# Tropical Fukaya Algebras, Part 2

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November 23, 2020

arXiv:2004.14314. Joint work with Chris Woodward.

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# Broken maps and broken manifolds

**Theorem**

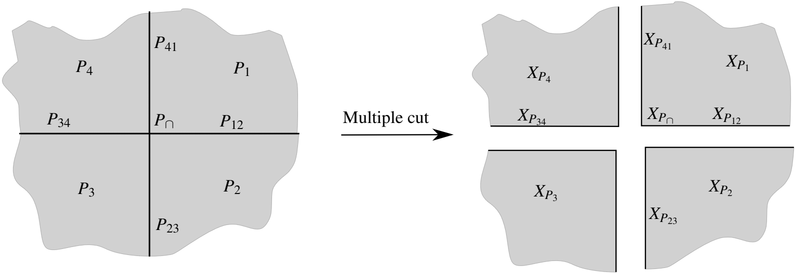
*Broken maps of index zero in* $\mathcal{X} \xleftrightarrow{\text{bijective}}$ *Unbroken maps of index zero in a neck-stretched manifold* $(X, J_\nu)$ *for large enough* $\nu$.

# Broken maps and broken manifolds

## Theorem

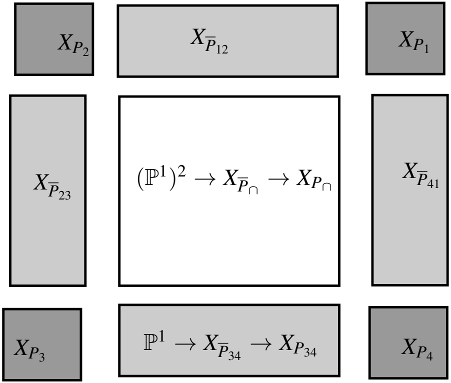
*Broken maps of index zero in* $\mathcal{X} \overset{\text{bijective}}{\longleftrightarrow}$ *Unbroken maps of index zero in a neck-stretched manifold* $(X, J_\nu)$ *for large enough* $\nu$.

- Broken maps live in a broken manifold. A broken manifold consists of **cut spaces**



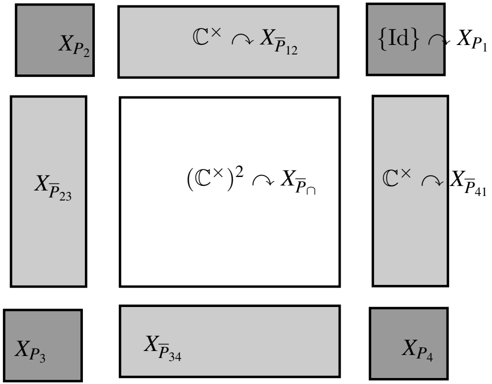
# Broken maps and broken manifolds

and **neck pieces**, which are relative submanifolds thickened into toric fibrations.



# Broken maps and broken manifolds

Pieces of the broken manifold have natural actions of complex tori:




Symmetry group on $X_{\overline{P}_\cap}$ $= T_{P_\cap} := (\mathbb{C}^\times)^2$

Symmetry group on $X_{\overline{P}_{ij}}$ $= T_{P_{ij}} := \mathbb{C}^\times$

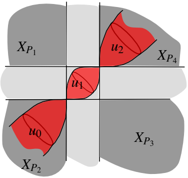
Symmetry group on $X_{P_i}$ $= T_{P_i} := \{\mathrm{Id}\}$

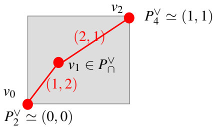
# Broken maps and broken manifolds

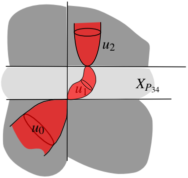
A broken map consists of

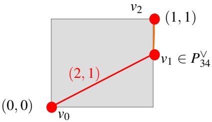
- map components in pieces of a broken manifold that satisfy a *matching condition* at nodes,
- and a *tropical graph* whose edge slopes are given by intersection multiplicities at nodes.

# Broken map : Example



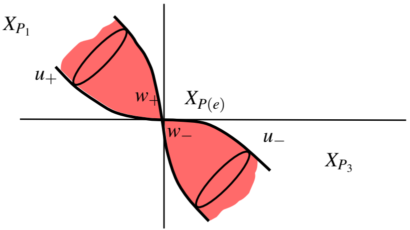



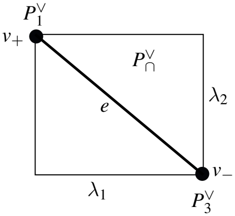
# Broken map : Example

At the node $w$ the intersection multiplicity with relative divisors on both lifts of the node $w$ are equal to the slope

$$\mu(e) = (\mu_1, \mu_2) \in (\mathbb{Z}_{>0})^2$$

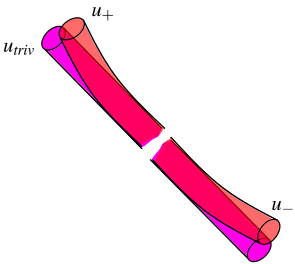
of the edge $e$ in the tropical graph.




