Differential Geometry III
Gauge Theory

Prof. Dr. Thomas Walpuski
Humboldt-Universität zu Berlin

2021-12-01

Contents

1 Covering theory

Here are some good references for covering theory: Jänich [Jän05, Kapitel 9], Fulton [Ful95, Parts VI and VII], May [May99, §1-§4], and Hatcher [Hat02, §1].

1.1 Covering maps

Definition 1.1. Let X, B be topological spaces. A **covering map** is a continuous map $p\colon X \to B$ such that: for every $b \in B$ there are an open neighborhood U of $b \in B$, a discrete space D, and a homeomorphism $\tau\colon p^{-1}(U) \to U \times D$ such that

$$\mathrm{pr}_1 \circ \tau = p.$$

X is a **covering space** of B if there is a covering map $p\colon X \to B$. •

Example 1.2. Let X be a topological space. Let D be a discrete space. The map $\mathrm{pr}_1\colon X \times D \to X$ is a covering map. ♠

Example 1.3. The projection maps $p\colon \mathbf{R} \to S^1 := \mathbf{R}/2\pi Z$ and $q\colon S^n \to \mathbf{R}P^n := S^n/\{\pm 1\}$ are covering maps. ♠

Example 1.4. For $k \in \mathbf{N}$ the map $p_k\colon S^1 \to S^1$ defined by $p_k([x]) = [kx]$ is a covering map. ♠

Example 1.5. `some covering spaces of the figure eight` ♠

Example 1.6. The map $\exp\colon \mathbf{C} \to \mathbf{C}^\times$ is a covering map. ♠

Example 1.7. Let $d \in \mathbf{N}_0$. Let $P_d \subset \mathbf{C}[x]$ be the subset of polynomials with distinct roots. Set $R_d := \{(x, p) \in \mathbf{C} \times P : p(x) = 0\}$. The map $\mathrm{pr}_2\colon R_d \to P_d$ is a covering map. ♠

Example 1.8. Let $n \in \mathbf{N}$. Let $p \in \mathbf{N}$ and $q_1, \dots, q_n \in \mathbf{Z}$ such that

$$\gcd(p, q_i) = 1.$$

Identify $\mathbf{R}^{2n} = \mathbf{C}^n$ and define $\phi \in \mathrm{SO}(2n)$ by

$$\phi(z_1, \dots, z_n) := (e^{2\pi i q_1/p} z_1, \cdots, e^{2\pi i q_n/p} z_1).$$

By construction, the subgroup $\langle \phi \rangle \subset \mathrm{SO}(2n)$ is cyclic of order p and acts freely on S^{2n-1}. The **lens space** $L(p; q_1, \dots, q_n)$ is the quotient

$$L(p; q_1, \dots, q_n) := S^{2n-1}/\langle \phi \rangle.$$

The projection map $p\colon S^{2n-1} \to L(p; q_1, \dots, q_n)$ is a covering map. ♠

Example 1.9. Denote by $\mathbf{H}$ the $\mathbf{R}$–algebra of the quaternions. Set $\mathrm{Sp}(1) := \{q \in \mathrm{Sp}(1) : |q| = 1\}$. The map $\mathrm{Ad}\colon \mathrm{Sp}(1) \to \mathrm{SO}(\mathrm{Im}\,\mathbf{H})$

$$\mathrm{Ad}(q)x := qxq^*$$

is a covering map. ♠

Proposition 1.10. *If $p\colon X \to B$ is a proper local homeomorphism, then it is a covering map.* ■

Exercise 1.11. Prove Proposition 1.10. ?

Proposition 1.12. *Let $p\colon X \to B$ be a covering space with B connected. For every $b_0, b_1 \in B$ there is a bijection $p^{-1}(b_0) \to p^{-1}(b_1)$.*

Proof. For $b_0 \in B$ set

$$S_{b_0} := \{b \in B : \text{there is a bijection } p^{-1}(b_0) \to p^{-1}(b)\}.$$

Trivially, $b_0 \in S_{b_0}$. If $b \in S_{b_0}$ and U is an open neighborhood of $b \in B$ as in Definition 1.1, then $U \subset S_{b_0}$. Therefore, S_{b_0} is open. Finally,

$$B \backslash S_{b_0} = \bigcup_{b \notin S_{b_0}} S_b.$$

Therefore, S_{b_0} is closed. Since B is connected, $S_{b_0} = B$. ■

Definition 1.13. Let $p\colon X \to B$ be a proper covering map. The **degree of** p is the map $\deg_{\cdot}(p)\colon B \to \mathbf{N}_0$ defined by

$$\deg_b(p) := \#p^{-1}(b).$$ •

1.2 Quotients and covering maps

Definition 1.14. Let G be a topological group. Let X be a topological space.

(1) A **left action** of G on X is a continuous map $L\colon\ G \times X \to X$ satisfying

$$L(\mathbf{1}, \cdot) = \mathrm{id}_X \quad \text{and} \quad L(g, L(h, x)) = L(gh, x).$$

Define $L_g \in \mathrm{Homeo}(X)$ by $L_g := L(g, \cdot)$ and abbreviate

$$g \cdot x := L(g, x).$$

(2) The **quotient of X by the L** is the topological space

$$G\backslash X := X/{\sim_L}$$

with $x \sim_L y$ if and only if $x = g \cdot y$ for some $g \in G$.

(3) A left action L of G on X is **properly discontinuous** if every $x \in X$ has a neighborhood U such that

$$U \cap (g \cdot U) \neq \varnothing \quad \text{if and only if} \quad g = \mathbf{1} \in G.$$ •

Proposition 1.15. *Let G be a group. Let X be a topological space. Let L be a properly discontinuous left action of G on X. Set $B := G\backslash X$.*

(1) *The canonical projection map $p\colon\ X \to B$ is a covering map.*

(2) *Let S be a discrete space. Let $\lambda\colon\ G \to \mathrm{Homeo}(S) = \mathrm{Bij}(S)$ be a group homomorphism. Set*

$$Y := X \times_G S := G\backslash X \times S$$

with $g \cdot (x, s) := (g \cdot x, \lambda(g)(x))$. The canonical projection map $q\colon\ Y \to B$ is a covering map. ■

Exercise 1.16. Prove Proposition 1.15. ?

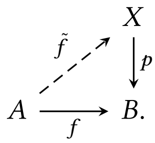
If B is not too pathological (e.g., if B is a manifold or CW complex), then (up to isomorphism) every covering map $p\colon\ X \to B$ arises from the construction in Proposition 1.15.

1.3 Lifting along covering maps, I

Definition 1.17. Let $p\colon\ X \to B$ and $f\colon\ A \to B$ be a continuous maps. A **lift of f along p** is a continuous map $\tilde{f}\colon\ A \to X$ such that

$$p \circ \tilde{f} = f.$$ •

The following diagram illustrates the situation in Definition 1.17:



The key to the theory of covering spaces is to solve the lifting problem: which maps f admit a lift along p?

Proposition 1.18. *Let $p\colon X \to B$ be a covering map. Let $f\colon A \to B$ be a continuous map. Let $a_0 \in A$ and $x_0 \in p^{-1}(f(a_0))$. If A is connected, then there is at most one lift $\tilde{f}\colon A \to X$ of f along p with $\tilde{f}(a_0) = x_0$.*

The proof is an immediate consequence of the following.

Lemma 1.19. *Let $p\colon X \to B$ be a covering map. Set*

$$X \times_B X \coloneqq \{(x, y) \in X \times X : p(x) = p(y)\} \quad \textit{and} \quad \Delta \coloneqq \{(x, x) \in X \times_B X\}.$$

The subset $\Delta \subset X \times_B X$ is open and closed.

Proof. For $x \in X$ denote by V_x an open neighborhood of $x \in X$ such that $p(V_x)$ is open and $p|_{V_x}\colon V_x \to p(V_x)$ is a homeomorphism. $(V \times V) \cap (X \times_B X)$ is an open neighborhood of $(x, x) \in X \times_B X$ and contained in Δ. Therefore, Δ is open.

Let $(x, y) \in (X \times_B X)\backslash\Delta$. Choose V_x, V_y as above with $V_x \cap V_y = \varnothing$. $(V_x \times V_y) \cap X \times_B X$ is an open neighborhood of $(x, y) \in X \times_B X$ and does not intersect Δ. Therefore, Δ is closed. ■

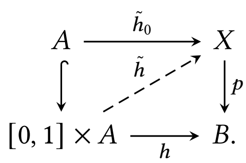
Proof of Proposition 1.18. Suppose $\tilde{f}_1, \tilde{f}_2\colon A \to X$ are lifts of f along p with $\tilde{f}_i(a_0) = x_0$. By Lemma 1.19, $S \coloneqq (\tilde{f}_1, \tilde{f}_2)^{-1}(\Delta)$ is open and closed. Since $a \in S$ and A is connected, $S = A$; hence: $\tilde{f}_1 = \tilde{f}_2$. ■

Example 1.20. The map $p_k\colon S^1 \to S^1$ does not admit a lift along the projection $p\colon \mathbf{R} \to S^1$. ♠

This illustrates that not every map $f\colon A \to B$ can be lifted along p. However, it is quite easy to see that this posssible if $A = [0, 1]$. It is quite easy to see that paths can be lifted along covering maps. In fact, this can be done in families.

Definition 1.21. Let A be a topological space. A continuous map $p\colon X \to B$ has the **homotopy lifting property (HLP) with respect to** A if for every homotopy $h\colon [0, 1] \times A \to B$ and lift $\tilde{h}_0\colon A \to X$ of $h(0, \cdot)$ there is a homotopy $\tilde{h}\colon [0, 1] \times A \to X$ which is a lift of h with $\tilde{h}(0, \cdot) = \tilde{h}_0$. •

The following diagram illustrates the situation in Definition 1.21:



Definition 1.22. A continuous map $p\colon X \to B$ is a **Hurewicz fibration** it has the HLP with respect to every topological space. •

Lemma 1.23. *If $p\colon X \to B$ is a covering map, then is a Hurewicz fibration.*

Proof. If D is a discrete space, then $\mathrm{pr}_1\colon B \times D \to B$ is a Hurewicz fibration. Consequently, every $b \in B$ has a neighborhood U such that $p|_{p^{-1}(U)}$ is a Hurewicz fibration.

Let A be a topological space. Let $h\colon [0,1] \times A \to B$ be a homotopy. For every $(t,a) \in [0,1] \times A$ choose a neighborhood $U_{t,a}$ of $h(t,a)$ as above. Since $[0,1]$ is compact, there are $0 = t_0 < t_1 < \cdots < t_n = 1$ and an open neighborhood V_a of $a \in A$ with $[t_i, t_{i+1}] \times V_a \subset h^{-1}(U_{t,a})$ for some $t \in [0,1]$. Let $\tilde{h}_0$ be a lift of $h(0,\cdot)$. Since $p|_{p^{-1}(h([t_i,t_{i+1}]\times V_a))}$ is a Hurewicz fibration, a finite induction argument constructs a lift $\tilde{h}_{V_a}$ of $h|_{[0,1]\times V_a}$ with $\tilde{h}_{V_a}(0,\cdot) = \tilde{h}_0|_{V_a}$.

By Proposition 1.18 and because $[0,1]$ is connected, $\tilde{h}_{V_a}$ and $\tilde{h}_{V_b}$ agree on $[0,1] \times (V_a \cap V_b)$. Therefore, they assemble into a lift $\tilde{h}$ of h with $\tilde{h}(0,\cdot) = \tilde{h}_0$. ■

1.4 The monodromy representation

The theory of covering maps is intricately intertwined with concept of fundamental group(oid).

Definition 1.24. Let X be a topological space. Let $\gamma, \delta\colon [0,1] \to X$ be paths.

(1) If $\gamma(1) = \delta(0)$, then the **concatenation** of γ and δ is the path $\gamma * \delta\colon [0,1] \to X$ defined by

$$(\gamma * \delta)(t) \coloneqq \begin{cases} \gamma(2t) & \text{if } t \in [0,1/2], \\ \delta(2t-1) & \text{if } t \in [1/2,1]. \end{cases}$$

(2) The **reverse** of γ is the path $\bar{\gamma}\colon [0,1] \to X$ defined by

$$\bar{\gamma}(t) \coloneqq \gamma(1-t).$$

(3) Let (X,x) be a pointed topological space. Its **fundamental group** is the set $\pi_1(X,x)$ of homotopy classes rel $\{0,1\}$ of paths $\gamma\colon [0,1] \to X$ with $\gamma(0) = \gamma(1)$ and the multiplication defined by

$$[\gamma] \cdot [\delta] \coloneqq [\delta * \gamma].$$

The unit $\mathbf{1} \in \pi_1(X,x)$ is the homotopy class of the constant path $[x]$ and $[\gamma]^{-1} = [\bar{\gamma}]$. •

The above definition of the group structure on $\pi_1(X,x)$ might appear backwards. This convention is justified by Proposition 1.25 (2). It is also the group structure inherited from the composition in a fundamental groupoid $\Pi_1(X)$.

Proposition 1.25. *Let $p\colon X \to B$ be a covering map.*

(1) *For every path $\gamma\colon [0,1] \to B$ there is a map $\mathrm{tra}_\gamma\colon \pi^{-1}(\gamma(0)) \to \pi^{-1}(\gamma(1))$ such that if $\tilde{\gamma}$ is a lift of γ, then*

$$\mathrm{tra}_\gamma(\tilde{\gamma}(0)) = \tilde{\gamma}(1).$$

(2) *If* $\gamma, \delta\colon [0,1] \to X$ *are paths with* $\gamma(1) = \delta(0)$, *then*

$$\mathrm{tra}_{\gamma * \delta} = \mathrm{tra}_\delta \circ \mathrm{tra}_\gamma .$$

(3) *For every path* $\gamma\colon [0,1] \to B$ *the map* tra_γ *is a bijection; indeed:*

$$\mathrm{tra}_{\bar\gamma} = \mathrm{tra}_\gamma^{-1} .$$

(4) *The map* tra_γ *depends only on the homotopy class of* γ *rel* $\{0,1\}$.

Proof. By Proposition 1.18, tra_γ is well-defined. This proves (1).

If $\tilde\gamma$ is a lift of γ and $\tilde\delta$ is a lift of δ with $\tilde\delta(0) = \tilde\gamma(0)$, then $\tilde\gamma * \tilde\delta$ is a lift of $\gamma * \delta$ with $(\tilde\gamma * \tilde\delta)(0) = \tilde\gamma(0)$. This proves (2).

If $\tilde\gamma$ is a lift of γ, then $\bar{\tilde\gamma}$ is a lift of $\bar\gamma$. This proves (3).

Let $\Gamma\colon [0,1] \times [0,1] \to B$ be a homotopy rel $\{0,1\}$. If $\tilde\Gamma$ is a lift of Γ along p, then $\tilde\Gamma(\cdot, 1)$ is constant it maps to $p^{-1}(\Gamma(0,1))$ and the latter is discrete. This proves (4). ■

Example 1.26. Consider Example 1.7 for $d = 2$. By the quadratic formula the roots of $p = x^2 + ax + b$ are

$$-\frac{1}{2}\left(a \pm \sqrt{a^2 - 4b}\right).$$

Therefore, $P_2 = \{x^2 + ax + b \in \mathbf{C}[x] : a^2 \neq 4b\}$. Consider the path $p\colon [0,1] \to P_2$ defined by

$$p(t) \coloneqq x^2 - e^{2\pi i t} = (x + e^{\pi i t})(x - e^{\pi i t}).$$

The lift $\tilde p\colon [0,1] \to R_2$ of p along $\mathrm{pr}_2\colon R_2 \to P_2$ with $\tilde p(0) = (1, p(t))$ is

$$\tilde p(t) = (e^{\pi i t}, p(t)).$$

Therefore, $\mathrm{tra}_p(1) = -1$. ♠

Definition 1.27. Let $p\colon X \to B$ be a covering map. Let $b \in B$. The **monodromy representation** of p is the homomorphism $\mathrm{tra}\colon \pi_1(B, b_0) \to \mathrm{Bij}(p^{-1}(b))$ defined by

$$\mathrm{tra}(\gamma) \coloneqq \mathrm{tra}_\gamma . \qquad \bullet$$

Definition 1.28. Let $p\colon (X, x_0) \to (B, b_0)$ be a pointed covering map. The **characteristic subgroup** of p and x_0 is the subgroup

$$C(p, x_0) \coloneqq \mathrm{im}(p_*\colon \pi_1(X, x_0) \to \pi_1(B, b_0)) < \pi_1(B, b_0). \qquad \bullet$$

Definition 1.29. Let G be a group. Let $H < G$ be a subgroup. The **normal core of** $H < G$ is the normal subgroup

$$\mathrm{Core}_G(H) \coloneqq \bigcap_{g \in G} gHg^{-1} \triangleleft G. \qquad \bullet$$

Proposition 1.30. *Let* $p\colon (X, x_0) \to (B, b_0)$ *be a pointed covering map.*

(1) *The homomorphism* $p_*\colon \pi_1(X,x_0) \to C(p,x_0) < \pi_1(B,b_0)$ *is an isomorphism.*

(2) *For every* $[\gamma] \in \pi_1(B,b_0)$ *and* $y := \mathrm{tra}([\gamma])(x)$

$$C(p,y) = [\gamma]C(p,x_0)[\gamma]^{-1}.$$

(3) *For every* $[\gamma] \in \pi_1(B,b_0)$

$$\mathrm{tra}([\gamma])(x) = x \quad \text{if and only if} \quad [\gamma] \in C(p,x_0).$$

(4) *If* X *is path-connected, then*

$$\mathrm{tra}(\cdot)(x)\colon \pi_1(B,b_0)/C(p,x_0) \to p^{-1}(b)$$

is a bijection; in particular,

$$\deg(p) = |C(p,x_0) : \pi_1(B,b_0)|.$$

(5) *If* X *is path-connected, then*

$$\ker(\mathrm{tra}\colon \pi_1(B,b_0) \to \mathrm{Bij}(p^{-1}(b_0)) = \mathrm{Core}_{\pi_1(B,b_0)}(C(p,x_0)).$$

Proof. If $[\tilde{\gamma}] \in \ker p_*$, then there is a homotopy $\Gamma\colon [0,1] \times [0,1] \to B$ rel $\{0,1\}$ with $\Gamma(0,\cdot) = \gamma := p \circ \tilde{\gamma}$ and $\Gamma(1,\cdot) = b$. Denote by $\tilde{\Gamma}\colon [0,1] \times [0,1] \to X$ the lift of Γ with $\tilde{\Gamma}(0,\cdot) = \tilde{\gamma}$. $\tilde{\Gamma}(\cdot,0)$, $\tilde{\Gamma}(\cdot,1)$, and $\tilde{\Gamma}(1,\cdot)$ are lifts of the constant path b; hence: equal to the constant path x. Therefore, $\tilde{\Gamma}$ is a homotopy rel $\{0,1\}$ and $H(1,\cdot) = x$ is the constant path x. Consequently, $[\tilde{\gamma}] = \mathbf{1} \in \pi_1(X,x_0)$. This proves (1).

