Bundles, connections, and curvature

Sunday, November 8, 2020 9:02 AM

These notes are already on the seminar's page.

§1 Fiber bundles

Let X be a manifold: the base manifold.

Def: A fiber bundle over X is a manifold F together with a smooth map

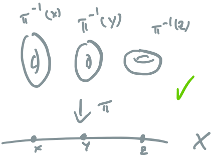
$$\pi : F \to X$$

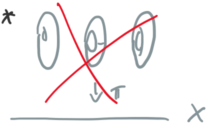
s.th. for every $x \in X$ there is an open neighborhood $\mathcal{U}$ of x and a diffeomorphism

$$\phi : \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times \pi^{-1}(x)$$

satisfying

$$\pi = pr_1 \circ \phi .$$

* 

* 

* vector bundles are fiber bundles

Ehresmann fibration theorem: If $\pi : F \to X$

is a proper submersion, then F is a fiber bundle.

Def: The vertical tangent bundle of F is

$$VF := \ker(T\pi : TF \to TX)$$

$$\subset TX$$

There is an exact sequence of vector bundles

$$(*) \qquad 0 \to VF \hookrightarrow TF \twoheadrightarrow \pi^* TX \to 0$$

but no canonical splitting.

Def: An Ehresmann connection on F is a subbundle

$$H \subset TF \qquad \text{"horizontal subbundle"}$$

s. th.

$$TF = VF \oplus H \quad \text{and} \quad T\pi : H \xrightarrow{\cong} \pi^* TX \text{ iso.}$$

Rmk: H defines a connection 1-form

$$A = A_H := pr_{VF} \in \Omega^1(F, TF) = \Gamma(\mathrm{Hom}(TF))$$

s. th

$$A \circ A = A \quad \text{and} \quad \mathrm{im}\, A = VF.$$

Every such A arises from an Ehresmann

connection.

Def: Let H be a connection on F. Let $\gamma : [a,b] \to X$ be a smooth path. A **horizontal lift** of γ is a smooth path

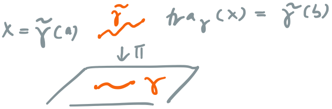
$\tilde{\gamma} : [a,b] \to F$ s.th

$$\pi \circ \tilde{\gamma} = \gamma \quad \text{and} \quad \dot{\tilde{\gamma}}(t) \in H.$$

The **parallel transport induced by H and γ** is the diffeomorphism $\mathrm{tra}_\gamma : \pi^{-1}(\gamma(a)) \to \pi^{-1}(\gamma(b))$ defined by

$$\mathrm{tra}_\gamma(\tilde{\gamma}(a)) = \tilde{\gamma}(b)$$

with $\tilde{\gamma}$ a horizontal lift of γ.



* The parallel transport is (usually) path-dependent, even if the base X is contractible.

* The vertical tangent bundle VF is always integrable, but H need not be (or cannot be).

Def: The curvature of H is the 2-form

$$F_H \in \Gamma(\mathrm{Hom}(\Lambda^2 H, VF)) \subset \Omega^2(F, VF)$$

defined by

$$F_H(\tilde{v}(x), \tilde{w}(x)) := -[\tilde{v}, \tilde{w}](x).$$

The Ehresmann connection H is flat if

$$F_H = 0.$$

($\iff$ H is integrable

$\iff$ tra_γ depends only on homotopy class of γ rel $\{a, b\}$)

§2 Principal bundles

(not: principle)

The above is too general for us. We need symmetry!

Def: A principal G-bundle over X is a manifold P together with a smooth map $\pi : P \to X$ and a free right G-action on P such that for every $x \in X$:

1. G preserves the fiber $P_x := \pi^{-1}(x)$ and its action on P_x is free and transitive.

2. There is an open neighborhood $\mathcal{U}$ of x and a G-equivariant diffeomorphism

$$\phi : P|_{\mathcal{U}} := \pi^{-1}(\mathcal{U}) \longrightarrow \mathcal{U} \times G$$

satisfying

$$\pi = pr_1 \circ \phi .$$

The slice theorem implies:

Prop: Let G be a compact Lie group. If G acts freely on the right of P, then $\pi : P \to X := P/G$ is a principal G-bundle.