$\mu(e)$ :=Slope of $e = (\mu_1, \mu_2)$

# Broken map : Matching condition




- Matching condition :

$$(u_+ \mod T_{\mu,\mathbb{C}})(w_+) = (u_- \mod T_{\mu,\mathbb{C}})(w_-)$$

in $Z_{P_\cap,\mathbb{C}}/T_{\mu,\mathbb{C}}$.

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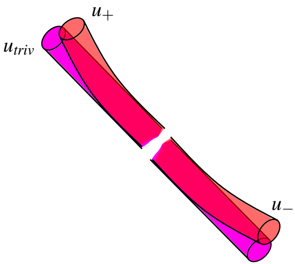


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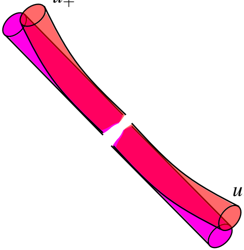


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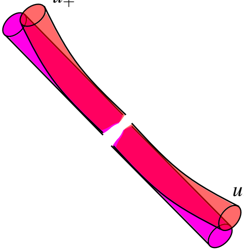


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- The trivial cylinder $u_{triv}$ is a $T_{\mu,\mathbb{C}}$-orbit.

Remark : The matching condition has codimension $\dim(X) - 2$. Therefore the presence of a node does not decrease the index of the map.

# Tropical symmetry group

- Let $\Gamma$ be the tropical graph of a broken map. The **tropical symmetry group**

$$T_{\mathrm{trop}}(\Gamma)$$

acts on the target space of the broken map and preserves the matching condition at nodes.

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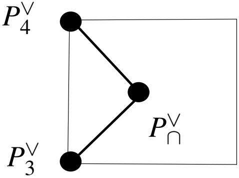
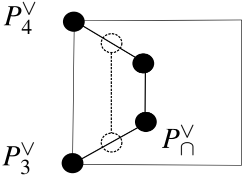
$$T_{\text{trop}}(\Gamma)$$

  acts on the target space of the broken map and preserves the matching condition at nodes.
- Symmetries of the tropical graph are the infinitesimal generators of the tropical symmetry group. These are the ways of moving the vertices in $\Gamma$ without changing the edge slope.




A rigid tropical graph:
$T_{\text{trop}}(\Gamma)$ is finite




$T_{\text{trop}}(\Gamma) = \mathbb{C}^\times$

# Gromov topology on broken maps

**Theorem (Gromov convergence)**

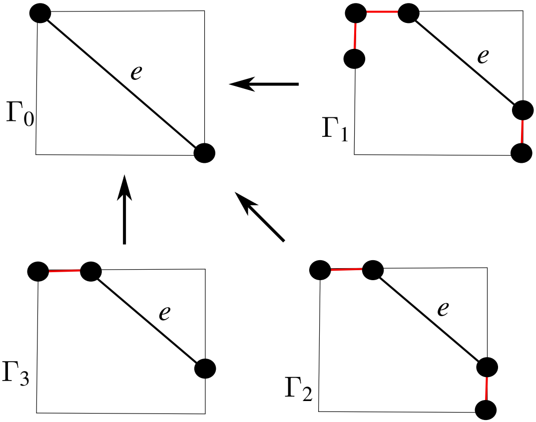
*Given a sequence $u_\nu$ with tropical graph $\Gamma$ with area $\leq E$, there is a subsequence that converges to a limit $u$ with tropical graph $\Gamma'$, and there is an* edge collapse *morphism $\Gamma' \to \Gamma$. Further $\operatorname{ind}(u) = \operatorname{ind}(u_\nu)$.*

We say that there is an edge collapse morphism $\Gamma' \to \Gamma$ if

- $\Gamma$ is obtained by collapsing edges in $\Gamma'$,
- and the slopes of the uncollapsed edges are the same in $\Gamma$ and $\Gamma'$.

# Gromov topology on broken maps: edge collapse

$\Gamma_1 \to \Gamma_0$, $\Gamma_2 \to \Gamma_0$, $\Gamma_3 \to \Gamma_0$ are edge collapse morphisms. In all graphs $\mathcal{T}(e) = (1, 1)$.



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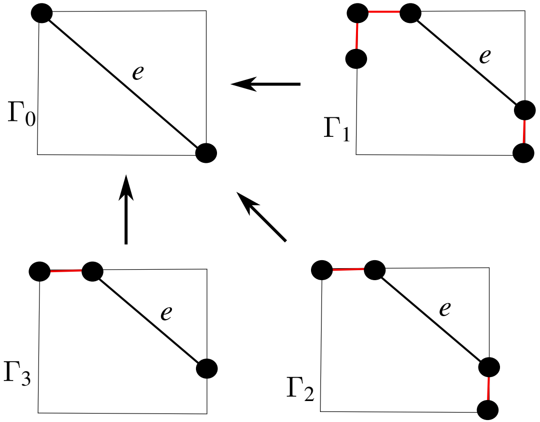


Remark : For an edge collapse $\Gamma' \to \Gamma$, $\Gamma' \neq \Gamma \implies T_{\text{trop}}(\Gamma) \subsetneq T_{\text{trop}}(\Gamma')$.