If $\tilde{\gamma}\colon [0,1] \to X$ is a path with $\tilde{\gamma}(0) = x$ and $\tilde{\gamma}(1) = y$, then the map $\pi_1(X,x_0) \to \pi(X,y)$ defined by

$$[\delta] \mapsto [\bar{\tilde{\gamma}} * \delta * \tilde{\gamma}]$$

is an isomorphism. This implies (2).

If $[\gamma] \in C(p,x_0)$, then $\mathrm{tra}([\gamma])(x_0) = x_0$. If $\mathrm{tra}([\gamma])(x_0) = x_0$, then the lift $\tilde{\gamma}$ of γ with $\tilde{\gamma}(0) = x_0$ satisfies $\tilde{\gamma}(1) = x_0$; hence: it defines an element $[\tilde{\gamma}] \in \pi_1(X,x_0)$. Evidently, $p_*[\tilde{\gamma}] = [\gamma]$. This proves (3).

By (3), $\mathrm{tra}(\cdot)(x_0)$ is injective. To prove that $\mathrm{tra}(\cdot)(x_0)$ is surjective, let $x \in p^{-1}(b_0)$. Since X is path-connected, there is path $\tilde{\gamma}\colon [0,1] \to X$ with $\tilde{\gamma}(0) = x_0$ and $\tilde{\gamma}(1) = x$. By construction, $\gamma := p \circ \tilde{\gamma}$ satisfies $\gamma(0) = \gamma(1) = b_0$, and $\mathrm{tra}([\gamma])(x_0) = x$. This proves (4)

By (3),

$$\ker \mathrm{tra} = \bigcap_{x \in p^{-1}(b_0)} C(p,x).$$

This, (2), and (4) imply (5). ■

1.5 Lifting along covering maps, II

Definition 1.31. A topological space X is **locally path connected** if for every $x \in X$ and every neighborhood U of $x \in X$ there is a path connected, open neighborhood $V \subset U$ of $x \in X$. •

Theorem 1.32 (Lifting criterion). *Let $p\colon X \to B$ be a covering map. Let $f\colon A \to B$ be a continuous map. Suppose that A is connected and locally path connected. Let $a \in A$ and $x \in p^{-1}(f(a))$. Set $b := p(x)$. There is a lift $\tilde{f}\colon A \to X$ of f along p with $\tilde{f}(a) = x$ if and only if*

$$\operatorname{im}(f_*\colon \pi_1(A, a_0) \to \pi_1(B, b_0)) < C(p, x_0).$$

Proof. If f admits as lift $\tilde{f}$ along p with $\tilde{f}(a) = x$, then

$$f_*\pi_1(A, a_0) = p_*\tilde{f}_*\pi_1(A, x_0) < p_*\pi_1(X, x_0) = C(p, x_0).$$

For $a \in A$ choose a path $\gamma\colon [0, 1] \to A$ with $\gamma(0) = a_0$ and $\gamma(1) = a$ and set

$$\tilde{f}(a) := \operatorname{tra}_{f\circ\gamma}(x_0).$$

This does not depend on the choice of γ. Indeed, if $\delta\colon [0, 1] \to A$ is path $\delta(0) = a_0$ and $\delta(1) = a$, then $f_*([\gamma]^{-1}[\delta]) \in \operatorname{im} p_*$. Therefore, by Proposition 1.30 (3), $\operatorname{tra}_{f\circ\gamma}(x_0) = \operatorname{tra}_{f\circ\delta}(x_0)$.

It remains to prove that $\tilde{f}$ is continuous. If $p = \mathrm{pr}_1\colon B \times D \to B$ with D discrete and $x_0 = (b_0, d_0)$, then the above construction yields the continuous map $\tilde{f} = (f, d_0)$. A moment's thought shows that repeating the above construction with a_0 and x_0 replaced with a and $\tilde{f}(a)$ respectively produces the same map $\tilde{f}$. Since A is locally path connected, every $b \in B$ has a path connected neighborhood V with $f(V) \subset B$ contained in an open subset U as in Definition 1.1. Therefore, $\tilde{f}$ is continuous. ∎

1.6 The classification of covering maps

If one wants to stick to the fundamental group (instead of the fundamental groupoid), then it is convenient to introduce base-points throughout and in the category of pointed topological spaces and pointed continuous map.

Definition 1.33. Two pointed covering maps $p\colon (X, x_0) \to (B, b_0)$ and $q\colon (Y, y_0) \to (B, b_0)$ are **isomorphic** if there is a pointed homeomorphism $\phi\colon (X, x_0) \to (Y, y_0)$ such that $q \circ \phi = p$. •

Theorem 1.34. *Let (B, b_0) be a connected, locally path-connected, pointed topological space. Two pointed covering maps $p\colon (X, x_0) \to (B, b_0)$ and $q\colon (Y, y_0) \to (B, b_0)$ with X, Y connected are isomorphic if and only if*

$$C(p, x_0) = C(q, y_0).$$

Proof. This is a straight-forward consequence of Theorem 1.32. ∎

Definition 1.35. A topological space X is **semi-locally simply-connected** if every $x \in X$ has a neighborhood such that every loop $\gamma\colon [0, 1] \to U$ with $\gamma(0) = \gamma(1) = x$ is homotopic rel $\{0, 1\}$ to a constant loop in X. •

Theorem 1.36. *Let (B, b_0) be a connected, locally path-connected, semi-locally simply-connecteed, pointed topological space. For every $C < \pi_1(B, b_0)$ there is a pointed covering map $p\colon (X, x_0) \to (B, b_0)$ with X connected and*

$$C(p, x_0) = C.$$

Proof.

Step 1. *Construction of the set X and $p\colon X \to B$.*

Denote by P_b the set of paths $\gamma\colon [0,1] \to B$ with $\gamma(0) = b$. Define the equivalence relation $\sim$ on P_b by $\gamma \sim \delta$ if and only if $\gamma(1) = \delta(1)$ and $[\gamma * \bar{\delta}] \in C$. Denote the equivalence class of γ with respect to $\sim$ by $\langle\gamma\rangle$ Set

$$X := P_b/{\sim}$$

and define $p\colon X \to B$ by

$$p(\langle\gamma\rangle) := \gamma(1).$$

Step 2. *Construction of the topology on X.*

For $\gamma\colon [0,1] \to B$ with $\gamma(0) = b$ and $s \in [0,1]$ define $\gamma_s\colon [0,1] \to P_b$ by $\gamma_s(t) := \gamma(st)$. By construction, $p(\langle\gamma_s\rangle) = \gamma(s)$. Let $U \subset X$ an path-connected open subset and $\langle\gamma\rangle \in p^{-1}(U)$. Denote by $V(U, [\gamma]) \subset X$ to be subset of elements of the form $\langle\gamma * \delta\rangle$ with $\delta\colon [0,1] \to U$ and $\delta(0) = \gamma(1)$. Evidently, these form the basis of a topology.

Step 3. *Proof that p is continuous and open.*

Since B is locally path-connected, p is continuous. Since U is path-connected, $p(V(U, [\gamma])) = U$. Therefore, p is open.

Step 4. *Proof that p is a covering map.*

Let $c \in B$. Let U be a path-connected neighborhood of $c \in B$ such that every $\varepsilon\colon [0,1] \to U$ with $\varepsilon(0) = \varepsilon(1) = b$ is homotopic rel $\{0,1\}$ to the constant loop c in B. Let $\langle\gamma\rangle, \langle\delta\rangle \in p^{-1}(b)$. If $V(U, [\gamma]) \cap V(U, [\delta]) \neq \varnothing$, then there are paths $\varepsilon, \phi\colon [0,1] \to U$ with $\varepsilon(0) = \gamma(1)$, $\phi(0) = \delta(1)$, and

$$[\gamma * \varepsilon * \bar{\phi} * \bar{\delta}] \in C.$$

By construction, $\varepsilon * \bar{\phi}$ is homotopic rel $\{0,1\}$ to the constant path c. Therefore, $[\gamma * \bar{\delta}]$; hence: $\langle\gamma\rangle = \langle\delta\rangle$. This proves that $p^{-1}(c)$ is discrete.

If $p(\langle\gamma\rangle) \in U$, then there is a path $\delta\colon [0,1] \to U$ with $\delta(0) = \gamma(1)$ and $\delta(1) = c$. Evidently $[\gamma] \in V(U, \langle\gamma * \delta\rangle)$. Therefore,

$$p^{-1}(U) = \coprod_{\langle\gamma\rangle \in p^{-1}(b)} V(U, \langle\gamma\rangle).$$

It remains to prove that $p\colon V(U, \langle\gamma\rangle) \to U$ is a homeomorphism. It already is continuous and open. Since U is path-connected, this map is surjective. Since every loop in U based at $\gamma(1)$ can be contracted, the map is injective.

Step 5. *Proof that X is path connected.*

X is path-connected since the maps $s \mapsto \langle \gamma_s \rangle$ are continuous.

Step 6. *Proof that* $C(p, x_0) = C$.

Finally, it remains to verify that $C(p, \langle b \rangle) = C$. A loop $\gamma \colon [0,1] \to B$ with $\gamma(0) = \gamma(1) = b$ represents an element of $C(p, \langle b \rangle)$ if and only if its lift $s \mapsto \langle \gamma_s \rangle$ to X satisfies $b = \langle \gamma_0 \rangle = \langle \gamma_1 \rangle = \langle \gamma \rangle$. By construction, the latter is equivalent to $[\gamma] \in C$. ■

1.7 Deck transformations

Definition 1.37. Let $p \colon X \to B$ be a covering map. A **deck transformation** of p is a homeomorphism $\phi \colon X \to X$ such that

$$p \circ \phi = p.$$

These form the **deck transformation group** denoted by

$$\mathrm{Deck}(p).$$ •

Proposition 1.38. *Let $p \colon X \to B$ be a covering map. If X is connected, then action of* $\mathrm{Deck}(p)$ *on X is properly discontinuous.*

Definition 1.39. Let G be a group. Let $H < G$ be a subgroup. The **normaliser** of $H < G$ is

$$N_G(H) \coloneqq \{g \in G : gHg^{-1} = H\}.$$ •

Proposition 1.40. *Let $p \colon X \to B$ be a covering map. Let $b \in B$ and $x, y \in p^{-1}(b)$.*

(1) *Let $\phi \in \mathrm{Deck}(p)$. If $\phi(x) = y$, then $C(p, x_0) = C(p, y_0)$.*

(2) *If X is connected and locally path-connected, and $C(p, x_0) = C(p, y_0)$, then there is a unique $\phi \in \mathrm{Deck}(p)$ with $\phi(x) = y$.*

(3) *If X is connected and locally path-connected, then there is a unique anti-isomorphism*

$$\tau \colon \mathrm{Deck}(p) \to N_{\pi_1(B,b_0)}(C(p,x_0))/C(p,x_0)$$

with the following property: if $\phi \in \mathrm{Deck}(p)$ and $\gamma \colon [0,1] \to X$ is a path with $\gamma(0) = x$ and $\gamma(1) = \phi(x)$, then

$$\tau(\phi) = [p \circ \gamma]. \tag{1.41}$$

(4) *If X is connected and locally path-connected, then* $\mathrm{Deck}(p)$ *acts transitively on $p^{-1}(b)$ if and and only if $C(p, x_0) < \pi_1(B, b_0)$ is normal.*

Proof. (1) and (2) are consequences of Theorem 1.32 and Proposition 1.18.

(1) and (2) the map $\mathrm{ev}_x \colon \mathrm{Deck}(p) \to \{y \in p^{-1}(x) : C(p, x_0) = C(p, y_0)\}$ defined by

$$\mathrm{ev}_x(\phi) = \phi(x)$$

is bijective. By Proposition 1.30 (2) and (4), the map

$$\mathrm{tra}(\cdot)(x)\colon\ N_{\pi_1(B,b_0)}(C(p,x_0))/C(p,x_0) \to \{y \in p^{-1}(x) : C(p,x_0) = C(p,y_0)\}$$

is bijective. The map

$$\tau := (\mathrm{tra}(\cdot)(x))^{-1} \circ \mathrm{ev}_x$$

satisfies (1.41) and is bijective.

To verify that τ is an anti-homomorphism, let $\phi, \psi \in \mathrm{Deck}(p)$. Let $\gamma, \delta\colon [0,1] \to X$ be a path with $\gamma(0) = \delta(0) = x$, $\gamma(1) = \phi(x)$, and $\delta(1) = \psi(x)$. Since $\varepsilon := \delta * (\psi \circ \gamma)$ satisfies $\varepsilon(0) = x$ and $\varepsilon(1) = \psi(\phi(x))$,

$$\tau(\psi\phi) = [p \circ (\delta * (\psi \circ \gamma))] = \tau(\phi)\tau(\psi).$$

Therefore, τ is an anti-homomorphism. This proves (3).

(4) is a direct consequence of Proposition 1.30 (4). ■

Exercise 1.42. Compute $\mathrm{Deck}(p)$ for Example 1.5. (This assumes that you already have some tools to understand the fundamental group in this case.) ?

Definition 1.43. A covering map $p\colon X \to B$ is **normal** if $C(p,x_0) < \pi_1(B,b_0)$ is normal. •

If $p\colon X \to B$ is a normal covering map with X connected and locally path-connected, then it induces a homeomorphism $\cong\backslash\mathrm{Deck}(p)B$.

1.8 Universal covering maps

Definition 1.44. A covering map $p\colon X \to B$ is **universal** if X is path-connected and simply-connected. •

Proposition 1.45. *If $p\colon X \to B$ is a universal covering map, then it is normal and for every $x \in X$ and $b := p(x)$ there is an isomorphism* $\mathrm{Deck}(p) \cong \pi_1(B,b_0)^{\mathrm{op}}$. ■

Exercise 1.46. Compute $\pi_1(S^1, [0])$. ?

Proposition 1.47. *Let (B,b_0) be a connected, locally path-connected, pointed topological space. Let $p\colon (X,x_0) \to (B,b_0)$ and $q\colon (Y,y_0) \to (B,b_0)$ be pointed covering maps with X, Y connected. Suppose that $C(p,x_0) < C(q,y_0)$. Denote by $f\colon (X,x_0) \to (Y,y_0)$ the unique lift of p along q.*

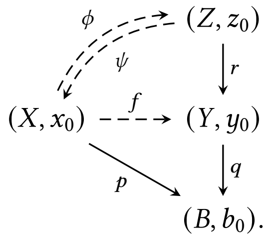
(1) *The map f is a covering map and* $\mathrm{Deck}(f) < \mathrm{Deck}(p)$.

(2) *The anti-isomorphism $\tau\colon\ \mathrm{Deck}(p) \to N_{\pi_1(B,b_0)}(C(p,x_0))/C(p,x_0)$ maps* $\mathrm{Deck}(f) < \mathrm{Deck}(p)$ *to $N_{C(q,y_0)}(C(p,x_0))/C(p,x_0)$.*

(3) *If p is normal, then f is normal. In particular,*

$$Y \cong \mathrm{Deck}(f)\backslash X \cong X \times_{\mathrm{Deck}(f)} S \quad \textit{with} \quad S := \mathrm{Deck}(p)/\mathrm{Deck}(f).$$

Proof. Denote by $r\colon (Z,z) \to (Y,y_0)$ a pointed covering map with Z connected and $C(r,z) = C(p,x_0) < C(q,y_0) \cong \pi_1(Y,y_0)$. Denote by ϕ the lift of f along r. Denote by ψ the lift of $q \circ r$ along p. The following diagram summarises this situation:



Since

$$p \circ \psi \circ \phi = q \circ r \circ \phi = q \circ f = p,$$

$\psi \circ \phi$ is the lift of p along p and thus agrees with id_X. Since

$$q \circ f \circ \psi = q \circ r,$$

$f \circ \psi$ is the lift of $q \circ r$ along q; but so is r; hence: $f \circ \psi = r$. Therefore,

$$r \circ \phi \circ \psi = f \circ \psi = r;$$

that is: $\phi \circ \psi$ is the lift of r along r and thus agrees with id_Z. This proves (1).

(2) and (3) are obvious. ∎

1.9 The classification of G–principal covering maps

Warning: here we use right instead of left actions.

Definition 1.48. Let G be a group. A G**–principal covering map** $p\colon X \to B$ is a normal covering map with a free right action $\rho\colon G \to \mathrm{Deck}(p)^{\mathrm{op}}$ such that p induces a homeomorphism $X/G \cong B$. •

Definition 1.49. Two pointed G–principal covering maps $(p\colon (X,x_0) \to (B,b_0), \rho)$ and $(q\colon (Y,y_0) \to (B,b_0), \sigma)$ are **isomorphic** if there is a homeomorphism $\phi\colon (X,x_0) \to (Y,y_0)$ with

$$q \circ \phi = p \quad \text{and} \quad \phi\rho\phi^{-1} = \sigma. \qquad \bullet$$

Definition 1.50. Let $(p\colon (X,x_0) \to (B,b_0), \rho)$ be a pointed principal G–covering map. The **monodromy representation** of (p,λ) is the homomorphism $\mu\colon \pi_1(B,b_0) \to G$ characterised by

$$\mathrm{tra}([\gamma])(x) = \rho(\mu([\gamma]))(x). \qquad \bullet$$

Example 1.51. Let (B,b_0) be a connected, locally path-connected, pointed topological space. Let $p\colon (X,x_0) \to (B,b_0)$ be a universal covering map. Proposition 1.40 (3) gives an isomorphism

$\tau\colon \pi_1(B,b_0) \cong \mathrm{Deck}(p)^{\mathrm{op}}$. This exhibits p is a $\pi_1(B,b_0)$–principal covering map. Let $\mu \in \mathrm{Hom}(\pi_1(B,b_0),G)$ be a homomorphism. Let $\pi_1(B,b_0)$ act on $X\times G$ via

$$(x,g)[\gamma] := (\tau([\gamma])(x), g\mu([\gamma])).$$

Set

$$Y := \pi_1(B,b_0)\backslash(X\times G) \quad \text{and} \quad y := [x,1].$$

The projection map $q\colon (Y,y_0) \to (B,b_0)$ is a pointed covering map. The map $\sigma\colon G \to \mathrm{Deck}(q)^{\mathrm{op}}$ is induced by right-multiplication on G makes q into a principal G–covering map. ♠

Theorem 1.52. *Let (B,b_0) be a connected, locally path-connected, semi-locally simply-connected, pointed topological space*

(1) *Two pointed G–principal covering maps are isomorphic if and only if their monodromy representations agree.*

(2) *Every $\mu \in \mathrm{Hom}(\pi_1(B,b_0),G)$ is the monodromy representation of a pointed G–principal covering map.*

Proof sketch. Fix a pointed universal covering map $r\colon (Z,z_0) \to (B,_0 b)$. Let $(p\colon (X,x_0) \to (B,b_0),\rho)$ be a pointed G–principal covering map. Let $(q\colon (Y,y_0) \to (B,b_0),\sigma)$ be the G–principal covering constructed in the previous example from r and the monodromy representation μ of p.

