Yang-Mills functional & Instantons

Prerequisits

- $P \xrightarrow{\pi} B$ is a G-principal bundle, iff G is a right action on P and
  - the action is free $\qquad (p \cdot g = p \Rightarrow g = e)$
  - the action is transitive on fibers $\qquad (p \cdot G = P_{\pi(p)})$
  - the local trivialization $\psi_u : \pi^{-1}(U) \to U \times G$ can be choosen s.t.
  $$\psi_u^{-1}(p, g \cdot h) = \psi_u^{-1}(p, g) \cdot h$$

- For $X \in \mathfrak{g}$ the fundamental vector field $\bar{X} \in \Gamma(TM)$ is
$$\bar{X}_p := dL_p(X) = \frac{d}{dt}\Big|_{t=0} \big(p \cdot \exp(tX)\big)$$

- The adjoint representation of a Lie group G is
$$Ad : G \to Aut(\mathfrak{g}) \qquad Ad_g := d_e \alpha_g ,$$
where $\alpha_g(h) := g \cdot h \cdot g^{-1}$ is the conjugation.
(For matrix Lie groups $Ad_g(X) = g \cdot X \cdot g^{-1}$).

- The adjoint representation of a Lie algebra $\mathfrak{g}$ is

$$\mathrm{ad}: \mathfrak{g} \to \mathrm{End}(\mathfrak{g}) \qquad \mathrm{ad}(X)(Y) := [X, Y]$$

and $d_e \mathrm{Ad} = \mathrm{ad}$.

- $\omega \in \Omega^1(P, \mathfrak{g}) = \Gamma(T^*P \otimes \mathfrak{g})$ is called connection 1-form, iff

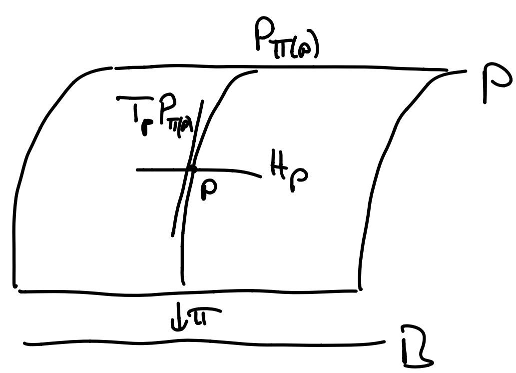
$$R_g^* \omega = \mathrm{Ad}_{g^{-1}} \circ \omega \qquad \forall g \in G$$
$$\omega(\overline{X}_p) = X \qquad \forall X \in \mathfrak{g}$$

The set of connection 1-forms is defined as

$$\mathcal{C}(P) := \{\omega \text{ connection 1-form}\}$$

- ω_p induces a splitting

$$T_pP = T_p P_{\pi(p)} \oplus H_p$$



with $H_p := \ker \omega_p$. Let

$$\pi_H : T_pP \longrightarrow H_p$$

denote the horizontal projection.

- The curvature form w.r.t. ω is $\Omega \in \Omega^2(P, \mathfrak{g})$

$$\Omega(X, Y) := d\omega(\pi_H(X), \pi_H(Y))$$

- Proposition: Let $[\omega,\eta](X,Y) := [\omega(X),\eta(Y)] - [\omega(Y),\eta(X)]$. Then

$$\Omega = d\omega + \frac{1}{2}[\omega,\omega]$$

- If G is abelian, then $[\cdot,\cdot] = 0$, and so $\Omega = d\omega$ and hence $d\Omega = dd\omega = 0$.

- In general $d\Omega|_{H\times H\times H} = 0$.

- Adjoint bundle: Let $P \xrightarrow{\pi} B$ be a G-principal bundle and $Ad: G \to Aut(\mathfrak{g})$ the adjoint representation. Then set

$$P\times_{Ad}\mathfrak{g} := P\times\mathfrak{g}/G,$$

where G acts on $P\times\mathfrak{g}$ by

$$(p,v)\cdot g := (p\cdot g, Ad_{g^{-1}}(v)).$$

Then $P\times_{Ad}\mathfrak{g} \to B$ is a vector bundle called adjoint bundle. Given a connection 1-form one can define a covariant derivative D^ω on $P\times_{Ad}\mathfrak{g}$ by

$$D^\omega_X[p(b),v(b)] := [p(b_0), \partial_X v(b_0) + Ad_*(p^*\omega(X))(v(b_0))]$$
$$= [p, \partial_X v + [p^*\omega(X), v]]$$

for $X\in T_{b_0}B$, which is well defined.

