Isolated singularities, minimal discrepancy and exact fillings

Michael B. Rothgang (he/him)

Symplectic geometry group
Humboldt-Universität zu Berlin

Working geometry group seminar
February 22, 2021

Outline

- Motivation: $\mathbb{RP}^{2n-1}$ is not exactly fillable
- Background: varieties, isolated singularities and their links
- Main results: minimal discrepancy and highest minimal index
- Outline of proof

Exact fillability of projective space

hierarchy of symplectic fillings: in order of strictness,

$$\text{tight} < \text{weak} < \text{strong} < \text{exact} < \text{Stein} = \text{Weinstein}.$$

Theorem (Zhou 2020)

$(\mathbb{RP}^{2n-1}, \xi_{std})$ is not exactly fillable for $n \neq 2^k$.

Consider the action of $\mathbb{Z}_k$ on $\mathbb{C}^n$ (multiply by $e^{2\pi i/k}$ in each component)

Theorem (Zhou 2020)

If k is prime and satisfies (an topological condition which implies $n > k$), the quotient $(\mathbb{S}^{2n-1}/\mathbb{Z}_k, \xi_{std})$ has no exact filling.

Exact fillability of projective space: about Zhou's proof

Theorem (Zhou 2020)

If k is prime and satisfies (an topological condition which implies $n > k$), the quotient $(\mathbb{S}^{2n-1}/\mathbb{Z}_k, \xi_{std})$ has no exact filling.

Proof outline.

- If W is an exact filling of $(\mathbb{S}^{2n-1}/\mathbb{Z}_k, \xi_{\text{std}})$ for $n > k$, $\oplus_i H^{2i}(W;\mathbb{R}) \leq k$ and $\oplus_i H^{2i+1}(W;\mathbb{R}) \leq k-2$. Uses neck-stretching + spectral sequence for a clever filtration of SH.
- Using the top. assumption, deduce a contradiction □

Symplectic part uses only $n \geq k+1$!

Putting Zhou's proof in context

- $\mathbb{C}^n/\mathbb{Z}_k$ is an (affine) algebraic variety, with an isolated singularity at 0
- $\mathbb{S}^{2n-1}/\mathbb{Z}_k$ is the *link* of the singularity at 0

Miracle

$n \geq k+1 \Leftrightarrow 0$ is a *terminal singularity* of $\mathbb{C}^n/\mathbb{Z}_k$.

Conjecture (Zhou 2020)

If $G \leqslant \mathrm{U}(n)$ finite and $\mathbb{C}^n/G$ has a terminal singularity at 0, its link has no (symp. aspherical or Calabi-Yau) filling.

Algebraic geometry concepts: algebraic varieties

- (complex) **affine space** is $A^n := \{(a_1, \dots, a_n) : a_i \in \mathbb{C}\}$
- **affine (algebraic) variety**

$$X = V(f_1, \dots, f_k) = \{a \in A^n : f_1(a) = \cdots = f_k(a) = 0\}$$

  for $f_k \in \mathbb{C}[x_1, \dots, x_n]$
- equivalently, consider $R := k[t_1, \dots, t_n]/\langle f_1, \dots, f_k \rangle$ is a finitely generated $\mathbb{C}$-algebra, coordinate-free definition
- X is **irreducible** iff there are no algebraic sets $Y, Z \subset X$ s.t. $X = Y \cup Z$.

Algebraic geometry concepts: singularities

Let $X = V(\langle g_1, \dots, g_r \rangle) \subset A^n$ be an algebraic variety.

