HUMBOLDT-UNIVERSITÄT ZU BERLIN

The Topology of 4 - Manifolds

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Homology and Cohomology

Some Essential Facts

Poincare duality:

Theorem
Let X be a closed orientable 4-manifold, then we have an isomorphism

$$PD : H^i(X;\mathbb{Z}) \xrightarrow{\cong} H_{2-i}(X;\mathbb{Z}).$$

Some Essential Facts

Theorem
Let X be a simply-connected closed oriented 4-manifold, then $H_2(X, \mathbb{Z})$ is a free abelian group.

Some Essential Facts

Proof.
This is a simple computation: We have:

$$H_2(X;\mathbb{Z}) \cong H^2(X;\mathbb{Z}).$$

And also

$$H_1(X;\mathbb{Z}) = \mathsf{Ab}(\pi_1(X)) = 0.$$

Thus by the universal coefficient theorem:

$$\begin{aligned} H^2(X,\mathbb{Z}) &= \mathsf{Ext}^1_{\mathbb{Z}}(H_1(X;\mathbb{Z}),\mathbb{Z}) \oplus \mathsf{Hom}(H_2(X;\mathbb{Z}),\mathbb{Z}) \\ &= \mathsf{Hom}(H_2(X;\mathbb{Z}),\mathbb{Z}). \end{aligned}$$

Since $H_2(X;\mathbb{Z})$ is fin. generated we have that $\mathsf{Hom}(H_2(X;\mathbb{Z}),\mathbb{Z})$ is free. □

Some Essential Facts

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- For $\alpha \in H^2(X;\mathbb{Z})$ choose a complex line bundle L s.t. $c_1(L) = \alpha$.
- Take a generic section σ
- We have an embedded surface $\Sigma_\alpha = \sigma^{-1}(0)$
- $[\Sigma_\alpha] = PD(\alpha)$

Note: Different construction using Eilenberg-MacLean spaces in appendix of notes.

Some Essential Facts

Next we will define an additional structures on $H^2(X;\mathbb{Z})$.

Intersection Forms

The Intersection Product

Cap Product:

$$\frown : H_p(X;\mathbb{Z}) \times H^q(X;\mathbb{Z}) \to H_{p-q}(X;\mathbb{Z}).$$

Kronecker Pairing:

$$\langle \cdot, \cdot \rangle : H^p(X;G) \times H_p(X;G) \to G.$$

Cup product:

$$\smile : H^i(X;\mathbb{Z}) \times H^j(X;\mathbb{Z}) \to H^{j+i}(X;\mathbb{Z}).$$

The Intersection Product

Now we have everything we need to make this definition:

Definition
Let X be a closed oriented **topological** 4-manifold. Then the bilinear map

$$Q : H^2(X;\mathbb{Z}) \times H^2(X;\mathbb{Z}) \longrightarrow \mathbb{Z}$$

given by

$$(\alpha, \beta) \mapsto \langle \alpha \smile \beta, [X] \rangle$$

is called (cohomology) *intersection form* of X.

The Intersection Product

This is a very algebraic definition. For a smooth four manifold we can interpret it in a more geometric way:

The Smooth Intersection Product

Theorem
Let X be closed oriented simply-connected smooth *4-manifold. Let $\alpha, \beta \in H^2(X;\mathbb{Z})$ and $[\Sigma_\alpha], [\Sigma_\beta] \in H_2(X;\mathbb{Z})$ be their duals. There are closed 2-forms ω_α and ω_β representing α, β such that*

$$Q(\alpha, \beta) = \langle \alpha \smile \beta, [X] \rangle = \Sigma_\alpha \cdot \Sigma_\beta = \int_X \omega_\alpha \wedge \omega_\beta.$$

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Since $H^2(X;\mathbb{Z})$ is torsion free we can go forth and back between integral and de Rahm cohomology.