Local description

- Let $\{U_\alpha\}_\alpha$ be an open cover of B such that $P|_{U_\alpha}$ is trivial. Let $s_\alpha : U_\alpha \to P|_{U_\alpha}$ be a section on U_α. Set $U_{\alpha\beta} := U_\alpha \cap U_\beta$. Then there are $g_{\alpha\beta} : U_{\alpha\beta} \to G$, s.t.

$$s_\alpha = g_{\alpha\beta} \cdot s_\beta ,$$

and they satisfy the cocycle conditions

$$g_{\alpha\alpha} = e , \qquad g_{\alpha\beta}\, g_{\beta\gamma} = g_{\alpha\gamma} .$$

- Let ω be a connection 1-form and Ω its curvature. We can set

$$\omega_\alpha := s_\alpha^* \omega \in \Omega^1(U_\alpha, \mathfrak{g}), \qquad \Omega_\alpha := s_\alpha^* \Omega \in \Omega^2(U_\alpha, \mathfrak{g}).$$

Then for $u_0 \in U_{\alpha\beta}$ one gets

$$\omega_\beta|_{u_0} = Ad_{g_{\alpha\beta}(u_0)^{-1}} \circ \omega_\alpha|_{u_0} + d\left(g_{\alpha\beta}(u_0)^{-1} \cdot g_{\alpha\beta}\right)\Big|_{u_0}$$

$$\Omega_\beta = Ad_{g_{\alpha\beta}(u_0)^{-1}} \circ \Omega_\alpha$$

- If $\omega, \bar{\omega}$ are connection 1-forms, then

$$\omega_\beta - \bar{\omega}_\beta = Ad_{g_{\alpha\beta}(u_0)^{-1}} \circ (\omega_\alpha - \bar{\omega}_\alpha).$$

So on $P \times_{Ad} \mathfrak{g}$ we set

$$[s_\beta, (\omega_\beta - \tilde{\omega}_\beta)(X)] = [s_\beta, Ad_{g_{\alpha\beta}(x_0)^{-1}} \circ (\omega_\alpha - \tilde{\omega}_\alpha)(X)]$$
$$= [s_\alpha, (\omega_\alpha - \tilde{\omega}_\alpha)(X)]$$

So any difference of two connection 1-forms induces a form in $\Omega^1(B, P \times_{Ad} \mathfrak{g})$. Furthermore $C(P)$ is an affine vector space.

- Analogously for Ω:

$$[s_\beta, \Omega_\beta(X, Y)] = [s_\alpha, \Omega_\alpha(X, Y)]$$

So Ω induces a globally defined 2-form $\overline{\Omega} \in \Omega^2(B, P \times_{Ad} \mathfrak{g})$.

Hodge Star operator

- In the following, let V be an $\mathbb{R}$-vector space with symmetric, non-degenerate bilinear form $\langle \cdot, \cdot \rangle$. (It does not need to be positive definite.)

- Let $e_1, \dots, e_n$ be a generalized orthonormal basis, i.e. it is a basis and

$$\langle e_i, e_j \rangle = \epsilon_j \, \delta_{ij} , \qquad \epsilon_j = \pm 1.$$

- We define a symmetric, non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ on $\Lambda^k V^*$ by

$$\langle \omega, \eta \rangle := \sum_{i_1 < \dots < i_k} \epsilon_{i_1} \cdots \epsilon_{i_k} \, \omega(e_{i_1}, \dots, e_{i_k}) \, \eta(e_{i_1}, \dots, e_{i_k})$$

which is a basis independent definition.

- Let V be oriented and $e_1, \dots, e_n$ be a positively oriented. We define the volume form

$$\mathrm{vol} := e_1^* \wedge \dots \wedge e_n^* \in \Lambda^n V^*,$$

which again is basis independent (up to orientation).