Ex: The *product bundle* is:

$$pr_1 : P = X \times G \longrightarrow X.$$

Ex: The *quaternionic Hopf map* is

$$\eta : S^7 \subset \mathbb{H}^2 \setminus \{0\} \longrightarrow \mathbb{HP}^1 \cong S^4$$

$$(x, y) \longmapsto [x : y].$$

Its fibres are precisely the orbits of the free right action of $Sp(1) = \{x \in \mathbb{H} : |x| = 1\}$.

This exhibits $\eta : S^7 \to S^4$ as a principal

Sp(1)-bundle.

Ex: let $E \to X$ be a vector bundle of rank r. The frame bundle of E is

$$\begin{tikzcd} Fr(E) := \coprod_{x\in X} \{ (v_1,\dots,v_r) \in E_x^{\oplus r} ; \; v_1,\dots,v_r \text{ basis of } E_x \}. \arrow[d, "\pi"] \\ X. \end{tikzcd}$$

$A = (a^i_j) \in GL(\mathbb{R}^r)$ acts on $Fr(E)$ by

$$(v_1,\dots,v_r)\cdot(a^i_j) := \left(\sum_{i=1}^r v_i a^i_1, \dots, \sum_{i=1}^r v_i a^i_r \right).$$

If E has more structure, then one can construct corresponding variations of $Fr(E)$.

Ex: The universal cover

$$\pi : \tilde{X} \to X \qquad X \text{ connected.}$$

is a principal $\underbrace{\pi_1(X)}_{\text{discrete topology}}$-bundle

§ 2A Principal connections

For $g \in G$ define $R_g \in \mathrm{Diff}(P)$ by

$$R_g(p) := p \cdot g.$$

For $\xi \in \mathfrak{g} := \mathrm{Lie}(G)$ denote by

$$v_\xi \in \mathrm{Vect}(P)$$

the vector field defined by

$$\sigma_\xi(p) := \frac{d}{dt}\Big|_{t=0} p\cdot\exp(t\xi).$$

This trivializes the vertical tangent bundle

$$P\times\mathfrak{g} \cong VP$$

$$(p,\xi)\mapsto\sigma_\xi(p).$$

Def: A (principal) connection (1-form) on P is a 1-form

$$A\in\Omega^1(P,\mathfrak{g})$$

satisfying

1. $A(\sigma_\xi) = \xi$
2. $R_g^* A = Ad(g)^{-1}\circ A.$

The set of connections on P is denoted by

$$\mathcal{A}(P).$$

Rmk: $$(R_g^* A)_p(\sigma_\xi(p)) = A_{pg}(\overbrace{\sigma_\xi(p)\cdot g}^{p\cdot\xi\cdot g \,=\, p\cdot g\cdot g^{-1}\xi\cdot g})$$

$$= Ad(g)^{-1}(\xi) \quad ✓$$

Ex: If $P = X\times G$, then

$$A_0 := pr_2^*\mu \qquad \text{"product connection"}$$

with $\mu\in\Omega^1(G,\mathfrak{g})$ denoting the Maurer-

Cartan form defined by

$$\mu_g(v) := dL_{g^{-1}}(v) = g^{-1}v$$

is a connection.

Ex: The universal cover $\pi : \tilde{X} \to X$ has a unique connection.

