Differential Geometry 1 (M13)
Exercise Sheet 5

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Try to solve the following problems by yourself before the tutorial on **2020-12-09**.

Problem 1.

1. Prove that
$$\mathrm{St}_k^*(\mathbf{R}^n) \coloneqq \{(v_1, \dots, v_k) \in (\mathbf{R}^n)^k : v_1, \dots, v_k \text{ are linearly independent}\}$$
is a submanifold of $(\mathbf{R}^n)^k$.

2. Prove that
$$\mathrm{St}_k(\mathbf{R}^n) \coloneqq \{(v_1, \dots, v_k) \in (\mathbf{R}^n)^k : \langle v_i, v_j \rangle = \delta_{ij}\}.$$
is a submanifold of $(\mathbf{R}^n)^k$.

   Hint: Use the regular value theorem.

3. Construct a surjective smooth map $r\colon \mathrm{St}_k^*(\mathbf{R}^n) \to \mathrm{St}_k(\mathbf{R}^n)$ satisfying $r \circ r = r$.

   Hint: https://en.wikipedia.org/wiki/Gram-Schmidt_process.

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Remark. Observe that $\mathrm{St}_n^*(n)$ is $\mathrm{GL}(\mathbf{R}^n)$ and $\mathrm{St}_n(n)$ is $\mathrm{O}(n)$. ♣

Problem 2. Denote by $\Delta \subset \mathbf{R}^{n\times n}$ the subspace of diagonal matrices. Define $f\colon \mathrm{O}(n) \times \Delta \to \mathrm{Sym}(\mathbf{R}^n)$ by

$$f(\Phi, \Lambda) \coloneqq \Phi\Lambda\Phi^t. \tag{1.1}$$

1. Determine the set of regular values of f.
2. For every $A \in \mathrm{Sym}(\mathbf{R}^n)$ determine $f^{-1}(A)$. For which A is $f^{-1}(A)$ is submanifold.

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Problem 3. Let X be a manifold without boundary and let $r\colon X \longrightarrow X$ be a smooth map satisfying

$$r \circ r = r. \tag{1.2}$$

Prove that $\operatorname{im} r \subset X$ is a submanifold.

Hint: Find an open neighborhood of $r(X)$ such that $T_x r$ has constant rank for $x \in U$. ◇

Problem 4. Let X, Y be finite-dimensional real vector spaces. For $r \in \mathbf{N}_0$ set

$$\mathscr{H}_r := \{L \in \operatorname{Hom}(X, Y) : \operatorname{rk} L = r\}.$$

Prove that $\mathscr{H}_r$ is a submanifold of codimension

$$\operatorname{codim} \mathscr{H}_r = (\dim X - r)(\dim Y - r).$$

Hint: Consider $L \in \mathscr{H}_r$. Decompose $X = X_1 \oplus X_2$ with $X_2 = \ker L$ and $Y = Y_1 \oplus Y_2$ with $Y_1 = \operatorname{im} L$. Decompose $\tilde{L} \in \mathscr{H}_r$ into blocks accordingly. For $\tilde{L}$ close to L, apply row and column operations to simplify $\tilde{L}$. Use the regular value theorem. ◇

Problem 5. Let $f\colon X \longrightarrow Y$ be smooth. The **graph of** f is the subset

$$f := \{(x, f(x) : x \in X\}.$$

Let $Z \subset Y$ be a submanifold. Prove that f is transverse to Z if and only if f is transverse to $X \times Z$. ◇