Differential Geometry 1 (M13)
Exercise Sheet 11

Prof. Thomas Walpuski

Try to solve the following problems by yourself before the tutorial on **2021-02-10**.

Problem 1. Using Cartan's magic formula compute $\mathscr{L}_v\alpha$ for the following v and α:

1. $v = x\partial_x + y\partial_y$ and $\alpha = x\mathrm{d}x + y\mathrm{d}y$;
2. $v = y\partial_x - x\partial_y$ and $\alpha = x^2\mathrm{d}x$;
3. $v = \partial_x$ and $\alpha = x\mathrm{d}y$;
4. $v = \partial_x$ and $\alpha = y\mathrm{d}x$. ◇

Problem 2. Let X be a smooth manifold. Prove that for $\alpha \in \Omega^k(X)$ and $v_1, \ldots, v_{k+1} \in \mathrm{Vect}(X)$

$$\begin{aligned}(\mathrm{d}\alpha)(v_1, \ldots, v_{k+1}) = &\sum_{i=1}^{k+1}(-1)^{i+1}\mathscr{L}_{v_i}[\alpha(v_1, \ldots, \widehat{v_i}, \ldots, v_{k+1})] \\ &+ \sum_{i<j=1}^{k+1}(-1)^{i+j}\alpha([v_i, v_j], v_1, \ldots, \widehat{v_i}, \ldots, \widehat{v_j}, \ldots, v_{k+1}).\end{aligned}$$ ◇

Problem 3.

1. Let (X, g) be a oriented Riemannian manifold with boundary. For $h, k \in C_c^\infty(X)$ and $v \in \mathrm{Vect}(X)$ prove the integration by parts formula

$$\int_M \Big((\mathscr{L}_v k)\cdot h + k\cdot(\mathscr{L}_v h) + k\cdot h\cdot \mathrm{div}(v)\Big)\,\mathrm{vol}_g = \int_{\partial M} k\cdot h\ i(v)\mathrm{vol}_g.$$

Consider $\mathbf{R}^m$ with a Riemannian metric g.

2. Define the functions g_{ij} by $g_{ij} \coloneqq g(\partial_{x_i}, \partial_{x_j})$. Show that the volume form is given by

$$\mathrm{vol}_g \coloneqq \sqrt{\det(g_{ij})}\cdot \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n$$

3. Show that if
$$v = \sum_{i=1}^{m} v^i \partial_{x_i},$$
then
$$\operatorname{div}(v) = \frac{1}{\sqrt{\det(g_{ij})}} \sum_{k=1}^{m} \partial_{x_k} \left(\sqrt{\det(g_{ij})} \cdot v^k \right). \qquad \diamond$$

Problem 4. Let (X, g) be an compact, connected, oriented Riemannian manifold with boundary. Given $f \in C^\infty(X)$, the **gradient** of f is the vector field $\nabla f = \nabla_g f \in \operatorname{Vect}(X)$ defined by
$$\mathrm{d}f(v) = g(\nabla f, v)$$
and the **Laplacian** of f is the function $\Delta f \in C^\infty(X)$ defined by
$$\Delta f := -\operatorname{div}(\nabla f).$$

The **outward pointing unit normal** is the vector field $n \in \Gamma(TX|_{\partial X})$ characterised by the following conditions: (a) $|n|_g = 1$, (b) $n(x) \perp T_x\partial X$, (c) if $(e_2, \ldots, e_m)$ is a positive basis of $T_x\partial X$, then $(-n, e_2, \ldots, e_m)$ is a positive basis of T_xX.

1. Prove **Green's identities**
$$\int_X h\Delta k \operatorname{vol}_{X,g} = \int_X \langle \nabla h, \nabla k \rangle \operatorname{vol}_{X,g} - \int_{\partial X} h\mathscr{L}_n k \operatorname{vol}_{\partial X,g}$$
and
$$\int_X (h\Delta k - k\Delta h) \operatorname{vol}_{X,g} = \int_{\partial X} (k\mathscr{L}_n h - h\mathscr{L}_n k) \operatorname{vol}_{\partial X,g}$$
with n denoting the outward-pointing unit normal.

2. Show that if $\partial X = \varnothing$, then $\Delta h = 0$ implies that h is constant.

3. Show that if $\partial X \neq \varnothing$, then $\Delta h = \Delta k = 0$ and $h|_{\partial X} = k|_{\partial X}$ implies that $h = k$. $\diamond$