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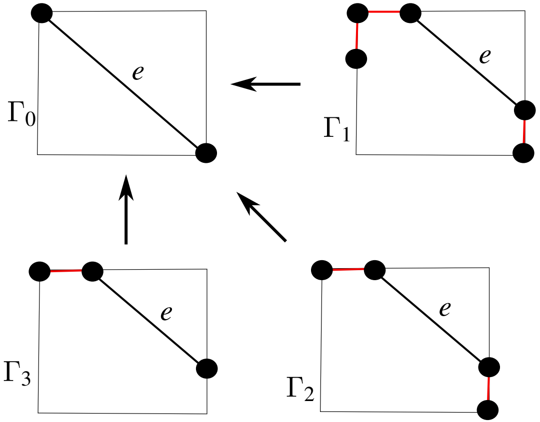


Remark : For an edge collapse $\Gamma' \to \Gamma$, $\Gamma' \neq \Gamma \implies T_{\text{trop}}(\Gamma) \subsetneq T_{\text{trop}}(\Gamma')$.

Corollary : In Gromov convergence, $\text{ind}(u_\nu) \leq 1$ implies that the limit has the same tropical type as $u_\nu$.

# Gromov topology on broken maps

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## Conjecture

*For generic perturbations, the space* $\mathcal{M}_d(\mathcal{X})^{<E}$

- *is compact,*
- *and is the coarse moduli space of a smooth Deligne-Mumford stack.*
- *In any connected component, generic points have a finite tropical symmetry group.*
- *For any tropical graph* $\Gamma$*, the moduli space* $\mathcal{M}_\Gamma(\mathcal{X})^{<E}$ *is a stratum of codimension* $\dim(T_{\text{trop},\mathbb{C}}(\Gamma))$ *in* $\mathcal{M}_d(\mathcal{X})^{<E}$.

# Gromov topology on broken maps : Observation

- Let $\mathcal{M}_\Gamma(\mathcal{X}) \subset \mathcal{M}_d^{\leq E}$ be an $n$-dimensional component where $\Gamma$ is rigid. Then there exists a sequence $u_\nu \in \mathcal{M}_\Gamma(\mathcal{X})$ that converges to a broken map with $n$-dimensional tropical symmetry.

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- Thus, the dimensions of the moduli space can be 'converted' to tropical symmetry by going towards an end of the moduli space in the right direction.

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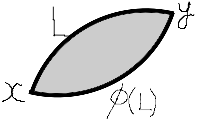
# Unobstructedness of Lagrangians : motivation

We recall the definition of Lagrangian intersection Floer cohomology.

- Let $L \subset (X, \omega)$ be a Lagrangian submanifold and let $\phi : X \to X$ be a 'small' Hamiltonian diffeomorphism such that $L$ and $\phi(L)$ intersect transversely.
- Let $CF(L)$ be a cochain complex generated by $L \cap \phi(L)$ with differential given by a weighted count of holomorphic strips

$$d : CF(L) \to CF(L), \quad x \mapsto \sum_y \#(\mathcal{M}(x, y))_0 y.$$

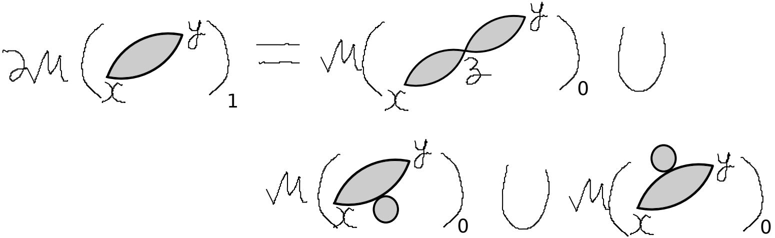
Here $\mathcal{M}(x, y)_0$ is the zero-dimensional component of the moduli space of holomorphic strips




$u : \mathbb{R} \times [0, 1] \to X$ with boundary in $L$, $\phi(L)$, and whose ends asymptote to $x$, $y$

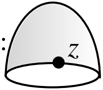
# Unobstructedness of Lagrangians : motivation

In general $d^2 \neq 0$ because of disk bubbling. The boundary of one-dimensional components is as follows.



# Unobstructedness of Lagrangians

- A Lagrangian $L$ is **unobstructed** if the boundary evaluation map

$$\mathrm{ev}_z : \Big\{ u :$$


$$\to (X, L) \Big\} \to L$$

  on the space of rigid one-pointed disks is a submersion.
- Unobstructedness implies $d^2 = 0$, because the second and third terms cancel out.

# Unobstructedness of Lagrangians : the toric case

- Let $X$ be a toric symplectic manifold with a moment map $\Phi : X \to \mathfrak{t}^\vee$, $T \simeq (S^1)^n$.
- For any regular value $c \in \mathfrak{t}^\vee$,

$$L := \Phi^{-1}(c)$$

  is a Lagrangian $T$-orbit.
- One way of obtaining the surjectivity of $d\mathrm{ev}_z$ is to use the torus-invariant almost complex structure $J_0$, and show that the space of index zero one-pointed $J_0$-holomorphic disks is $T$-invariant.

