Seminar Eichtheorie / Gauge Theory

Constructing the Moduli Space of ASD Instantons

Joshua P. Egger

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- Let G be a Lie group (usually $G = SO(3)$ or $G = SU(2)$), $P \xrightarrow{\pi} X$ be a principal G bundle over a manifold X with Lie algebra valued connection A. Given $V \in$ Vect with representation $\rho : G \to GL(V)$, the usual **associated vector bundle** is $E := P \times_G V$.

- $G \curvearrowright V$ via ρ, and the connection A on $P \xrightarrow{\pi} X$ induces a connection resp. covariant derivative E ∇_A on E

• Note that while the connection A on $P \xrightarrow{\pi} X$ lives in $\Omega^1(P, \mathfrak{g})$, the induced connection on E, for simplicity also denoted by A, is best understood in a local trivialisation U_α.

• On each U_α, the connection 1-form A_α is a $gl(V) := \mathrm{Lie}(GL(V))$-valued one-form

• In any other trivialisation U_β, with $g_{\alpha\beta}$ the transition maps of the bundle E, A transforms via

$$A_\beta = g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta} + i g_{\alpha\beta}^{-1} d g_{\alpha\beta}$$

• The representation $\rho : G \to GL(V)$ induces a representation of Lie algebras $\rho_* : \mathfrak{g} \to gl(V)$ • For simplicity denote $\rho_*(\mathfrak{g}) = \mathfrak{g}$. The

adjoint action $G \curvearrowright \rho_*(\mathfrak{g})$ is also defined va the representation ρ.

• Recall that the adjoint bundle $\mathfrak{g}_E$ is the subbundle of $\mathrm{End}(E)$ defined by

$$\mathfrak{g}_E := P \times_G \mathfrak{g}$$

• Ex: if $G = SU(2)$ and V corresponds to fundamental

representation (from Lie theory), then $\mathfrak{g}_E$ consists of the Hermitian, trace-free endomorphisms of the assoc. bundle E.

• In light of transformation rule, one can show that the difference of two connections is a one form with values in the adjoint bundle, i.e. lives in $\Omega^1(\mathfrak{g}_E)$. The space of all connections $\mathcal{A}$ is then an affine space with tangent space given by $T_A\mathcal{A} = \Omega^1(\mathfrak{g}_E)$.

• The curvature F_A of the of the associated bundle E can also be defined in terms of local trivialisations: on U_α the curvature F_α is a $gl(V)$-valued two-form which transforms via

$$F_\beta = g_{\alpha\beta}^{-1} F_\alpha g_{\alpha\beta}$$

• This shows that the curvature can be seen as an adjoint

bundle-valued two-form, $F_A \in \Omega^2(\mathfrak{g}_E)$.

• Recall that gauge transformations are automorphisms of the associated bundle E as above which preserve the fiber structure and descend to the identity on X. They can be viewed as sections of the automorphism bundle $\mathrm{Aut}(E)$, and form an infinite-dimensional Lie group which we denote by $\mathcal{G}$– the group structure being pointwise multiplication.

• The Lie algebra of $\mathcal{G} := \Gamma(\mathrm{Aut}(E))$ is given by the adjoint-bundle valued zero forms, $\mathrm{Lie}(\mathcal{G}) = \Omega^0(\mathfrak{g}_E)$. This is seen by looking at local charts: on an open set U_α the gauge transformation is given by a map $u_\alpha : U_\alpha \to G$, where G acts through the representation ρ. In this language, gauge transformations thus act on connections according to

$$u^*(A_\alpha) = u_\alpha A_\alpha u_\alpha^{-1} + i d u_\alpha u_\alpha^{-1} = A_\alpha + i(\nabla_A u_\alpha) u_\alpha^{-1}$$

where the covariant derivative has the form

$$\nabla_A u_\alpha = du_\alpha + i[A_\alpha, u_\alpha]$$

Gauge transformations also act on curvature via

$$u^*(F_\alpha) = u_\alpha F_\alpha u_\alpha^{-1}$$

• For analytical purposes, throughout this talk we will always think of $\mathcal{A}$ as the space of $W^{2,l-1}$ connections on E for $l > 2$ and $\mathcal{G}$ as consisting of class $W^{2,l}$ gauge transformations. Later however, we will see that these spaces are completely independent of the choice of $l > 2$.