The Smooth Intersection Product

Proof.
First we notice:

$$Q(\alpha, \beta) = \langle \alpha \smile \beta, [X] \rangle = \langle \alpha, [X] \frown \beta \rangle$$
$$= \langle \alpha, PD(\beta) \rangle = \langle \alpha, [\Sigma_\beta] \rangle$$

Switching to de Rahm cohomology:

$$\langle \alpha, [\Sigma_\alpha] \rangle = \int_{\Sigma_\beta} \omega_\alpha$$

Proof.
Now we have to show:

$$\int_{\Sigma_\beta} \omega_\alpha = \Sigma_\alpha \cdot \Sigma_\beta$$

Choose $\Sigma_\alpha \pitchfork \Sigma_\beta$. Then we have a finite number of intersection points. Since ω_α vanishes away from Σ_α it is enough to compute the integral at the intersection points.

Proof.
Around any intersection point choose $\mathfrak{U}$ and oriented local coordinates x_1, x_2, x_3, x_4 s.t.

$$\mathfrak{U} \cap \Sigma_\alpha = \{x_3 = x_4 = 0\} \qquad \mathfrak{U} \cap \Sigma_\beta = \{x_1 = x_2 = 0\}$$

and $\mathfrak{U} \cap \Sigma_\alpha$ is oriented by $dx_1 \wedge dx_2$. Then

$$\omega_\alpha = f(x_3, x_4) dx_3 \wedge dx_4$$

for a bump function $f : \mathbb{R}^2 \to \mathbb{R}$. Then

$$\int_{\mathfrak{U} \cap \Sigma_\beta} f(x_3, x_4) dx_3 \wedge dx_4 = \pm 1$$

depending on orientation.

Proof.
By summing over all intersection points we get:

$$\int_{\Sigma_\beta} \omega_\alpha = \Sigma_\alpha \cdot \Sigma_\beta.$$

For the last equality we have:

$$\langle \omega, [N] \rangle = \int_N \omega$$
$$[\omega_1 \wedge \omega_2] = [\omega_1] \smile [\omega_2]$$

Giving us

$$Q(\alpha, \beta) = \int_X \omega_\alpha \wedge \omega_\beta.$$

□

Unimodularity

Theorem
Let X be closed oriented simply-connected 4-manifold. Then Q_X is unimodular, i.e. $a \to Q(\cdot, a)$ and $b \to Q(b, \cdot)$ are isomorphisms.

Unimodularity

Proof.
By the universal coefficient theorem

$$H^2(X;\mathbb{Z}) \to \mathrm{Hom}(H_2(X;\mathbb{Z}))$$
$$\alpha \mapsto \langle \alpha, \cdot \rangle$$

is an isomorphism. This suffices, as

$$Q(\alpha, \beta) = \langle \alpha, PD(\beta) \rangle$$

and Q is symmetric. □

Example

Example
Consider

$$X = S^2 \times S^2.$$

Then

$$H^2(X;\mathbb{Z}) = \langle PD^{-1}([\{\mathrm{pt}\} \times S^2]), PD^{-1}([\{\mathrm{pt}\} \times S^2]) \rangle.$$

And

$$Q \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

In the Presence of Torsion

What happens if $H^2(X;\mathbb{Z})$ is not free?

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What happens if $H^2(X;\mathbb{Z})$ is not free? Let $\alpha \in H^2(X;\mathbb{Z})$ s.t. $n \cdot \alpha = 0$, then

$$nQ(\alpha, \beta) = Q(n \cdot \alpha, \beta) = Q(0, \beta) = 0.$$

In the Presence of Torsion

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$$nQ(\alpha, \beta) = Q(n \cdot \alpha, \beta) = Q(0, \beta) = 0.$$

So we can define

$$\tilde{Q} : \left({H^2(X;\mathbb{Z})}\big/{\mathrm{Ext}^1_{\mathbb{Z}}(H_1(X;\mathbb{Z}), \mathbb{Z})} \right)^2 \longrightarrow \mathbb{Z}$$

and use the arguments there.