- $a \in X$ is **regular** iff the Jacobian $(\frac{\partial g_i}{\partial x_j}(a))$ has maximal rank, otherwise a singular point or **singularity**
- **tangent space** of $a \in X$ is $T_a X = \{v \in \mathbb{C}^n : J(a)v = 0\}$, where $J(a) = (\frac{\partial g_i}{\partial x_j}(a))_{ij}$ is the Jacobian of the g_i
- X has **dimension** $\dim X = n - \mathrm{rk}(J(a)) = n - \dim T_a X$, where $a \in X$ is any regular point.
- **singular set** $\mathrm{Sing}(X) = \{a \in X : \text{singular}\} \subset X$ is (Zariski) closed proper subset, hence an algebraic subvariety

$\Rightarrow$ $X \setminus \mathrm{Sing}(X) \subset X$ is an open dense subset

Key concepts: link of a singularity

$A \subset \mathbb{C}^N$ irreducible affine (algebraic) variety with $\dim_{\mathbb{C}} A = n$
$0 \in A$ isolated singularity (perhaps smooth, i.e. a regular point)

- **link** of A is $L_A := A \cap \{\sum_{i=1}^N |z_i|^2 = \epsilon^2\}$ for small $\epsilon > 0$.
- **Fact.** L_A depends only on the germ of A near 0; in particular, L_A is independent of the choice of ϵ.
- **Fact.** L_A is a differentiable manifold of (real) dimension $2n - 1$.
- **Observation.** Near 0, A is homeomorphic to a cone over L_A.
- **Trivial Example.** If A is smooth at 0, then L_A is diffeo to a sphere.
- **Fact.** $\xi_A := \xi_{\text{std}}|_{TL_A}$ is a contact structure on L_A.
- Observe that $\xi_A = TL_A \cap J_{\text{std}}(TL_A)$

A peek at different kinds of singularities

- (regular points)
- **normal** singularities $\longrightarrow$ normalisation (then: $\operatorname{codim} \operatorname{Sing}(X) \geq 2$)
- **topologically smooth** singularities: $L_A \cong_{\text{diff}} \mathbb{S}^{2n-1}$
- For an isolated singularity in $\dim_{\mathbb{C}}(A) \geq 2$,

  num. $\mathbb{Q}$-Gorenstein $\supset$ $\mathbb{Q}$-Gorenstein $\supset$ complete intersection sing.;

  0 is numerically $\mathbb{Q}$-Gorenstein $\Leftrightarrow c_1(\xi_A) = c_1(TA|_{L_A})$ is torsion.
- **canonical** singularity: numerically Q-Gorenstein and $\operatorname{md}(A, 0) \geq 0$
- **terminal** singularity: numerically Q-Gorenstein and $\operatorname{md}(A, 0) > 0$

Capturing local behaviour: local rings

- type of singularity is "local behaviour"
  capture local behaviour near $x \in X$ using the **local ring at** x
- R non-zero unital communitative ring
  - $I \subset R$ is an **ideal** of R iff $I \leqslant (R, +)$ and $ri = ir \in I$ for all $i \in I, r \in R$
  - a proper ideal $I \subset R$ is **prime** iff $ab \in I$ implies $a \in I$ or $b \in I$
  - a proper ideal $I \subset R$ is **maximal** iff $\nexists$ ideal J s.t. $I \subsetneq J \subsetneq R$
  - maximal ideals are prime
- **Fact.** For $a = (a_1, \ldots, a_n) \in A^n$, each $\mathfrak{m}_a := \langle x_1 - a_1, \ldots, x_n - a_n \rangle \subset \mathbb{C}[x_1, \ldots, x_n]$ is a maximal ideal of $\mathbb{C}[x_1, \ldots, x_n]$, and every maximal ideal is of this form.

