Recall

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1.12.20

Uhlenbeck's gauge fixing theorem

Recall: Let $\pi : P \to M$ be a principal G-bundle over a compact 4-mfld M with G compact. $\boxed{G \subseteq U(n)}$

A connection 1-form $\alpha \in \Omega^1(P, \mathfrak{g})$ satisfying

$$\begin{cases} \alpha(V_\xi) = \xi & \text{for } \xi \in \mathfrak{g} \\ R_g^* \alpha = Ad_g^{-1} \circ \alpha \end{cases}$$

determines a G-invariant splitting of TP:

$$\ker \pi_* \oplus \ker \alpha$$

- The space of such objects is locally modelled on the affine space $\Omega^1(M, P \times_{ad} \mathfrak{g})$ $\left(\cong \Omega^1_{Hor}(P, \mathfrak{g})^{(G, Ad)}\right)$
- $\alpha = d + A$, so we refer to α by A.
- A has an associated curvature form $F_A \in \Omega^2(M, P \times_{ad} \mathfrak{g})$ given by $F_A = dA + A \wedge A$.

Recall from Niklas' talk:

- The Hodge star $* : \Omega^2(M) \circlearrowleft$ gives a decomposition

$$\Omega^2(M) = \Omega^+(M) \oplus \Omega^-(M)$$

into ± 1 eigenspaces, and this extends to

$$\Omega^2(M, P \times_{ad} \mathfrak{g}) = \Omega^+(M, P \times_{ad} \mathfrak{g}) \oplus \Omega^-(M, P \times_{ad} \mathfrak{g})$$

$$\left(= C^\infty(M, \Lambda^2_\pm T^*M \otimes (P \times_{ad} \mathfrak{g})) \right)$$

$\left(= C^\infty(M, \Lambda^2 T^*M \otimes (P \times_{ad} \mathfrak{g})\right)$

- We say a connection A is anti-self dual if

$$\boxed{F_A^+ = 0}.$$

(End recall)

Ellipticity

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Ellipticity and the ASD equation

- Recall from Solveig's talk: Ellipticity of a linear operator $D: W^{k,p}(M,V) \longrightarrow W^{k-m,p}(M,W)$ depends only on the principal symbol of D.

  (V, W vector bundles, m = order of D)

- Major fact: Let $\sigma_m^D(x): T_x^*M \times V_x \to W_x$ denote the principal symbol of D. If $\sigma_m^D(x)\xi$ is invertible $\forall x \in M$ and $0 \neq \xi \in T_x^*M$ (D is elliptic), then D is Fredholm.

- However, the ASD equation is invariant under gauge transformation:

$$F_{u^*A} = \tilde{u}^{-1} \circ F_A \circ \tilde{u}, \text{ where } \tilde{u} \in C^\infty(P, G)^{(G, \mathrm{conj})}$$

- This does not effect the "Form part" of F_A,

  So the Hodge eigenvalue decomposition is invariant under change of gauge.

  Hence, $F_{u^*A}^+ = 0 \iff F_A^+ = 0$.

- Since the gauge group is infinite dimensional, (the linearization of) the ASD eq. cannot have f.d. kernel

  $\Rightarrow$ not elliptic

- Note the leading order part of ASD is

$$\frac{1}{2} d^+ : \Omega^1(M, P\times \mathfrak{g}) \longrightarrow \Omega^+(M, P\times\mathfrak{g})$$

where $d^+ := (Id + *)\circ d$ with $(Id + *) : \Omega^2 \longrightarrow \Omega^+$

How to turn $A \longmapsto F_A^+$ into an elliptic operator?

Restrict to case $M = B_1(\mathbb{R}^4) =: \mathbb{B}_1$ and trivial bundle.

Exercise: $(d + d^*)^2 = \Delta : \bigoplus \Omega^i(\mathbb{B}_1) \longrightarrow \bigoplus \Omega^i(\mathbb{B}_1)$

We have $(d^+ + d^*)^* = d^* + d$, and

$$(d^* + d)(d^+ + d^*) = d^* d^+ + d d^* = d^*(d + *d) + dd^*$$
$$= \Delta + d^* * d$$
$$= \Delta + (-1)^j * d * * d$$
$$= \Delta$$

Since Δ is elliptic and principal symbols satisfy

$$\sigma_2^{\Delta} = \sigma_1^{d^* + d} \circ \sigma_1^{d^+ + d^*} = \left(\sigma_1^{d^+ + d^*}\right)^* \circ \sigma_1^{d^+ + d^*}$$

we have that $d^+ + d^*$ is elliptic, and $\frac{1}{2} d^+, d^*$ form an elliptic system.