Intersection Form Invariants

- **Parity:**
  If $Q(\alpha, \alpha) \in 2\mathbb{Z} \forall \alpha \in H^2(X; \mathbb{Z})$ we call Q **even**. Otherwise it is called **odd**.

Intersection Form Invariants

- **Parity:**
  If $Q(\alpha, \alpha) \in 2\mathbb{Z} \forall \alpha \in H^2(X; \mathbb{Z})$ we call Q **even**. Otherwise it is called **odd**.
- **Definiteness:**
  If $Q(\alpha, \alpha) > 0 \forall \alpha \in H^2(X; \mathbb{Z})$ we call Q **positive-definite.** If $Q(\alpha, \alpha) < 0 \forall \alpha \in H^2(X; \mathbb{Z})$ we call Q **negative-definite.** Otherwise it is called **indefinite**.

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- **Rank:**
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- **Rank:**
  The second Betti number $b_2(X)$ is called the **rank** of Q.
- **Signature:**
  Over $\mathbb{R}$ Q has b_2^+ positive and b_2^- negative eigenvalues. We call

  $$\operatorname{sign} Q = b_2^+ - b_2^-$$

  the **signature** of Q.

Hasse-Minkowski Classification

Theorem (Hasse-Minkowski)
Let H be a free $\mathbb{Z}$ module. If $Q : H \times H \to \mathbb{Z}$ is an odd indefinite bilinear form then

$$Q \cong l(1) \oplus m(-1)$$

with $l, m \in \mathbb{N}_0$. If $Q : H \times H \to \mathbb{Z}$ is an even indefinite bilinear form then

$$Q \cong l \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus mE_8$$

with $l, m \in \mathbb{N}_0$.

Hasse-Minkowski Classification

$$E_8 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

Hasse-Minkowski Classification

What about definite forms?

- No easy classification

Hasse-Minkowski Classification

What about definite forms?

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- Many exotic forms

Hasse-Minkowski Classification

What about definite forms?

- No easy classification
- Many exotic forms
- Number of unique even definite forms of some ranks:

Rank	8	16	24
#	1	2	5

Diagonalizability

Warning: Any intersection form is diagonalizable over $\mathbb{Q}$ but might not be over $\mathbb{Z}$.

Exercise
Show that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is not diagonalizable over $\mathbb{Z}$.

Homotopy Type

Milnor's Theorem

We will now find a direct link between homotopy type and intersection form of four manifolds.

Milnor's Theorem

Theorem (Milnor (1958))
The oriented homotopy type of a simply-connected closed oriented 4-manifold is determined by its intersection form.

Milnor's Theorem

Proof.
Define $X' = X \setminus B^4$. Then

$$H_k(X';\mathbb{Z}) = \begin{cases} H_2(X) & k = 2 \\ 0 & k = 1,3,4 \end{cases}.$$

By Hurewicz's theorem:

$$f : S^2 \vee ... \vee S^2 \to X'$$

represents $\pi_2(X) \cong H_2(X';\mathbb{Z})$. This induces an isomorphism

$$H_k(S^2 \vee ... \vee S^2;\mathbb{Z}) \cong H_k(X';\mathbb{Z})$$

for every k.

Milnor's Theorem

Proof.
Thus

$$X \simeq (S^2 \vee ... \vee S^2) \cup_h e^4$$

with $[h] \in \pi_3(S^2 \vee ... \vee S^2)$. Left to show: $[h]$ depends only on Q.
Complete proof can be found in: [1, p.141ff]

Milnor's Theorem

Proof.
Sketch:

- $[X] \in H_4(X;\mathbb{Z})$ corresponds to
  $[e^4] \in H_4((S^2 \vee \ldots \vee S^2) \cup_h e^4;\mathbb{Z})$

Milnor's Theorem

Proof.
Sketch:

- $[X] \in H_4(X;\mathbb{Z})$ corresponds to $[e^4] \in H_4((S^2 \vee ... \vee S^2) \cup_h e^4; \mathbb{Z})$
- $S^2 \vee \cdots \vee S^2 = \mathbb{CP}^1 \vee \cdots \vee \mathbb{CP}^1 \subset \mathbb{CP}^\infty \times \cdots \times \mathbb{CP}^\infty$

Milnor's Theorem

Proof.
Sketch:

- $[X] \in H_4(X; \mathbb{Z})$ corresponds to $[e^4] \in H_4((S^2 \vee ... \vee S^2) \cup_h e^4; \mathbb{Z})$
- $S^2 \vee \cdots \vee S^2 = \mathbb{CP}^1 \vee \cdots \vee \mathbb{CP}^1 \subset \mathbb{CP}^\infty \times \cdots \times \mathbb{CP}^\infty$
- Long exact sequence on relative homotopy groups:

$$\pi_4(\times_m \mathbb{CP}^\infty) \longrightarrow \pi_4(\times_m \mathbb{CP}^\infty, \vee_m S^2) \longrightarrow \pi_3(\vee_m S^2) \longrightarrow \pi_3(\times_m \mathbb{CP}^\infty)$$

Milnor's Theorem

Proof.
Sketch:

- $[X] \in H_4(X; \mathbb{Z})$ corresponds to $[e^4] \in H_4((S^2 \vee ... \vee S^2) \cup_h e^4; \mathbb{Z})$
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$$\pi_4(\times_m \mathbb{CP}^\infty) \longrightarrow \pi_4(\times_m \mathbb{CP}^\infty, \vee_m S^2) \longrightarrow \pi_3(\vee_m S^2) \longrightarrow \pi_3(\times_m \mathbb{CP}^\infty)$$

- $\mathbb{CP}^\infty$ is $K(\mathbb{Z}, 2) \implies$

$$\pi_3(\vee_m S^2) \cong \pi_4(\times_m \mathbb{CP}^\infty, \vee_m S^2) \cong H_4(\times_m \mathbb{CP}^\infty, \vee_m S^2)$$

Milnor's Theorem

Proof.

- $\pi_3(\vee_m S^2) \cong \pi_4(\times_m \mathbb{CP}^\infty, \vee_m S^2) \cong H_4(\times_m \mathbb{CP}^\infty, \vee_m S^2)$
- Oriented manifold: $[h]$ is determined by $\alpha_k(h_*([e^4]))$ for $\alpha_1, ..., \alpha_l$ basis of $H^4(\times_m \mathbb{CP}^\infty)$.

Milnor's Theorem

Proof.

- $\pi_3(\vee_m S^2) \cong \pi_4(\times_m \mathbb{CP}^\infty, \vee_m S^2) \cong H_4(\times_m \mathbb{CP}^\infty, \vee_m S^2)$
- Oriented manifold: $[h]$ is determined by $\alpha_k(h_*([e^4]))$ for $\alpha_1, ..., \alpha_l$ basis of $H^4(\times_m \mathbb{CP}^\infty)$.
- Basis is given by cupping $PD^{-1}([S_i^2])$. Since

$$H^2(\times_m \mathbb{CP}^\infty) \cong H^2(\vee_m S^2) \cong H^2(X'; \mathbb{Z}) = H^2(X; \mathbb{Z})$$

  these classes can be seen in X.

Milnor's Theorem

Proof.

- $\pi_3(\vee_m S^2) \cong \pi_4(\times_m \mathbb{CP}^\infty, \vee_m S^2) \cong H_4(\times_m \mathbb{CP}^\infty, \vee_m S^2)$
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  these classes can be seen in X.
- We are done since $\langle PD^{-1}([S_i^2]) \smile PD^{-1}([S_j^2]), h_*([e^4]) \rangle = \langle \omega_i, \omega_j, [X] \rangle = Q(\omega_i, \omega_j)$

$\square$

Milnor's Theorem

Exercise
Fill in the gaps in the proof sketch.