Rmk: $A \in \Omega^1(P, \mathfrak{g}) \subset \Gamma(\mathrm{End}(TP))$ satisfies $A \circ A$ by (1), and $H := \ker A$ satisfies $R_g^* H = H$ by (2). "principal Ehresmann connection"

Def: A differential form $\alpha \in \Omega^\bullet(P)$ is horizontal if

$$i_{v_\xi} \alpha = 0 \quad \text{for every } \xi \in \mathfrak{g}.$$

The subset of horizontal forms is denoted by

$$\Omega^\bullet_{hor}(P).$$

Def: Let $\rho : G \to GL(V)$ be a representation. Let G act on $\Omega^\bullet(P, V)$ by

$$g \cdot \alpha := \rho(g) \circ R_g^* \alpha.$$

Prop: $\mathcal{A}(P)$ is an affine space modelled on

$$\Omega^1_{hor}(P, \mathfrak{g})^{(G, \mathrm{Ad})}.$$

§2B Curvature

Prop: Define $d_A : \Omega^\bullet(P,V) \to \Omega^{\bullet+1}(P,V)$ by

$$\alpha \mapsto d\alpha + \underbrace{[\mathrm{Lie}(S)\circ A]}_{\in\, \Omega^1(P,\mathrm{End}(V))} \wedge \alpha .$$

1. d_A maps $\Omega^\bullet_{hor}(P,V)^{(G,S)}$ to $\Omega^{\bullet+1}_{hor}(P,V)^{(G,S)}$

2. There is a 2-form

$$F_A = dA + \tfrac{1}{2}[A\wedge A] \in \Omega^2_{hor}(P,\mathfrak{g})^{(G,Ad)}$$

s.th:

$$d_A \circ d_A = \underbrace{[\mathrm{Lie}(S)\circ F_A]}_{\Omega^2_{hor}(P,\mathrm{End}(V))^{(G,\mathrm{End}(S))}} \wedge -$$

3. F_A agrees with the curvature of the Ehresmann connection $H := \ker A$.

4. $F_{A+a} = F_A + d_A a + \frac{1}{2}[a\wedge a]$.

5. $d_A F_A = 0$ "Bianchi identity"

Pf: 1. let $\alpha \in \Omega^k_{hor}(P,V)^{(G,S)}$.

$$* \; i_{\sigma_\xi} d_A \alpha = i_{\sigma_\xi} d\alpha + d\, \overset{\overset{0}{\shortparallel}}{i_{\sigma_\xi}\alpha}$$

$$+ [\mathrm{Lie}(S)\circ \underset{\xi}{A(\cancel{\sigma_\xi})}]\cdot \alpha$$

$$- \mathrm{Lie}(S)\circ A \wedge \cancel{i_{\sigma_\xi}\alpha} = 0$$

$$= \mathcal{L}_{\sigma_\xi}\alpha + [\mathrm{Lie}(S)(\xi)]\cdot\alpha$$

$$\Vert$$

$$\frac{d}{dt}\Big|_{t=0} R^*_{\exp(t\xi)}\,\alpha$$

$$= \frac{d}{dt}\Big|_{t=0} \rho(\exp(-t\xi))\cdot\alpha$$

$$= -[\mathrm{Lie}(\rho)(\xi)]\cdot\alpha$$

$$= 0. \quad ✓$$

$$* \quad R_g^* d_A\alpha = R_g^*\{d\alpha + [\mathrm{Lie}(\rho)\circ A]\cdot\alpha\}$$

$$= d\,R_g^*\alpha + [\mathrm{Lie}(\rho)\circ R_g^* A]\cdot R_g^*\alpha$$

$$= -\,d[\rho(g)\cdot\alpha]$$

$$-\,[\underbrace{\mathrm{Lie}(\rho)\,(\mathrm{Ad}(g)^{-1}\circ A)}_{\rho(g)\circ[\mathrm{Lie}(\rho)\circ A]\circ\rho(g)^{-1}}]\;\rho(g)\cdot\alpha$$