Capturing local behaviour: local rings

- type of singularity is "local behaviour"
  capture local behaviour near $x \in X$ using the **local ring at** x
- R non-zero unital communitative ring
  - $I \subset R$ is an **ideal** of R iff $I \leqslant (R, +)$ and $ri = ir \in I$ for all $i \in I, r \in R$
  - a proper ideal $I \subset R$ is **prime** iff $ab \in I$ implies $a \in I$ or $b \in I$
  - a proper ideal $I \subset R$ is **maximal** iff $\nexists$ ideal J s.t. $I \subsetneq J \subsetneq R$
  - maximal ideals are prime
- **Fact.** For $a = (a_1, \ldots, a_n) \in A^n$, each $\mathfrak{m}_a := \langle x_1 - a_1, \ldots, x_n - a_n \rangle \subset \mathbb{C}[x_1, \ldots, x_n]$ is a maximal ideal of $\mathbb{C}[x_1, \ldots, x_n]$, and every maximal ideal is of this form.
- given a prime ideal $\mathfrak{p} \subset R$, **localisation** at $\mathfrak{p}$ is $R_\mathfrak{p} := \{r/s : r \in R, s \in R \setminus \mathfrak{p}\}/\sim$, equivalence by cancellation.
- **Definition.** The **local ring** of a variety $X \subset A^n$ at $a \in X$ is the localisation $k[X]_{\mathfrak{m}_a}$ of the coordinate algebra $k[X]$ of X at the maximal ideal $\mathfrak{m}_a$ corresponding to a.
- local ring $\mathcal{O}_p(X)$ encodes local properties of X at p

Normal singularities

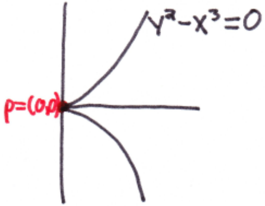
- **Definition**. Let $\phi : R \to S$ be a ring homomorphism ("S is an R-algebra"). $x \in S$ is **integral** over R iff $f(x) = 0$ for some monic polynomial $f \in R[t]$
- **Fact.** The set of integral elements of S is a subalgebra of S, called the **normalisation** of S.
- **Definition.** An integral domain R is **normal** iff it equals its normalisation in its quotient field.
- **Definition.** An affine variety X is **normal at** $x \in X$ if the local ring at this point is normal. X is **normal** iff it is normal at every point.

Normal singularities (cont.)

X irreducible affine variety

- **Definition.** X is **normal at** $x \in X$ if the local ring at this point is normal. X is **normal** iff it is normal at every point.
- **Theorem.** X is normal at every regular point.
- **Theorem.** The **singular locus** $\mathrm{Sing}(X) = \{a \in X : X \text{ singular at } a\}$ is a proper algebraic subset of X.
- **Proposition.** If X is normal, $\dim \mathrm{Sing}(X) \leq \dim X - 2$.

Normal singularities: geometric intuition



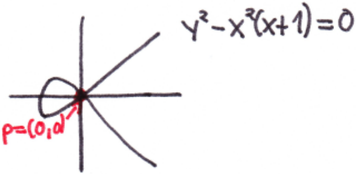


Figure: Pictures reproduced from Eisenbud: Commutative algebra (1995), p. 128.

- Consider $f = y^2 - x^3$ resp. $f = y^2 - x^2(x+1) \in \mathbb{C}[x, y]$
- compute: $X = V(f)$ has one singular point, $p = (0, 0)$
- consider $y/x \in \mathcal{O}_p(X)$: bounded along X near p
- algebraically: y/x is integral, e.g. $(y/x)^2 - x = 0$ (left)

Normal singularities: geometric intuition

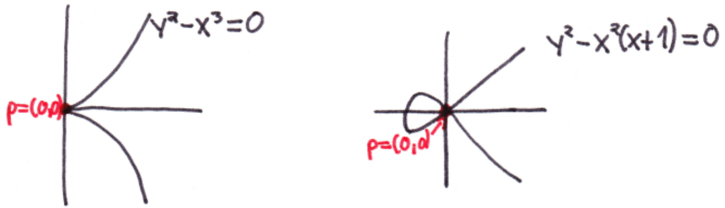


Figure: Pictures reproduced from Eisenbud: Commutative algebra (1995), p. 128.