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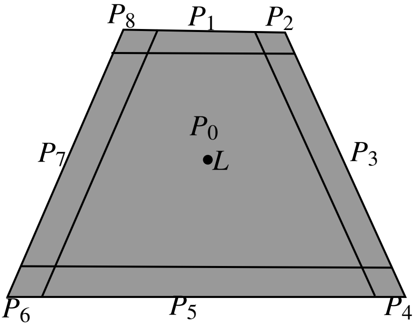
- The approach fails if there are negative index $J_0$-holomorphic spheres in the torus-invariant divisors (and therefore $J_0$ is not regular). In that case $J_0$ has to be perturbed, and we lose the $T$-invariance of the moduli space.

# Unobstructedness of Lagrangians : the toric case

Since the bad spheres lie on torus-invariant divisors we 'cut away' these divisors. Thus we have one cut space $X_{P_0}$ which is symplectomorphic to $X$, but with a 'smaller' symplectic form. All the torus-invariant divisors of $X_{P_0}$ are relative divisors, so there are no spheres in these divisors.



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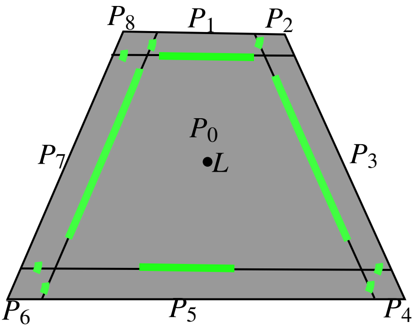
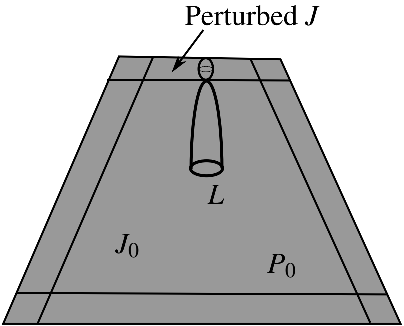



Figure: There are no spheres in the relative divisors

# Unobstructedness of Lagrangians : the toric case




The moduli space of broken disks is not $T$-invariant, because the almost complex structure is not the standard one in all pieces. So we proceed to eliminate the matching condition at nodes.

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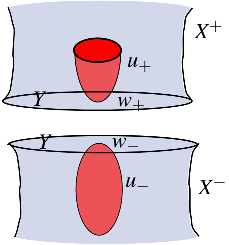
# Degenerating the matching condition : single cut

Recall that in a single cut the matching condition is

$$u_+(w_+) = u_-(w_-) \quad \text{in } Y,$$

where $Y := Z/S^1$ is the relative divisor.



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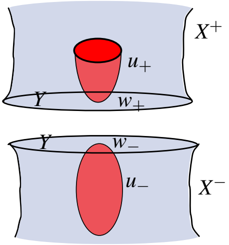


The moduli space of broken maps is

$$\mathcal{M}(X_+) \times_Y \mathcal{M}(X_-)$$

where $\mathrm{ev}_\pm : \mathcal{M}(X_\pm) \to Y$ is the evaluation map at the node.

# Degenerating the matching condition : single cut

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Step 1 : For any $t \in \mathbb{R}$, there is a homotopy equivalence :

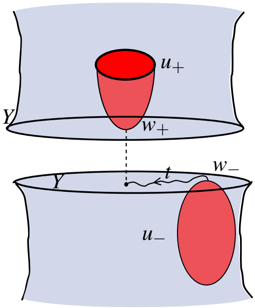
$$\mathcal{M}^{\text{brok}}(\mathcal{X}, L) \simeq \mathcal{M}^{t\text{-def}}(\mathcal{X}, L).$$

$t$-Deformed map

Matching condition :

$u_+(w_+) = \phi_t^Y u_-(w_-)$

$\phi_t^Y$ : Time $t$ gradient flow of a Morse function $H : Y \to \mathbb{R}$.



# Degenerating the matching condition : single cut

Step 2 : Taking limit $t \to \infty$, we get a homotopy equivalence

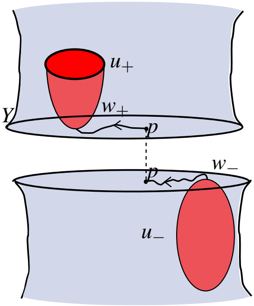
$$\mathcal{M}^{\text{brok}}(\mathcal{X}, L) \simeq \mathcal{M}^{\infty\text{-def}}(\mathcal{X}, L).$$

$\infty$-Deformed map

Matching condition :

$u_+(w_+)$, $u_-(w_-)$ are connected by a broken flow line.