Denote by $f\colon (Z,z_0) \to (X,x_0)$ the lift of r along p. A moments thought shows that

$$f(\tau([\gamma])x) = \rho(\mu([\gamma]))^{-1}f(x).$$

Define $\Phi\colon Z\times G \to X$ by

$$\Phi(\zeta,g) = \rho(g)(f_{x_0}(\zeta)).$$

Let $[\gamma] \in \pi_1(B,b_0)$. Denote by $\tau\colon \pi_1(B,b_0) \cong \mathrm{Deck}(r)^{\mathrm{op}}$ the isomorphism from Proposition 1.40 (3). The map Φ is $\pi_1(B,b_0)$–invariant; indeed:

$$\begin{aligned}\Phi((x,g)[\gamma]) &= \Phi(\tau([\gamma])(x), g\mu([\gamma]))\\ &= \rho(g\cdot\mu([\gamma]))f_x(\tau([\gamma])x)\\ &= \rho(g)f_x(x)\\ &= \Phi(x,g).\end{aligned}$$

Therefore, Φ induces a continuous map $\phi\colon (Y,y_0) \to (X,x_0)$. Evidently,

$$q\circ\phi = p.$$

A moment's thought shows that ϕ is an isomorphism of pointed G–principal covering maps.

To complete prove it remains to verify that the construction in the preceding example indeed gives a G–principal covering map with monodromy given by μ. This is an exercise. ■

1.10 The Seifert–van Kampen theorem

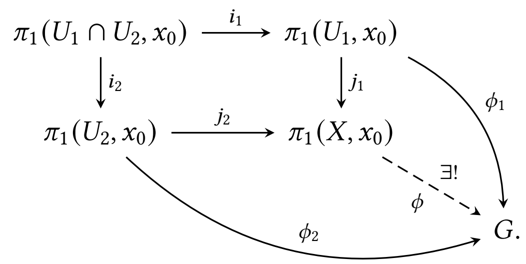
Theorem 1.53 (Seifert–van Kampen). *Let X be a topological space. Let $\{U_1, U_2\}$ be an open cover of X. Suppose that $X, U_1, U_2, U_1 \cap U_2$ are connected, locally path-connected, semi-locally simply-connected. Let $x \in U_1 \cap U_2$. Denote by*

$$\iota_i\colon \pi_1(U_1 \cap U_2, x_0) \to \pi_1(U_i, x_0) \quad \textit{and} \quad \jmath_i\colon \pi_1(U_i, x_0) \to \pi_1(X, x_0)$$

the maps induced by inclusion. The fundamental group $\pi_1(X, x_0)$ has the following universal property: if $\phi_i\colon \pi_1(U_i, x_0) \to G$ are homomorphism with $\phi_1 \circ \iota_1 = \phi_2 \circ \iota_2$, then there is a unique $\phi\colon \pi_1(X, x_0) \to G$ such that

$$\phi \circ \jmath_i = \phi_i.$$

The following diagram illustrates Theorem 1.53:



Exercise 1.54. Proof this from the classification of pointed G–principal covering maps. ?

1.11 The topological proof of the Nielsen–Schreier Theorem

Definition 1.55. Let S be a set. The **free group on** S is the group $F(S)$ generated by S. A group G is **free** if it is isomorphic to $F(S)$ for some S. The **rank** of G is $\mathrm{rk}(G) := \#S$. •

Theorem 1.56 (Nielsen–Schreier Theorem). *If G is a free group, then every subgroup $H < G$ is free. If $\mathrm{rk}(G) = r \in \mathbf{N}_0$ and $|H : G| = i \in \mathbf{N}$, then $\mathrm{rk}(H) = i(r-1) + 1$.*

The proof relies on realising G as a fundamental group and covering theory.

Definition 1.57.

(1) A **graph** is a triple $\Gamma = (V, E, \alpha)$ with V a set, E a set of unordered pairs, and a map $\alpha\colon \bigcup E \to V$. The **vertices** and **edges** of Γ are the elements of V and E respectively. An edge e **connects** $x, y \in V$ if $\alpha(e) = \{x, y\}$.

(2) For every unordered pair $e = \{x, y\}$ set

$$I_e := (e \times [0,1])/{\sim}$$

with $\sim$ denoting the equivalence relation generated by $(x, t) \sim (y, 1-t)$.

(3) The **topological realisation** of Γ is

$$X(\Gamma) := \left(V \amalg \coprod_{e\in E} I_e\right)\Big/ \sim$$

with $\sim$ denoting the equivalence realation generated by $[x,0] \sim \alpha(x)$. •

Example 1.58. Let S be a set. Set $V := \{*\}$ and $E := \{0,1\} \times S$. There is a unique map $\alpha\colon E \to V$. The graph $\Gamma = (V, E, \alpha)$ has a unique vertices $\star$ and an edge connecting $*$ to itself for every $s \in S$. The topological realisation $X(\Gamma)$ of Γ is homeomorphic to a **bouquet of circles** indexed by S:

$$X(\Gamma) \cong \bigvee_{s\in S} \{s\} \times S^1 := \left(\coprod_{s\in S} \{s\} \times S^1\right)\Big/ \sim$$

with $\sim$ denoting the equivalence relation generated by $(s,[0]) \sim (t,[0])$. By Theorem 1.53,

$$\pi_1(X(\Gamma), *) \cong F(S).$$ ♠

Definition 1.59. Let $\Gamma = (V, E, \alpha)$ be a graph.

(1) A **subgraph** of a graph $\Gamma = (V, E, \alpha)$ is a graph $\Delta = (W, F, \beta)$ with $W \subset V$, $F \subset E$, and $\beta = \alpha|_F$.

(2) A **path** in Γ is a is a sequence of vertices $v_0, \dots, v_n$ together with a sequence of edges $e_0, \dots, e_n$ such that e_i connects v_i and v_{i+1}. A **cycle** in Γ is a path with $n \geqslant 1$, $v_0 = v_n$, and $e_i \neq e_{i+1}$.

(3) Γ is **connected** if for every $v, w \in V$ there is a path with $v_0 = v$ and $v_n = w$.

(4) A **forest** is a graph without cycles. A **tree** is a connected forest. •

Proposition 1.60. *Let Γ be a connected graph. $X(\Gamma)$ is homotopy equivalent to a bouquet of circles.*

Proof sketch. Denote by $\mathscr{T}$ the set of subgraphs of Γ which are trees. There is an obvious order on $\mathscr{T}$. Use Zorn's lemma to construct a maximal $T \in \mathscr{T}$. A moment's thought shows that T has the same vertices as Γ. The subspace $X(T) \subset X(\Gamma)$ is contractible. $X(\Gamma)/X(T)$ is homeomorphic to a bouquet of circles. Finally, the projection $X(\Gamma) \to X(\Gamma)/X(T)$ is a homotopy equivalence. ■

Proposition 1.61. *Let Γ be a graph. If $p\colon Y \to X(\Gamma)$ is a covering map, then Y is homeomorphic to $X(\Delta)$ for some graph Δ.*

Proof. Exercise. ■

Proof of Theorem 1.56. Let G be a free group. Construct a graph Γ with $\pi_1(X(\Gamma)) \cong G$. If $H < G$ is a subgroup, then there is a covering map $p\colon Y \to X(\Gamma)$ with characteristic subgroup isomorphic to H. By the above, $\pi_1(Y)$ is free.

If F has rank r and $|H : G| = i$, then $\deg(p) = i$; hence:

$$1 - \mathrm{rk}(H) = \chi(Y) = i\chi(X(\Gamma)) = i(1-r).$$

This implies $\mathrm{rk}(H) = i(r-1) + 1$. ■

2 Fibre bundles

The purpose of this section is to develop the theory of Ehresmann connections on fibre bundles. [Ste51] is the classical reference of the topological theory of fibre bundles. [Hus94] is a more modern reference. The theory of connections of fibre bundles is due to Ehresmann [Ehr51]. Kolář, Michor, and Slovák [KMS93]

2.1 Definition and examples

Definition 2.1. Let X, B be smooth manifolds. A **fibre bundle** is a smooth map $p\colon X \to B$ such that for every $b \in B$ there are an open subset $b \in U \subset B$, a smooth manifold F, and a diffeomorphism $\tau\colon p^{-1}(U) \to U \times F$ such that

$$\mathrm{pr}_1 \circ \tau = p;$$

that is: the diagram

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\tau} & U \times F \\ & \searrow^{p} & \downarrow{\scriptstyle \mathrm{pr}_1} \\ & & U \end{array}$$

commutes. The **total space** of p is X. The **base space** of p is B. For $b \in B$ the **fibre of** p **over** b is

$$X_b \coloneqq p^{-1}(b). \qquad \bullet$$

Definition 2.2. Let $p\colon X \to B$ and $q\colon Y \to C$ fibre bundles. A **morphism of fibre bundles** $(\phi, f)\colon p \to q$ is a pair of smooth maps $\phi\colon X \to Y$ and $f\colon B \to C$ such that

$$q \circ \phi = f \circ p;$$

that is: the diagram

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ B & \xrightarrow{f} & C \end{array}$$

commutes. If $B = C$, then a **morphism of fibre bundles over** B $\phi\colon p \to q$ is a smooth map $\phi\colon X \to Y$ such that (ϕ, id_B) is a morphism of fibre bundles. •

Example 2.3. Let B, F be smooth manifolds. The **trivial fibre bundle** over B with fibre F is the projection map $\mathrm{pr}_1\colon B \times F \to B$. ♠

Example 2.4. The **Hopf bundle** is the projection $p\colon S^{2n+1} \subset \mathbf{C}^{n+1}\backslash\{0\} \to \mathbf{C}P^n$. ♠

Example 2.5. Let X, B be smooth manifolds. If $p\colon X \to B$ is a covering map, then it is a fiber bundle. ♠

Example 2.6. Let B be a smooth manifold. If $p\colon E \to B$ is a vector bundle, then it is a fiber bundle. ♠

Example 2.7. Let B be a manifold. Let V be an Euclidean vector bundle over B. The **sphere bundle**

$$p\colon S(V) \to B \quad \text{with} \quad S(V) := \{v \in V : |v| = 1\}.$$

is a fibre bundle. ♠

Example 2.8. Let B be a smooth manifold and $p\colon V \to B$ be a vector bundle. For $r \in \mathbf{N}_0$ denote by

$$\mathrm{Gr}_r(V) := \{(b, \Pi) : b \in B, \Pi \subset V_b \text{ with } \dim \Pi = r\}$$

the **Grassmannian of r–planes in** V. $\mathrm{Gr}_r(V)$ admits the structure of a smooth manifold such that the map $q\colon \mathrm{Gr}_r(V) \to B$ obtained by restriction of pr_1 is a fibre bundle. ♠

Example 2.9. Let B be a manifold. Let $p\colon V \to B$ be a vector bundle. Denote by

$$\mathrm{Fr}(V) := \{(b, \phi) : b \in B, \phi\colon \mathbf{R}^{\mathrm{rk}_b V} \to V_b \text{ isomomorphism}\}$$

the **frame bundle of** V. $\mathrm{Fr}(V)$ admits the structure of a smooth manifold such that the map $q\colon \mathrm{Fr}(V) \to B$ obtained by restriction of pr_1 is a fibre bundle. ♠

Example 2.10. Let $k \in \mathbf{N}_0$. Let X, Y be smooth manifolds. Let U, V be open neighborhoods of $x \in X$. Two maps $f \in C^\infty(U, Y)$ and $g \in C^\infty(V, Y)$ have the same **k–jet at** x if $f(x) = g(x) =: y$ and for every chart ϕ on X with $\phi(x) = 0$ and ψ on Y with $\psi(y) = 0$ the maps $\tilde{f} := \psi \circ f \circ \phi^{-1}$ and $\tilde{g} := \psi \circ g \circ \phi^{-1}$ satisfy

$$\partial^\alpha \tilde{f}(0) = \partial^\alpha \tilde{g}(0)$$

for every $\alpha \in \mathbf{N}_0^{\dim_x X}$ with $|\alpha| \leqslant k$ and every chart ϕ on X and ψ on Y. Having the same k–jet at x is an equivalence relation. Denote the set of equivalence classes by $J^k_x(X, Y)$. An element of $J^k_x(X, Y)$ is a **k–jet at** x. The **k–jet space space** of maps $X \to Y$ is

$$J^k(X, Y) := \coprod_{x \in X} J^k_x(X, Y)$$

$J^k(X, Y)$ admits the structure of a smooth manifold such that the canonical projection maps $p\colon J^k(X, Y) \to X$, $q\colon J^k(X, Y) \to Y$ are fibre bundle. ♠

Exercise 2.11. Construct the above mentioned structures of a smooth manifold on $S(V)$, $\mathrm{Gr}_r(V)$, $\mathrm{Fr}(V)$, and $J^k(X, Y)$. ?

Exercise 2.12. Construct diffeomorphism $J^0(X, Y) \to X \times Y$ and $J^1(X, \mathbf{R}) \to T^*X \times \mathbf{R}$. ?

Proposition 2.13. *Let $p\colon X \to B$ be a fiber bundle. If B is connected and $b_0, b_1 \in B$, then X_{b_0} and X_{b_1} are diffeomorphic.* ■

Theorem 2.14 (Ehresmann fibration theorem). *Let X, B be smooth manifolds. If $p\colon X \to B$ is a proper submersion, then p is a fibre bundle.*

2.2 Constructions: Product, disjoint union, pullback

Proposition 2.15. *Let $p\colon X \to B$ and $q\colon Y \to B$ be fibre bundles.*

(1) *The subset*
$$X \times_B Y \coloneqq \{(x,y) \in X \times Y : p(x) = q(y)\}$$
is a submanifold and the projection map $q\colon X \times_B Y \to B$ is a fiber bundle.

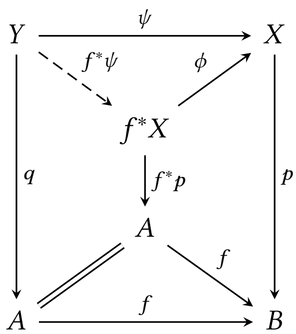
(2) *The map $p \amalg q\colon X \amalg Y \to B$ is a fiber bundle.*

■

Proposition 2.16. *Let $p\colon X \to B$ be a fibre bundle. Let $f\colon A \to B$ a smooth map.*

(1) *The subset*
$$f^*X \coloneqq \{(a,x) \in A \times X : f(a) = p(x)\}$$
is a smooth submanifold and the map $f^*p\colon f^*X \to A$ obtained as the restriction of pr_1 is a fibre bundle.

(2) *Denote by $\phi\colon f^*X \to X$ the restriction of pr_2. If $q\colon Y \to A$ is a fibre bundle and $(\psi, f)\colon q \to p$ is a morphism, then $f^*\psi \coloneqq (q,\psi)\colon q \to f^*p$ is the unique morphism over A such that*
$$(\phi, f) \circ (f^*\psi, \mathrm{id}_A) = (\phi, f);$$
that is: the diagram



commutes. ■

Definition 2.17. Let $p\colon X \to B$ be a fibre bundle. The fibre bundle $f^*p\colon f^*X \to A$ is the **pullback of** p **via** f. •

Definition 2.18. Let $p\colon X \to B$ be a fibre bundle. Let $f\colon A \to B$ be a smooth map. A smooth map $\tilde{f}\colon A \to X$ is a **lift of** f **along** p if $p \circ \tilde{f} = f$. A **section of** p is lift of id_B along p; that is: a smooth map $s\colon B \to X$ satisfying $p \circ s = \mathrm{id}_B$. •

Applying Proposition 2.16 with $q = \mathrm{id}_A$ yields.

Corollary 2.19. *Let $p\colon X \to B$ be a fibre bundle. Let $f\colon A \to B$ a smooth map. Denote by $f^*p\colon f^*X \to A$ the pullback of p via f. Let $\phi\colon f^*X \to X$ be as above. Composition with ϕ induces a bijection*
$$\{\text{sections of } f^*X\} \to \{\text{lifts of } f\}, s \mapsto \phi \circ s.$$
■

2.3 Ehresmann connections

It is not terribly difficult to prove the following.

Theorem 2.20. *If $p\colon X \to B$ is a fibre bundle, then it is a Hurewicz fibration.* ■

However, fibre bundles usually lack the unique path lifting property. This defect is can be overcome by choosing an Ehresmann connections.