- Consider $f = y^2 - x^3$ resp. $f = y^2 - x^2(x+1) \in \mathbb{C}[x, y]$
- compute: $X = V(f)$ has one singular point, $p = (0, 0)$
- consider $y/x \in \mathcal{O}_p(X)$: bounded along X near p
- algebraically: y/x is integral, e.g. $(y/x)^2 - x = 0$ (left)

Theorem. An element $p(x)/q(x)$ of the quotient field is integral over $\mathbb{C}[X]$ iff each $x \in X$ has a neighbourhood U s.t. $|\frac{p(x)}{q(x)}|$ is bounded at all points of U where q is non-zero.

Normalisation and resolution of varieties

- normalise a variety X using its coordinate algebra $R := \mathbb{C}[X]$
- **Recall.** anti-equivalence of categories

$$\{\text{affine algebraic varieties}\} \longleftrightarrow \{\text{finitely generated } \mathbb{C}\text{-algebras}\},$$
$$\text{variety } X \longmapsto \text{coordinate algebra } \mathbb{C}[X]$$

- normalisation $\widetilde{R}$ of R corresponds to the **normalisation** $\widetilde{X}$ of X
- natural inclusion $R \hookrightarrow \widetilde{R}$ into normalisation $\widetilde{R}$
- induces a birational map $\pi\colon \widetilde{X} \to X$
- A **resolution** of an algebraic variety X is a non-singular variety $\widetilde{X}$ together with a proper birational map $\pi\colon \widetilde{X} \to X$.
- **Theorem (Hironaka '64).** Every variety has a resolution.

Normalisation: geometric intuition

consider $X = V(f)$ for $f = y^2 - x^3$ or $f = y^2 - x^2(x+1) \in \mathbb{C}[x, y]$

Figure: Normalisation of the curves from the previous example.
Pictures reproduced from Eisenbud: Commutative algebra (1995), p. 141.

algebraically: normalisation of $R = \mathbb{C}[X]$ is $\mathbb{C}[t]$
geometrically: normalisation $\widetilde{X} \cong \mathbb{C}$

Known results about singularities and their links

- **Theorem (Mumford '61).** In complex dimension two, every normal topologically smooth singularity is smooth.
- Many counterexamples in dimension ≥ 3, such as $A := \{x^2 + y^2 + z^2 + w^2 = 0\} \subset \mathbb{C}^4$.
- **Theorem (Ustilovski '99).** For each $m > 0$, there are infinitely many singularities with links diffeomorphic to $\mathbb{S}^{4m+1}$, but not contactomorphic.
- **Theorem (Kwon-van Koert '16).** For weighted homogeneous hypersurface singularities $\{\sum z_j^{k_j} = 0\}$, (L_A, ξ_A) determines whether $\sum_j 1/k_j > 1 \Leftrightarrow 0$ is a canonical singularity.

The highest minimal index

- $(C^{2n-1}, \xi = \ker\alpha)$ co-oriented contact manifold
  $\rightarrow$ symplectic vector bundle $(d\alpha|_\xi, \xi)$
- first Chern class $c_1(\xi) := c_1(\xi, J) \in H^2(C;\mathbb{Z})$ for J compatible acs on $d\alpha|_\xi$
- Suppose $Nc_1(\xi) = 0$ and $H^1(C;\mathbb{Q}) = 0$
  $\longrightarrow$ Conley-Zehnder index $CZ(\gamma) \in \frac{1}{N}\mathbb{Z}$ of a Reeb orbit γ
- **lower SFT index**

$$\mathrm{lSFT}(\gamma) := CZ(\gamma) + (n-3) - \frac{1}{2}\dim\ker(D_{\gamma(0)}\phi_L|_{xi} - id)$$

- **minimal SFT index** $\mathrm{mi}(\alpha) := \inf_\gamma \mathrm{lSFT}(\gamma)$
- **highest minimal SFT index** $\mathrm{hmi}(C,\xi) := \sup_\alpha \mathrm{mi}(\alpha)$.
- **Observation.** $\mathrm{hmi}(C,\xi)$ is a contact invariant.