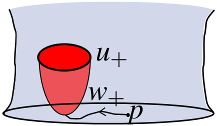
$p$ is a critical point of $H : Y \to \mathbb{R}$



# Degenerating the matching condition : single cut

The moduli space of $\infty$-deformed maps is a sum of products :

$$\sum_{p \in \mathrm{crit}(Y)} \Bigg($$

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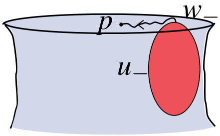


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The Morse deformation approach fails in the case of multiple cuts because stable/unstable manifolds may be contained in relative divisors. Generically there are additional bubbles in the limit $t \to \infty$.

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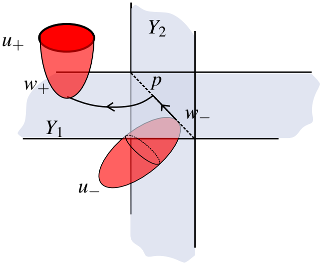


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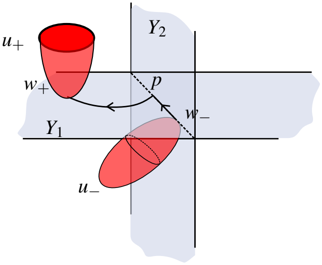


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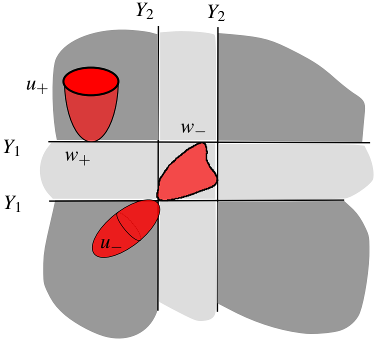


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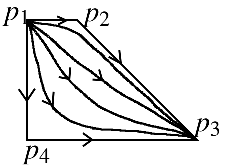

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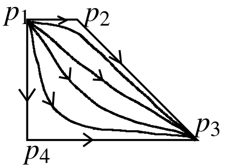
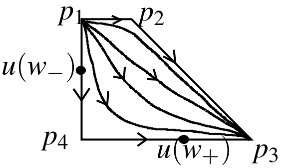



Then there is a decomposition $T_{\mathbb{C}} = T_{\mathbb{C}}^{+} \times T_{\mathbb{C}}^{-}$ such that the evaluation cycle $[\mathrm{ev}_{w_\pm}]$ intersects a $T_{\mathbb{C}}^{\pm}$-orbit transversely.



# Towards tropical Fukaya algebra : degenerating the matching condition at nodes

- Recall that the matching condition at the node is of the form

$$(u_+ \mod T_{\mu,\mathbb{C}})(w_+) = (u_- \mod T_{\mu,\mathbb{C}})(w_-)$$

in $(\mathbb{C}^\times)^n / T_{\mu,\mathbb{C}}$, where $\mu \in (\mathbb{Z}_{\geq 0})^n$ is the intersection multiplicity vector.

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  in $(\mathbb{C}^\times)^n/T_{\mu,\mathbb{C}}$, where $\mu \in (\mathbb{Z}_{\geq 0})^n$ is the intersection multiplicity vector.
- A **deformed map** is a version of the broken map where the matching condition is replaced by

$$(u_+ \mod T_{\mu,\mathbb{C}})(w_+) = \tau(u_- \mod T_{\mu,\mathbb{C}})(w_-)$$

  for some $\tau \in (\mathbb{C}^\times)^n/T_{\mu,\mathbb{C}}$.

# Towards tropical Fukaya algebra : degenerating the matching condition at nodes

- Let $\tau_\nu \in (\mathbb{C}^\times)^n / T_{\mu,\mathbb{C}}$ be a sequence of deformation parameters, $\tau_\nu \to \infty$, and let $u_\nu$ be a $\tau_\nu$-deformed map.

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- Let $\tau_\nu \in (\mathbb{C}^\times)^n / T_{\mu,\mathbb{C}}$ be a sequence of deformation parameters, $\tau_\nu \to \infty$, and let $u_\nu$ be a $\tau_\nu$-deformed map.
- We prove that the sequence $u_\nu$ converges to a **split map**.
- A split map is a version of a broken map with **no matching condition on nodes**, but with a **non-trivial tropical symmetry group**. In particular, the codimension of the matching condition is equal to the dimension of the tropical symmetry group.

# Towards tropical Fukaya algebra : degenerating the matching condition at nodes

**Result**

*The moduli space of split disks modulo the action of the tropical symmetry group is homotopy equivalent to the moduli space of broken disks:*

$$\mathcal{M}^{\text{split}}(\mathcal{X}, L)/T_{\text{trop},\mathbb{C}} \simeq \mathcal{M}^{\text{brok}}(\mathcal{X}, L).$$

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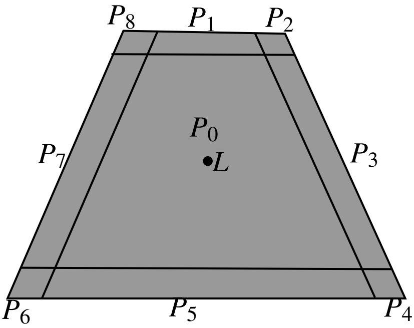
## Theorem (VW)

*There is a homotopy equivalence* $CF_{\mathrm{trop}}(\mathcal{X}, L) \simeq CF_{\mathrm{brok}}(\mathcal{X}, L)$.