Uhlenbeck's gauge fixing (statement)

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- We consider connections on the trivial bundle V^n over $B_1 \subseteq \mathbb{R}^4$. For $x \in T_x B_1$, let $A_r = \sum \left(\frac{x_i}{r}\right) A_i$ denote the radial component of A.

- Boundary conditions: We say that
$\lim_{|x| \to 1} A_r = 0$ if it holds in a distributional sense on the sphere.

Theorem (Uhlenbeck gauge): There are constants $\varepsilon, N > 0$ so that any $U(n)$-connection A on the trivial bundle $B_1 \times \mathbb{R}^n \to B_1 \subseteq \mathbb{R}^4$ with $\|F_A\|_{L^2} < \varepsilon$ is gauge equivalent to a connection $\tilde{A}$ with

$$(1) \quad \begin{cases} d^* \tilde{A} = 0 \\ \lim_{|x| \to 1} \tilde{A}_r = 0 \\ \|\tilde{A}\|_{W^{1,2}} \leq M \|F_{\tilde{A}}\|_{L^2} \end{cases}$$

Moreover, for suitable constants ε, N, $\tilde{A}$ is uniquely determined up to a constant gauge transformation $\tilde{A} \mapsto u_0 \tilde{A} u_0^{-1}$ for $u_0 \in U(n)$.

Proof strategy:

- Work over S^4
- Use "continuity method": show that the set of connections in a 1-parameter family satisfying

- Use "continuity method": show that the set of connections in a 1-parameter family satisfying properties similar to (i) is open & closed.

- Restrict to $\mathbb{B}$, via a smoothing of the projection onto a hemisphere.

A priori estimate

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Lemma 1: (A priori estimate)

Let B be a connection on the trivial bundle over S^4 with $d^*B = 0$. There are constants $\eta, N > 0$ s.t.

$$\|B\|_{L^4} \leq \eta \implies \|B\|_{W^{1,2}} \leq N \|F_B\|_{L^2}.$$

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Sketch: Recall from Solveig's talk the fundamental elliptic estimate for $\bar{\partial}$. Similarly, since $H^1(S^4) = 0$ and $d^*B = 0$ there is an elliptic estimate

$$\|B\|_{W^{1,2}} \leq C_1 \left[\|dB\|_{L^2} + \|d^*B\|_{L^2}\right] = C_1 \|dB\|_{L^2}$$

- Exercise: Show this by showing $B \perp \ker(d + d^*)$ (use $H^1 = \ker \Delta$), and then use Fredholm property of $d + d^*$.
- Since $F_B = dB + B \wedge B$, it only remains to estimate $\|B \wedge B\|_{L^2}$ and "rearrange".
- Remark: L^4 comes from Sobolev embedding $W^{1,2} \to L^4$

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Continuity method (statement)

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Method of Continuity:

Proposition 1: There is a constant $\eta > 0$ s.t. if B_t' $t \in [0,1]$ is a one-parameter family of connections on the trivial bundle over S^4 with $\|F_{B_t'}\|_{L^2} < \eta$ and B_0' the product connection, then for each t there exists a gauge transformation u_t such that $u_t(B_t') = B_t$ satisfies

$$\begin{cases} d^*B = 0 \\ \|B_t\|_{W^{1,2}} < 2N \|F_{B_t}\|_{L^2} \end{cases}$$

where N is from Lemma 1. //

- Notation/Definition: We say a connection with $d^*B = 0$ is in "Coulomb gauge".

Technical lemmas

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First, we need a few results:

- Let $Q_k(B) = \|F_B\|_{L^\infty} + \sum_{i=1}^{k} \|\nabla_B^{(i)} F_B\|_{L^2}$

  where $\nabla_B^{(i)} := \nabla_B \circ \cdots \circ \nabla_B$.

Lemma 2 (Higher derivative estimates): There is an $\eta' > 0$ such that if the connection B of Lemma 1 has $\|B\|_{L^2} < \eta'$ then for each $k \geq 1$, we have

$$\|B\|_{W^{k+1,2}} \leq f_k(Q_k(B))$$

for a universal continuous function f_k with $f_k(0) = 0$. //

See Donaldson-Kronheimer.

Lemma 3 (Convergence of gauge equivalence):

If A_i, B_i are C^∞ bounded sequences of connections on a G-bundle over X (G compact), and if $B_i = u_i(A_i)$ for a sequence of gauge transformations u_i, then $\exists\, u, A, B$ s.t.

$$u_i \to u \text{ in } C^\infty$$
$$A_i \to A$$
$$B_i \to B$$

and $B = u(A)$.