Short refresher:

• The Yang-Mills functional $YM(\omega)$ of a connection 1-form ω splits

$$YM(\omega) \int_X |F_\omega|^2 d\mu = \int_X \left(|F_\omega^+|^2 + |F_\omega^-|^2\right) d\mu$$

μ being the Riemannian volume element. The connections with $F_\omega = F_\omega^-$ are the **anti-self-dual instantons**, their most salient property being that they minimise the Yang-Mills action

$$S_{YM} = \frac{1}{2} \int_X F \wedge \star F$$

• The anti-self-dual condition is a non-linear diffeq for non-abelian gauge connections, and defines a subspace of the infinite dimensional space of connections which can be regarded as the zero set of the section

$$\sigma : \mathcal{A} \to \Omega^{2,+}(\mathfrak{g}_E)$$

given by

$$\sigma(A) = F_A^+$$

• The goal of this talk is to define a finite-dimensional moduli space, starting from the zero set $\sigma^{-1}(0)$ of σ. The section σ is equivariant with respect to the action of the gauge group,

$$\sigma(u^*(A)) = u^*(\sigma(A)$$

meaning that if a gauge connection is ASD, then it will remain ASD under any gauge transformation.
• The idea is that we obtain a finite-dimensional moduli space by quotienting out $\sigma^{-1}(0)$ by the action of the gauge group $\mathcal{G}$. Due to the $\mathcal{G}$-equivariance of σ, we can define the moduli space of ASD connections $\mathcal{M}_{ASD}$ as

$$\mathcal{M}_{ASD} := \{[A] \in \mathcal{A}/\mathcal{G} : \sigma(A) = 0\}$$

with $[A]$ being the gauge equivalence class of the connection A, well-definedness coming from $\mathcal{G}$-equivariance.

- The L^2 metric on $\mathcal{A}$

$$||A_1 - A_2|| = \left(\int_X |A_1 - A_2|^2 d\mu \right)^{1/2}$$

where $d\mu$ denotes the Riemannian volume element, is preserved by the action of the gauge group, and therefore descends to a fairly natural 'distance function' on the space $\mathcal{B}$, given by

$$d([A],[B]) := \inf_{g \in \mathcal{G}} ||A - g(B)||$$

- All of the metric properties follow fairly readily, except nondegeneracy, it's not immediately clear that $d([A],[B]) = 0 \implies [A] = [B]$, so let's prove this.

Proof:
Suppose that $[A], [B] \in \mathcal{B}$ and $d([A], [B]) = 0$, and let B_α be a sequence of connections in $\mathcal{A}$, all gauge equivalent to B, and converging in L^2 to A. We need to show that A and B are gauge equivalent. Now since the B_α are all gauge equivalent to B there exist gauge transformations $\{u_\alpha\}$ such that $B_\alpha = u_\alpha(B)$.

$$d_B u_\alpha = (B - B_\alpha)u_\alpha$$

This follows from the formula $B_\alpha = u_\alpha A u_\alpha^{-1} - du_\alpha u_\alpha^{-1}$ for the action of gauge transformations, which can be explored further in Donaldson Kronheimer 2.3.7. The u_α are uniformly bounded due to compactness of the structure group G. This also shows that the first derivatives $d_B u_\alpha$ are bounded in L^2, so taking a subsequence, we can suppose that the u_α, if we regard them as sections of the vector bundle $\mathrm{End}(E)$ converge weakly in $W^{1,2}$, and converge strongly in L^2 to a limit u which also satisfies the linear equation

$$d_B u = (B - A)u$$

because if $\varphi \in \Gamma(\mathrm{End}(E))$ is any smooth test section, we have

$$\langle d_b u, \varphi \rangle = \lim_\alpha \langle d_b u_\alpha, \varphi \rangle = \lim_\alpha \langle (B - B_\alpha) u_\alpha, \varphi \rangle = \langle (B - A)u, \varphi \rangle$$

since $B_\alpha u_\alpha \mapsto Au$ in L^1. This equation for u is an overdetermined elliptic equation with $W^{l-1,2}$ coefficients, so we can bootstrap to get that $u \in W^{l,2}$ (for those unfamiliar, bootstrapping refers to the inference of regularity for weak solutions to differential operators, for example $\Delta u \in W^{k,2} \implies u \in W^{k+2,2}$ for a generalised Laplace operator Δ). u is clearly a unitary section in $\mathrm{End}(E)$. $\square$

