Differential Geometry 1 (M13)
Exercise Sheet 13

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Try to solve the following problems by yourself before the tutorial on **2021-02-24**.

Problem 1. Let X and Y be closed, oriented, smooth manifolds of dimension M. Suppose that Y is connected. Let $f\colon X \to Y$ be a smooth map. For $\mu \in \Omega^m(Y)$ with $\int_Y \mu \neq 0$ define

$$\deg_\mu f := \frac{\int_X f^*\mu}{\int_Y \mu}.$$

1. Prove that $\deg_\mu$ does not depend on the choice of μ.
2. Prove that $\deg_\mu f$ agrees with the degree as defined in the lecture.
3. Let Z be a further closed, connected, smooth manifold of dimension m. Let $g\colon Y \to Z$ be a smooth map. Prove that

$$\deg(g \circ f) = \deg g \cdot \deg f.$$ ◇

Problem 2. Let $p(z) = \sum_{k=0}^d a_k z^k$ be a complex polynomial of degree d. Define $f\colon \mathbf{C}P^1 \to \mathbf{C}P^1$ by

$$f([z : w]) := \left[\sum_{k=0}^d a_k z^k w^{d-k} : w^d\right].$$

Prove that $\deg(f) = d$ and derive the fundamental theorem of algebra. ◇

Problem 3. Denote by D^m the closed unit-ball in $\mathbf{R}^m$. Use the degree to prove that there is no smooth map $f\colon D^m \to S^{m-1}$ with $f(x) = x$ for every $x \in \partial D^m$. ◇