Main results: relating minimal discrepancy and hmi

Main Theorem (McLean '15)

Suppose A has a normal isolated singularity at 0 that is numerically $\mathbb{Q}$-Gorenstein with $H^1(L_A; \mathbb{Q}) = 0$. Then,

- if $\mathrm{md}(A, 0) \geq 0$ then $\mathrm{hmi}(L_A, \xi_A) = 2\,\mathrm{md}(A, 0)$,
- if $\mathrm{md}(A, 0) < 0$, then $\mathrm{hmi}(L_A, \xi_A) < 0$.

Main results: relating minimal discrepancy and hmi

Main Theorem (McLean '15)

Suppose A has a normal isolated singularity at 0 that is numerically $\mathbb{Q}$-Gorenstein with $H^1(L_A; \mathbb{Q}) = 0$. Then,

- if $\mathrm{md}(A, 0) \geq 0$ then $\mathrm{hmi}(L_A, \xi_A) = 2\,\mathrm{md}(A, 0)$,
- if $\mathrm{md}(A, 0) < 0$, then $\mathrm{hmi}(L_A, \xi_A) < 0$.

- **Recall.** 0 is canonical if $\mathrm{md}(A, 0) \geq 0$, terminal if $\mathrm{md}(A, 0) > 0$

Main results: relating minimal discrepancy and hmi

Main Theorem (McLean '15)

Suppose A has a normal isolated singularity at 0 that is numerically $\mathbb{Q}$-Gorenstein with $H^1(L_A;\mathbb{Q}) = 0$. Then,

- if $\mathrm{md}(A,0) \geq 0$ then $\mathrm{hmi}(L_A, \xi_A) = 2\,\mathrm{md}(A,0)$,
- if $\mathrm{md}(A,0) < 0$, then $\mathrm{hmi}(L_A, \xi_A) < 0$.

- **Recall.** 0 is canonical if $\mathrm{md}(A,0) \geq 0$, terminal if $\mathrm{md}(A,0) > 0$
- Conley-Zehnder indices on L_A determine whether 0 is canonical or terminal

Main results: relating minimal discrepancy and hmi

Main Theorem (McLean '15)

Suppose A has a normal isolated singularity at 0 that is numerically $\mathbb{Q}$-Gorenstein with $H^1(L_A; \mathbb{Q}) = 0$. Then,

- if $\text{md}(A, 0) \geq 0$ then $\text{hmi}(L_A, \xi_A) = 2\,\text{md}(A, 0)$,
- if $\text{md}(A, 0) < 0$, then $\text{hmi}(L_A, \xi_A) < 0$.

- **Recall.** 0 is canonical if $\text{md}(A, 0) \geq 0$, terminal if $\text{md}(A, 0) > 0$
- Conley-Zehnder indices on L_A determine whether 0 is canonical or terminal

Main results: relating minimal discrepancy and hmi

- **Definition.** If (M, ξ) is contactomorphic to some link (L_A, ξ_A), it is **Milnor fillable**, and A is a **Milnor filling** of M.
- **Example.** $(\mathbb{S}^{2n-1}, \xi_{\text{std}})$ is Milnor fillable; its Milnor filling is $\mathbb{C}^n$.
- **Corollary.** If A is normal and (L_A, ξ) is contactomorphic to $(\mathbb{S}^5, \xi_{\text{std}})$, then A is smooth at 0.

$\Rightarrow$ $(\mathbb{S}^5, \xi_{\text{std}})$ has a unique smooth Milnor filling up to normalization.
Extends Mumford's results to complex dimension three.

- **Observation.** Milnor fillable contact structures are strongly fillable.
- **Conjecture (Shukorov '02).** If A is normal and numerically $\mathbb{Q}$-Gorenstein with $\text{md}(A, 0) = n - 1$, then A is smooth at 0.
- **Corollary.** *If* the conjecture holds, A is normal and $(L_A, \xi_A) \cong (\mathbb{S}^{2n-1}, \xi_{\text{std}})$ (any n), then A is smooth at 0.