This fact now allows us to conclude that $\mathcal{B}$ is Hausdorff in the quotient topology.

Reducible and Irreducible Connections

• In order to analyse the moduli space $\mathcal{M}_{ASD}$ we first consider the map

$$\mathcal{G} \times \mathcal{A} \to \mathcal{A}$$

and the quotient space $\mathcal{A}/\mathcal{G}$ of connections by the sections $\mathcal{G}$ of $\mathrm{Aut}(E)$. If the action of $\mathcal{G}$ on $\mathcal{A}$ is not free, then there will be singularities in the quotient space, so we make things work by introducing the isotropy group of a connection $A \in \mathcal{A}$:

$$\Gamma_A := \{u \in \mathcal{G} : u(A) = A\}$$

measuring the extent to which the action $\mathcal{G} \curvearrowright \mathcal{A}$ of $\mathcal{G}$ on $\mathcal{A}$ is not free. If the isotropy group is the center of the group $Z(G) := \{z \in G : \forall g \in G, zg = gz\}$, then the action is free, in which case we say that the connection A is **irreducible**. If the isotropy group is not the center $Z(G)$, the connection A is **reducible**.

• Reducibility of a connection A on a G principal bundle is equivalent to the statement that for each point $x \in X$, the holonomy maps T_γ of loops based at x lie in a proper subgroup of the automorphism group of the bundle at each point, $\mathrm{Aut}(E_x) \cong G$.

• Recall that given a rank k vector bundle E, a connection A on E, and a piecewise smooth loop $\gamma \in X^{[0,1]}$ based at $x \in X$, we have the parallel transport map $P_\gamma : E_x \to E_x$ induced by the connection on the fibre which lives in $GL(E_x)$, and the holonomy group of A based at x is defined as

$$Hol_x(A) := \{P_\gamma \in GL(E_x) : \gamma \in X^{[0,1]} \text{ is a loop based at } x\}$$

• the **holonomy map** for a loop γ is the map sending γ to $hol(\gamma) \in \mathrm{Aut}(E_x)$, giving the linear transformation of vectors after to parallel transport around the loop γ.

• If the base space is connected, then it's not to difficult to show that we can restrict attention to a single fibre and obtain a holonomy group $H_A \subseteq G$, which can be shown to be a closed Lie subgroup of G.

• In physics, reducible connections are well known, as they correspond to gauge configurations in which the gauge symmetry is broken to a smaller subgroup. For example, the $SU(2)$ connection

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & -\alpha \end{bmatrix}$$

is in actuality a $U(1)$ connection. More on this can be found in the physics literature.

• The following lemma which we state without proof gives us the relationship between the isotropy group of a connection and its holonomy group.

Lemma: Given any connection A over a connected base X, the isotropy group Γ_A of A is isomorphic to the center of the holonomy group H_A of A in G.

• If you try to prove this, regard both H_A and Γ_A as subgroups of the automorphism group $\mathrm{Aut}(E_x)$ for $x \in X$, and to note that the center $Z(G)$ is always contained in the isotropy group Γ_A.

• Let's denote the the open subset of $\mathcal{A}$ consisting of *irreducible* connections by $\mathcal{A}^*$. Since $\mathcal{A}^*$ consists of connections whose isotropy group is minimal, we can write

$$\mathcal{A}^* := \{A \in \mathcal{A} : \Gamma_A = Z(G)\}$$

• By definition then, the reduced group of gauge transformations

$\widehat{\mathcal{G}} := \mathcal{G}/Z(G)$ acts freely on the space $\mathcal{A}^*$ of irreducible connections.