- There is a unique linear map $* : \Lambda^k V^* \longrightarrow \Lambda^{n-k} V^*$ called Hodge-star operator, s.t.

$$\omega \wedge \eta = \langle * \omega, \eta \rangle \cdot \mathrm{vol}$$

for all $\omega \in \Lambda^k V^*$, $\eta \in \Lambda^{n-k} V^*$

- Properties: Let $\omega, \eta \in \Lambda^k V^*$ and let $p := \mathrm{ind} \langle \cdot, \cdot \rangle$, then

$$* * \omega = (-1)^{k(n-k)+p} \omega \qquad (0)$$

$$\langle * \omega, * \eta \rangle = (-1)^p \langle \omega, \eta \rangle \qquad (00)$$

- Let V be a euclidean vector space of dimension 4. Then $* : \Lambda^2 V^* \longrightarrow \Lambda^2 V^*$ satisfies

$$* \circ * = (-1)^{2 \cdot (4-2) + 0} \cdot \mathrm{id} = \mathrm{id}$$

By (00), we get that $*$ is an isometry and hence $*$ has Eigenvalues ± 1. We get the decomposition

$$\Lambda^2 V^* = \Lambda^2_+ V^* \oplus \Lambda^2_- V^*$$

with the (anti-) self dual 2-forms

$$\Lambda^2_\pm V^* := \{ \omega \in \Lambda^2 V^* \mid * \omega = \pm \omega \}$$

- Let $g = \langle \cdot, \cdot \rangle$ and $g' = \lambda^2 g$ be a rescaling with $\lambda > 0$. Then $e_i' := \frac{1}{\lambda} e_i$ is a generalized ONB of V for g'. And $e_i'^{*} = \lambda e_i^{*}$. Hence

$$\mathrm{vol}_{g'} = \lambda^n \mathrm{vol}_g.$$

For $\omega \in \Lambda^k V^*$, $\eta \in \Lambda^{n-k} V^*$ we have

$$\begin{aligned}
\langle \omega, * \eta \rangle \mathrm{vol}_g = \omega \wedge \eta &= \langle \omega, *' \eta \rangle' \cdot \mathrm{vol}_{g'} \\
&= \lambda^{-2k} \langle \omega, *' \eta \rangle \cdot \lambda^n \mathrm{vol}_g \\
&= \langle \omega, \lambda^{n-2k} *' \eta \rangle \mathrm{vol}_g
\end{aligned}$$

Hence $*' = \lambda^{2k-n} *$.

- In particular if $n = 2k$, then $*' = *$. So the Hodge-star operator is said to be conformally invariant.

Yang-Mills functional

- Physical motivation: $SU(2)$ ~ weak interaction
  $SU(3)$ ~ strong interaction.

- In the following, let M be a (compact) oriented Riemannian 4-manifold and $P \to M$ an $SU(N)$ bundle.

$$G = SU(N) = \{ A \in GL(n, \mathbb{C}) \mid A^* A = 1,\ \det A = 1 \}$$
$$\mathfrak{g} = su(N) = \{ A \in Mat(n \times n, \mathbb{C}) \mid A^* = -A,\ tr(A) = 0 \}$$

- We define $\lambda : su(N) \times su(N) \to \mathbb{R}$

$$\lambda(A, B) := - tr(A \cdot B)$$

which is an Ad-invariant, positive definite, symmetric bilinear form, i.e.

$$\lambda(Ad_g(A), Ad_g(B)) = \lambda(A, B) \qquad \forall g \in SU(2).$$

- It yields a Riemannian metric λ on $P \times_{Ad} \mathfrak{g}$ by

$$\lambda([p, A], [p, B]) := \lambda(A, B).$$

If ω is a connection 1-form, then the covariant derivative ∇^ω on $P \times_{Ad} \mathfrak{g}$ is a metric connection w.r.t. $\lambda(\cdot, \cdot)$.

- Let $\bar{\Omega} \in \Omega^2(M, P \times_{Ad} \mathfrak{g})$ be the curvature form of ω. Then

$$d^\omega \bar{\Omega} = 0 \qquad \text{„Bianchi identity"}$$

where $d^\omega := d^{D^\omega}$ is the exterior derivative on a vector bundle:

$$(d^{D^\omega} \eta)(X_0, \dots, X_k) := \sum_{i=0}^{k} D^\omega_{X_i} \eta(X_0, \dots, \hat{X}_i, \dots, X_k)$$
$$+ \sum_{i<j} (-1)^{i+j} \eta([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_k)$$

Locally the Bianchi identity reads

$$d\Omega_\alpha + [\omega_\alpha, \Omega_\alpha] = 0$$

which is a non-linear equation for ω.