# Example : deforming the node matching condition

We return to the example of a toric variety.

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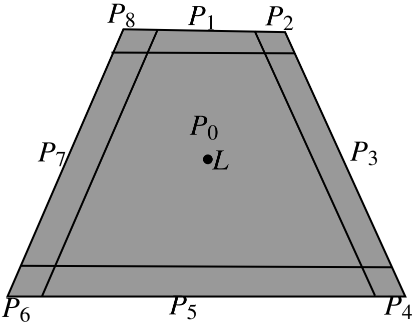


- Let $X$ be a toric variety. Neighborhoods of torus-invariant divisors are separated using cuts.
- There is one cut space $X_{P_0}$ which is diffeomorphic to $X$, but with a 'smaller' symplectic form. All the torus-invariant divisors of $X_{P_0}$ are relative divisors.

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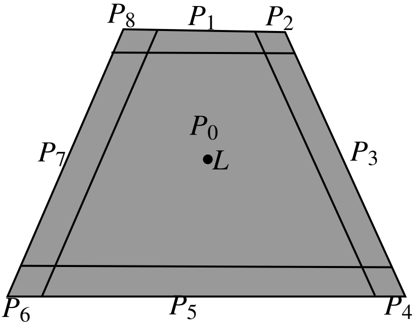


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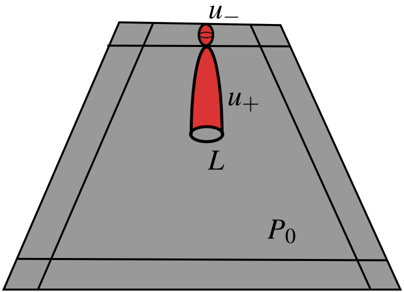
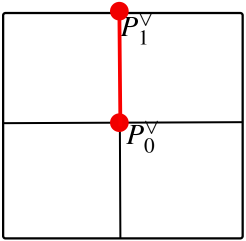


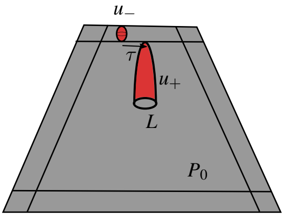

Broken map $u : C \to \mathcal{X}$
of Maslov index 2



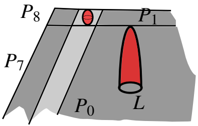

Tropical graph

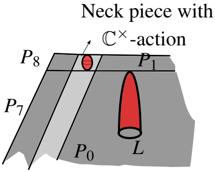
# Example : deforming the node matching condition

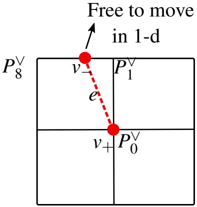
A $\tau$-deformed $u_\tau = (u_+, u_-^\tau)$ disk in $\mathcal{X}$ :




As $\tau_\nu \to \infty$, the sequence of $\tau_\nu$-deformed maps converge to a map with no matching condition on the node.





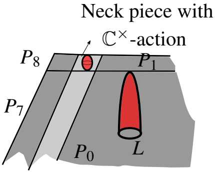



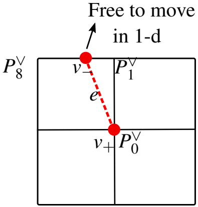

Tropical graph

Slope of $e$:
$\mu_e = (0, 1) \in \mathbb{Z}^2$

- Codimension of matching condition = Dimension of tropical symmetry group of $u_\infty$=2.

# Example : Split disk

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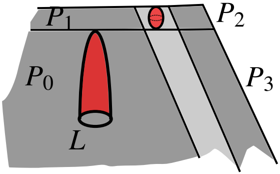
- Codimension of matching condition = Dimension of tropical symmetry group of $u_\infty$=2.
- Observe : we have chosen the direction in which deformation parameters go to infinity as
  $\eta := \pi^\perp_{\mu_e}(1,0) \in \mathbb{R}^2/\langle \mu_e \rangle$.
  Here $\pi^\perp_{\mu_e} : \mathbb{R}^2 \to \mathbb{R}^2/\langle \mu_e \rangle$ is the projection map.