$$= -\,\rho(g)\cdot d\alpha$$

$$-\,\rho(g)\cdot\mathrm{Lie}(\rho)\circ A\cdot\alpha$$

$$= -\,\rho(g)\,d_A\alpha. \quad ✓$$

2. $d_A d_A\alpha = d_A(d\alpha + (\mathrm{Lie}(\rho)\circ A)\wedge\alpha)$

$$= d\overset{=0}{\cancel{d}}\alpha + (\mathrm{Lie}(\rho)\circ A)\wedge d\alpha$$

$$\Sigma = 0$$

$$+ (\mathrm{Lie}(\rho)\circ dA)\wedge\alpha - \mathrm{Lie}(\rho)\circ A\wedge d\alpha$$

$$+ (\mathrm{Lie}(\rho)\circ A)\wedge(\mathrm{Lie}(\rho)\circ A)\wedge\alpha$$

$$= \mathrm{Lie}(\rho)\circ\left(dA + \tfrac{1}{2}[A\wedge A]\right)\wedge\alpha \quad ✓$$

Ex: Check that $F_A = dA + \frac{1}{2}[A \wedge A]$ is

hint: Maurer-Cartan eq'n

horizontal and G-invariant!

$$3. \quad \underset{\substack{\uparrow \\ \text{horizontal}}}{F_A(\tilde v, \tilde w)} = dA(\tilde v, \tilde w) + [A(\cancel{\tilde v}), A(\cancel{\tilde w})]$$

$$= \mathcal{L}_{\tilde v} A(\overset{=0}{\cancel{\tilde w}}) - \mathcal{L}_{\tilde w} A(\overset{=0}{\cancel{\tilde v}})$$

$$\underset{\substack{\uparrow \\ \text{explains sign in } F_H.}}{-} A([\tilde v, \tilde w]) = F_H(\tilde v, \tilde w)$$

4. & 5. exercise! $\blacksquare$

Ex: The product connection has vanishing curvature: it is flat.

Ex: The unique connection on $\tilde X$ has vanishing curvature.

§2c Gauge transformations

Def: Let $\pi : P \to X$ and $\rho : Q \to X$ be principal G-bundles. An isomorphism from P to Q is a G-equivariant smooth map

$$\phi : P \to Q$$

s.th.

$$\pi = \phi \circ u.$$

These are automatically diffeomorphisms and u^{-1} is an isomorphism $Q \to P$.

A gauge transformation of P is an isomorphism $u : P \to P$. The gauge group of P is

$$\mathcal{G}(P) := \{ u : P \to P \text{ gauge transformation} \}.$$

Rmk: * P is trivial if it is isomorphic to the product bundle.

* By definition, every P is locally trivial.

* P is trivial if and only if it admits a section; i.e.: a smooth map $s : X \to P$ satisfying $\pi \circ s = id_X$.

Prop: If $A \in \mathcal{A}(P)$ and $u : P \to Q$ is an isomorphism, then

$$u_* A = (u^{-1})^* A \in \mathcal{A}(Q).$$

Cor: $\mathcal{G}(P) \circlearrowright \mathcal{A}(P)$

Prop:

1. If $u \in \mathcal{G}(P)$, then there is a unique $\tilde{u} : P \to G$ s.th. for all $p \in P$

$$u(p) = p \cdot \tilde{u}(p).$$

Moreover: $\tilde{u}$ is G-equivariant w.r.t.

$C : G \to \mathrm{Aut}(G)$ def. by

$$C(g)\,h := g h g^{-1}.$$

2. A and $u_* A$ are related by Maurer-Cartan

$$u_* A = \mathrm{Ad}(\tilde{u}) \circ A + (\tilde{u}^{-1})^* \overset{\downarrow}{\mu}$$

$$\left.\begin{array}{l}\text{less formal}\\ \text{but more}\\ \text{enlightening}\end{array}\right\{ \begin{aligned} &= \tilde{u} \cdot A \cdot \tilde{u}^{-1} + \tilde{u}\; T(\tilde{u}^{-1}) \\ &= \tilde{u} \cdot A\, \tilde{u}^{-1} - (T\tilde{u})\, \tilde{u}^{-1}. \end{aligned}$$

