Differential Geometry 1 (M13)
Exercise Sheet 9

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Try to solve the following problems by yourself before the tutorial on **2021-01-27**.

Definition. A **Lie group** is a smooth manifold G together with a group structure such that the multiplication map $m\colon G \times G \to G$ and the inversion map $\mathrm{inv}\colon G \to G$ are smooth. •

Problem 1. Recall that $\mathrm{SO}(n) = \{A \in \mathbf{R}^{n\times n} : A^t A = \mathrm{id}, \det(A) = 1\}$ is a submanifold of $\mathbf{R}^{n\times n}$. Matrix multiplication makes $\mathrm{SO}(n)$ into a group. Prove that $\mathrm{SO}(n)$ is a Lie group. ◇

Definition. Let G be a Lie group. Let X be a smooth manifold. A **smooth action of** G **on** X is a smooth map $\rho\colon G \times X \to X$ such that

$$\rho(h, \rho(g, x)) = \rho(hg, x) \quad \text{and} \quad \rho(\mathbf{1}, x) = x$$

for every $h, g \in G, x \in X$. •

It is often convenient to set $\rho_g(x) \coloneqq \rho(g, x)$–or even: $gx \coloneqq \rho(g, x)$.

Problem 2. Prove that the following are smooth actions of G on itself.

1. The left-multiplication $L\colon G \times G \to G$ defined by

$$L(g, h) \coloneqq gh$$

2. The right-multiplication with the inverse $R\colon G \times G \to G$ defined by

$$R(g, h) \coloneqq hg^{-1}.$$

3. The conjugation $C\colon G \times G \to G$ defined by

$$C(g, h) \coloneqq ghg^{-1}.$$

◇

Problem 3. Prove that $R\colon \mathrm{SO}(n+1)\times S^n \to S^n$ defined by

$$R(A,x) \coloneqq Ax$$

is a smooth action. ◇

Definition. Let $\rho\colon G\times X \to X$ be a smooth action. A vector field $v \in \mathrm{Vect}(X)$ is ρ**–invariant** if it is ρ_g–related to itself for every $g \in G$; concretely, for every $g \in G$ and $x \in X$

$$v(\rho_g(x)) = T_x\rho_g v(x).$$

The space of ρ–invariant vector fields is denoted by $\mathrm{Vect}(M)^\rho$. •

Definition. Let G be a Lie group. The **Lie algebra of** G is

$$\mathrm{Lie}(G) = \mathfrak{g} \coloneqq \mathrm{Vect}(G)^L.$$ •

Problem 4.

1. Prove that the map $\mathrm{ev}_{\mathbf{1}}\colon \mathfrak{g} \to T_{\mathbf{1}}G$ defined by

   $$\mathrm{ev}_{\mathbf{1}}(\xi) \coloneqq \xi(\mathbf{1})$$

   is an isomorphism of vector space. (In particular; $\dim\mathfrak{g} = \dim G$).

2. Let $\xi,\eta \in \mathfrak{g}$. Prove that the $[\xi,\eta] \in \mathfrak{g} \subset \mathrm{Vect}(G)$.

3. Since $C_g(\mathbf{1}) = g\mathbf{1}g^{-1} = \mathbf{1}$, $T_gC\colon T_{\mathbf{1}}G \to T_{\mathbf{1}}G$. Define $\mathrm{Ad}\colon G \to \mathrm{End}(\mathfrak{g})$ by

   $$\mathrm{Ad}(g) = \mathrm{Ad}_g \coloneqq \mathrm{ev}_1 \circ T_gC \circ \mathrm{ev}_1^{-1}.$$

   Prove that

   $$\mathrm{Ad}_g[\xi,\eta] = [\mathrm{Ad}_g\,\xi, \mathrm{Ad}_g\,\eta].$$

4. Define $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ by

   $$\mathrm{ad}(\xi) = \mathrm{ad}_\xi \coloneqq T_{\mathbf{1}}\,\mathrm{Ad} \circ \mathrm{ev}_{\mathbf{1}}^{-1}.$$

   Prove that

   $$\mathrm{ad}_\xi(\eta) = [\xi,\eta]$$ ◇