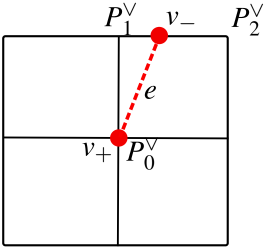
# Direction of approach for split maps

If the deformation parameters approached infinity in the opposite direction, i.e.

$$\pi^{\perp}_{\mu_e}(-(1,0)) \in \mathbb{R}^2/\langle \mu_e \rangle$$

then as $\tau_\nu \to \infty$ we would obtain the following different limit map :

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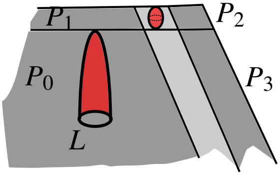


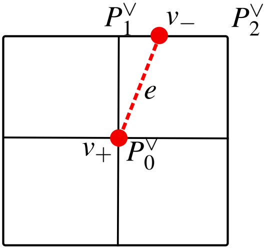


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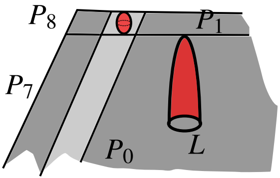
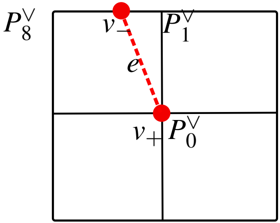
To prevent an over-count we should not count both this map, and the one in the previous page. Therefore the **direction of approach** $\eta$ is part of the datum of a split map, which is similar to a choice of Morse function.

# Example : Split disk, continued




Slope of $e$:
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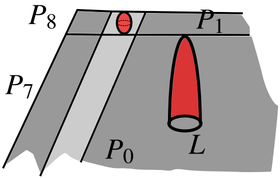
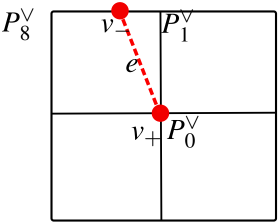
- Direction of approach $= \eta := \pi_{\mu_e}^{\perp}(1, 0) \in \mathbb{R}^2/\langle \mu_e \rangle$.

# Example : Split disk, continued

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$$\textbf{Discrepancy cone } \mathcal{W} := \pi^\perp_{\mu_e}(\{\mathcal{T}(v_+) - \mathcal{T}(v_-)\}) = \pi^\perp_{\mu_e}(\{(1,0)t : t \geq 0\}) \subset \mathbb{R}^2/\mu_e.$$

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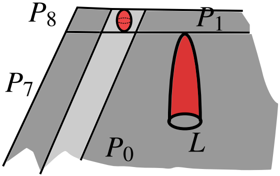


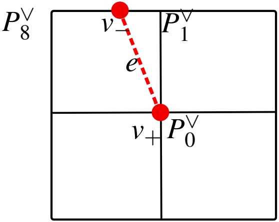


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- Remark : $\mathcal{W}$ is a cone containing $\eta$.

The remark is true in general :

**Result**

*If a sequence $u_\nu$ of $\nu\eta$-deformed maps converge to $u$ as $\nu \to \infty$. Then the set of discrepancies $\mathcal{W}_u$ is a cone containing $\eta$.*

# Definition of a split disk

**Definition**

Given a direction of approach $\eta \in \mathbb{R}^n/\mu_e$, a split disk $u : C \to \mathcal{X}$

- is a version of a broken map with no matching condition on the edge $e$,
- and for which the set of discrepancies of tropical weights across $e$

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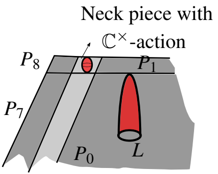


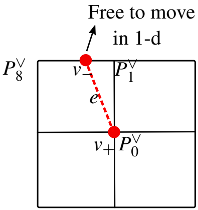


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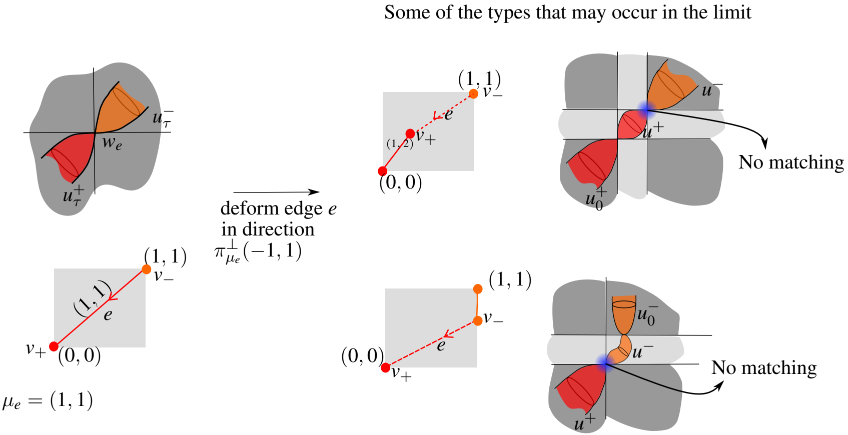
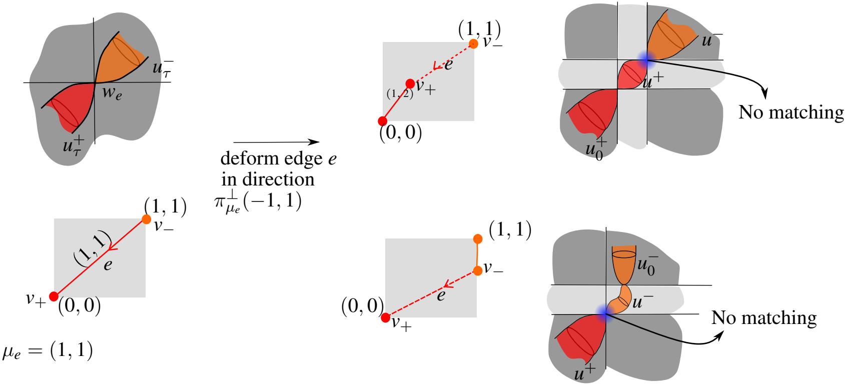
# Example of split map : 2 dimensions