3. The curvature of A and $u_* A$ are related by

$$F_{u_* A} = \mathrm{Ad}(\tilde{u}) \circ F_A .$$

Pf: 1. & 3. exercise? ∎

2. Since $u(p) = p \cdot \tilde{u}(p) = R_{\tilde{u}(p)}\, p$,

$$T_p u(v) = d_p R_{\tilde{u}(p)}(v) + \underset{\shortparallel}{``p \cdot [d_p \tilde{u}(v)]"}$$

$$p \cdot \tilde{u}(p) \cdot \underbrace{\tilde{u}^{-1}(p)\, T_p \tilde{u}(v)}_{(\tilde{u}^* \mu)(v)} .$$

Thus

$$(u^* A)_p(v) = A(T_p u(v))$$

$$= A(d_p R_{\tilde{u}(p)}(v))$$

$$+ A(v_{\tilde{u}^*\mu(v)}(p))$$

$$= Ad(\tilde{u}(p))^{-1} \circ A + \tilde{u}^*\mu. \checkmark \quad \boxtimes$$

§2D Associated bundles

Def: Let F be a manifold and $\rho: G \to \mathrm{Diff}(F)$ be a homomorphism. The fibre bundle associated with F and ρ is

$$P \times_\rho F = P \times_G F := (P \times F)/G$$

with

$$g \cdot (p, x) := (p \cdot g^{-1}, \rho(g)(x)).$$

Rmk: If $\rho: G \to GL(V)$ is a finite-dim. representation, then $P \times_\rho V$ is a vector bundle.

Prop: Let A be a connection 1-form on P and set $H_A := \ker A$. There is an Ehresmann connection H_A^ρ on $P \times_\rho F$ satisfying

$$H^\rho_{A,[p,x]} = d_{p,x}\,\mathrm{pr}(H_{A,p})$$

$$\begin{tikzcd} P \times G \arrow[d, "\mathrm{pr}"] \\ P \times_\rho G \end{tikzcd}$$

$$= \left\{ \frac{d}{dt}\Big|_{t=0} [\tilde{\gamma}(t), x] : \tilde{\gamma} \text{ horizontal path through } p \text{ in } P \right\}.$$

(Moreover: if ρ is a representation, then H_A^ρ is linear.)

Exercise: Figure out how F_A and $F_{H_A^\rho}$ are related?

Prop:

1. There is a unique bijection

$$\hat{\ } : \Gamma(X, P \times_\rho F) \to C^\infty(P, F)^{(G,\rho)}$$

satisfying

$$s \circ \pi(p) = [p, \hat{s}(p)].$$

Moreover, if $\rho : G \to GL(V)$ is a representation, then:

2. There is a unique bijection

$$\hat{\ } : \Omega^\bullet(X, P \times_\rho V) \to \Omega^\bullet_{hor}(P, V)^{(G,\rho)}$$

satisfying

$$\pi^* \alpha = [p, \hat{\alpha}].$$

3. If $\tilde{A}$ denotes the linear connection

on $P\times_\rho V$ induced by A, then

$$\begin{tikzcd}
\Omega^\bullet(X, P\times_\rho V) \arrow[r, "d\tilde{A}"] \arrow[d, "\wedge"] & \Omega^{\bullet+1}(X, P\times_\rho V) \arrow[d, "\Lambda"] \\
\Omega^\bullet_{hor}(P, V)^{(G,\rho)} \arrow[r, "d_A"'] & \Omega^{\bullet+1}_{hor}(P,V)^{(G,\rho)}
\end{tikzcd}$$

commutes.

Pf: exercise ? ∎

Rmk: * $\mathcal{G}(P) \cong C^\infty(P, G)^{(G,c)} \cong \Gamma(X, \mathcal{C}(P))$

with $\mathcal{C}(P) := P\times_c G$.