Canonical bundles and $\mathbb{Q}$-Cartier divisors

- **Definition.** X *non-singular* algebraic variety with $\dim_C X = n$. The **canonical bundle** of X is $\Omega = \Lambda^n T^* X$.
- X normal variety. A **(Weil) $\mathbb{Q}$-divisor** is a finite formal linear combination $D = \sum_{j=1}^{k} a_j E_j$ with $a_j \in \mathbb{Q}$, $E_j \subset X$ irreducible codimension 1 subvariety.
- A $\mathbb{Q}$-divisor D is $\mathbb{Q}$-**Cartier** if we can choose the E_j to be locally defined by one equation.
- **Fact.** If X is non-singular, every $\mathbb{Q}$-divisor is $\mathbb{Q}$-Cartier.
- **Fact.** Every line bundle on a normal variety X is the class of some Cartier divisor.

Numerically $\mathbb{Q}$-Gorenstein singularities

A (irreducible) algebraic variety with an isolated singularity at 0

- A **smooth normal crossings divisor** is a Cartier divisor whose components only intersect transversely. Near each point, the divisor looks like the intersection of coordinate hyperplanes.
- Take a resolution $\pi\colon \widetilde{A} \to A$ of A s.t. $\pi^{-1}(0) = \bigcup_i E_i$ for smooth normal crossing divisors E_i, and π is an isomorphism away from these divisors.
- **Definition.** A is **numerically $\mathbb{Q}$-Gorenstein** iff there exists a $\mathbb{Q}$-Cartier divisor $K^{\mathrm{num}}_{\widetilde{A}/A} := \sum_j E_j$ s.t. $C \cdot (K^{\mathrm{num}}_{\widetilde{A}/A} - K_{\widetilde{A}}) = 0$ for any projective algebraic curve $C \subset \pi^{-1}(0)$.

Defining the minimal discrepancy

- **Definition.** A is **numerically** $\mathbb{Q}$-**Gorenstein** iff there exists a $\mathbb{Q}$-Cartier divisor $K^{\mathrm{num}}_{\widetilde{A}/A} := \sum_j E_j$ s.t. $C \cdot (K^{\mathrm{num}}_{\widetilde{A}/A} - K_{\widetilde{A}}) = 0$ for any projective algebraic curve $C \subset \pi^{-1}(0)$.
- **Fact.** The $a_j \in \mathbb{Q}$ are unique; a_j is called the **discrepancy** of E_j.
- **Definition.** The **minimal discrepancy** $\mathrm{md}(A, 0)$ of A is the infimum of a_j over all resolutions π.
- **Proposition.** If π is a fixed resolution, not the identity, then

$$\mathrm{md}(A,0) = \begin{cases} \min_j a_j & \text{if } a_j \geq -1 \ \ \forall j \in \{1, \dots, l\} \\ -\infty & \text{otherwise} \end{cases}$$

If A is smooth at 0, we have $\mathrm{md}(A, 0) = \dim_{\mathbb{C}} A - 1$.

Strategy of McLean's proof

- easier part: $\mathrm{hmi}(L_A, \xi_A) \geq 2\,\mathrm{md}(A, 0)$
- harder parts: If $\mathrm{md}(A, 0) \geq 0$ then $\mathrm{hmi}(L_A, \xi_A) \leq 2\,\mathrm{md}(A, 0)$; if $\mathrm{md}(A, 0) < 0$ then $\mathrm{hmi}(L_A, \xi_A) < 0$.
- model case: A is the cone over a projective variety X; we skip explaining the proof in the general case