• Let $u \in \mathcal{G}$ be a section of $\mathrm{Aut}(E)$, and let us recall how u acts on connections:

$$u^*(A_\alpha) = A_\alpha + i(\nabla_A u_\alpha)u_\alpha^{-1}$$

we then see that the isotropy group can be written as

$$\Gamma_A = \{u \in \mathcal{G} : \nabla_A u = 0\}$$

• That is, the isotropy group at a connection A is given by the covariantly constant sections of the automorphism bundle of the associated bundle E. Γ_A is a Lie group (as a closed subgroup of G) whose elements are the covariantly constant sections of the bundle $\mathrm{Aut}(E)$, and has Lie algebra given by

$$\mathrm{Lie}(\Gamma_A) := \{f \in \Omega^0(\mathfrak{g}_E) : \nabla_A f = 0\}$$

• Therefore, a useful way of detecting whether Γ_A is bigger than the center $Z(G)$ (i.e. has positive dimension, which occurs precisely when there exist nontrivial covariantly constant sections), is to study the kernel of the covariant derivative ∇_A in the $\mathfrak{g}_E$ valued zero forms $\Omega^0(\mathfrak{g}_E)$ on $X-$ the reducible connections then correspond to a nontrivial kernel of

$$\nabla_A : \Omega^0(\mathfrak{g}_E) \to \Omega^1(\mathfrak{g}_E)$$

• As an example, in the case of structure group being the special unitary or special orthogonal groups $SU(2)$, $SO(3)$, which are the most common structure groups appearing physical contexts, reducible connections have exactly the form

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & -\alpha \end{bmatrix}$$

and have isotropy group given by the circle group $\Gamma_A/Z(G) = U(1)$.

- Topologically, this means that a $SU(2)$ bundle E splits as

$$E = L \oplus L^{-1}$$

where L is a complex line bundle, wheras a reducible $SO(3)$ bundle splits into a direct sum of a complex line bundle C with the trivial rank-one real bundle $\mathcal{R}$ over the manifold X.

$$V = \mathcal{R} \oplus C$$

- This can be derived by considering the real part of the

symmetric tensor product $\mathrm{Sym}^2(E)$ on E (the symmetric tensor product is the space of symmetric, contravariant rank-2 tensors on E, spanned by the basis derived from a basis $\{e_\alpha\}$ of E given by $\{e_\alpha \odot e_\beta\}$where $x \odot y = \frac{1}{2}(x \otimes y + y \otimes x)$)

• We now want to construct a local model for the moduli space. That is, we want to characterise its tangent space at a point.

• The way in which we will do this, is by considering the tangent space at an ASD connection $A \in \mathcal{A}$ which is isomorphic to $\Omega^1(\mathfrak{g}_E)$, and look for the directions in the vector space which preserve the ASD condition, and are not gauge orbits, since we're in any case quotienting out by $\mathcal{G} = \Gamma(\mathrm{Aut}(E))$

• Before we do this however, let's first, as promised, obviate the need for worrying about the index l in the Sobolev classes $W^{2,l-1}$ and $W^{2,l}$ of $\mathcal{A}$ and $\mathcal{G}$ respectively.

• For the following proposition, let's temporarily denote the orbit space by $\mathcal{B}(l)$, so that for each $l > 2$ and fixed G-bundle E we have a moduli space $\mathcal{M}(l) \subseteq \mathcal{B}(l)$ of $W^{2,l-1}$ ASD connections mod $W^{2,l}$ gauge transformations. A priori both of these spaces, both as sets and as topological spaces *do* depend on l, however this proposition alleviates our working memories slightly:

• **Proposition:** The natural inclusion of $\mathcal{M}(l+1)$ in $\mathcal{M}(l)$ is a homeomorphism.

• The essence of this proposition is the statement that if A is an ASD connection of Sobolev class $W^{2,l-1}$ for $l > 2$, there exists a a Sobolev class $W^{2,l}$ gauge transformation $u \in \mathcal{G}$ such that the image $u(A)$ is of class $W^{2,l}$. I will not take the time here to prove this, but a full proof can be found in Donaldson and Kronheimer 4.2.3.