Definition 2.21. Let $p\colon X \to B$ be a fibre bundle. The **vertical tangent bundle** of p is the vector bundle

$$V_p := \ker(Tp\colon TX \to p^*TB) \to X.$$ •

Remark. There is a mild abuse of notation in the above: usually Tp is a map $TX \to TB$. It would be more correct to write p^*Tp for the above map (but that makes the notation quite heavy). ♣

Definition 2.22. Let $p\colon X \to B$ be a fibre bundle. An **Ehresmann connection** on p is a splitting of the exact sequence

$$0 \to V_p \xrightarrow{\iota} TX \xrightarrow{Tp} p^*TB \to 0;$$

that is: an isomorphism

$$A\colon TX \to V_p \oplus p^*TB$$

such that the diagram

$$\begin{array}{ccccc} V_p & \xrightarrow{\iota} & TX & \xrightarrow{Tp} & p^*TB \\ \| & & \downarrow{\scriptstyle A} & & \| \\ V_p & \longrightarrow & V_p \oplus p^*TB & \longrightarrow & p^*TB \end{array}$$

commutes. The **horizontal subbundle of** A is the subbundle $H_A \subset TX$ defined by

$$H_A := A^{-1}(p^*TB).$$

The **connection** 1**–form of** A is the 1–form $\theta_A \in \Omega^1(X, V_p)$ defined by

$$\theta_A := \mathrm{pr}_{V_p} \circ A.$$

The set of Ehresmann connections on p is denoted by $\mathscr{A}(p)$. •

Example 2.23. Let B, F be smooth manifolds. For the trivial fibre bundle $\mathrm{pr}_B\colon B \times F \to B$

$$V_{\mathrm{pr}_B} = \mathrm{pr}_F^*TF.$$

The **product connection** A_0 is the Ehresmann connection

$$A_0 := (T\mathrm{pr}_F, T\mathrm{pr}_B)\colon TX \to \mathrm{pr}_F^*TF \oplus \mathrm{pr}_B^*TB.$$

The horizontal subbundle of A_0 is $H_{A_0} = \ker T\mathrm{pr}_F$. The connection 1–form of A_0 is $\theta_{A_0} = T\mathrm{pr}_F$. ♠

Example 2.24. Continue with the above situation. For every $a \in \mathrm{Hom}(\mathrm{pr}_B^* TB, \mathrm{pr}_F^* TF_B)$

$$A := \begin{pmatrix} \mathbf{1} & a \\ 0 & \mathbf{1} \end{pmatrix} \circ A_0$$

defines an Ehresmann connection on pr_B. In fact, every Ehresmann connection on pr_B is of this form. ♠

Example 2.25. Let X, B be smooth manifolds. If $p\colon X \to B$ is a covering map, then $V_p = 0$; hence: p admits a unique Ehresmann connection. ♠

Proposition 2.26. *Let $p\colon X \to B$ be a fibre bundle.*

(1) *A subbundle $H \subset TX$ is the horizontal subbundle of an Ehresmann connection if and only if the map $Tp\colon H \to p^*TB$ is an isomorphism.*

(2) *A 1–form $\theta \in \Omega^1(X, V_p)$ is the connection 1–form of an Ehresmann connection if and only if*

$$\theta \circ \iota = \mathrm{id}_{V_p}.$$

Proof. If $Tp\colon H \to p^*TB$ is an isomorphism, then $TX = V_p \oplus H$ and $A := \mathrm{id}_{V_p} \oplus Tp\colon TX = V_p \oplus H \to V_p \oplus p^*TB$ is an Ehresmann connection with $H_A = H$.

If $\theta \circ \iota = \mathrm{id}_{V_p}$, then $A = (\theta, Tp)\colon TX \to V_p \oplus p^*TB$ is an Ehresmann connection with $\theta_A = \theta$. ■

Definition 2.27. Let $p\colon X \to B$ be a fibre bundle. Let V be a vector bundle over X. A differential form $\alpha \in \Omega^\bullet(X, V)$ is **horizontal** if for every $v \in V_p$

$$i_v\alpha = 0.$$

The subspace of basic of basic differential forms is denoted by

$$\Omega^\bullet_{\mathrm{hor}}(X, V).$$ •

Proposition 2.28. *Let $p\colon X \to B$ be a fibre bundle. There is an Ehresmann connection A on p. If A_0, A are Ehresmann connections on p, then there is an $a \in \Omega^1_{\mathrm{basic}}(X, V_p) \subset \mathrm{End}(TX)$ such that*

$$A = A_0 + a;$$

moreover, for every $a \in \Omega^1_{\mathrm{hor}}(X, V_p)$, $A_0 + a$ is an Ehresmann connection.

Proof. Choose an open cover $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ of X such that for every $\alpha \in A$ there are a smooth manifold F_α and a diffeomorphism $\tau_\alpha\colon p^{-1}(U_\alpha) \to U_\alpha \times F_\alpha$ such that

$$\mathrm{pr}_1 \circ \tau_\alpha = p.$$

For every $\alpha \in A$ the 1–form $\theta_\alpha \in \Omega^1(p^{-1}(U_\alpha), V_p)$ defined by

$$\theta_\alpha := \tau_\alpha^*(T\mathrm{pr}_2).$$

Denote by $\{\chi_\alpha : \alpha \in A\}$ a partition of unity subordinate to $\mathscr{U}$. The 1–form θ defined by

$$\theta := \sum_{\alpha \in A} \chi_\alpha \cdot \theta_\alpha$$

satisfies $\theta \circ \iota = \mathrm{id}_{V_p}$; hence: it is the connection 1–form of an Ehresmann connection.

Let A_0 be an Ehresmann connection. $A \in \mathrm{Hom}(TX, VP \oplus p^*TB)$ defines an Ehresmann connection if and only if

$$A \circ A_0^{-1} = \begin{pmatrix} \mathbf{1} & * \\ 0 & \mathbf{1} \end{pmatrix}$$

This proves the remaining assertions. ∎

Remark. Here is a different proof of the existence part: Choose a Riemannian metric on X and let H be the orthogonal complement of V_p. ♣

Proposition 2.29. *Let $p\colon X \to B$ be a fibre bundle. Let $A \in \mathscr{A}(p)$. Let $f\colon A \to B$ be a smooth map. Denote by $f^*p\colon f^*X \to A$ the pullback of p via f and denote by $(\phi, f)\colon f^*p \to p$ the associated morphism.*

(1) *The map $T\phi\colon Tf^*X \to \phi^*TX$ induces an isomorphism*

$$\Psi\colon V_{f^*p} \to \phi^*V_p.$$

(2) *The 1–form*

$$\Psi^{-1} \circ \phi^*\theta_A \in \Omega^1(f^*X, V_{f^*p})$$

is the connection 1–form of an Ehresmann connection f^*A.

Proof. Since $p \circ \phi = f \circ (f^*p)$, $T\phi(V_{f^*p} \subset \phi^*V_p$. Therefore, $T\Phi$ induces a morphism $\Psi\colon V_{f^*p} \to V_p$. Since ϕ (tautologically) induces an diffeomorphism $(f^*p)(a) \to p^{-1}(f(a))$, Ψ is an isomorphism. This proves (1).

To prove (2), let $v \in V_{f^*p}$ and compute

$$(\Psi^{-1} \circ \phi^*\theta_A)(v) = \Phi^{-1}\theta_A(T\phi(v)) = \Phi^{-1}\theta_A(\Psi(v)) = \Phi^{-1}(\Psi(v)) = v.$$ ∎

Definition 2.30. In the situation of Proposition 2.29, f^*A is the **pullback of A via f**. •

2.4 Parallel transport

Definition 2.31. Let $p\colon X \to B$ be a fibre bundle. Let $A \in \mathscr{A}(p)$. Let $b_0 \in B$ and $x_0 \in p^{-1}(x_0)$. Let $v \in T_{b_0}B$. The A–**horizontal lift** of v to x_0 is the unique $\tilde{v} \in H_{A,x_0}$ with $T_{x_0}p(\tilde{v}) = v$. •

Definition 2.32. Let $p\colon X \to B$ be a fibre bundle. Let $A \in \mathscr{A}(p)$. A smooth map $f\colon A \to X$ is a A–**horizontal** if

$$\mathrm{im}\, T_x f \subset f^*H_A;$$

or, equivalently,

$$f^*\theta_A = 0.$$ •

The correspondence between lifts and sections of the pullback from Corollary 2.19 interacts with being horizontal as follows.

Proposition 2.33. *Let $p\colon X \to B$ be a fibre bundle. Let $f\colon A \to B$ be a smooth map. Denote by $f^*p\colon f^*X \to A$ the pullback of p. Denote by $\phi\colon f^*X \to X$ the canonical map. Let $A \in \mathscr{A}(p)$ and denote its pullback via f by f^*A. Composition with ϕ induces a bijection*

$$\{(f^*A)\text{–horizontal sections of } f^*X\} \to \{A\text{–horizontal lifts of } f\}, s \mapsto \phi \circ s.$$

Proof. A section $s\colon A \to f^*X$ of f^*p is f^*A–horizontal if and only if

$$0 = s^*\theta_{f^*A} = \Psi^{-1} \circ s^*\phi^*\theta_A = \Psi^{-1} \circ (\phi \circ s)^*\theta_A.$$

Since Ψ is an isomorphism, this is equivalent to $(\phi \circ s)^*\theta_A = 0$; that is: $(\phi \circ s)$ being A–horizontal. ■

Proposition 2.34. *Let $p\colon X \to B$ be a fibre bundle. Let $A \in \mathscr{A}(p)$. Let $\gamma\colon [0,1] \to B$ be a smooth path. For every $x_0 \in p^{-1}(\gamma(0))$ there is at most one horizontal lift $\tilde{\gamma}\colon [0,1] \to X$ with $\tilde{\gamma}(0) = x_0$.*

Proof. By Proposition 2.33, it suffices to consider $B = [0,1]$ and $\gamma = \mathrm{id}_B$. There is a unique A–horizontal vector field $v_A \in \Gamma(H_A) \subset \mathrm{Vect}(X)$ which is p–related to $\partial_t \in \mathrm{Vect}([0,1])$. A section $s\colon [0,1] \to X$ is A–horizontal if and only if it is an integral curve of v_A. The assertion therefore follows from the Picard–Lindelöf Theorem. ■

The above proof also tell us that constructing horizontal lifts of paths amounts to integrating a vector field.

Definition 2.35. Let $p\colon X \to B$ be a fibre bundle. An Ehresmann connection $A \in \mathscr{A}(p)$ is **complete** if for every smooth path $\gamma\colon [0,1]$ and every $x_0 \in p^{-1}(\gamma(0))$ there is an A–horizontal lift $\tilde{\gamma}\colon [0,1] \to X$ with $\tilde{\gamma}(0) = x_0$. •

Remark 2.36. The theory of Ehresmann connections developed so far does not require p to be a fibre bundle. It would suffice that p is a submersion so that $V_p = \ker Tp$ is a vector bundle of rank complementary to TB. However,if p admits a complete Ehresmann connection, then p is a fibre bundle. One can also prove that if p is a fibre bundle, then it admits a complete Ehresmann connection [dHoy16]. Of course, if p is proper, then every Ehresmann connection on p is complete. ♣

Definition 2.37. Let $p\colon X \to B$ be a fibre bundle. Let $A \in \mathscr{A}(p)$ be a complete Ehresmann connection. Let $\gamma\colon [0,1] \to B$ be a smooth path. The **parallel transport along** γ is the diffeomorphism $\mathrm{tra}_\gamma^A\colon p^{-1}(\gamma(0)) \to p^{-1}(\gamma(1))$ characterised by

$$\mathrm{tra}_\gamma^A(\tilde{\gamma}(0)) := \tilde{\gamma}(1)$$

for every A–horizontal lift $\tilde{\gamma}$ of γ. •

Example 2.38. For covering maps this reconstructs the transport map. ♠

Example 2.39 (Parallel transport and line integrals). Let B be a smooth manifold. Set $X := B \times \mathbf{R}$. Consider the trivial bundle $\mathrm{pr}_B\colon X \to B$. Identifying $V_{\mathrm{pr}_B} = \underline{\mathbf{R}}$ every $\alpha \in \Omega^1(B)$ defines a connection 1–form

$$\theta_A := T\mathrm{pr}_{\mathbf{R}} + \alpha.$$

A smooth path $\tilde{\gamma} =: (I, \gamma)\colon [0,1] \to X$ is A–horizontal if and only if

$$0 = \theta(\dot{\tilde{\gamma}}) = \dot{I} + \alpha(\dot{\gamma}) = \dot{I} + \gamma^*\alpha$$

or, equivalently,

$$I(t) = I(0) - \int_{[0,t]} \gamma^*\alpha.$$

Consequently,

$$\mathrm{tra}_\gamma^A(x_0, I_0) = \left(x_0, I_0 - \int_{[0,1]} \gamma^*\alpha\right). \qquad \spadesuit$$

Definition 2.40. Let $p\colon X \to B$ be a fibre bundle. Let $A \in \mathscr{A}(p)$. Let $b_0 \in B$. The **holonomy group of** A **based at** b_0 is the subgroup $\mathrm{Hol}_{b_0}(A) < \mathrm{Diff}(p^{-1}(b_0))$ defined by

$$\mathrm{Hol}_{b_0}(A) := \left\{\mathrm{tra}_\gamma^A : \gamma\colon [0,1] \to B \text{ piecewise smooth with } \gamma(0) = \gamma(1) = b_0\right\}. \qquad \bullet$$

2.5 Curvature

Example 2.39 illustrates that the parallel transport tra_γ^A depends on γ and not just its homotopy class rel $\{0,1\}$.

This illustrates that parallel transport depends on γ and not just its homotopy class rel $\{0,1\}$. It does not factor through the fundamental groupoid $\Pi_1(B)$. Of course, tra_γ^A is invariant under reparametrisation. Piecewise smooth paths up to reparametrisations build up the thin path groupoid $P_1(B)$. Parallel transport factors though $P_1(B)$.

Proposition 2.41. *Let $p\colon X \to B$ be a fibre bundle. Let $A \in \mathscr{A}(p)$. There is a unique horizontal 2–form $F_A \in \Omega^2_{\mathrm{hor}}(X, V_p)$ such that for every $v, w \in \mathrm{Vect}(X)$*

$$F_A(v, w) = -\theta_A([v - \theta_A(v), w - \theta_A(w)]).$$

F_A is the ***curvature of*** *A.*

Proof. For $f \in C^\infty(X)$ and $v, w \in \Gamma(H_A)$

$$\theta_A([fv, w]) = f\theta_A([v, w]) - \theta_A(\mathscr{L}_w f \cdot v) = f\theta_A([v, w]),$$

and, similarly, $\theta_A([v, fw]) = f\theta_A([v, w])$. Therefore, $v \wedge w \mapsto \theta_A([v, w])$ is tensorial. ∎

Example 2.42. Consider the Hopf bundle $p\colon S^{2n+1} \subset \mathbf{C}^{n+1}\backslash\{0\} \to \mathbf{C}P^n$. The vertical tangent bundle is spanned by the vector field $\partial_\alpha \in \mathrm{Vect}(S^{n+1})$ defined by

$$\partial_\alpha(z) = iz.$$

Define a connection A by

$$H_A := \{v \in TS^{2n+1} : v \perp \partial_\alpha\}.$$

Let $v, w \in \mathrm{Vect}(S^{2n+1})$ The curvature of A is

$$\begin{aligned} F_A(v, w) &= -\langle [v, w], iz\rangle \otimes iz \\ &= -\langle \nabla_v w - \nabla_w v, iz\rangle \otimes iz \\ &= -2\langle v, iw\rangle \otimes iz \\ &= -2\pi \cdot p^*\omega_{\mathrm{FS}} \otimes \partial_\alpha. \end{aligned}$$

Here $\omega_{\mathrm{FS}} \in \Omega^2(\mathbf{C}P^n)$ is the Fubini–Study form on $\mathbf{C}P^n$. ♠

Exercise 2.43. Let $U_a := \{[z_0, \dots, z_n] \in \mathbf{C}P^n : z_a \neq 0\}$ and define $\phi_a \colon U_a \to \mathbf{C}^n$ by

$$\phi([z_0, \dots, z_n]) := [z_0/z_a : \dots : \widehat{z_a/z_a} : \dots : z_n/z_a].$$

Prove that there is a unique 2–form ω_{FS} on $\mathbf{C}P^n$ satisfying

$$(\phi_a)_*\omega_{\mathrm{FS}} = \frac{i}{2\pi}\left(\sum_{b=1}^{n} \frac{\mathrm{d}z_b \wedge \mathrm{d}\bar{z}_b}{1 + |z|^2} - \sum_{b,c=1}^{n} \frac{\bar{z}_c\mathrm{d}z_c \wedge z_b\mathrm{d}\bar{z}_b}{(1 + |z|^2)^2}\right).$$

Prove that the above formula for F_A indeed holds. ?

Definition 2.44. Let $p\colon X \to B$ be a fibre bundle. Let $A \in \mathscr{A}(p)$. $F_A \in \Omega^2_{\mathrm{hor}}(X, V_p)$ **curvature of** A. The Ehresmann connection A is **flat** if $F_A = 0$. •

Proposition 2.45 (Flat connections and covering maps). *Let $p\colon X \to B$ be a fibre bundle. If $A \in \mathscr{A}(p)$ is complete and flat, then there is a covering map $q\colon S \to B$ and a bijective immersion $\iota\colon S \looparrowright X$ such that $q = p \circ \iota$ and $T\iota\colon TS \hookrightarrow \iota^*TX$ induces an isomorphism $TS \cong \iota^*H_A$; in particular:* tra_γ^A *depends only on the homotopy class rel $\{0, 1\}$ of γ.*

$$\begin{array}{ccc} S & \xrightarrow{\ \iota\ } & X \\ & \searrow^{q} & \downarrow p \\ & & B. \end{array}$$

Proof of Proposition 2.45. A is flat if and only if the distribution H_A is involutive. Frobenius's theorem guarantees the existence of a bijective immersion $\iota\colon S \looparrowright X$ such that $T\iota\colon TS \hookrightarrow \iota^*TX$ induces an isomorphism $TS \cong \iota^*H_A$.

To prove that $q := p \circ \iota$ is a covering map, let $b_0 \in B$ and let U be a connected, simply-connected, open neighborhood of b_0. Let V be a connected component of $S \cap \iota^{-1}(U)$. It remains to prove that $q|_V\colon V \to U$ is a diffeomorphism. By construction, $q|_V$ is a local diffeomorphism. Since U is path-connected, $q|_V$ is surjective. To prove that $q|_V$ is injective, let $s_0, s_1 \in q^{-1}(b_0) \cap V$. Since V is path-connected, there is a smooth path $\tilde{\gamma}\colon [0, 1] \to S$ with $\tilde{\gamma}(0) = s_0$ and $\tilde{\gamma}(1) = s_1$. Since U is simply-connected, there is a smooth homotopy

$\Gamma\colon [0,1]\times[0,1]\to B$ rel $\{0,1\}$ with $\Gamma(0,\cdot) = q\circ\tilde\gamma$ and $\Gamma(1,\cdot) = b_0$. The task at hand is to find an A–horizontal lift $\tilde\Gamma\colon [0,1]\times[0,1]\to X$ of Γ along p with $\tilde\Gamma(0,0) = x_0 := \iota(s_0)$.