# Example of split map : 2 dimensions

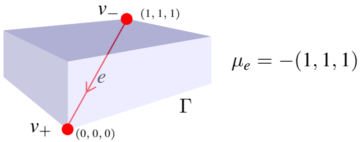
Some of the types that may occur in the limit




In both these examples $\dim \mathcal{W} = 1$, and therefore it is a top-dimensional cone in $\mathbb{R}^2/\langle \mu_e \rangle$.

# An example in 3 dimensions

Suppose we deform maps modelled on the graph $\Gamma$ in the direction $\eta := \pi^{\perp}_{(1,1,1)}(2,1,0) \in \mathbb{R}^3/\langle(1,1,1)\rangle$.



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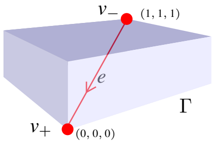
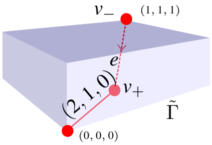



$\mu_e = -(1,1,1)$

A possible type of limit map :



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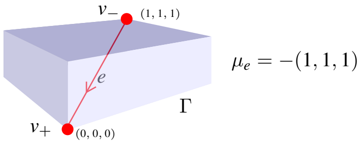


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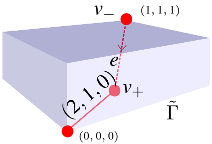


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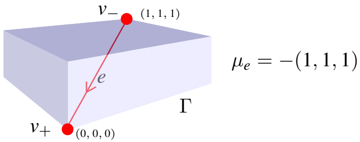


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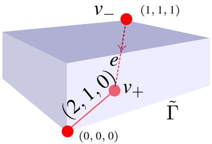


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- To get around this issue, we take the direction of approach to be 'generic'.

# Deforming in a generic direction

Let us deform maps modelled on the graph $\Gamma$ in the direction
$\eta := \pi^{\perp}_{(1,1,1)}(r, 1, 0) \in \mathbb{R}^3/\langle(1, 1, 1)\rangle$,
where $1 < r < 2$ is a fixed irrational number.

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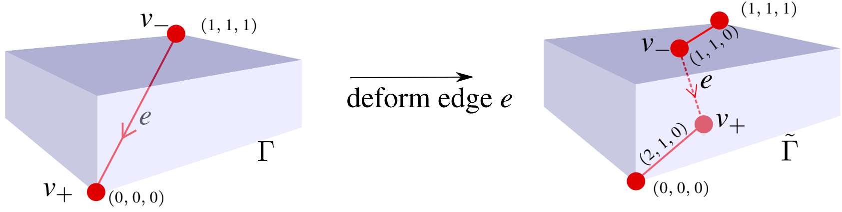
One of the possible limit maps is $\tilde{u}$ modelled on the tropical graph is $\tilde{\Gamma}$ :




The discrepancy cone is
$\mathcal{W} = \{\mathcal{T}(v_+) - \mathcal{T}(v_-)\} = \{(2, 1, 0)t_1 + (1, 1, 0)t_2 : t_1, t_2 \geq 0\}$,
which contains the direction of deformation $\eta$, and is top-dimensional in
$\mathbb{R}^3/\langle(1, 1, 1)\rangle$.

# Application: unobstructedness

**Theorem (VW)**

*Suppose $\mathcal{X}$ is a broken manifold. Let $X_{P_0} \subset \mathcal{X}$ be a toric component. That is, the tropical moment map is an honest moment map on $X_{P_0}$. For any Lagrangian torus orbit $L \subset X_{P_0}$, $CF_{\text{brok}}(\mathcal{X}, L)$ is unobstructed.*

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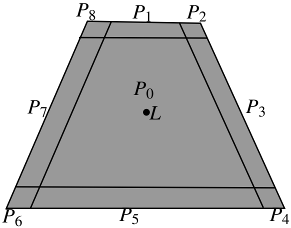

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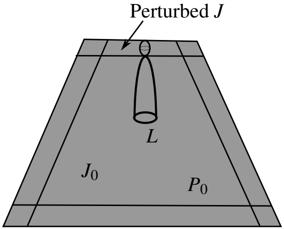


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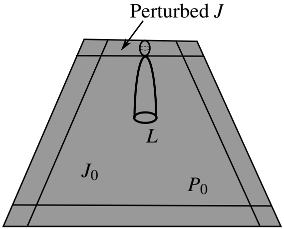


For a split map $u$, the cone condition implies that for any $t \in T_{\mathbb{C}}$, the tropical symmetry orbit of $u$ contains a $t$-deformed map.
Consequently, in a split map, the moduli space of the disk part is invariant under the action of the compact torus, leading to unobstructedness.