Model case: cone singularity

- Model case: $A \subset \mathbb{C}^N$ is the cone of a smooth connected projective variety $X \subset \mathbb{CP}^{N-1}$
- resolution $\widetilde{A}$ by blowing up at the origin; $\mathcal{O}(-1) = (\tilde{\pi} \colon \widetilde{A} \to X)$ is the tautological line bundle
- numerically $\mathbb{Q}$-Gorenstein $\Leftrightarrow c_1(K_{\widetilde{A}}|_{L_A}; \mathbb{Q}) = 0$
- $L_A \to \widetilde{A} \setminus X$ is a homotopy equivalence: $c_1(K_{\widetilde{A}}|_{\widetilde{A} \setminus X}; \mathbb{Q}) = 0$
- for some $N > 0$, $K_{\widetilde{A}}^{\otimes N}$ has a smooth section s which is transverse outside a compact set
- discrepancy of A is the $a \in \mathbb{Q}$ satisfying

$$[s^{-1}(0)] = aN(X) \in H_{2n-2}(\widetilde{A}; \mathbb{Q}) = H_{2n-2}(X; \mathbb{Q}),$$

minimal discrepancy $\mathrm{md}(A, 0)$ is a if $a \geq -1$, otherwise $-\infty$.

Model case: proof of easier statement

want to show: $\mathrm{hmi}(L_A, \xi_A) \geq 2\,\mathrm{md}(A, 0)$

- goal: find a contact form α_A for ξ_A s.t. $\mathrm{md}(\alpha_A) = 2\,\mathrm{md}(A, 0)$
- $\mathcal{O}(-1)$ is a Hermitian line bundle,
  link L_A is the radius ϵ circle bundle on $\mathcal{O}(-1)$
- $\pi = \tilde{\pi}|_{L_A}$ makes L_A a circle bundle over X
- consider the contact form $\alpha_A := -\frac{1}{4\pi\epsilon^2} d^c(\sum_j |z_j|^2)|_{L_A}$
- all Reeb orbits are of the form
  $\gamma : \mathbb{R}/k\mathbb{Z} \to L_A, \gamma(t) = B(t, p)$ for $k \in \mathbb{Z}^+, p \in L_A$

Model case: proof of easier statement

want to show: $\mathrm{hmi}(L_A, \xi_A) \geq 2\,\mathrm{md}(A, 0)$

- goal: find a contact form α_A for ξ_A s.t. $\mathrm{md}(\alpha_A) = 2\,\mathrm{md}(A, 0)$
- $\mathcal{O}(-1)$ is a Hermitian line bundle,
  link L_A is the radius ϵ circle bundle on $\mathcal{O}(-1)$
- $\pi = \tilde{\pi}|_{L_A}$ makes L_A a circle bundle over X
- consider the contact form $\alpha_A := -\frac{1}{4\pi\epsilon^2} d^c(\sum_j |z_j|^2)|_{L_A}$
- all Reeb orbits are of the form
  $\gamma : \mathbb{R}/k\mathbb{Z} \to L_A, \gamma(t) = B(t, p)$ for $k \in \mathbb{Z}^+, p \in L_A$
- compute: $CZ(\gamma) = 2(a+1)k$
  - F be the fiber containing γ, s_F a non-zero section of $K_{\widetilde{A}}^{\otimes N}$.
  - define
    $Q_F : \mathbb{R}/k\mathbb{Z} \to U(1), t \mapsto [z \mapsto P(B_K(t, s_F(\gamma(0)))/s_F(\gamma(t)))]$
  - compute: $\deg Q_F = -kN$, $s^{-1}(0)|_F] = aN$

$\Rightarrow$ $\mathrm{ISFT}(\gamma) = 2(a+1)k - \frac{1}{2}(2n-2) + (n-3) = 2(a+1)k - 2$

$\Rightarrow$ $\mathrm{mi}(a_\alpha) = 2\,\mathrm{md}(A, 0)$

Model case: proof of harder statement

to show: any contact form β for ξ_A admits a Reeb orbit γ with $\mathrm{ISFT}(\gamma) < 0$ or $\mathrm{ISFT}(\gamma) \leq 2\,\mathrm{md}(A,0)$