By Proposition 2.33, it suffices to consider $B = [0,1]\times[0,1]$ and $\gamma = \mathrm{id}_B$. Denote by v_1, v_2 the A–horizontal lifts of ∂_1, ∂_2. The lift

$$\tilde\Gamma(t_1,t_2) := \mathrm{flow}_{v_1}^{t_1}\circ\mathrm{flow}_{v_2}^{t_2}(x_0)$$

maps into the maximal integral submanifold through x_0; hence, it is A–parallel. ■

Proposition 2.46. *Let $p\colon X\to B$ be a fibre bundle. Let $f\colon A\to B$ be a smooth map. Denote by $f^*p\colon f^*X\to A$ the pullback of p. Denote by $\phi\colon f^*X\to X$ the canonical map. Let $A\in\mathscr{A}(p)$ and denote its pullback via f by f^*A. The curvature of A and f^*A are related by*

$$F_{f^*A} = \Psi^{-1}\circ\phi^*F_A.$$

Proof sketch. Let $v,w\in H_{f^*A,x_0}$ and set $\tilde v := T\phi(v)$, $\tilde w := T\phi(w)$. Extend v,w and $\tilde v,\tilde w$ to ϕ–related vector field in H_{f^*A} and H_A. (This part of the proof actually is a little fishy. It is not obvious how to construct these. A clean way of doing this is to prove it only after the introduction of the Fröhlicher–Nijenhuis bracket; see Proposition 2.69.) Since $[v,w]$ and $[\tilde v,\tilde w]$ are ϕ–related, $F_{f^*A}([v,w]) = -\theta_{f^*A}([v,w]) = -\Psi^{-1}\circ\theta_A([\tilde v,\tilde w]) = \Psi^{-1}\circ\phi^*F_A(v,w)$. ■

Here is another (more direct) way to see that $F_A = 0$ implies homotopy-independence of parallel transport.

Lemma 2.47. *Let $p\colon X\to B$ be a fibre bundle. Let $A\in\mathscr{A}(p)$ be a complete Ehresmann connection. Let $b_0\in B$ and $x_0\in p^{-1}(b_0)$ Let $\Gamma\colon [0,1]\times[0,1]\to B$ with $\Gamma(0,0) = b_0$. Define $\tilde\Gamma\colon [0,1]\times[0,1]\to X$ by*

$$\tilde\Gamma(s,t) := \mathrm{tra}^A_{\Gamma(\cdot,t)|_{[0,s]}}\circ\mathrm{tra}^A_{\Gamma(0,\cdot)|_{[0,t]}}(x_0).$$

The derivative of $\tilde\Gamma(1,\cdot)$ satisfies

$$\partial_t\tilde\Gamma(1,\cdot) = \widetilde{\partial_t\Gamma}\circ\tilde\Gamma(1,\cdot) + \int_0^1 ((\mathrm{flow}^\sigma_{\widetilde{\partial_s\Gamma}})_*F_A(\widetilde{\partial_s\Gamma},\widetilde{\partial_t\Gamma}))\circ\tilde\Gamma(1,\cdot)\,\mathrm{d}\sigma.$$

Remark 2.48. The integrals in Lemma 2.47 measures the deviation of $\tilde\Gamma(1,\cdot)$ from being horizontal. ♣

Proof of Lemma 2.47. It suffices to prove this for $B = [0,1]^2$ and $\Gamma = \mathrm{id}_B$. Since $\tilde\Gamma(s,t) = \mathrm{flow}^s_{\tilde\partial_s}\circ\mathrm{flow}^t_{\tilde\partial_t}(x_0)$,

$$\partial_t\tilde\Gamma(1,\cdot) = ((\mathrm{flow}^1_{\tilde\partial_s})_*\tilde\partial_t)\circ\tilde\Gamma(1,\cdot).$$

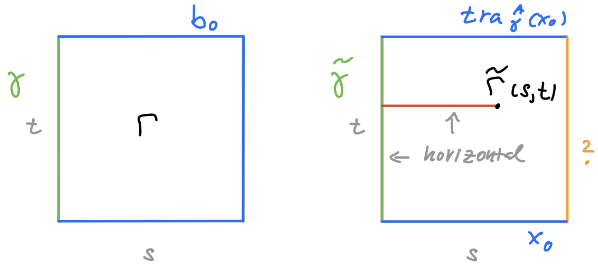


Figure 1: An illustration of Lemma 2.47 for a null-homotopy Γ.

Since $[\partial_s, \partial_t] = 0$,

$$\begin{aligned}
(\mathrm{flow}^1_{\tilde\partial_s})_*\tilde\partial_t &= \widetilde{(\mathrm{flow}^1_{\partial_s})_*\partial_t} + \theta_A((\mathrm{flow}^1_{\tilde\partial_s})_*\tilde\partial_t) \\
&= \tilde\partial_t + \int_0^1 \partial_\sigma \theta_A((\mathrm{flow}^\sigma_{\tilde\partial_s})_*\tilde\partial_t)\,\mathrm{d}\sigma \\
&= \tilde\partial_t - \int_0^1 \theta_A((\mathrm{flow}^\sigma_{\tilde\partial_s})_*[\tilde\partial_s, \tilde\partial_t])\,\mathrm{d}\sigma \\
&= \tilde\partial_t - \int_0^1 (\mathrm{flow}^\sigma_{\tilde\partial_s})_*\theta_A([\tilde\partial_s, \tilde\partial_t])\,\mathrm{d}\sigma \\
&= \tilde\partial_t + \int_0^1 (\mathrm{flow}^\sigma_{\tilde\partial_s})_*F_A(\tilde\partial_s, \tilde\partial_t)\,\mathrm{d}\sigma.
\end{aligned}$$ ■

Corollary 2.49. *Let $p\colon X \to B$ be a fibre bundle. Let $A \in \mathscr{A}(p)$. The parallel transport tra^A_γ depends only on the homotopy class rel $\{0,1\}$ of γ if and only if A is flat.* ■

2.6 Digression: Ehresmann connections and Riemannian metrics

Question 2.50. Let $p\colon X \to B$ be a fibre bundle. Let g be a Riemannian metric on X. Denote by A the Ehresmann connection with $H_A = V_p^\perp$. What is F_A?

Let us take a slightly broader perspective. Let X be a smooth manifold. Let $p\colon E \to X$ be a vector bundle equipped with a covariant derivative $\nabla\colon \Gamma(E) \to \Omega^1(X, E)$. Suppose a direct sum decomposition $E = E' \oplus E''$ given. Denote by $P' \in \mathrm{End}(E)$ the projection onto E' respectively. ∇ induces a covariant derivative

$$\nabla' := P' \circ \nabla\colon \Gamma(E') \to \Omega^1(X, E')$$

on E' respectively. If $s \in \Gamma(E')$, then

$$\nabla s = \nabla' s + \mathrm{II}' s \quad \text{with} \quad \mathrm{II}' := \nabla P' \in \Omega^1(X, \mathrm{Hom}(E', E'')).$$

To see this, observe that $(\mathbf{1} - P')\nabla s = \nabla((1 - P')s) - (\nabla(\mathbf{1} - P'))s$.

If $E = TX$ and ∇ is torsion-free, then for $v, w \in \Gamma(E') \subset \mathrm{Vect}(X)$

$$[v, w] = \nabla_v w - \nabla_w v = \nabla'_v w - \nabla'_w v + \mathrm{II}'(v)w - \mathrm{II}'(w)v.$$

To answer the initial question set $E' = H_A$, $E'' = V_p$, and choose ∇ to be the Levi-Civita connection of g. The above formula implies

$$F_A(v, w) = -\mathrm{II}'(v)w + \mathrm{II}'(w)v.$$

Therefore, A is flat if and only if the section of $\mathrm{Hom}(H_A \otimes H_A, V_p)$ induced by II' is symmetric. In fact, if A is flat, then the latter is the second fundamental form of the integral submanifold of H_A.

I recommend to read [Kar99] for more on the Riemannian geometry computations on fibre bundles. Karcher also gives a neat proof of Frobenius' theorem.

2.7 The Gauß–Manin connection

Let X be a manifold. The **de Rham complex** is $(\Omega^\bullet(X), \mathrm{d})$. Its cohomology $\mathrm{H}^\bullet(\Omega^\bullet(X), \mathrm{d})$ is the **de Rham cohomology** of X:

$$\mathrm{H}^k_{\mathrm{dR}}(X) := \frac{\ker(\mathrm{d}\colon \Omega^k(X) \to \Omega^{k+1}(X))}{\mathrm{im}(\mathrm{d}\colon \Omega^{k-1}(X) \to \Omega^k(X))}.$$

If $f\colon X \to Y$ is a smooth map, then $f^*\colon \Omega^*(Y) \to \Omega^*(X)$ descends to

$$f^*\colon \mathrm{H}^k_{\mathrm{dR}}(Y) \to \mathrm{H}^k_{\mathrm{dR}}(X).$$

The latter depends only on the homotopy class of f. If f is a diffeomorphism, then f^* is an isomorphism.

Let $p\colon X \to B$ be a fibre bundle. Set

$$\mathscr{H}^k_{\mathrm{dR}}(p) := \coprod_{b \in B} \mathrm{H}^k_{\mathrm{dR}}(p^{-1}(b)).$$

Denote by $q\colon \mathscr{H}^k_{\mathrm{dR}}(p) \to B$ the canonical projection. This can be given the structure of a flat vector bundle.

Let $\{U_\alpha : \alpha \in A\}$ be an open cover of B, and for every $\alpha \in A$ let F_α be a smooth manifold and let $\tau_\alpha\colon p^{-1}(U_\alpha) \to U_\alpha \times F_\alpha$ be a diffeomorphism such that $\mathrm{pr}_{U_\alpha} \circ \tau_\alpha = p$. Define

$$\phi_\alpha\colon q^{-1}(U_\alpha) = \coprod_{b \in U_\alpha} \mathrm{H}^k_{\mathrm{dR}}(p^{-1}(b)) \to U_\alpha \times \mathrm{H}^k_{\mathrm{dR}}(F_\beta).$$

by

$$\phi_\alpha(b, [\alpha]) := (b, (\tau_\alpha^{-1}(b, \cdot))^*[\alpha]).$$

These maps are bijections and the transition maps $\phi_\beta \circ \phi_\alpha^{-1}\colon (U_\alpha \cap U_\beta) \times F_\alpha \to (U_\alpha \cap U_\beta) \times F_\beta$ satisfy

$$\phi_\beta \circ \phi_\alpha^{-1}(b, [\alpha]) = \Big(b, \big(\tau_\alpha(b, \cdot) \circ \tau_\beta^{-1}(b, \cdot)\big)^*[\alpha]\Big).$$

The map $\psi_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{Hom}(\mathrm{H}^k_{\mathrm{dR}}(F_\alpha), \mathrm{H}^k_{\mathrm{dR}}(F_\alpha))$, $b \mapsto \big(\tau_\alpha(b, \cdot) \circ \tau_\beta^{-1}(b, \cdot)\big)^*$ is locally constant by the homotopy invariance of de Rham cohomology. Therefore, the ϕ_α define a smooth structure on $\mathscr{H}^k_{\mathrm{dR}}(p)$ which makes q into a flat vector bundle. Explicitly, the flat covariant derivative $\nabla\colon \Gamma(\mathscr{H}^k_{\mathrm{dR}}(p)) \to \Omega^1(B, \mathscr{H}^k_{\mathrm{dR}}(p))$ is defined as follows. Let $s \in \Gamma(\mathscr{H}^k_{\mathrm{dR}}(p))$. For $x \in U_\alpha$

$$\phi_\alpha \circ s(x) = (x, \tilde{s}_\alpha(x))$$

for some map $\tilde{s}_\alpha\colon U_\alpha \to \mathrm{H}^k_{\mathrm{dR}}(F_\alpha)$. These are related by

$$\tilde{s}_\beta(x) = \psi_{\beta\alpha}\tilde{s}_\alpha(x).$$

Since $\mathrm{d}\psi_{\beta,\alpha} = 0$,

$$\mathrm{d}\tilde{s}_\beta(x) = \psi_{\beta\alpha}(x)\mathrm{d}\tilde{s}_\alpha(x).$$

Therefore, there is a $\nabla s \in \Omega^1(X, \mathscr{H}^k_{\mathrm{dR}}(p))$ such that for $x \in U_\alpha$

$$\phi_\alpha \circ \nabla s(x) = (x, \mathrm{d}\tilde{s}_\alpha(x)).$$

The covariant derivative defined in this way is the **Gauß–Manin connection**.

Example 2.51. Let F be smooth manifold. Let $f \in \mathrm{Diff}(F)$. Denote X_f the **mapping torus** of f; that is:

$$X_f := ([0,1] \times F)/\sim$$

with denoting the equivalence relation generated by $(0, x) \sim (1, f(x))$. X_f is a smooth manifold and the projection map $p\colon X \to S^1 = \mathbf{R}/\mathbf{Z}$ is a fibre bundle. The monodromy of the Gauß–Manin connection on $\mathscr{H}^\bullet_{\mathrm{dR}}(p)$ is precisely the action of $\mathbf{Z}$ on $\mathrm{H}^\bullet_{\mathrm{dR}}(X)$ generated by f^*. ♠

Let $p\colon X \to B$ be a fibre bundle. Let $b_0 \in B$ and $\gamma\colon [0,1] \to B$ be a smooth loop based at b_0. The pullback $\gamma^*p\colon \gamma^*X \to [0,1]$ is isomorphic to a trivial bundle $\mathrm{pr}_{[0,1]}\colon [0,1] \times F \to [0,1]$. The chain of diffeomorphisms

$$p^{-1}(b_0) \cong (\gamma^*p)^{-1}(0) \cong F \cong (\gamma^*p)^{-1}(1) \cong p^{-1}(b_0)$$

defines a diffeomorphism $f \in \mathrm{Diff}(p^{-1}(b_0))$. The monodromy of the Gauß–Manin connection around γ is precisely $f^* \in \mathrm{End}(\mathrm{H}^\bullet_{\mathrm{dR}}(p^{-1}(b_0)))$.

Remark 2.52. A **local system** on X is a vector bundle E equipped with a flat connection ∇. For every local system $(\Omega^\bullet(X, E), \mathrm{d}_\nabla)$ is the **twisted de Rham complex**. Its cohomology is

$$\mathrm{H}^k_{\mathrm{dR}}(X, E) := \frac{\ker(\mathrm{d}_\nabla\colon \Omega^k(X,E) \to \Omega^{k+1}(X,E))}{\mathrm{im}(\mathrm{d}_\nabla\colon \Omega^{k-1}(X,E) \to \Omega^k(X,E))}.$$

The above shows that $\mathscr{H}^\bullet_{\mathrm{dR}}(p)$ is a local system on B. Its tempting to ask: what is the relation between $\mathrm{H}^\bullet_{\mathrm{dR}}(X)$ and $\mathrm{H}^\bullet_{\mathrm{dR}}(B, \mathscr{H}^\bullet_{\mathrm{dR}}(p))$? The answer to this question is (an instance of) the Leray–Serre spectral sequence. It will lead us to do far astray to discuss this in detail. An excellent reference is [BT82, §14]. ♣

2.8 Graded derivations of the exterior algebra

Our next goal is to understand how the de Rham complex $(\Omega^\bullet(X), \mathrm{d})$ refines if X is the total space of a fibre bundle equipped with an Ehresmann connection. To prepare this task it is helpful to first ponder graded derivations of $\Omega^\bullet(X)$. [KMS93, §8] contains a more detailed treatment of the material discussed below.

Definition 2.53. Let $k \in \mathbf{Z}$. A **graded derivation of degree** k on $\Omega^\bullet(X)$ is an $\mathbf{R}$–linear map $\delta \colon \Omega^\bullet(X) \to \Omega^{\bullet+k}(X)$ satisfying the **graded Leibniz rule**

$$\delta(\alpha \wedge \beta) = (\delta\alpha) \wedge \beta + (-1)^{k\cdot\ell}\alpha \wedge (\delta\beta)$$

for every $\alpha \in \Omega^\ell(X), \beta \in \Omega^\bullet(X)$. The graded derivations of $\Omega^\bullet(X)$ form a graded Lie algebra $\mathrm{Der}_\bullet(\Omega^\bullet(X))$. •

Exercise 2.54. Verify the last sentence in the above definition. ?

Example 2.55. The exterior derivative d is a graded derivation of degree 1 of $\Omega^\bullet(X)$. If $v \in \mathrm{Vect}(X)$, then i_v is a graded derivation of degree -1 of $\Omega^\bullet$. By Cartan's formula, their graded commutator is the Lie derivative

$$\mathscr{L}_v = \mathrm{d}i_v + i_v\mathrm{d} = [i_v, \mathrm{d}];$$

itself a graded derivation of degree 0. ♠

The derivations of $C^\infty(X)$ are precisely the vector fields on X: $\mathrm{Der}(C^\infty(X)) \cong \mathrm{Vect}(X)$. Is there an analogous result for $\mathrm{Der}_\bullet(\Omega^\bullet(X))$?

Definition 2.56. Let $k \in \mathbf{N}_0$. Denote by $i_\cdot \colon \Omega^{k+1}(X, TX) \to \mathrm{Der}_k(\Omega^\bullet(X))$ the unique linear map satisfying

$$i_{\xi\otimes v}\alpha = \xi \wedge i_v\alpha \quad \text{for} \quad \xi \in \Omega^{k+1}(X) \quad \text{and} \quad v \in \mathrm{Vect}(X).$$

Define $\mathscr{L}_\cdot \colon \Omega^k(X, TX) \to \mathrm{Der}_k(\Omega^\bullet(X))$ by

$$\mathscr{L}_\Xi := [i_\Xi, \mathrm{d}].$$ •

Exercise 2.57. Prove that ι and $\mathscr{L}$ are injective. ?

Exercise 2.58. For $\Xi = \sum_i \xi_i \otimes v^i \in \Omega^k(X, TX)$ prove that

$$\mathscr{L}_\Xi\alpha = \sum_i \xi_i \wedge \mathscr{L}_{v^i}\alpha + (-1)^k(\mathrm{d}\xi_i) \wedge i_{v^i}\alpha.$$?