- We define the Yang-Mills Lagrangian

$$\mathcal{L}_{YM} : C(P) \to \Omega^4(M; \mathbb{R}) \qquad \omega \mapsto \tfrac{1}{2} \lambda(\bar{\Omega} \wedge * \bar{\Omega})$$

The Yang-Mills action then is

$$S_{YM}(\omega) = \int_M \mathcal{L}_{YM}(\omega)$$

Here for $\omega, \eta \in \Omega^2(M, E) = \Gamma(T^*M \otimes E)$ with

$$\omega = \sum_{i,j} \omega_{i,j} \otimes dx^i \wedge dx^j \quad , \quad \eta = \sum_{k,l} \eta_{k,l} \otimes dx^k \wedge dx^l$$

we set

$$\lambda(\omega \wedge \eta) := \sum_{ijkl} \lambda(\omega_{ij}, \eta_{kl})\, dx^i \wedge dx^j \wedge dx^k \wedge dx^l.$$

- Due to the Hodge star operator we get

$$\lambda(\bar{\Omega} \wedge * \bar{\Omega}) = \langle \bar{\Omega}, \bar{\Omega} \rangle \cdot vol,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product on $\Lambda^2 T^*M \otimes (P \times_{Ad} \mathfrak{g})$.
Hence

$$\int_M \mathcal{L}_{YM}(\omega) = \frac{1}{2} \|\bar{\Omega}\|_{L^2}^2 \geq 0.$$

- Gauge transformations: $\varphi : P \to P$ diffeomorphism, s.t. $\varphi(p \cdot g) = \varphi(p) \cdot g$ and $\varphi|_{P_b} : P_b \to P_b$.

- Gauge invariance of $\mathcal{L}_{YM}$: Let $\omega \in C(P)$ and $\varphi : P \to P$ be a gauge transformation. Then let $\omega' := \varphi^* \omega$, $X, Y \in TM$

$$\begin{aligned} \bar{\Omega}'(X,Y) &= [s, \Omega'(ds(X), ds(Y))] \\ &= [s, \varphi^*\Omega(ds(X), ds(Y))] \\ &= [\varphi^{-1} \circ s', (s')^* \Omega(X,Y)] \qquad (s' := \varphi \circ s) \\ &= [s', Ad_{g^{-1} \circ s'} \circ (s')^* \Omega(X,Y)] \end{aligned}$$

where $g : P \to G$ is such that $\varphi(p) = p \cdot g(p)$. Hence

$$\bar{\Omega}' = Ad_{g^{-1}} \circ \bar{\Omega}$$

and by the Ad-invariance of λ we get

$$\lambda(\bar{\Omega}' \wedge * \bar{\Omega}') = \lambda(\bar{\Omega} \wedge * \bar{\Omega}).$$

and therefore $\mathcal{L}_{YM}(\omega') = \mathcal{L}_{YM}(\omega)$. $\square$

- Conformal invariance: $\mathcal{L}_{YM}(\omega, g) = \mathcal{L}_{YM}(\omega, \lambda^2 g)$, by conformal invariance of the Hodge star operator for $n = 4 = 2 \cdot 2 = 2 \cdot k$.

The Euler-Lagrange equations

- A connection 1-form ω is called Yang-Mills connection, iff ω is critical for $\int_M \mathcal{L}_{YM}(\omega)$.

- Let $\omega_t = \omega + t\eta$ be a variation of a connection 1-form, i.e. $\omega, \omega_t \in C(P)$ and $\eta \in \Omega^1_{Ad}(P, su(N))$ is an Ad-invariant 1-form. We find

$$
\begin{aligned}
\Omega_t &= d\omega_t + \tfrac{1}{2}[\omega_t, \omega_t] \\
&= d\omega + t\,d\eta + \tfrac{1}{2}[\omega,\omega] + \tfrac{1}{2}t\left([\omega,\eta] + [\eta,\omega]\right) + O(t^2) \\
&= \Omega + t\left(d\eta + [\omega,\eta]\right) + O(t^2)
\end{aligned}
$$

