac operators (with examples). Prove the Weitzenböck formula. Introduce the Sobolev spaces H^s (with $s \in \mathbf{R}$), e.g, using Fourier analysis. Sketch the proof of the Sobolev embedding theorem and Rellich's theorem. Establish interior elliptic estimates for Dirac operators. Explain why Dirac operators induce Fredholm operators. Sketch the proof of elliptic regularity.

There are plenty of sources for the above material, e.g., [LM89; Gil95; BGV92; Wel08].

5.5 **Unbounded operators and abstract boundary value problems**. (TBD)

Review the theory of unbounded operators on Hilbert spaces. Prove that if $D\colon \operatorname{dom}(D) \to H$ is symmetric, closed, and densely defined, then the quotient $\check{H} := \operatorname{dom}(D^*)/\operatorname{dom}(D)$ is a symplectic Hilbert space Explain that closed extensions of D correspond to closed subspaces $B \subset \check{H}$. Characterize self-adjoint and Fredholm extensions. Explain why this applies to the minimal extension $D_{\min}\colon \operatorname{dom}(D_{\min}) := H^1_0\Gamma(S) \to L^2\Gamma(S)$. Time permitting, review the spectral theory of self-adjoint unbounded operators with compact resolvent.

There are plenty of sources for the theory of unbounded operators, e.g., [BS18, §6; Ped89, Chapter 5]. The space $\check{H}$ is discussed as the Gelfand–Robbin quotient in [MS98, Exercise 2.17] and as the von Neumann space in [BLZ, §6.2.4].

Although the abstract theory seems quite satisfactory, a more concrete description of $\check{H}$ is required. The next few talks realise $\check{H}$ as a space of sections of $E|_{\partial X}$, and construct a restriction map $\operatorname{res}\colon \operatorname{dom}(D_{\max}) \to \check{H}$ and an extension map $\operatorname{ext}\colon \check{H} \to \operatorname{dom}(D_{\max})$ with $D_{\max} := D^*_{\min}$. Afterwards, questions of regularity theory and index theory can be addressed.

12.5 **The model operator and the trace theorem**. (TBD)

State and prove the trace theorem. Discuss [BB12, §2.1, §4, §5 up to Facts 5.4].

19.5 The day after Ascension (Himmelfahrt).

26.5 **The restriction and the extension map**. (TBD)

Discuss [BB12, §5 after Facts 5.4, §6 up to Lemma 6.3].

2.6 **Boundary regularity theory**. (TBD)

Discuss [BB12, §6 after Lemma 6.3, the key result is Theorem 6.11, possibly skip the proof of Lemma 6.4].

9.6 **Elliptic boundary conditions, I.** (TBD)

Discuss [BB12, §7 up to Theorem 7.10].

16.6 **Elliptic boundary conditions, II.** (TBD)

Discuss [BB12, §7.3 after Theorem 7.10, §7.4].

23.6 **(Pseudo-)local boundary conditions and examples**. (TBD)

Discuss [BB12, §7.5, §7.6].

30.6 **Index theory, I.** (TBD)

Discuss [BB12, §8.1, §8.2, §8.3].

7.7 **Index theory, II.** (TBD)

Discuss [BB12, §8.4, §8.5].

14.7 **Hodge theory on manifolds with boundary and applications.** (TBD)

Discuss the absolute and relative boundary conditions for the Dirac operator $D = d + d^*\colon \Omega(X) \to \Omega(X)$. Prove the Hodge theorem for $\mathrm{H}_{\mathrm{dR}}(X)$ and $\mathrm{H}_{\mathrm{dR}}(X, \partial X)$. Derive Poincaré–Lefschetz duality for de Rham cohomology. Introduce the concept of bordism. Use Poincaré–Lefschetz duality to prove that the bordism invariance of the signature.

It is not difficult to do this by using the theory developed above in any of the standard treatments of the Hodge theorem, e.g., in [Wel08]. I'm not aware of a good reference that does this explicitly. [Sch95] does prove the Hodge theorem, but not using the the above theory.

21.7 **Option 1: Spectral flow and the Maslov index. Option 2: The Atiyah–Patodi–Singer index theorem. Option 3: the BVP theory of general first order operator** (TBD)

[RS95] [BF98] [Mel93] XXX

References

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