• Now the condition that the directions in the tangent space at a connection A are not gauge orbits amounts for us to finding slices of the action of the reduced group of gauge transformation $\widehat{\mathcal{G}} := \mathcal{G}/Z(G)$. The procedure is then to consider the derivative of the map $\mathcal{G} \times \mathcal{A} \to \mathcal{A}$ mentioned earlier with respect to the $\mathcal{G}$ variable at a point $A \in \mathcal{A}^*$ (that is, at an irreducible connection A), which gives a map

$$C : \mathrm{Lie}(\mathcal{G}) \to T_A\mathcal{A}$$

which coincides precisely with the covariant derivative

$$C = \nabla_A : \Omega^0(\mathfrak{g}_E) \to \Omega^1(\mathfrak{g}_E)$$

• Since there is a natural metric on $\Omega^*(\mathfrak{g}_E)$ (recall that one can always take an inner product g on a vector space V and define one on the k-fold tensor product via $g(\otimes_i v_i, \otimes_i w_i) := \frac{1}{k!}\prod_i g(v_i, w_i)$), we can look at the formal adjoint operator

$$C^* : \Omega^1(\mathfrak{g}_E) \to \Omega^1(\mathfrak{g}_E)$$

• Now it is a fact that given a linear map $T : X \to Y$ between two finite-dimensional Hilbert spaces, there is always a decomposition of the codomain Y into the image of T and the kernel of its adjoint T^*, $Y = \mathrm{Im}(T) \oplus \ker(T^*)$. This follows from the facts that $\ker(T*) = (\mathrm{Im}(A))^\perp$ and $Y = \mathrm{Im}(T) \oplus \mathrm{Im}(T)^\perp$ by the definition of the orthogonal compliment.

• We can thus orthgonally decompose the tangent space at A into the gauge orbit $\mathrm{Im}(C)$ and its compliment

$$\Omega^1(\mathfrak{g}_E) = \mathrm{Im}(C) \oplus \mathrm{Ker}(C^*)$$

• Locally this means that a neighbourhood of the equivalence class of a connection $[A]$ in $\mathcal{A}^*/\mathcal{G}$ can be modelled by the kernel of the adjoint of the covariant derivative ∇_A, i.e. by $\mathrm{Ker}(\nabla_A^*) \subseteq T_A\mathcal{A}$.

• Furthermore, the isotropy group Γ_A acts naturally on $\Omega^1(\mathfrak{g}_E)$ by adjoint multiplication, i.e. in the same way gauge transformations act on the curvature as mentioned earlier: $u^*(F_\alpha) = u_\alpha F_\alpha u_\alpha^{-1}$.

• If the connection $A \in \mathcal{A}$ is reducible, then the moduli space is locally modelled on $(\mathrm{Ker}\nabla_A^*)/\Gamma_A$.

• We also have the useful proposition: If A is an ASD connection over X, then a neighbourhood of $[A]$ in the moduli space is modelled on a quotient $f^{-1}(0)/\Gamma_A$, where

$$f : \mathrm{Ker}\delta_A \to \mathrm{Coker}(d_A^+)$$

is a Γ_A-equivariant map.

• What we've done thus far is obtain a local model for the *orbit space* $\mathcal{A}^*/\mathcal{G}$, but it still remains to enforce the ASD condition in order to obtain a local model for the moduli space of ASD connections mod gauge transformations.

• To that end, let $A \in \mathcal{A}^*$ be an irreducible ASD connection, i.e. $F_A^+ = 0$, and let $A + a$ for $a \in \Omega^1(\mathfrak{g}_E)$ be another ASD connection. The condition we obtain on a when we start from $F_{A+a}^+ = 0$ is $\pi^+(\nabla_A a + a \wedge a) = 0$ where π^+ denoted the projection on to the self-dual part of a two-form. Expanding linearly we have that $\pi^+ \nabla_A a = 0$.

