Differential Geometry 1 (M13)
Exercise Sheet 3

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Try to solve the following five problems by yourself before the tutorial on **2020-11-25**.

Problem 1. In the lecture, we defined two smooth atlases $\mathscr{A}$ and $\mathscr{B}$ on S^n:

1. The charts of $\mathscr{A}$ are the homeomorphisms $\phi_{i,\pm} \colon H_{i,\pm} \to B_1(0) \subset \mathbf{R}^n$ defined by
$$\phi_{i,\pm}(x_1, \dots, x_{n+1}) \coloneqq (x_1, \dots, \widehat{x_i}, \dots, x_{n+1})$$
with
$$H_{i,\pm} \coloneqq \{(x_1, \dots, x_{n+1}) \in S^n : \pm x_i > 0\}.$$

2. The charts of $\mathscr{B}$ are the homeomorphisms
$$\sigma_\pm(x) \coloneqq \frac{(x_1, \dots, x_n)}{1 \mp x_{n+1}}.$$
with
$$U_\pm \coloneqq S^n \backslash (0, \dots, 0, \pm 1).$$

Prove that $\mathscr{A}$ and $\mathscr{B}$ induce the same smooth structure on S^n. ◇

Problem 2. Denote by $\iota \colon S^n \to \mathbf{R}^{n+1}$ the inclusion. Prove that with respect to the identification $T_x\mathbf{R}^{n+1} = \mathbf{R}^{n+1}$ for every $x \in S^n$
$$\operatorname{im} T_x\iota = \left\{v \in \mathbf{R}^{n+1} : \langle x, v \rangle = 0\right\}.$$ ◇

Problem 3. Let $k \in \mathbf{N}$. Let X be a smooth manifold without boundary of dimension two. Let $f \in \mathrm{Diff}(X)$ be a diffeomorphism of X such that
$$f^k = \underbrace{f \circ \cdots \circ f}_{k \text{ times}} = \mathrm{id}_X.$$

Suppose that $x \in X$ is a fixed-point of f; that is: $f(x) = x$. Prove that there are a $\mu \in \mathbf{C}$ with $\mu^k = 1$ and a chart $\phi \colon U \to B_1(0) \subset \mathbf{R}^2 = \mathbf{C}$ with $\phi(x) = 0$ such that $\phi(U) = U$
$$\phi \circ f \circ \phi^{-1}(z) = \mu \cdot z.$$ ◇

Definition. Let X and Y be smooth manifolds and let $f: X \to Y$ be a smooth math. A point $x \in X$ is said to be a **critical point of** f if $T_x f$ is surjective, otherwise it is said to be **regular**. A point $y \in X$ is said to be a **critical value of** f if it is the image of a critical point. It is a **regular value of** f if it is not a critical value. •

Problem 4. Define $f: \mathbf{R}^{n+1} \to \mathbf{R}$ by

$$f(x_1, \ldots, x_{n+1}) := |x|^2 = \sum_{i=1}^{n+1} x_i^2$$

Prove that 1 is a regular value of f. ◇

Problem 5. Define $f: \operatorname{End}(\mathbf{R}^n) \to \operatorname{Sym}(\mathbf{R}^n)$ by

$$f(A) := AA^t.$$

Prove that $\mathbf{1}$ is a regular value of f. ◇