Hence $\bar{\Omega}_t = \bar{\Omega} + t\,d^\omega \bar{\eta}$. For $\bar{\eta} \in \Omega^1(M, su(N))$ with $\operatorname{supp}(\bar{\eta}) \subset\subset M$ we get

$$
\begin{aligned}
\frac{d}{dt}\Big|_{t=0} \int_M \mathcal{L}_{YM}(\omega_t) &= -\frac{1}{2}\frac{d}{dt}\Big|_{t=0} \int_M \operatorname{tr}\left(\bar{\Omega}_t \wedge * \bar{\Omega}_t\right) \\
&= -\frac{1}{2}\int_M \operatorname{tr}\left(\bar{\Omega} \wedge *(d^\omega \bar{\eta}) + (d^\omega\bar{\eta}) \wedge * \bar{\Omega}\right) \\
&= -\int_M \operatorname{tr}\left((d^\omega \bar{\eta}) \wedge * \bar{\Omega}\right) \\
&= \underbrace{-\int_M d\left(\operatorname{tr}\left(\bar{\eta} \wedge * \bar{\Omega}\right)\right)}_{=0 \text{ by Stokes}} + \operatorname{tr}\left(\bar{\eta} \wedge d^\omega(*\bar{\Omega})\right)
\end{aligned}
$$

Hence: $\boxed{\omega \text{ critical for } \mathcal{L}_{YM} \iff d^\omega(*\bar{\Omega}) = 0.}$

Instantons and Energy bounds

- If $\omega \in C(P)$ is (anti-) self dual, i.e. $\bar\Omega \in \Omega^2_\pm(M, P\times_{Ad}\mathfrak{g})$, then ω is a Yang-Mills connection.

  Proof: $d^\omega * \bar\Omega = \pm\, d^\omega \bar\Omega = 0$ by Bianchi identity.

- An (anti-) self dual connection 1-form is called (A)SD-instanton.

- We can decompose

$$\bar\Omega = \bar\Omega_+ + \bar\Omega_-$$

where $\bar\Omega_\pm \in \Omega^2_\pm(M, P\times_{Ad}\mathfrak{g})$. Then

$$\begin{aligned}
&\lambda(\bar\Omega \wedge * \bar\Omega) \\
&= \lambda\Big(\bar\Omega_+ \wedge * \bar\Omega_+ + \underbrace{\cancel{\bar\Omega_+ \wedge * \bar\Omega_-}}_{-\bar\Omega_+ \wedge \bar\Omega_-} + \underbrace{\cancel{\bar\Omega_- \wedge * \bar\Omega_+}}_{\bar\Omega_- \wedge \bar\Omega_+} + \bar\Omega_- \wedge * \bar\Omega_-\Big) \\
&= \lambda(\bar\Omega_+ \wedge * \bar\Omega_+ + \bar\Omega_- \wedge * \bar\Omega_-) \\
&= (\langle \bar\Omega_+, \bar\Omega_+\rangle + \langle \bar\Omega_-, \bar\Omega_-\rangle)\cdot \mathrm{vol}
\end{aligned}$$

Hence

$$S_{YM} = \int_M \mathcal{L}_{YM} = \int_M \left(\|\bar\Omega_+\|^2 + \|\bar\Omega_-\|^2\right)\cdot \mathrm{vol}$$

- On the other hand

$$\Theta := \lambda(\bar\Omega \wedge \bar\Omega) = \lambda\big(\bar\Omega_+ \wedge \bar\Omega_+ + \underbrace{\cancel{\bar\Omega_+ \wedge \bar\Omega_-}}_{* \bar\Omega_+ \wedge * \bar\Omega_- = -\bar\Omega_+ \wedge \bar\Omega_-} + \cancel{\bar\Omega_- \wedge \bar\Omega_+} + \bar\Omega_- \wedge \bar\Omega_-\big)$$

$$= \lambda\big(\bar\Omega_+ \wedge * \bar\Omega_+ - \bar\Omega_- \wedge * \bar\Omega_-\big)$$

$$= \big(\langle \bar\Omega_+, \bar\Omega_+ \rangle - \langle \bar\Omega_-, \bar\Omega_- \rangle\big) \cdot \mathrm{vol}$$

- We set

$$c := \int_M \lambda(\bar\Omega \wedge \bar\Omega) = \int_M \big(\|\bar\Omega_+\|^2 - \|\bar\Omega_-\|^2\big) \cdot \mathrm{vol}.$$

Then (Exercise):

$$S_{YM} \geq |c|$$

with equality if and only if $\bar\Omega_\pm = 0$. Then $S_{YM} = \mp c$.

- Let M be a smooth manifold. The k-th de Rham cohomology of M is

$$H^k_{dR}(M) := \frac{\ker\big(d : \Omega^k(M,\mathbb{R}) \to \Omega^{k+1}(M,\mathbb{R})\big)}{\mathrm{im}\big(d : \Omega^{k-1}(M,\mathbb{R}) \to \Omega^k(M,\mathbb{R})\big)}$$

- $\Theta = \lambda(\bar\Omega \wedge \bar\Omega)$ is a 4-form. By dimensional reasons Θ is closed and therefore defines a de Rham class $[\Theta] \in H^4_{dR}(M)$.