• But the map $\pi^+ \nabla_A$ is actually just the linearisation of the section $\sigma : \mathcal{A} \to \Omega^{2,+}(\mathfrak{g}_E)$, $\sigma(A) = F_A^+$ we introduced at the start,

$$\pi^+ \nabla_A = d\sigma : T_A \mathcal{A} \to \Omega^{2,+}(\mathfrak{g}_E)$$

• The kernel of this linearisation then corresponds precisely to the

tangent vectors satisfying the ASD condition. We can now describe the tangent space to $\mathcal{M}_{ASD}$ at A: we would like to take the directions which *are* in $\mathrm{Ker}(d\sigma)$ but *not* in the image of the gauge orbit $\mathrm{Im}(C)$.

• First note that since σ is gauge equivariant, $(\sigma(u^*(A)) = u^*(\sigma(A))$ we have that $\mathrm{Im}(C) \subseteq \mathrm{Ker}(d\sigma)$, which can be checked via direct computation,

$$\pi^+ \nabla_A \nabla_A \varphi = [F_A^+, \varphi] = 0$$

for $\varphi \in \Omega^0(\mathfrak{g}_E)$ since A is anti-self-dual. Now taking into account the decomposition $\Omega^1(\mathfrak{g}_E) = \mathrm{Im}(C) \oplus \mathrm{Ker}(C^*)$, we finally arrive at

$$T_{[A]}\mathcal{M}_{ASD} \cong (\mathrm{Ker}(d\sigma)) \cap \mathrm{Ker}(\nabla_A^*)$$

which can also be regarded as the kernel of the operator

$$D : \Omega^1(\mathfrak{g}_E) \to \Omega^0(\mathfrak{g}_E) \oplus \Omega^{2,+}(\mathfrak{g}_E)$$

given by $D = d\sigma \oplus \nabla_A^*$.

• Now because $\mathrm{Im}(C) \subseteq \mathrm{Ker}(d\sigma)$, there is a short exact sequence called alternately the **Atiyah-Hitchin-Singer complex** or the **instanton deformation complex** which gives an elegant local model for $\mathcal{M}_{ASD}$:

$$0 \to \Omega^0(\mathfrak{g}_E) \xrightarrow{C} \Omega^1(\mathfrak{g}_E) \xrightarrow{d\sigma} \Omega^{2,+}(\mathfrak{g}_E) \to 0$$

• We have in particular that

$$T_{[A]}\mathcal{M}_{ASD} = H_A^1 =: \frac{\mathrm{Ker}(d\sigma)}{\mathrm{Im}(C)}$$

• The **index** of the Atiyah-Hitchin-Singer (AHS) complex is given

by

$$\mathrm{ind} = \dim H_A^1 - \dim H_A^0 - \dim H_A^2$$

or alternatively

$$\text{ind} = \dim H^1_A - \dim \text{Ker}(C) - \dim \text{Coker}(d\sigma)$$

as $H^0_A = \text{Ker}\ (C)$ and $H^2_A = \text{Coker}\ (d\sigma)$. The index is often called the *virtual* dimension of $\mathcal{M}_{ASD}$, and coincides with the dimension of the moduli space in the case where A is an irreducible connection $(\text{Ker}(\nabla_A) = 0)$ and $H^2_A = 0$.

• In this case, A is called a **regular** connection. The AHS index can be computed for any group G via the Atiyah-Singer index theorem.

• Important: $\mathcal{M}_{ASD}$ turns out to be a smooth manifold of **dimension 5** away from the singular points for a generic metric on the base manifold.

• The idea of the proof is to construct a slice of the $\mathcal{G}$-action of the space of connections away from the reducible connections, this will show that the orbit space is a manifold, however it is not necessarily the case that it is a manifold for arbitrary choice of metric.

• It turns out to in fact be true that $\mathcal{M}_{ASD}$ is a smooth manifold for a generic metric, which is the content of the Freed-Uhlenbeck generic metrics theorem, more on this can be found at: https://www.math.stonybrook.edu/ milivojevic/instantons-and-four-manifolds.pdf or in Instantons and Four Manifolds.