Proposition 2.59. *Let X be a smooth manifold. Let $k \in \mathbf{Z}$. The map $\mathscr{L} + \iota \colon \Omega^k(X, TX) \oplus \Omega^{k+1}(X, TX) \to \mathrm{Der}_k(\Omega^\bullet(X))$ is an isomorphism. Moreover, $\delta \in \mathrm{im}\,\mathscr{L}$ if and only if $[\delta, \mathrm{d}] = 0$; and $\varepsilon \in \mathrm{im}\,\mathscr{L}$ if and only if $\varepsilon(\Omega^0(X)) = 0$.*

Proof. Every $\delta \in \mathrm{Der}_k(\Omega^\bullet(X))$ is determined by its restriction to $\Omega^0(X) \oplus \Omega^1(X)$. If $v_1, \ldots, v_k$, then the map

$$f \mapsto (\delta f)(v_1, \ldots, v_k)$$

is a derivation of $\Omega^0(X) = C^\infty(X)$. Hence, there is a unique vector field $\Xi(v_1, \ldots, v_k)$ such that

$$(\delta f)(v_1, \ldots, v_k) = \mathscr{L}_{\Xi(v_1,\ldots,v_k)} f.$$

A moment's thought shows that $(v_1, \ldots, v_k) \mapsto \Xi(v_1, \ldots, v_k)$ is tensorial. Therefore, it defines a $\Xi \in \Omega^k(X, TX)$. The derivation $\varepsilon := \delta - \mathscr{L}_\Xi$ vanishes on $\Omega^0(X)$.

If $f \in C^\infty(X)$ and $\alpha \in \Omega^1(X)$, then

$$\varepsilon(f\alpha) = f \cdot \varepsilon\alpha;$$

that is: $\varepsilon\colon \Omega^1(X) \to \Omega^k(X)$ is tensorial. Therefore, there is a $\Theta \in \Omega^k(X, TX)$ with $\varepsilon = i_\Theta$.

By construction, $\delta = i_\Theta + \mathscr{L}_\Xi$ on $\Omega^0(X) \oplus \Omega^1(X)$. This proves the first assertion. The vanishing criterion for Θ is obvious. A brief computation shows that $[\mathscr{L}_\Xi, \mathrm{d}] = 0$. Since

$$[i_\Theta, \mathrm{d}] = \mathscr{L}_\Theta,$$

the final assertion follows. ■

Exercise 2.60. What are Θ and Ξ for $\delta = \mathrm{d}$? ?

Exercise 2.61. Use Proposition 2.59 to prove Cartan's formula $\mathscr{L}_v = \mathrm{d}i_v + i_v\mathrm{d}$. ?

The identification can be used to define the Lie bracket $[\cdot,\cdot]$ on $\mathrm{Vect}(X)$. Since

$$[[\mathscr{L}_\Theta, \mathscr{L}_\Xi], \mathrm{d}] = 0,$$

one obtains a graded Lie bracket on $\Omega^\bullet(X, TX)$.

Definition 2.62. The **Fröhlicher–Nijenhuis bracket** is the map

$$[\cdot,\cdot]\colon \Omega^\bullet(X, TX) \otimes \Omega^\bullet(X, TX) \to \Omega^\bullet(X, TX)$$

characterised by

$$[\mathscr{L}_\Theta, \mathscr{L}_\Xi] = \mathscr{L}_{[\Theta,\Xi]}.$$

•

It turns out (somewhat miraculously in my opinion) that Fröhlicher–Nijenhuis bracket consistently shows up as an obstruction to integrability.

Proposition 2.63. *If $\theta \in \Omega^1(X, TX)$, then the* ***Nijenhuis tensor***

$$N_\theta := -\frac{1}{2}[\theta, \theta] \in \Omega^2(X, TX)$$

satisfies

$$N_\theta(v, w) = -\theta(\theta([v, w])) - [\theta(v), \theta(w)] + \theta([\theta(v), w] + [v, \theta(w)]).$$

The proof relies in the following

Proposition 2.64. *For $\theta, \alpha \in \Omega^1(X, TX)$ and $v, w \in \mathrm{Vect}(X)$*

$$(\mathscr{L}_\theta \alpha)(v, w) = \mathscr{L}_{\theta(v)}(\alpha(w)) - \mathscr{L}_{\theta(w)}(\alpha(v)) - \alpha([\theta(v), w]) - \alpha([v, \theta(w)]) + \alpha(\theta([v, w])).$$

Proof. Let $\theta, \alpha \in \Omega^1(X, TX)$ and $v, w \in \mathrm{Vect}(X)$. Since

$$(\mathrm{d}\alpha)(v, w) = \mathscr{L}_v(\alpha(w)) - \mathscr{L}_w(\alpha(v)) - \alpha([v, w]),$$

by definition of $\mathscr{L}_\theta$,

$$\begin{aligned}(\mathscr{L}_\theta \alpha)(v, w) &= (i_\theta \mathrm{d}\alpha - \mathrm{d}i_\theta a)(v, w) \\ &= \mathscr{L}_{\theta(v)}(\alpha(w)) - \mathscr{L}_w(\alpha(\theta(v))) - \alpha([\theta(v), w]) \\ &\quad + \mathscr{L}_v(\alpha(\theta(w))) - \mathscr{L}_{\theta(w)}(\alpha(v)) - \alpha([v, \theta(w)]) \\ &\quad - \mathscr{L}_v(\alpha(\theta(w))) + \mathscr{L}_w(\alpha(\theta(v))) + \alpha(\theta([v, w])) \\ &= \mathscr{L}_{\theta(v)}(\alpha(w)) - \alpha([\theta(v), w]) \\ &\quad - \mathscr{L}_{\theta(w)}(\alpha(v)) - \alpha([v, \theta(w)]) \\ &\quad + \alpha(\theta([v, w])).\end{aligned}$$

■

Proof of Proposition 2.63. $N_\theta \in \Omega^2(X, TX)$ is determined by the action on $\Omega^0(X) = C^\infty(X)$. For $\theta \in \Omega^k(TX, X)$

$$(\mathscr{L}_\theta f)(v_1, \ldots, v_k) = \mathscr{L}_{\theta(v_1, \ldots, v_k)} f;$$

in particular, $(\mathscr{L}_\theta f)(v) = \mathscr{L}_{\theta(v)} f$. Therefore, using Proposition 2.64,

$$\begin{aligned}(\mathscr{L}_\theta \mathscr{L}_\theta f)(v, w) &= \mathscr{L}_{\theta(v)} \mathscr{L}_{\theta(w)} f - \mathscr{L}_{\theta([\theta(v), w])} f \\ &\quad - \mathscr{L}_{\theta(w)} \mathscr{L}_{\theta(v)} f - \mathscr{L}_{\theta([v, \theta(w)])} f \\ &\quad + \mathscr{L}_{\theta(\theta([v, w]))} f \\ &= \mathscr{L}_{[\theta(v), \theta(w)]} f - \mathscr{L}_{\theta([\theta(v), w])} f \\ &\quad - \mathscr{L}_{\theta([v, \theta(w)])} f + \mathscr{L}_{\theta(\theta([v, w]))} f.\end{aligned}$$

This implies the assertion. ■

Remark 2.65. If $J \in \mathrm{End}(TX)$ is an almost complex structure (that is: $J^2 = -\mathbf{1}$), then the vanishing of N_J characterises the integrability of J. Indeed, the Newlander–Nirenberg theorem asserts that $N_J = 0$ if and only if X admits a holomorphic structure which induces the almost complex structure J. ♣

Exercise 2.66. Let $p \colon X \to B$ be fibre bundle. Let $A \in \mathscr{A}(p)$. Regard the connection 1–form $\theta_A \colon \Omega^1(X, V_p)$ as TX–valued 1–form. Prove that

$$F_A = N_{\theta_A}$$

?

Remark 2.67. The graded Jacobi identity implies the **Bianchi identity**

$$[\theta_A, F_A] = 0.$$

♣

Definition 2.68. Let X, Y be smooth manifolds Let $f\colon X \to Y$ be a smooth map.
$\Theta \in \Omega^k(X, TX)$ and $\Xi \in \Omega^k(Y, TY)$ are f**–related** if for every $x \in X, v_1, \ldots, v_k \in T_xX$

$$T_x f(\Theta(v_1, \ldots, v_k)) = \Xi(T_x f(v_1), \ldots, T_x f(v_k)). \qquad \bullet$$

Proposition 2.69. *Let X, Y be smooth manifolds Let $f\colon X \to Y$ be a smooth map. Let $\Theta_1, \Theta_2 \in \Omega^\bullet(X, TX)$ and $\Xi_1, \Xi_2 \in \Omega^\bullet(Y, TY)$. If Θ_i and Ξ_i are f–related, then $[\Theta_1, \Theta_2]$ and $[\Xi_1, \Xi_2]$ are f–related.*

Proof. Exercise. ■

2.9 Differential forms on fibre bundles

Definition 2.70. Let $p\colon X \to B$ be a fibre bundle. Let $A \in \mathscr{A}(p)$.

(1) For $p, q \in \mathbf{N}_0$ set

$$\Lambda_A^{p,q} T^*X := \Lambda^p H_A^* \otimes \Lambda^q V_p^* \quad \text{and} \quad \Omega_A^{p,q}(X) := \Gamma(\Lambda_A^{p,q} T^*X).$$

These define bi-gradings on $\Lambda^\bullet T^*X$ and $\Omega^\bullet(X)$.

(2) For $p, q \in \mathbf{Z}$ denote by $\mathrm{d}_A^{p,q}$ the component of d of bidegree (p, q).

(3) For every $b \in B$ denote by $\mathrm{res}_b\colon \Omega_A^{p,q}(X) \to \Lambda^p T_b^* B \otimes \Omega^q(p^{-1}(b))$ the composition of the maps

$$\Omega_A^{p,q}(X) \to \Gamma(p^{-1}(b), \Lambda^p H_A^* \otimes \Lambda^q V_p^*) \cong \Gamma(\Lambda^p T_b^* B \otimes \Lambda^q V_p^*) \cong \Lambda^p T_b^* B \otimes \Omega^q(p^{-1}(b)). \quad \bullet$$

Remark 2.71. Dualizing the exact sequence defining V_p yields

$$0 \to p^* T^* B \to T^* X \to V_p^* \to 0.$$

The image of p^*T^*B in T^*X is V_p^0, the annihilator of V_p. Consequently, $\Omega_A^{p,0}(X)$ is independent of A. Indeed: $\Omega_A^{p,0}(X) = \Omega_{\mathrm{hor}}^p(X)$. ♣

Proposition 2.72. *Let $p\colon X \to B$ be a fibre bundle. Let $A \in \mathscr{A}(p)$.*

(1) *The exterior derivative decomposes into three components of bidegree $(1, 0)$, $(0, 1)$, and $(2, -1)$:*

$$\mathrm{d} = \mathrm{d}_A^{1,0} + \mathrm{d}_A^{0,1} + \mathrm{d}_A^{2,-1}.$$

(2) *These components satisfy*

$$\mathrm{d}_A^{1,0} = \mathscr{L}_{\mathrm{id}_{TX} - \theta_A} - 2 i_{F_A}, \quad \mathrm{d}_A^{0,1} = \mathscr{L}_{\theta_A} + i_{F_A}, \quad \textit{and} \quad \mathrm{d}_A^{2,-1} = i_{F_A}.$$

(3) *For every* $b \in B$

$$\begin{array}{ccc}
\Omega_A^{p,q}(X) & \xrightarrow{\mathrm{d}_A^{0,1}} & \Omega_A^{p,q+1}(X) \\
\downarrow{\scriptstyle \mathrm{res}_b} & & \downarrow{\scriptstyle \mathrm{res}_b} \\
\Lambda^p T_b^* B \otimes \Omega^q(p^{-1}(b)) & \xrightarrow{\mathrm{id}\otimes \mathrm{d}} & \Lambda^p T_b^* B \otimes \Omega^{q+1}(p^{-1}(b)).
\end{array}$$

(4) *The operators* $\mathrm{d}_A^{1,0}$, $\mathrm{d}_A^{0,1}$, $\mathrm{d}_A^{2,-1}$ *satisfy*

$$\begin{aligned}
\left(\mathrm{d}_A^{1,0}\right)^2 + \mathrm{d}_A^{0,1}\mathrm{d}_A^{2,-1} + \mathrm{d}_A^{2,-1}\mathrm{d}_A^{0,1} &= 0, \\
\left(\mathrm{d}_A^{0,1}\right)^2 &= 0, \\
\left(\mathrm{d}_A^{2,-1}\right)^2 &= 0, \\
\mathrm{d}_A^{1,0}\mathrm{d}_A^{0,1} + \mathrm{d}_A^{0,1}\mathrm{d}_A^{1,0} &= 0, \quad \textit{and} \\
\mathrm{d}_A^{1,0}\mathrm{d}_A^{2,-1} + \mathrm{d}_A^{2,-1}\mathrm{d}_A^{1,0} &= 0.
\end{aligned}$$

Proof. The proof has several steps.

Step 1. $\mathrm{d}_A^{p,q} \in \mathrm{Der}_{p+q}(\Omega^\bullet(X))$ *and vanishes unless* $(p,q) \in \{(2,-1),(0,1),(1,0),(2,-1)\}$.

Since d is a graded derivation of $\Omega^\bullet(X)$, so are its components $\mathrm{d}_A^{p,q}$. Evidently, $\mathrm{d}_A^{p,q} = 0$ vanishes unless $p + q = 1$. A graded derivation of $\Omega^\bullet(X)$ is determined by its restriction to $\Omega^0(X) \oplus \Omega^1(X)$. Therefore, $\mathrm{d}_A^{p,q} = 0$ unless $p, q \geqslant -1$.

Step 2. $\mathrm{d}_A^{-1,2} = 0$ *and* $\mathrm{d}_A^{2,-1} = i_{F_A}$.

It suffices to verify these identities on $C^\infty(X)$ and $\Omega^1(X)$. For $f \in C^\infty(X)$

$$\mathrm{d}_A^{-1,2} f = 0 = \mathrm{d}_A^{2,-1} f$$

for degree reasons. Similarly, $\mathrm{d}_A^{-1,2}\alpha = 0$ for $\alpha \in \Omega_A^{1,0}(X)$ and $\mathrm{d}_A^{2,-1}\alpha = 0 = i_{F_A}\alpha$ for $\alpha \in \Omega_A^{0,1}(X)$. For $\alpha \in \Omega^{1,0}(X)$ and $v, w \in \mathrm{Vect}(X)$, since $\alpha \circ \theta_A = 0$,

$$\begin{aligned}
(\mathrm{d}_A^{-1,2}\alpha)(v,w) &= (\mathrm{d}\alpha)(\theta_A(v), \theta_A(w)) \\
&= -\alpha([\theta_A(v), \theta_A(w)]).
\end{aligned}$$

The latter vanishes since $[\theta_A(v), \theta_A(w)]$ is a vertical vector field. For $\alpha \in \Omega^{0,1}(X)$ and $v, w \in \mathrm{Vect}(X)$, since $\alpha \circ \theta_A = \alpha$,

$$\begin{aligned}
(\mathrm{d}_A^{-1,2}\alpha)(v,w) &= (\mathrm{d}\alpha)(v - \theta_A(v), v - \theta_A(w)) \\
&= -\alpha(\theta_A[v - \theta_A(v), w - \theta_A(w)]) \\
&= (i_{F_A}\alpha)(v,w).
\end{aligned}$$

The next two steps determine explicit formulae for $\mathrm{d}_A^{1,0}$ and $\mathrm{d}_A^{0,1}$. The computations are longish and not particularly illuminating. `Skip the proofs in class.`

Step 3. $\mathrm{d}_A^{1,0} = \mathscr{L}_{\mathrm{id}_{TX}-\theta_A} - 2i_{F_A}$.

It suffices to verify the identity on $C^\infty(X)$ and $\Omega^1(X)$.

For $f \in C^\infty(X)$

$$\begin{aligned}\mathrm{d}_A^{1,0} f &= \mathrm{d}f \circ (\mathrm{id}_{TX} - \theta_A) \\ &= (\mathscr{L}_{\mathrm{id}_{TX}-\theta_A} - 2i_{F_A})f.\end{aligned}$$

For $\alpha \in \Omega^1(X)$ and $v, w \in \mathrm{Vect}(X)$

$$\begin{aligned}(\mathscr{L}_{\mathrm{id}_{TX}-\theta_A}\alpha)(v,w) &= \mathscr{L}_{v-\theta_A(v)}(\alpha(w)) - \mathscr{L}_{w-\theta_A(w)}(\alpha(v)) \\ &\quad - \alpha([v-\theta_A(v), w]) - \alpha([v, w-\theta_A(w)]) + \alpha([v,w] - \theta_A([v,w])).\end{aligned}$$

For $\alpha \in \Omega_A^{1,0}(X)$ and $v, w \in \mathrm{Vect}(X)$, since $\alpha \circ \theta = 0$,

$$\begin{aligned}(\mathrm{d}_A^{1,0}\alpha)(v,w) &= \mathrm{d}\alpha(v - \theta_A(v), w - \theta_A(w)) \\ &= \mathscr{L}_{v-\theta_A(v)}(\alpha(w)) - \mathscr{L}_{w-\theta(w)}(\alpha(v)) - \alpha([v-\theta_a(v), w-\theta_a(w)]) \\ &= (\mathscr{L}_{\mathrm{id}_{TX}-\theta_A}\alpha - 2i_{F_A}\alpha)(v,w) \\ &\quad + \alpha([v-\theta_A(v), w]) + \alpha([v, w-\theta_A(w)]) \\ &\quad - \alpha([v,w]) - \alpha([v-\theta_a(v), w-\theta_a(w)]).\end{aligned}$$

The sum of the last four term vanishes because

$$[v-\theta_A(v), w] + [v, w-\theta_A(w)] - [v,w] - [v-\theta_a(v), w-\theta_a(w)] = -[\theta_A(v), \theta_A(w)]$$

is a vertical vector field.

For $\alpha \in \Omega_A^{0,1}(X)$ and $v, w \in \mathrm{Vect}(X)$, since $\alpha \circ \theta = \alpha$,

$$\begin{aligned}\mathrm{d}_A^{1,0}\alpha(v,w) &= \mathrm{d}\alpha(\theta_A(v), w-\theta_A(w)) + \mathrm{d}\alpha(v-\theta_A(v), \theta_A(w)) \\ &= \mathscr{L}_{v-\theta_A(v)}(\alpha(w)) - \mathscr{L}_{w-\theta_A(w)}(\alpha(v)) \\ &\quad - \alpha([\theta_a(v), w-\theta_a(w)]) - \alpha([v-\theta_a(v), \theta_a(w)]) \\ &= \big(\mathscr{L}_{\mathrm{id}_{TX}-\theta_A}\alpha - 2i_{F_A}\alpha\big)(v,w) \\ &\quad + \alpha([v-\theta_A(v), w]) + \alpha([v, w-\theta_A(w)]) \\ &\quad - 2\alpha([v-\theta_A(v), w-\theta(w)] \\ &\quad - \alpha([\theta_a(v), w-\theta_a(w)]) - \alpha([v-\theta_a(v), \theta_a(w)]).\end{aligned}$$

The sum of the last five terms vanishes.