- On a compact connected 4-manifold, the integration of forms yields an isomorphism

$$H^4_{dR}(M) \xrightarrow{\cong} \mathbb{R} \qquad [\omega] \mapsto \int_M \omega.$$

- So the class $[\theta]$ yields a unique number in $\mathbb{R}$. But θ could depend on the connection 1-form $\omega \in C(P)$. However, this is not the case. Let $\omega_0, \omega_1 \in C(P)$. Then

$$\omega_0 + t(\omega_1 - \omega_0) \in C(P) \qquad (C(P) \text{ is affine space})$$

Then $\bar{\Omega}_t = d\omega_t + \frac{1}{2}[\omega_t, \omega_t]$

$$\begin{aligned}
\frac{d}{dt}\bar{\Omega}_t &= d(\omega_1 - \omega_0) + [\omega_0, \omega_1 - \omega_0] + t[\omega_1 - \omega_0, \omega_1 - \omega_0] \\
&= d(\omega_1 - \omega_0) + [\omega_t, \omega_1 - \omega_0] \\
&= d^{\omega_t}(\omega_1 - \omega_0)
\end{aligned}$$

Then $\theta_t = \lambda(\bar{\Omega}_t \wedge \bar{\Omega}_t)$

$$\begin{aligned}
\frac{d}{dt}\theta_t &= 2\lambda(\bar{\Omega}_t \wedge \tfrac{d}{dt}\bar{\Omega}_t) = 2\lambda(\bar{\Omega}_t \wedge d^{\omega_t}(\omega_1 - \omega_0)) \\
&= 2\, d\lambda(\bar{\Omega}_t \wedge (\omega_1 - \omega_0)) - 2\lambda(\underbrace{d^{\omega_t}\bar{\Omega}_t}_{=0} \wedge (\omega_1 - \omega_0))
\end{aligned}$$

Hence by integration

$$\theta_1 - \theta_0 = \int_0^1 \frac{d}{dt}\theta_t \, dt = 2\, d\left(\int_0^1 \lambda(\bar{\Omega}_t \wedge (\omega_1 - \omega_0))\, dt\right)$$

and so in cohomology $[\theta_1] = [\theta_0]$.

- In fact this is the first Pontrjagin number (for $SU(2)$ bundle)

$$p_1(P \times_{Ad} su(2))[M] := -\frac{1}{4\pi^2} \int_M \lambda(\Omega \wedge \Omega)$$

and so

$$-c = 4\pi^2 \, p_1(P \times_{Ad} su(2))[M] = 8\pi^2 \, c_2(P \times_{Ad} su(2))[M]$$

where $c_2(P \times_{Ad} su(2))[M]$ is the 2^{nd}-Chern number.

- One can show that the first Pontrjagin number

$$k := p_1(P \times_{Ad} su(2)) \qquad \text{Instanton number}$$

is an integer. Hence $S_{YM} \geq 4\pi^2 |k|$.

- Theorem (Atiyah - Hitchin - Singer): Let M be a compact self-dual Riemannian 4-manifold with positive scalar curvature. Let $P \to M$ be a G-principal bundle, where G is compact, semi-simple. Then

$$\mathrm{ind}(D) = p_1(P \times_{Ad} \mathfrak{g}) - \frac{1}{2}(\underbrace{\chi}_{\text{Euler characteristic}} - \overbrace{\tau}^{\text{signature}}) \dim G,$$

where $\mathrm{ind}(D)$ is the index of the Dirac operator (or more detailed the dimension of the moduli space of irreducible self-dual connections on P.)

- Assume $\Omega_- = 0$, then

$$S_{YM} = \int_M \|\Omega_+\|^2 \,\mathrm{vol} = 4\pi^2 |k| = 4\pi^2 \left(\mathrm{ind}(D) + \tfrac{1}{2}(\chi - \tau) \dim G \right)$$

BPST - Instanton

- We construct an instanton on euclidean $\mathbb{R}^4$.

- For the convergence of the Yang-Mills-functional, the field strength vanishes sufficiently fast at infinity.