The “Big” Structure Theorems

Freedman's Theorem

Theorem (Freedman)
Let Q be an quadratic (i.e. unimodular symmetric bilinear) form over $\mathbb{Z}$, then there exists a topological *4-manifold M s.t. Q is (up to isomorphism) the intersection form of M. If Q is even, then M is unique.*

Rohlin's Theorem

Theorem (Rohlin)
Let X be a simply-connected closed oriented smooth *4-manifold with $w_2(X) = 0$. Then*

$$\text{sign}\, Q_X \in 16\mathbb{Z}.$$

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Let X be a simply-connected closed oriented smooth *4-manifold with $w_2(X) = 0$. Then*

$$\operatorname{sign} Q_X \in 16\mathbb{Z}.$$

The original proof by Rohlin is very involved. Simpler proof due to Atiyah and Singer using the Atiyah-Singer index theorem.
Reference: [2, Theorem 29.9]

Rohlin's Theorem

Corollary
Let X be a simply-connected closed oriented smooth *4-manifold with even intersection form Q_X. Then*

$$\operatorname{sign} Q_X \in 16\mathbb{Z}.$$

Corollary
There exists a simply-connected closed 4-manifold E_8 with intersection form E_8 that has no smooth structure.

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There exists a simply-connected closed 4-manifold E_8 with intersection form E_8 that has no smooth structure.

Proof.
E_8 is a negative definite even form with signature -8. The existence is given by Freedman's theorem. □

Donaldson's Theorem

Theorem (Donaldson)
Let X be a simply-connected closed smooth *4-manifold. If Q is definite, Q is diagonalizable over $\mathbb{Z}$.*

Donaldson's Theorem

Corollary
Let X be a simply-connected closed smooth *4-manifold. If Q is positive-definite then*

$$X \cong \#_k \mathbb{CP}^2$$

as topological manifolds.

Whitney Disks and the Failure of the h-Cobordism Principle in Dimension Four

Definition
Let M and N be closed simply-connected manifolds and W be a cobordism between them (i.e. $\partial W = M \cup \bar{N}$). If the inclusions $M \to W$ and $N \to W$ are homotopy equivalences, then M and N are called *h-cobordant.*

h-Cobordisms

Theorem (Wall)
Two simply-connected four-manifolds with isomorphic intersection form are h-cobordant.

The h-Cobordism Theorem

Theorem (Smale (1961))
Let M and N be cobordant smooth n-manifolds with $n > 4$. Then M and N are diffeomorphic.

Warning: This theorem only holds for $n \geq 5$.

The Culprit

Why does the (smooth) h-cobordism principle fail in dimension four?

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- Short answer:

> The statement
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Why does the (smooth) h-cobordism principle fail in dimension four?

- Short answer:

> The statement
> $$2 + 2 < 4$$
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- That is not so helpful, we are looking for a longer answer.

The Culprit

Strategy of the proof in higher dimensions:

- Goal: Show that $W \cong M \times [0,1]$

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- Goal: Show that $W \cong M \times [0,1]$
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- Choose a Morse function $f : W \to [0,1]$ with $f(M) = 0$ and $f(N) = 1$
- If f has no critical values we are done!
- Idea: Modify f s.t. all critical values disappear

The Culprit

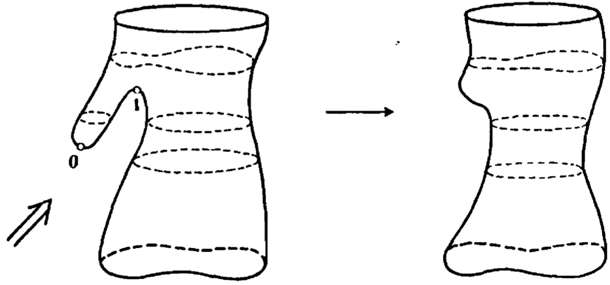

Fig. 4

Figure 1: Cancelling an index 0 critical point with an index 1 critical point. From [3]

The Culprit

Removing critical points of index $0, 1, 4, 5$ works in dimension four.
But: Canceling critical points of index 3 and 2 does not work (with this method).