Step 4. $\mathrm{d}_A^{0,1} = \mathscr{L}_{\theta_A} + i_{F_A}$.

For $f \in C^\infty(X)$

$$\begin{aligned} \mathrm{d}_A^{0,1} f &= \mathrm{d}f \circ \theta_A \\ &= (\mathscr{L}_{\theta_A} + i_{F_A})f. \end{aligned}$$

For $\alpha \in \Omega^1(X)$ and $v, w \in \mathrm{Vect}(X)$

$$\begin{aligned} (\mathscr{L}_{\theta_A}\alpha)(v,w) = \mathscr{L}_{\theta_A(v)}(\alpha(w)) - \mathscr{L}_{\theta_A(w)}(\alpha(v)) \\ - \alpha([\theta_A(v), w]) - \alpha([v, \theta_A(w)]) + \alpha(\theta_A([v,w])). \end{aligned}$$

For $\alpha \in \Omega_A^{1,0}(X)$ and $v, w \in \mathrm{Vect}(X)$, since $\alpha \circ \theta = 0$,

$$\begin{aligned} (\mathrm{d}_A^{0,1}\alpha)(v,w) &= \mathrm{d}\alpha(\theta_A(v), w - \theta_A(w)) + \mathrm{d}\alpha(v - \theta_A(v), \theta_A(w)) \\ &= \mathscr{L}_{\theta_A(v)}(\alpha(w)) - \mathscr{L}_{\theta_A(w)}(\alpha(v)) \\ &\quad - \alpha([\theta_A(v), w - \theta_A(w)]) - \alpha([v - \theta_A(v), \theta_A(w)]) \\ &= (\mathscr{L}_{\theta_A}\alpha + i_{F_A}\alpha)(v,w) \\ &\quad + \alpha([\theta_A(v), w]) + \alpha([v, \theta_A(w)]) \\ &\quad - \alpha([\theta_A(v), w - \theta_A(w)]) - \alpha([v - \theta_A(v), \theta_A(w)]). \end{aligned}$$

The sum of the last four term vanishes because

$$[\theta_A(v), w] + [v, \theta_A(w)] - [\theta_A(v), w - \theta_A(w)] - [v - \theta_A(v), \theta_A(w)] = 2[\theta_A(v), \theta_A(w)]$$

is a vertical vector field.

For $\alpha \in \Omega_A^{0,1}(X)$ and $v, w \in \mathrm{Vect}(X)$, since $\alpha \circ \theta = \alpha$,

$$\begin{aligned} (\mathrm{d}_A^{0,1}\alpha)(v,w) &= \mathrm{d}\alpha(\theta_A(v), \theta_A(w)) \\ &= \mathscr{L}_{\theta_A(v)}(\alpha(w)) - \mathscr{L}_{\theta(w)}(\alpha(v)) - \alpha([\theta_A(v), \theta_A(w)]) \\ &= (\mathscr{L}_{\theta_A}\alpha + i_{F_A}\alpha)(v,w) \\ &\quad + \alpha([\theta_A(v), w]) + \alpha([v, \theta_A(w)]) - \alpha([v,w]) \\ &\quad + \alpha([v - \theta_A(v), w - \theta_A(w)]) - \alpha([\theta_A(v), \theta_A(w)]). \end{aligned}$$

The sum of the last five term vanishes.

~

At this point (1) and (2) are established. (3) is obvious and (4) is a consequence of $\mathrm{d}^2 = 0$ and degree considerations. ■

It is evident from Proposition 2.72 that

$$E_1^{p,q} := \frac{\ker(\mathrm{d}_A^{0,1} \colon \Omega_A^{p,q}(X) \to \Omega_A^{p,q+1}(X))}{\mathrm{im}(\mathrm{d}_A^{0,1} \colon \Omega_A^{p,q-1}(X) \to \Omega_A^{p,q}(X))} \cong \Omega^p(B, \mathscr{H}_{\mathrm{dR}}^q(p)).$$

Since
$$\mathrm{d}_A^{1,0}\mathrm{d}_A^{0,1} + \mathrm{d}_A^{0,1}\mathrm{d}_A^{1,0} = 0 \quad \text{and} \quad (\mathrm{d}_A^{1,0})^2 = -(\mathrm{d}_A^{0,1}\mathrm{d}_A^{2,-1} + \mathrm{d}_A^{2,-1}\mathrm{d}_A^{0,1}),$$
$\mathrm{d}_A^{1,0}$ descends to an operator $\mathrm{d}_1^{0,1}$ on $E_1 \coloneqq \bigoplus_{p,q} E_1^{p,q}$ and satisfies $\left(\mathrm{d}_1^{0,1}\right)^2 = 0$. Indeed, $\mathrm{d}_1^{1,0}$ corresponds to d_∇ on $\Omega^p(B, \mathscr{H}^q_{\mathrm{dR}}(p))$ arising from the Gauß–Manin connection. The cohomology
$$E_2 \coloneqq \mathrm{H}(E_1, \mathrm{d}_1^{1,0}) \cong \mathrm{H}^\bullet_{\mathrm{dR}}(B, \mathscr{H}^\bullet_{\mathrm{dR}}(p))$$
does not typically compute $\mathrm{H}^\bullet_{\mathrm{dR}}(X)$. There is a differential $\mathrm{d}_2^{2,-1}$ on E_2 (whose computation requires a bunch of work). $E_3 \coloneqq \mathrm{H}(E_2, \mathrm{d}_2^{2,-1})$ does not typically compute $\mathrm{H}^\bullet_{\mathrm{dR}}(X)$ either. This procedure can be repeated indefinitely, after finitely many steps $\mathrm{d}_k^{k,k-1} = 0$, and $E_k = E_{k+1} = \cdots$ does compute $\mathrm{H}^\bullet_{\mathrm{dR}}(X)$. This is called the Leray–Serre spectral sequence. An excellent treatment for the Leray–Serre spectral sequence can be found in [BT82, §14]. The lecture notes [Bun] are also outstanding.

It is tempting to guess that the differential $\mathrm{d}_2^{2,-1}$ on E_2 corresponds to i_{F_A} and, therefore, if A is flat, then the Leray–Serre spectral sequence terminates at E_2. **This is false!** If $F_A = 0$, then $\mathrm{d} = \mathrm{d}_A^{0,1} + \mathrm{d}_A^{0,1}$ and $\Omega^\bullet(X)$ is a double complex. The cohomology of a double complex typically does not agree with the iterated cohomology.

Exercise 2.73. Define a double complex by
$$C^{p,q} \coloneqq \begin{cases} \mathbf{R} & \text{if } (p,q) \in \{(0,1),(0,2),(1,0),(1,1)\} \\ 0 & \text{otherwise} \end{cases},$$
with $\mathrm{d}^{1,0}\colon C^{0,1} \to C^{1,1}$, $\mathrm{d}^{0,1}\colon C^{0,1} \to C^{0,2}$, $\mathrm{d}^{0,1}\colon C^{1,0} \to C^{1,1}$ agreeing with $\mathrm{id}_\mathbf{R}$ and with zero otherwise. Set $\mathrm{d} \coloneqq \mathrm{d}^{1,0} + \mathrm{d}^{0,1}$. Compute
$$H(C,\mathrm{d}), \quad H(H(C,\mathrm{d}^{1,0}),\mathrm{d}^{0,1}), \quad \text{and} \quad H(H(C,\mathrm{d}^{0,1}),\mathrm{d}^{1,0}). \qquad ?$$

Example 2.74 ([Ban13, §6], [MS74, Appendix C]). $\mathbf{PSL}_2(\mathbf{R})$ acts on $H \coloneqq \{z \in \mathbf{C} : \operatorname{Im} z > 0\}$ by Möbius transformations.
$$\lambda_g(z) \coloneqq \frac{az+b}{cz+d} \quad \text{for} \quad g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$
Define $P\colon STH \to H \times (\mathbf{R} \cup \{\infty\})$ by
$$P(z,v) \coloneqq \lim_{t\to\infty} \exp_x(tv).$$
Here $\exp_x$ is computed with respect to the hyperbolic metric g_{-1} on H. `Draw a picture to illustrate this.` A brief computation reveals that
$$P(z,v) = \begin{cases} \infty & \text{if } v = i, \\ 0 & \text{if } v = -i, \\ z + \frac{\operatorname{Im} z}{\operatorname{Re} v}(1 - iv) & \text{otherwise.} \end{cases}$$
But that is not very helpful. $\mathbf{PSL}_2(\mathbf{R})$ acts on STH by
$$\Lambda_g(z,v) \coloneqq (\lambda_g(z), T_z\lambda_g(v))$$

and on $\mathbf{R} \cup \{\infty\}$ by Möbius transformations:

$$\lambda_g^\infty(x) := \frac{ax+b}{cx+d} \quad \text{for} \quad g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

A moments thought shows that P is $\mathbf{PSL}_2(\mathbf{R})$–equivariant; that is:

$$P \circ \Lambda_g = (\lambda_g \times \lambda_g^\infty) \circ P.$$

Let Σ be a Riemann surface of genus $g \geqslant 2$. By the uniformization theorem there is a $\Gamma < \mathbf{PSL}(2, \mathbf{R})$ such that $\Sigma = \Gamma \backslash H$. Evidently, $ST\Sigma = \Gamma \backslash STH$. By the above, P induces an fibre bundle isomorphism

$$\Gamma \backslash STH \cong H \times_\Gamma (\mathbf{R} \cup \{\infty\})$$

The latter inherits flat connection.

Using, for example, the Gysin sequence it can be proved that $H^1(ST\Sigma) \cong H^1(\Sigma)$. Therefore, the Leray–Serre spectral sequence can collapse at E_2. ♠

Despite all of the above, it is true that the Leray–Serre spectral sequences terminates at E_2 if X is compact and for some Riemannian metric on $F := p^{-1}(b_0)$ the monodromy of A acts by Riemannian isometries [Ban13, Theorem 5.2]. (If A has finite monodromy group G, then this can be derived from Künneth's formula and using G–invariance.)

Exercise 2.75. Prove that the Hopf bundle $p\colon S^{2n+1} \to \mathbf{C}P^n$ does not admit a flat connection. ?

2.10 Fibre integration

Definition 2.76. Let $p\colon X \to B$ be a fibre bundle. A **fibre orientation** on p is a an orientation on the line bundle $\det(V_p) \to X$. If B is oriented, then a fibre orientation on p induces an orientation on X via the above isomorphism. •

Proposition 2.77. *Let $d \in \mathbf{N}_0$. Let $p\colon X \to B$ be a fibre bundle with compact fibres of dimension d.*

(1) *There is a unique linear map $p_*\colon \Omega^\bullet(X) \to \Omega^{\bullet-d}(B)$ such that for every $\alpha \in \Omega^{d+k}(X)$, $b \in B$, and $\tilde v_1, \dots, \tilde v_k \in \Gamma(TX|_{p^{-1}(b)})$ lifts of $v_1, \dots, v_k \in T_bB$*

$$(p_*\alpha)(v_1, \dots, v_k) = \int_{p^{-1}(b)} i_{\tilde v_k} \cdots i_{\tilde v_1} \alpha|_{p^{-1}(b)}. \tag{2.78}$$

(2) *For every $\alpha \in \Omega^\bullet(X)$ and $\beta \in \Omega^\bullet(B)$*

$$p_*(\alpha \wedge p^*\beta) = p_*\alpha \wedge \beta.$$

(3) *Suppose that B is oriented. For every $\alpha \in \Omega^\bullet(X)$*

$$\int_X \alpha = \int_B p_*\alpha$$

(4) *Suppose that $\partial B = \varnothing$. Set $\partial p \coloneqq p|_{\partial X}\colon \partial X \to B$. For every $\alpha \in \Omega^\bullet(X)$*

$$p_*\mathrm{d}\alpha - (-1)^d \mathrm{d}p_*\alpha = \partial p_*\alpha.$$

Proof. The right-hand side of (2.78) is independent of the lifts $\tilde{v}_1, \ldots, \tilde{v}_k$. To verify that (2.78) does define the map p_* it suffices to it suffices the require smoothness. It is enough to verify this for $\mathrm{pr}_B\colon B \times F \to B$. This proves (1).

(2) is evident from the construction. (3) follows from Fubini's theorem.

(4) is a consequence of Stokes' theorem; indeed: for every $\alpha \in \Omega^{k+d}(X)$ and $\beta \in \Omega^\bullet(B)$

$$\begin{aligned}\int_B (p_*\mathrm{d}\alpha) \wedge \beta &= \int_B p_*(\mathrm{d}\alpha \wedge p^*\beta) \\ &= \int_X \mathrm{d}\alpha \wedge p^*\beta \\ &= \int_{\partial X} \alpha \wedge p^*\beta + (-1)^{k+d} \int_X \alpha \wedge p^*\mathrm{d}\beta \\ &= \int_{\partial X} \alpha \wedge p^*\beta + (-1)^{k+d} \int_B (p_*\alpha) \wedge \mathrm{d}\beta \\ &= \int_{\partial X} \alpha \wedge p^*\beta + (-1)^{d} \int_B (\mathrm{d}p_*\alpha) \wedge \beta.\end{aligned}$$ ∎

Definition 2.79. In the situation of Proposition 2.77, the map p_* is the **fibre integration**. •

If $\partial X = \partial B = \varnothing$, then p_* descends to de Rham homology $p_*\colon \mathrm{H}^\bullet_{\mathrm{dR}}(X) \to \mathrm{H}^{\bullet-d}_{\mathrm{dR}}(B)$. Set $K^\bullet \coloneqq \ker p_*\colon \Omega^\bullet(X) \to \Omega^{\bullet-d}(B)$. The short exact sequence

$$0 \to K^\bullet \to \Omega^\bullet(X) \to \Omega^{\bullet-d}(B) \to 0$$

induces a long exact sequence

$$\cdots \to \mathrm{H}^k(K^\bullet) \to \mathrm{H}^k_{\mathrm{dR}}(X) \to \mathrm{H}^{k-d}_{\mathrm{dR}}(B) \xrightarrow{\delta} \mathrm{H}^{k+1}(K^\bullet) \cdots.$$

If p is an oriented sphere bundle, then the map $p^*\colon \mathrm{H}^k(B) \to \mathrm{H}^k(K^\bullet)$ is an isomorphism. The Bockstein homomorphism δ arises from taking the wedge product with the **Euler class** of p (more about that later). This instance of the above elong exact sequence is the **Gysin sequence**.

3 Lie groups

In this section I will introduce (review?) the concept of a Lie group, that is, a group in the category of manifolds. For the purpose of this course Lie groups will be a tool and a source of examples of manifolds. The theory of Lie groups is a vast subject and we will not even scrape the surface. A good reference is Bump [Bum13].

3.1 Definition

Definition 3.1. A **Lie group** is a smooth manifold G together with a group structure on G such that the maps $m\colon G \times G \to G$ defined by $m(g,h) \coloneqq g \cdot h$, and $i\colon G \to G$ defined by $i(g) \coloneqq g^{-1}$ are smooth. Let G and H be Lie groups. A **Lie group homomorphism** from G to H is a smooth group homomorphism $\rho\colon G \to H$. •

Example 3.2. $S^1 = \mathbf{R}/\mathbf{Z}$, $\mathrm{GL}_n(\mathbf{R})$, $\mathrm{GL}_n(\mathbf{C})$, $\mathrm{O}(n)$, $\mathrm{U}(n)$, $\mathrm{SO}(n)$, $\mathrm{SU}(n)$ are Lie groups. ♠

Example 3.3. Let V be a vector space. If $\omega \in \Lambda^2 V^*$ is a non-degenerate 2–form on V, then $H = H(V,\omega)$, the **Heisenberg group** of (V,ω), is defined by $H \coloneqq \mathrm{U}(1) \times V$ with the group operation

$$(e^{i\alpha}, v) \cdot (e^{i\beta}, w) \coloneqq (e^{i\alpha + i\beta + 2\pi i \omega(v,w)}, v + w).$$

H is a Lie group. ♠

Theorem 3.4 (`reference?`). *Let G be a Lie group. Let $H < G$ be a subgroup. If H is closed, then it is a submanifold; hence: H is a Lie group.*

Theorem 3.5 (Yamabe [Yam50]; see also Goto [Got69]). *Let G be a Lie group. Let $H < G$ be a subgroup. If H is path-connected, then H is an immersed submanifold; hence: H is a Lie group.*

3.2 Lie group actions

Definition 3.6. Let X be a smooth manifold. Let G be a Lie group.

(1) A **(left) action** of G on X is a smooth map $L\colon G \times X \to X$ satisfying

$$L(\mathbf{1}, \cdot) = \mathrm{id}_X \quad \text{and} \quad L(g, L(h, \cdot) = L(gh, \cdot).$$

Define $L_g \in \mathrm{Diff}(X)$ by $L_g \coloneqq L(g, \cdot)$ and abbreviate $g \cdot x = L(g,x)$.

(2) The **orbit** of $x \in X$ is

$$G \cdot x \coloneqq \{g \cdot x : g \in G\}.$$

(3) The **stabiliser** of $x \in X$ is

$$G_x \coloneqq \{g \in G : g \cdot x = x\}.$$

(4) The action of G on X is **free** if $G_x = \mathbf{1}$ for every $x \in X$.

(5) The action of G on X is **proper** if the map $(L, \mathrm{pr}_X)\colon G \times X \to X \times X$ is proper. •

(6) A **right action** of G on X is a smooth map $R\colon X \times G \to G$ satisfying

$$R(\cdot, \mathbf{1}) = \mathrm{id}_X \quad \text{and} \quad R(R(\cdot, g), h) = R(\cdot, gh).$$

Set $R_g(\cdot) \coloneqq R(\cdot, g)$ and abbreviate $x \cdot g \coloneqq R(x, g)$. If R is a right action, then $L(g,x) \coloneqq R(x, g^{-1})$ defines a left action. The notions orbit, stabiliser, free, proper carry over to right actions in the obvious way.

In this section, actions are assumed to be left actions unless explicitly stated otherwise.