- We identify $\mathbb{R}^4$ with the quaternions $\mathbb{H}$ by

$$\mathbb{R}^4 \ni x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \mapsto x_1 + x_2 i + x_3 j + x_4 k \in \mathbb{H}.$$

We denote by x also the corresponding quaternion and set

$$\mathrm{Re}(x) = x_1 \qquad \mathrm{Im}(x) = x_2 i + x_3 j + x_4 k$$

The Euclidean inner product agrees with the quaternionic inner product $x \cdot y = \mathrm{Re}(x \cdot \bar{y})$. The Norm is $|x|^2 = \mathrm{Re}(x \cdot \bar{x})$

- The Lie group $SU(2)$ is isomorphic to

$$Sp(1) := \{ x \in \mathbb{H} \mid |x|^2 = 1 \}$$

and $\quad \mathfrak{su}(2) \cong \mathfrak{sp}(1) := \{ x \in \mathbb{H} \mid \mathrm{Re}(x) = 0 \}.$

- We write

$$dx = dx_1 + i\,dx_2 + j\,dx_3 + k\,dx_4$$
$$d\bar{x} = dx_1 - i\,dx_2 - j\,dx_3 - k\,dx_4$$

- Let $P = \mathbb{R}^4 \times SU(2) \cong \mathbb{H} \times Sp(1)$ the trivial bundle. Then $\omega \in \Omega^1(\mathbb{R}^4; sp(1))$,

$$\omega(x) := \operatorname{Im}\left(\frac{x}{1+|x|^2}\, d\bar{x}\right)$$

is a self dual connection 1-form. The curvature form is

$$\Omega(x) = \frac{1}{(1+|x|^2)^2}\, dx \wedge d\bar{x}$$

And the instanton number is

$$k = \frac{1}{4\pi^2}\int_{\mathbb{R}^4} \bar{\Omega} \wedge \bar{\Omega} = \frac{1}{4\pi^2}\int_{\mathbb{R}^4} \frac{24}{(1+|x|^2)^4}\, d^4x = 1.$$

- $\mathbb{R}^4$ is conformally equivalent to S^4 with a point removed.

- Since the (anti-) self duality conditions are conformally equivalent, an instanton on $\mathbb{R}^4$ induces an instanton on S^4.

- An instanton on $\mathbb{R}^4$ induces an instanton on $S^7 \to S^4$ as an $SU(2)$-principal bundle. (Analogously to Hopf fibration $S^3 \to S^2$).

Electrodynamics

- Let M be Lorentzian 4-manifold with signature $(-,+,+,+)$.

- Let $P \to M$ be a $U(1)$-principal bundle. Then $\mathfrak{g} = \mathfrak{u}(1) = i\mathbb{R}$.

- Let $\omega \in \mathcal{C}(P)$ a connection 1-form and $\bar{\Omega} \in \Omega^2(M, i\mathbb{R})$ its curvature. We write $\Omega = iF$ for $F \in \Omega^2(M,\mathbb{R})$.

- Let (t,x,y,z) be local coordinates on M, s.t. $\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \rangle < 0$ and $\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \rangle, \langle \frac{\partial}{\partial y}, \frac{\partial}{\partial y} \rangle, \langle \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \rangle > 0$. We write

$$
\begin{aligned}
F = {} & E_x\, dx \wedge dt + E_y\, dy \wedge dt + E_z\, dz \wedge dt \\
& + B_x\, dy \wedge dz + B_y\, dz \wedge dx + B_z\, dx \wedge dy
\end{aligned}
$$

and we set $\vec{E} := (E_x, E_y, E_z)$, $\vec{B} := (B_x, B_y, B_z)$. Then

$$
dF = 0 \iff \begin{cases} \operatorname{div} \vec{B} = 0 & \text{„Gauß Law"} \\ \operatorname{rot} \vec{E} = -\frac{\partial \vec{B}}{\partial t} & \text{„Faraday's Law"} \end{cases}
$$

which are the first two Maxwell equations.

- The „bare" Yang-Mills action $S_{YM} = \frac{1}{2}\int_M F \wedge {*F}$ would lead to the Euler-Lagrange equation $d{*F} = 0$. However this is just a vacuum field. We want to couple it to matter!