Sketch:

Arzela-Ascoli & writing du_i

Arzela-Ascoli & writing du_i

in terms of A_i, B_i, u_i:

$$B_i = u_i A_i u_i^{-1} - du_i u_i^{-1} \quad \text{(transformation law)}$$

$$\Rightarrow \quad du_i = u_i A_i + B_i u_i .$$

$\Rightarrow$ get bounds for u_i in C^{r+1} assuming bounds in C^r

(!! C^0 bounds follow from compactness of G)

$\Rightarrow$ Apply Arzela-Ascoli. //

Closedness

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Closedness of condition in Proposition 1

- Let $S \subseteq [0,1]$ denote the set of t s.t. Prop 1 holds.

$\boxed{C = \text{Sobolev const.}}$

- Take $\delta > 0$ small enough so that $2CN\delta < \eta, \eta'$ of Lemmas (A priori est.) and (Higher deriv. est.).
- For $t \in S$,

$\|B_t\|_{L^4} \leq C\|B_t\|_{W^{1,2}} \leq 2NC\|F_{B_t}\| \leq 2NC\delta$,

thus by Lemma (A priori),

$$\|B_t\|_{W^{1,2}} \leq N\|F_{B_t}\|_{L^2} \quad (1)$$

(a closed condition).

- By (Higher deriv. est.) and the invariance of $\|\nabla^{(i)}_{B_t} F_{B_t}\|$ under unitary gauge transformation, it follows that we have a universal bound on the derivatives of B_t $\forall t \in S$.
- Suppose $t_i \in S$, $t_i \to t$ with B'_{t_i}, u_i such that $u_i(B'_{t_i}) =: B_{t_i}$ satisfies the conclusion.
- Lemma (Conv. of gauge equiv) gives limiting B'_t, B_t, u s.t. $u(B'_t) = B_t$
- Since $d^*B_{t_i} = 0$ and the estimate (1) are closed

- Since $d^* B_{t_i} = 0$ and the estimate (1) are closed conditions, it follows that $t \in S$. //

Openness

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Openness by implicit function theorem:

- We apply the IFT to

$$d^*(u_r B_r') = d^*(u_r B_r u_r^{-1} - du_r u_r) = 0$$

- Fix $t \in S$ and WLOG let $B_t = B_t' =: B$
- $B_{t+r} =: B + b_r$
- Want to find u_{t+r} in the form

$$u_{t+r} = \exp(x_r) =: e^{x_r}$$

- Apply IFT to solve

$$H(x,b) := d^*\left(e^{x}(B+b)e^{-x} + d(e^{x})e^{x}\right) = 0$$

locally for x near $(0,0)$

- $H : E_k \times F_{k-1} \to E_{k-2}$, $k \geq 3$, smooth,

where $E_k = W^{k,2}(S^4, \mathfrak{g})_0$

$F_k = W^{k,2}(S^4, T^*S^4 \otimes \mathfrak{g})$

- The "0" means functions with 0 integral:

$f = d^*\alpha = *d*\alpha \Rightarrow *f = f\, d\mathrm{vol}_g \in \mathrm{Im}\, d$

By Stokes $\int_{S^4} f\, d\mathrm{vol} = 0$.

- If the partial derivative $(\partial_x H)_0 : E_k \to E_{k-2}$

is surjective, then for small b in F_{k-1}

is surjective, then for small b in F_{k-1} there is a solution x to $H(x,b)=0$, which would show openness near t.

- Exercise: $(\partial_x H)_0 \xi = d^* d_B \xi$
- Claim: $d^* d : E_k \to E_{k-2}$ is elliptic, hence Fredholm.

$\omega \in T^*S^4$, $\sigma_1^{d}(\omega) f = f \wedge \omega$, $\sigma_1^{d^*}(\omega)\alpha = \alpha(\omega^\vee)$

$\Rightarrow \sigma_2^{d^*d}(\omega) f = \sigma_1^{d^*}(\omega) f \wedge \omega = \cancel{f(\omega^\vee)\wedge\omega} + f\,\omega(\omega^\vee) = f$

$\Rightarrow$ isomorphism, hence $d^* d_B$ is Fredholm by (major fact).

- Suppose that it weren't surjective. Since the image is closed, there is a splitting $\operatorname{Im} d^*d_B \oplus \operatorname{coker} d^*d_B$, so we can assume that $\exists \omega$ s.t. $\langle d^*d_B \xi, \omega\rangle = 0 \quad \forall \xi$.

