Differential Geometry 1 (M13)
Exercise Sheet 2

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Try to solve the following seven problems by yourself before the tutorial on **2020-11-18**.

Problem 1. Show that the closed disk

$$D := \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 \leqslant 1\}$$

can be equipped with a smooth structure. ◇

Problem 2. Let X be a smooth manifold. Let $f, g \in C^\infty(X)$ and $\lambda \in \mathbf{R}$. Prove that

$$f \cdot g \quad \text{and} \quad f + \lambda g$$

are smooth. ◇

Problem 3. Prove that the **Segre embedding** $\sigma\colon \mathbf{R}P^2 \to \mathbf{R}P^5$ defined by

$$\sigma([x_0 : x_1 : x_2]) := [x_0^2 : x_1^2 : x_2^2 : x_0x_1 : x_0x_2 : x_1x_2] \tag{1.1}$$

is smooth. ◇

Problem 4. Let $m, n \in \mathbf{N}_0$. Let $U \subset \mathbf{R}^m$ and $V \subset \mathbf{R}^n$ non-empty open subsets. Prove that if $f\colon U \to V$ is bijective, f, and f^{-1} are smooth, then $n = m$. (Prove this directly: do not use Brouwer's theorem.) ◇

Problem 5.

1. Prove that there is a homeomorphism $f\colon [0, \infty) \times [0, \infty) \to [0, \infty) \times \mathbf{R}$.
2. Prove that there is no bijection $f\colon [0, \infty) \times [0, \infty) \to [0, \infty) \times \mathbf{R}$ such that f and f^{-1} are both smooth. ◇

Problem 6. Define $h\colon S^2 \to \mathbf{R}$ by

$$h(x_1, x_2, x_3) \coloneqq x_1.$$

This map is smooth. Determine the **critical points** of h; that is, those $x \in S^2$ such that the derivative $T_x h$ vanishes. ◇

Problem 7. Let $A \in \mathrm{Sym}(\mathbf{R}^3)$ be a symmetric 3×3 matrix. Define $q\colon \mathbf{R}P^2 \to \mathbf{R}$ by

$$q([x]) \coloneqq \frac{\langle Ax, x\rangle}{|x|^2}.$$

This map is smooth. Determine its critical points. ◇