• Pick a „background" connection 1-form $\omega_0 \in \mathcal{C}(P)$. Then for any $\omega \in \mathcal{C}(P)$ the difference $\omega - \omega_0$ induces a 1-form $iA = i\,A(\omega,\omega_0) \in \Omega(M, i\mathbb{R})$. It satisfies $dA = F - F_0$. Let $j \in \Omega^3(M,\mathbb{R})$. We define

$$\mathcal{L} : \mathcal{C}(P) \to \Omega^4(M,\mathbb{R}), \quad \mathcal{L}(\omega) := \frac{1}{2} F \wedge * F + A \wedge j.$$

The Euler-Lagrange equations are

$$d * F + j = 0$$

• The Lagrangian is dependent on the choice of background connection ω_0, but not the Euler-Lagrange equation. If $\tilde{\omega}_0$ is a different background connection, then

$$\mathcal{L}(\omega) - \tilde{\mathcal{L}}(\omega) = A(\omega_0, \tilde{\omega}_0) \wedge j$$

which after integration is just a constant.

• For $j \in \Omega^3(M,\mathbb{R})$ we write

$$j = \rho\, dx \wedge dy \wedge dz - j_x\, dt \wedge dy \wedge dz - j_y\, dt \wedge dz \wedge dx - j_z\, dt \wedge dx \wedge dy$$

where ρ is the electric charge density and $\vec{j} := (j_x, j_y, j_z)$ the electric current density. Then

$$dj = d d * F = 0$$

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} + \operatorname{div} \vec{j} = 0 \qquad \text{„Continuity equation"}$$

- In Minkowski space $\mathbb{R}^{1,3}$ we can calculate the Hodge-star operator explicitly and find

$$d * F + j = 0 \iff \begin{cases} \operatorname{div} \vec{E} = \rho & \text{„Coulomb's law"} \\ \operatorname{rot} \vec{B} = \vec{j} + \frac{\partial \vec{E}}{\partial t} & \text{„Ampère's law"} \end{cases}$$

- To summarize

$$dF = 0 \quad \text{and} \quad d * F + j = 0$$

are the Maxwell equations with continuity equation $dj = 0$.

- For Einstein gravity one uses the Lagrangian

$$\mathcal{L}(g) := -\frac{1}{2}\, \mathrm{scal}_g \cdot \mathrm{vol}_g$$

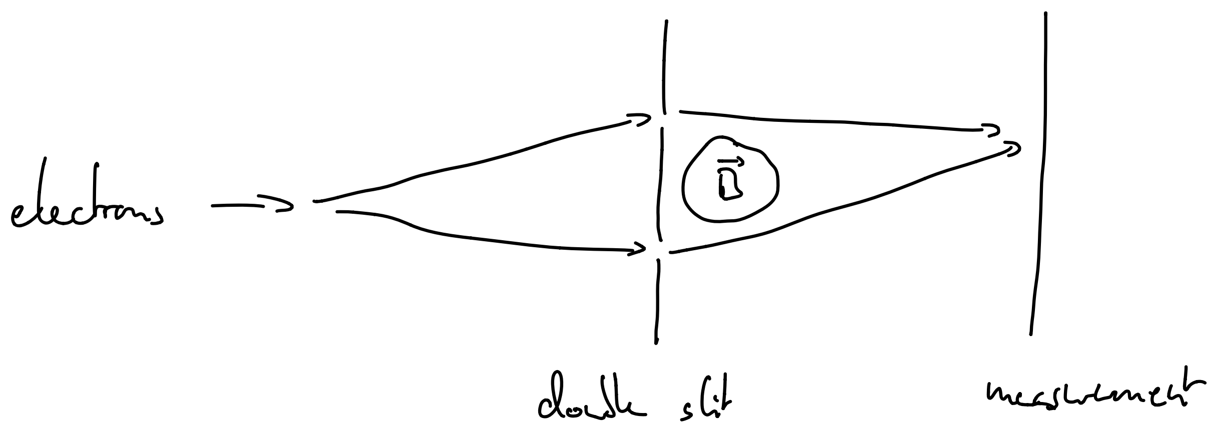
where the metric is to be considered a dynamical variable.

Aharonov - Bohm - Effect

- Why do we need gauge theory physically?

- We can only measure the $\vec{E}$ and $\vec{B}$ field in F.
  Not the connection 1-form A!

- However A does have an effect on systems:



- Outside of the zylinder we have $\vec{B}=0$, but if $\vec{B}\neq 0$ inside the cylinder, then $A\neq 0$ outside the cylinder.

- The potential A shifts the pase of the wavefunction of the electron path dependently.

  $\Rightarrow$ Outcome of the experiment is dependent on the value of $\vec{B}$ inside the cylinder.