- Set $\xi = \omega$.

$$0 = \langle d_B\omega, d\omega\rangle = \|d\omega\|_{L^2}^2 + \langle [B\wedge\omega], d\omega\rangle \qquad (2)$$

- Exercise: Show that functions ω with $\int \omega = 0$ are orthogonal to kernel of $d + d^*$. Hence we have an elliptic estimate:

$$\|\omega\|_{W^{1,2}} \le C\|d\omega + d^*\omega\|_{L^2} = C\|d\omega\|, \text{ and}$$

$$\begin{aligned} |\langle B\wedge\omega, d\omega\rangle| &\le C\|d\omega\|_{L^2}\|B\wedge\omega\|_{L^2} \\ &\le C\|d\omega\|_{L^2}\|B\|_{L^4}\|\omega\|_{L^4} \\ &\le C\|d\omega\|_{L^2}^2\|B\|_{W^{1,2}} \qquad (3)\end{aligned}$$

$$\leq C \| dw \|_{L^2} \, \| B \|_{W^{1,2}} \qquad (3)$$

(Sobolev embedding + elliptic estimate)

- Thus $\| dw \|_{L^2} \leq C \| B \|_{W^{1,2}}$ by (2) and (3).
- We can choose $\eta \ll 1$ in the statement of Proposition 1 to violate this. $\#$ $\square$

Transferring to ball

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Transfering back to $\mathbb{B}'$

- Let $\delta_t : \mathbb{R}^4 \longrightarrow \mathbb{R}^4 \quad \delta_t(x) = tx \quad t \in [0,1]$ be dilation.

- $A_t := \delta_t^* A$ a connection on $\mathbb{B}_1$
- Recall from Niklas' talk: $\|F_A\|_{L^2}$ is conformally invariant. Thus:

$$\|F_{A_t}\|^2_{L^2(\mathbb{B}_1)} = \int_{|x|\le t} |F_A|^2 dx \le \|F_A\|^2_{L^2(\mathbb{B}_1)}$$

- Let $\mathbb{B}_1 \subseteq S^4$ be a hemisphere and $p : S^4 \longrightarrow \mathbb{B}_1$ the projection that fixes equatorial $S^3 \subseteq S^4$

- Lipschitz, a.e. differentiable.
- For α a connection on $\mathbb{B}_1$, we can consider the (non-smooth) double $B := p^* A$ and

$$\|F_B\|^2_{L^2(S^4)} = 2\|F_A\|^2_{L^2(\mathbb{B}_1)}$$

- If p were smooth, we could do the following:
- Applying continuity method: Take connection A over $\mathbb{B}_1$

- Applying continuity method: Take connection A over $\mathbb{B}$, with $\|F_A\| \le \frac{\eta}{\sqrt{2}}$, η from Prop. 1

Let $B_t' := p^* A_t$.

- Then $\|F_{B_t'}\| \le \eta$ by above estimate. Get u_t s.t. $B_t := u_t^* B_t'$ satisfies

$$d^* B_t = 0, \quad \|B_t\|_{W^{1,2}} \le 2N \|F_{B_t}\|_{L^2}$$

- Let $\tilde{A} := B_1 \big|_{\mathbb{B}_1}$, then $d^* \tilde{A} = 0$ and $\|\tilde{A}\|_{W^{1,2}} \le \sqrt{2} N \|F_{\tilde{A}}\|_{L^2}$ with $\tilde{A} = u_1^* A$.

Smoothness

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- "Smoothing" the above argument:

  Choose $\rho_\varepsilon : S^4 \to \overline{\mathbb{B}}_1$ converging to ρ uniformly $\varepsilon \to 0$ with $\nabla \rho_\varepsilon$ uniformly bounded, $\rho_\varepsilon = \rho$ outside ε-nbhd of equator.

$$\int_{S^4} |F(\rho_\varepsilon^*(\alpha))|^2 \leq 2 \int_{\mathbb{B}_1} |F(\alpha)|^2 + \varepsilon C \|F_\alpha\|^2_{L^\infty(\mathbb{B}_1)}$$

- To see this, $\int_{\varepsilon\text{-nbhd}} |F(\rho^*\alpha)|^2 = \int_{\varepsilon\text{-nbhd}} |\rho^* d\alpha + \rho^*\alpha \wedge \alpha|^2$

$$= \int_{\varepsilon\text{-collar}} |\nabla \rho| \, |F(\alpha)|^2$$

$$\leq \varepsilon C \|F(\alpha)\|^2_{L^\infty(\mathbb{B}_1)}$$

- Apply proposition 1 to $\rho_\varepsilon^*(A_k)$ to get $\tilde{B}^\varepsilon$ Coulomb rep. for $\rho_\varepsilon^*(A)$
- Apply Lemmas (2) & (3) over domain $D \subseteq S^4 \setminus S^3$ to get $\tilde{B}|_D$ as $\varepsilon \to 0$.
- Let $\hat{A}_D$ be the restriction to $\mathbb{B}_1 \cap D$. Increase D to $\tilde{A}$ for A over $\mathbb{B}_1$.