The Culprit

- Suppose f has two critical points: p of index 2 and q of index 3

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- Let p and q be separated by $Z_{1/2} = f^{-1}(\frac{1}{2})$

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- Suppose f has two critical points: p of index 2 and q of index 3
- Let p and q be separated by $Z_{1/2} = f^{-1}(\frac{1}{2})$
- Fact: p and q can be canceled if there is exactly one flow line from p to q

The Culprit

We define

$$S_+ = \{x \in Z_{1/2} \mid x \text{ flows to } p \text{ as } t \to \infty\}$$
$$S_- = \{x \in Z_{1/2} \mid x \text{ flows to } q \text{ as } t \to -\infty\}.$$

These are embedded spheres. If $S_- \pitchfork S_+$ is a single point we can glue the flow lines and are done.

The Culprit

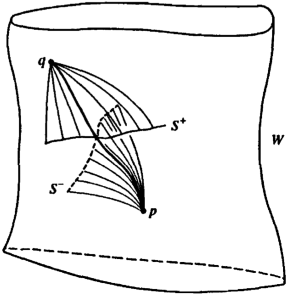

Fig. 6

Figure 2: Analogy in dimension three showing S_+ and S_- intersection transversely and the resulting flow line. From [3]

The Culprit

The algebraic intersection number is 1 because W is h-cobordism. Problem: The geometric intersection number might not agree! We need an isotopy to correct this.

The Whitney Disk

Usual procedure:

- Choose intersection points with opposite signs, e.g. x and y
- Find path $\alpha \subset S_+$ and $\beta \subset S_-$ joining them
- W simply-connected $\implies \alpha \cup \beta$ inessential
- There is a disk $D \subset W$ with $\partial D = \alpha \cup \beta$
- If the disk lies outside S_+ and S_- we get an isotopy removing the intersection points

The Culprit

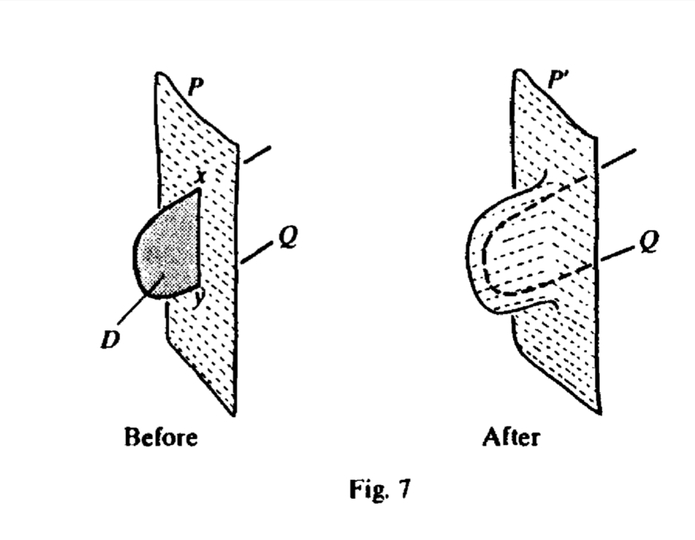

Figure 3: Removing intersection points in pairs. From [3]

The Whitney Disk

In dimension $n \geq 5$:

- D is generically embedded
- D generically does not intersect S_+ and S_- in any interior points

The Whitney Disk

In dimension $n \geq 5$:

- D is generically embedded
- D generically does not intersect S_+ and S_- in any interior points

In dimension four on the other hand both is not true! The intersection form makes this clear.

Final Note

With the existence of non-smooth manifolds one the one hand and the failure of the h-cobordism on the other hand, we see that the topology and geometry of four-manifolds is quite unique.

Thank you for your attention!

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