- Compactify $\tilde{\pi}\colon \tilde{A} \to x$ to a $\mathbb{CP}^1$-bundle $\check{S} := P(\widetilde{A} \oplus \mathbb{C})$.
- embed (L_A, ξ_A) as a contact hypersurface inside $\check{S}$.
- neck-stretching: shows L_A admits a Reeb orbit
  in fact, limiting curve has negative ends asymptotic to Reeb orbits γ_i,
- lives in a moduli space of virtual dimension
  $2\,\mathrm{md}(A,0) - \sum_i \mathrm{ISFT}(\gamma_i) \geq 0$
- Thus, $2\,\mathrm{md}(A,0) < 0$ implies $\mathrm{ISFT}(\gamma_i) < 0$ for some i;
  $\mathrm{md}(A,0) \geq 0$ implies $\mathrm{ISFT}(\gamma_i) \leq 2\,\mathrm{md}(A,0)$ for some i.

Technical apparatus for the proof

- contact-type hypersurface L_A in symplectic manifold $\check{S}$
- symplectic dilation (similar procedure to neck-stretching) $\to$ contact embedding of L_A into $\check{S}$
- Gromov-Witten theory: L_A admits a special holomorphic curve (dim $M \leq 6 \to$ rigorous transversality results)
- neck-stretching: L_A admits a Reeb orbit
- dimension computation

Neck-stretching step

(M, ω) compact symplectic manifold which has a contact type hypersurface $C \subset M$ so that

1. $M \setminus C$ has two connected components M_- and M_+.
2. There are codimension 2 submanifolds $Q_\pm \subset M_\pm$, and $[A] \in H_2(M; \mathbb{Z})$ s.t. $[A] \cdot [Q_\pm] \neq 0$.
3. For every compatible acs J, there exists a compact genus 0 J-holomorphic curve $u \colon \Sigma \to M$ representing [A].

Then C has at least one Reeb orbit.

Proof sketch.

- Choose a collar neighbourhood of C and a curve u as in (3)
- Stretched curves u_i converge to some s. inj. limit u_∞
- since $[u] = A$, each u_i must intersect the manifolds $Q_\pm$
- in particular, u_i intersects M_- and M_+, hence $u_i|_{u^{-1}(M_+)}$ is a proper map with non-compact domain for all i

$\Rightarrow$ the domain of u_∞ is not compact; C has a Reeb orbit. □

Gromov-Witten invariants

Theorem

Let (M,ω) compact symplectic manifold, $[A] \in H_2(M;\mathbb{Z})$ satisfying $c_1(M,\omega)([A]) + n - 3 = 0$. There is an invariant $GW_0(M,[A],\omega) \in \mathbb{Q}$ satisfying the following properties,

1. *If $GW_0(M,[A],\omega) \neq 0$, for any compactible acs J there exists a compact nodal J-holomorphic curve representing $[A]$.*
2. *Given a smooth family of symplectic forms $(\omega_t)_{t\in[0,1]}$ on M with $\omega_0 = \omega$, then $GW_0(M,[A],\omega_0) = GW_0(M,[A],\omega_1)$.*
3. *Suppose (M,ω) admits a compatible acs J so that (M,J) is biholomorphic to a complex manifold and for all genus 0 J-holomorphic curves $u\colon \Sigma \to M$, the domain of u is biholomorphic to $\mathbb{CP}^1$ and u^*TM is a direct sum of complex line bundles of degree ≥ -1.*
   Then $GW_0(M,[A],\omega)$ counts unparametrized connected genus 0 J-holomorphic curves representing $[A]$. □

Conclusions

1. Algebro-geometric properties of an isolated singularity relate to symplectic filling properties of its link.
2. The link of an isolated singularity in an affine variety carries a contact structure.
3. The minimal discrepancy is strongly related to computing Conley-Zehnder indices on the link. For instance, this computations determines if the singularity is canonical or terminal.