⚠ $\mathcal{C}(P)$ is not a principal G-bundle ⁉

(unless G is abelian)

* $F_A \in \Omega^2_{hor}(P, \mathfrak{g})^{(G,Ad)} \cong \Omega^2(X, Ad(P))$

with

$$Ad(P) := P\times_{Ad}\mathfrak{g} \quad \text{'adjoint bundle'}$$

* $\mathcal{A}(P)$ is an affine space modelled on

$\Omega^1_{hor}(P, \mathfrak{g})^{(G,Ad)} \cong \Omega^1(X, Ad(P))$.

* When people say a fiber bundle has structure group G this means that it is (isomorphic to) a associated

bundle of a principal G-bundle.

§2E The cocycle picture

The following is very useful for doing analysis in gauge theory:

- Cover P by open subsets $\{\mathcal{U}_\alpha : \alpha \in A\}$ such that there are isomorphisms

$$\phi_\alpha : P|_{\mathcal{U}_\alpha} \longrightarrow \mathcal{U}_\alpha \times G.$$

- There are $g^\alpha_\beta \in C^\infty(\mathcal{U}_{\alpha\beta}, G)$ with $\mathcal{U}_{\alpha\beta} := \mathcal{U}_\alpha \cap \mathcal{U}_\beta$ defined by

$$\phi_\beta \circ \phi_\alpha^{-1}(x, h) =: (x, g^\alpha_\beta(x) \cdot h).$$

- These satisfy the cocycle conditions:

$$\cdot\ g^\alpha_\alpha = 1_G \qquad \text{on } \mathcal{U}_\alpha$$

$$\cdot\ g^\beta_\gamma \cdot g^\alpha_\beta = g^\alpha_\gamma \qquad \text{on } \mathcal{U}_{\alpha\beta\gamma} := \mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma$$

- P can be recovered (up to isomorphism) from $\{g^\alpha_\beta\}$:

$$P \xrightarrow[\{\phi_\alpha\}]{\cong} \coprod_{\alpha \in A} \mathcal{U}_\alpha \times G / \sim$$

with

$$(x_\alpha, h_\alpha) \sim (x_\beta, h_\beta)$$

$$\Updownarrow$$

$$x_\alpha = x_\beta =: x \quad \text{and} \quad h_\beta = g_\beta^\alpha(x) \cdot h_\alpha .$$

* If A is a connection on P, then $(\phi_\alpha)_* A$ is a connection on $U_\alpha \times G$; hence:

$$(\phi_\alpha)_* A =: A_\alpha =: A_0 + \underbrace{a_\alpha}_{\in \Omega^1(U_\alpha, \mathfrak{g})}$$

and on $U_{\alpha\beta}$

$$a_\alpha = g_\alpha^\beta \cdot a_\beta \cdot (g_\alpha^\beta)^{-1} + g_\alpha^\beta \, d(g_\alpha^\beta)^{-1}$$

$$= g_\alpha^\beta \cdot a_\beta \cdot (g_\alpha^\beta)^{-1} - d g_\alpha^\beta \cdot (g_\alpha^\beta)^{-1}$$

* Moreover:

$$(\phi_\alpha)_* F_A = F_{A_\alpha}$$

$$= \overset{=0}{\cancel{F_{A_0}}} + \underbrace{d a_\alpha + \tfrac{1}{2}[a_\alpha \wedge a_\alpha]}_{\in \Omega^2(U_\alpha, \mathfrak{g})}$$

and

$$F_{A_\alpha} = g_\alpha^\beta \cdot F_{A_\beta} \cdot (g_\alpha^\beta)^{-1} .$$

* If $u \in \mathcal{G}(P)$, then there are $u_\alpha \in C^\infty(U_\alpha, G)$ s.th

$$\phi_\alpha \circ u \circ \phi_\alpha^{-1}(x, h) = (x, u_\alpha(x) h)$$

and, on $U_{\alpha\beta}$,

$$u_\beta = g^\alpha_\beta \cdot u_\alpha \cdot (g^\alpha_\beta)^{-1}.$$

UPSHOT: After fixing a presentation of P as a 1-cocycle:

- A : $\{a_\alpha \in \Omega^1(U_\alpha, \mathfrak{g})\}$ s.th

$$a_\beta = g^\alpha_\beta \cdot a_\alpha \cdot (g^\alpha_\beta)^{-1} - (d g^\beta_\alpha) \cdot (g^\beta_\alpha)^{-1}$$

- F_A : $\{ F_{A\alpha} = d a_\alpha + \frac{1}{2}[a_\alpha \wedge a_\alpha] \in \Omega^2(U_\alpha, \mathfrak{g})\}$ s.th.

$$F_{A\beta} = g^\alpha_\beta \cdot F_{A\alpha} \cdot (g^\alpha_\beta)^{-1}$$

- $u = \{ u_\alpha \in C^\infty(U_\alpha, G)\}$ s.th.

$$u_\beta = g^\alpha_\beta \cdot u_\alpha \cdot (g^\alpha_\beta)^{-1}.$$

§3 The Chern-Weil homomorphism

Let $\pi : P \to X$ be a principal G-bundle.

Let $p \in \mathbb{C}[\mathfrak{g}]^G$ be an Ad-invariant polynomial on $\mathfrak{g}$ of degree k.

Ex: ($\mathfrak{g} = \mathfrak{u}(n)$)

$$p(\xi) = \mathrm{tr}(\xi^k), \quad p(\xi) = \det(\xi),$$

$$p(\xi) = \underset{\underset{\text{symmetric polynomial}}{\uparrow}}{\sigma_k}(\underbrace{\lambda_1, \dots, \lambda_n}_{\mathrm{Spec}(\xi)})$$

Identify

$$\bigoplus_{k\in\mathbb{N}_0} \mathrm{Hom}(S^k\mathfrak{g}, \mathbb{C})^G \cong \mathbb{C}[\mathfrak{g}]^G$$

$$\lambda \mapsto (\xi \mapsto \lambda(\xi \cdots \xi))$$

Chevalley's Theorem:

$$\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[\mathfrak{t}]^W$$

with $\mathfrak{t}$ maximal torus and W Weyl group

Observe:

* If $A \in \mathcal{A}(P)$, then

$$p(F_A^{\wedge 2k}) \in \Omega^{2k}_{hor}(P, \mathbb{C})^G$$

is closed because $d_A F_A = 0$.

* If $a \in \Omega^1_{hor}(P, \mathfrak{g})^{(G, \mathrm{Ad})}$, then

$$p(F_{A+a}^{\wedge 2k}) - p(F_A^{\wedge 2k}) = d\tau$$

with

check this

$$\tau = \sum_{m=1}^{k} \binom{k}{m} p(a \wedge (d_A a)^{\wedge m-1} \wedge F_A^{\wedge (k-m)})$$

$$\in \Omega^{2d-1}_{hor}(P, \mathbb{C})^G$$

* Therefore, the cohomology class

$$\gamma_P(p) := [p(F_A^{\wedge 2k})]$$

$$\in H^{2k}_{dR}(X, \mathbb{C})$$

depends only on $\underline{P}$ and p.

The Chern-Weil homomorphism is the map

$$\gamma_{\underline{P}} : \mathbb{C}[\mathfrak{g}]^G \to H^\bullet_{dR}(X, \mathbb{C}).$$

Ex: ($\mathfrak{g} = \mathfrak{gl}(\mathbb{C}^n)$) The total Chern class of $\underline{P}$

is

$$\begin{aligned} c(\underline{P}) &= \gamma_{\underline{P}}\left(\det\left(\mathbb{1} - \frac{\bullet}{2\pi i}\right)\right) \\ &\parallel \\ c_0(\underline{P}) &+ c_1(\underline{P}) + \cdots \end{aligned}$$