Example 3.7. If G is a Lie group, then G acts on itself on the left by left multiplication $L\colon\ G\times G \to G$,

$$L(g,h) \coloneqq g\cdot h.$$

The same formula also defines the action of G on itself on the right by right multiplication $R\colon\ G\times G$,

$$R(h,g) = h\cdot g.$$

These actions commute and G acts on itself on the left by conjugation $C\colon\ G\to G\to G$,

$$C(g,h) \coloneqq ghg^{-1}.$$ ♠

Exercise 3.8. Let G be a Lie group. Let $H < G$ be a closed subgroup. Prove that the action of H on G is free and proper. ?

Example 3.9. $\mathrm{U}(1)$ acts on S^{2n+1} via $e^{i\alpha}\cdot z = e^{i\alpha}z$. ♠

Example 3.10. Let $\theta\in\mathbf{R}$. $\mathbf{R}$ acts on $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ via $L(t,[x,y]) \coloneqq [x+t, y+\theta t]$. ♠

3.3 The slice theorem

Definition 3.11. Let X be a manifold. Let G be a Lie group acting on X. A **quotient** of X by G is a smooth manifold X/G together with a smooth G–invariant map $p\colon\ X\to X/G$ such that every G–invariant map $f\colon\ X\to Y$ uniquely factors though p. •

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle p} & \nearrow & \\ X/G. & & \end{array}$$

$$C^\infty(X/G,\cdot) \cong C^\infty(X,\cdot)^G$$

Which actions admit quotients?

Proposition 3.12. *Let X be a manifold. Let G be a Lie group. If G acts freely and properly on X, then it admits a quotient.*

Proof assuming that G is compact. Denote by X/G the topological quotient space and by $p\colon\ X\to X/G$ the projection map. X/G is paracompact and Hausdorff, and p is open. (Exercise!)

Let $x\in X$. The map $G\to X, g\mapsto gx$ is a proper injective immersion. Therefore, the orbit $G\cdot x\subset X$ is a submanifold. Choose a G–invariant metric g on X. (This is a red herring. The proof requires no Riemannian geometry, but it psychologically helpful.) Identify

$$N_x(G\cdot x)\cong T_x(G\cdot x)^\perp\subset T_xX$$

For $\varepsilon > 0$ set $V_x := B_\varepsilon(0) \subset N_x(G \cdot x)$ and define $\jmath_x \colon G \times V_x \to X$ by

$$\jmath_x(g, v) := g \exp_x(v).$$

Provided $\varepsilon \ll 1$, $\jmath$ is a G–equivariant embedding. Set $S_x := \jmath_x(\{\mathbf{1}\} \times V_x)$ and $U_x := p(\tilde{U}_x)$. The map $p|_{S_x} \colon S_x \to U_x$ is a homeomorphism. Define $\phi_x \colon U_x \to V_x$ by

$$\phi_x := \mathrm{pr}_{V_x} \circ \jmath_x^{-1} \circ \left(p|_{S_x}\right)^{-1}.$$

The task at hand is to prove that the maps ϕ_x form a smooth atlas. Let $x, y \in X$. $U_x \cap U_y \neq \varnothing$ if and only if $(G \cdot S_x) \cap S_y \neq \varnothing$. By construction, $(g \cdot S_x) \cap S_x = \varnothing$ unless $g = 1$. Therefore, there is a unique map $\gamma_x^y \colon (G \cdot S_x) \cap S_y \to G$ satisfying $\gamma_x^y(z) \cdot z \in S_x$ or, equivalently, $\mathrm{pr}_G \circ \jmath_y^{-1}(\gamma_x^y(z) \cdot z) = \mathbf{1}$. By the implicit function theorem, γ_x^y is smooth. A moment's thought shows that the transition map $\phi_x \circ \phi_y^{-1}$ satisfies

$$\phi_x \circ \phi_y^{-1}(z) = \mathrm{pr}_{V_x} \circ \jmath_x\big(\gamma_x^y(\jmath_y^{-1}(\mathbf{1}, z)) \cdot \jmath_y^{-1}(\mathbf{1}, z)\big).$$

Therefore, it is smooth. This finishes the construction of the smooth atlas on X/G.

The universal property is evident from the construction. ■

Remark 3.13. For non-compact G one first proves that $G \cdot x$ is a submanifold and then produces an S_x in some (quite arbitrary way). ♣

Definition 3.14. A **homogeneous space** is a smooth manifold X together with a transitive G action. •

Proposition 3.15. *If X is a homogeneous space, then the map $G/G_{x_0} \to X$ defined induced by $g \mapsto g \cdot x_0$ is a diffeomorphism.* ■

Example 3.16. $\mathbf{C}P^n \cong S^{2n+1}/\mathrm{U}(1)$. ♠

Example 3.17. $\mathrm{Gr}_r(\mathbf{R}^n) \cong O(n)/(\mathrm{O}(r) \times \mathrm{O}(n-r))$. ♠

`For future versions: One could do a lengthy discussion of the slice theorem and its consequences here.`

3.4 Lie algebra

Proposition 3.18. *Let G be a Lie group. Denote by*

$$\mathrm{Lie}(G) := \mathrm{Vect}(G)^L := \{\xi \in \mathrm{Vect}(G) : L_g^*\xi = \xi \text{ for every } g \in G\}.$$

the space of left-invariant vector fields on G.

(1) $\mathrm{Lie}(G) \subset \mathrm{Vect}(G)$ *is a Lie subalgebra.*

(2) *For $g \in G$ and $\xi \in \mathrm{Lie}(G)$, $R_g^*\xi \in \mathrm{Lie}(G)$.*

Proof. (1) is obvious. (2) holds because R_g and L_g commute. ■

Definition 3.19. Let G be a Lie group. The **Lie algebra** is the Lie algebra of left-invariant vector fields:

$$\mathrm{Lie}(G) := \mathrm{Vect}(G)^L$$

The **adjoint representation** $\mathrm{Ad}\colon G \to \mathrm{End}(\mathrm{Lie}(G))$ is defined by

$$\mathrm{Ad}(g)\xi := R_g^*\xi.$$

The **adjoint representation** $\mathrm{ad}\colon \mathrm{Lie}(G) \to \mathrm{End}(\mathrm{Lie}(G))$ is defined by

$$\mathrm{ad}(\xi)\eta := [\xi, \eta].$$

•

Proposition 3.20. *Let G be a Lie group.*

(1) *The evaluation map* $\mathrm{ev}_{\mathbf{1}}\colon \mathrm{Vect}(G)^L \to T_{\mathbf{1}}G$ *is an isomorphism.*

(2) *For* $g \in G$ *and* $\xi \in \mathrm{Vect}(G)^L$

$$\mathrm{Ad}(g)\xi = \mathrm{ev}_{\mathbf{1}}^{-1} \circ T_{\mathbf{1}}C_g \circ \mathrm{ev}_{\mathbf{1}}(\xi).$$

(3) *For* $\xi, \eta \in \mathrm{Vect}(G)^L$

$$T_{\mathbf{1}}\,\mathrm{Ad}(\mathrm{ev}_{\mathbf{1}}(\xi))\eta = [\xi, \eta].$$

(4) *If* $\rho\colon G \to H$ *is a Lie group homomorphism, then* $\mathrm{Lie}(\rho)\colon \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ *defined by*

$$\mathrm{Lie}(\rho) = \mathrm{ev}_{\mathbf{1}}^{1} \circ T_{\mathbf{1}}\rho \circ \mathrm{ev}_{\mathbf{1}}$$

is a Lie algebra homomorphism.

Proof. A left-invariant vector field v satisfies

$$v_g = T_{\mathbf{1}}L_g(v_{\mathbf{1}}).$$

Therefore, it is determined by $v_{\mathbf{1}}$. Conversely, the above formula defines a left-invariant vector field. This proves (1).

To prove (2), compute

$$\begin{aligned}\mathrm{ev}_{\mathbf{1}}(R_g^*\xi) &= T_gR_{g^{-1}}(\xi_g)\\ &= T_gR_{g^{-1}}T_{\mathbf{1}}L_g(\xi_{\mathbf{1}})\\ &= T_{\mathbf{1}}C_g(\xi_{\mathbf{1}}).\end{aligned}$$

To prove (3), observe that

$$\begin{aligned}\mathrm{flow}_\xi^t(g) &= \mathrm{flow}_\xi^t(L_g(\mathbf{1}))\\ &= L_g(\mathrm{flow}_\xi^t(\mathbf{1}))\\ &= R_{\mathrm{flow}_\xi^t(\mathbf{1})}g.\end{aligned}$$

Therefore,

$$\begin{aligned} T_1 \operatorname{Ad}(\mathrm{ev}_1(\xi))\eta &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} R^*_{\mathrm{flow}^t_\xi(1)}(\eta) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\mathrm{flow}^t_\xi)^*(\eta) \\ &= [\xi, \eta]. \end{aligned}$$

To prove (4), observe that by (2)

$$\operatorname{Ad}(\rho(g)) \circ \operatorname{Lie}(\rho)(\xi) = \operatorname{Lie}(\rho) \circ \operatorname{Ad}(g)(\xi).$$

By (3), this implies that $\operatorname{Lie}(\rho)$ is a Lie algebra homomorphism. ■

The following gadget turns out to be important for us later.

Definition 3.21. Let G be a Lie group. The **Maurer–Cartan form** $\mu \in \Omega^1(G, \operatorname{Lie}(g))$ is defined by

$$\mu_g(\xi) \coloneqq \mathrm{ev}_1^{-1} \circ T_g L_{g^{-1}}(\xi). \qquad \bullet$$

Proposition 3.22. *Let G be a Lie group.*

(1) *The Maurer–Cartan form μ satisfies $\mu(\xi) = \xi$ for every $\xi \in \operatorname{Lie}(G)$.*

(2) *For every $g \in G$*

$$R_g^*\mu = \operatorname{Ad}(g^{-1}) \circ \mu.$$

(3) *The Maurer–Cartan form μ satisfies the **Maurer–Cartan equation***

$$\mathrm{d}\mu + \tfrac{1}{2}[\mu \wedge \mu] = 0$$

Proof. (1) is obvious.

To prove (2), for $g \in G$ and $\xi \in \operatorname{Lie}(G)$ compute

$$(R_g^*\mu)(\xi) = \mu((R_g)_*\xi) = (R_{g^{-1}})^*\xi = \operatorname{Ad}(g^{-1})\xi.$$

To prove (3), compute

$$\begin{aligned} \big(\mathrm{d}\mu + \tfrac{1}{2}[\mu \wedge \mu]\big)(\xi, \eta) &= \mathscr{L}_\xi(\mu(\eta)) - \mathscr{L}_\eta(\mu(\xi)) - \mu([\xi, \eta]) \\ &\quad + \frac{1}{2}([\mu(\xi), \mu(\eta)] - [\mu(\xi), \mu(\eta)]) \\ &= 0. \end{aligned}$$

■

Exercise 3.23. Let $\rho\colon G \to H$ be a Lie group homomorphism. Prove that

$$\rho^*\mu_H = \operatorname{Lie}(\rho) \circ \mu_G. \qquad ?$$

3.5 Exponential map

Definition 3.24. Let G be a Lie group. The **exponential map** $\exp\colon \operatorname{Lie}(G) \to G$ is defined by

$$\exp(\xi) := \operatorname{flow}^1_\xi(\mathbf{1}).$$

•

Exercise 3.25. (1) Prove that exp is well-defined.

(2) Let $\rho\colon G \to H$ be a Lie group homomorphism. Prove that

$$\rho \circ \exp(\xi) = \exp \circ \operatorname{Lie}(\rho)(\xi).$$

(3) Prove that

$$C_g \circ \exp = \exp \circ \operatorname{Ad}_g .$$

?

Definition 3.26. Let X be smooth manifold. Let G be a Lie group. Let $L\colon G\times X \to X$ be a smooth left action. The **infinitesimal action** of $\operatorname{Lie}(G)$ on X is the map $v = v^L\colon \operatorname{Lie}(G) \to \operatorname{Vect}(X)$ defined by

$$v_\xi(x) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} L_{\exp(t\xi)}(x).$$

•

Proposition 3.27. *Let X be smooth manifold. Let G be a Lie group. Let $L\colon G\times X \to X$ be a smooth left action. Denote by $v\colon \operatorname{Lie}(G) \to \operatorname{Vect}(X)$ the corresponding infinitesimal action.*

(1) *For every $\xi \in \operatorname{Lie}(G)$*

$$L_{\exp(t\xi)} = \operatorname{flow}^t_{v_\xi} .$$

(2) *For every $g \in G$ and $\xi \in \operatorname{Lie}(G)$*

$$v_{\operatorname{Ad}(g)\xi} = L^*_{g^{-1}} v_\xi$$

(3) *The infinitesimal action v is an Lie algebra anti-isomorphism; that is: for every $\xi, \eta \in \operatorname{Lie}(G)$*

$$v_{[\xi,\eta]} = -[v_\xi, v_\eta].$$

Remark 3.28. If R is a right action and L is the corresponding left-action, then $v^R = -v^L$. In particular, v^R it is a Lie algebra homomorphism. ♣

Proof. (1) is obvious.

To prove (2), compute

$$\begin{aligned} v_{\operatorname{Ad}(g)\xi}(x) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} L_{g\exp(t\xi)g^{-1}}(x) \\ &= T_{L_g(x)}L_g\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} L_{\exp(t\xi)}L_{g^{-1}}(x)\right) \\ &= T_{L_g(x)}L_g\Big(v_\xi(L_{g^{-1}}(x))\Big) \\ &= (L^*_{g^{-1}}v_\xi)(x). \end{aligned}$$

To prove (3), differentiate

$$v_{\operatorname{Ad}(\exp(t\xi))\eta} = L^*_{\exp(-t\xi)}v_\eta = \left(\operatorname{flow}^t_{v_\xi}\right)^* v_\eta.$$

■

3.6 Haar volume form

Proposition 3.29. *Let G be a Lie group. Set $d \coloneqq \dim G$. There is a unique left-invariant volume form up to multiplication by a non-zero constant:*

$$\dim \Omega^d(G)^L \coloneqq \{\nu \in \Omega^d(G) : L_g^*\nu = \nu\} = 1.$$

Definition 3.30. Let G be a Lie group. A **Haar volume form** on G is a left-invariant volume form ν on G. ν is **normalised** if $\int_G \nu = 1$. •

Proof of Proposition 3.29. If $\nu \in \Omega^d(G)$ is left-invariant, then

$$\nu_g = \nu_1 \circ \Lambda^d T_g L_{g^{-1}}.$$

Therefore, ν is uniquely determined by $\nu_1 \in \Lambda^d T_1^* G$. Conversely, every $\nu_1 \in \Lambda^d T_1^G$ determines a left-invariant $\nu \in \Omega^d(G)$. ■

Exercise 3.31. Let G be a Lie group. Let ν be a Haar volume form on G. For every $g \in G$, $R_g^*\nu$ is a Haar volume form. The **modular function** of G is the function $\Delta \in C^\infty(G, \mathbf{R}^\times)$ defined by

$$\Delta(g) \coloneqq \frac{R_g^*\nu}{\nu}.$$

(1) Prove that $\Delta = 1$ if and only if G admits a right-invariant Haar measure. These groups are **unimodular**.

(2) Prove that $\Delta\colon G \to \mathbf{R}^\times$ is a Lie group homomorphism.

(3) Prove that $\Delta = 1$ if G is compact.

(4) Define $i\colon G \to G$ by $i(g) \coloneqq g^{-1}$. Prove that $i^*\nu = \Delta\nu$.

(5) Consider the Lie group

$$G \coloneqq \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} : x > 0, y \in \mathbf{R} \right\}.$$

Compute modular function of G. ?

3.7 The Killing form

Definition 3.32. Let $\mathfrak{g}$ be a Lie algebra. The **Killing form** $B \in S^2\mathfrak{g}^*$ is defined by

$$B(\xi, \eta) \coloneqq \operatorname{tr}(\operatorname{ad}(\xi) \circ \operatorname{ad}(\eta)).$$ •

Exercise 3.33. Prove that

$$B([\xi, \eta], \zeta) = B(\eta, [\xi, \zeta]).$$?

Definition 3.34. A Lie algebra is called **semisimple** if B is negative definite. G **semisimple** if $\operatorname{Lie}(G)$ is semisimple. •

Exercise 3.35. Prove that if $\mathfrak{g} = \mathfrak{gl}(n)$, then

$$B(\xi, \eta) = 2n \operatorname{tr}(\xi\eta) - 2 \operatorname{tr}(\xi) \operatorname{tr}(\eta).$$?

Exercise 3.36. Prove that if $\mathfrak{g} = \mathfrak{su}(n)$, then

$$B(\xi, \eta) = 2n \operatorname{tr}(\xi\eta).$$?

3.8 de Rham cohomology of manifolds with G–actions

Let X be a manifold. Let G be a Lie group. Let $L\colon G \times X \to X$ be an action. Such an action can tremendously simplify the computation of $\mathrm{H}^\bullet_{\mathrm{dR}}(X)$. To see this define

$$\Omega^\bullet(X)^L \coloneqq \{\alpha \in \Omega^\bullet(X) : L_g^*\alpha = \alpha \text{ for every } g \in G\}.$$

Exercise 3.37. Prove that $\mathrm{d}\Omega^\bullet(X)^L \subset \Omega^\bullet(X)^L$. ?

We have an inclusion of cochain complexes $i\colon \Omega^\bullet(X)^L \to \Omega^\bullet(X)$.

Proposition 3.38. *If G is connected and compact, then i induces an isomorphism $H^\bullet(i)\colon \mathrm{H}^\bullet(\Omega^\bullet(X)^L) \cong \mathrm{H}^\bullet_{\mathrm{dR}}(X)$.*

Proof. Let ν be a normalized Haar volume form on G. Define $\mathrm{av}\colon \Omega^\bullet(X) \to \Omega^\bullet(X)^L$ by

$$\mathrm{av}(\alpha) \coloneqq \int_G L_g^*\alpha\, \nu(g)$$

This is a cochain map and

$$\mathrm{av} \circ i = \mathrm{id}_{\Omega^\bullet(X)^L};$$

hence, $H^\bullet(\mathrm{av})$ is a left inverse of $H^\bullet(i)$.

To show that $H^\bullet(\mathrm{av})$ also is a right inverse we proceed as follows. Denote by

$$\int_G\colon \Omega^\bullet(G \times X) \to \Omega^{\bullet - \dim G}(X)$$

the fibre integration map defined by the property that

$$\left(\int_G \alpha\right)_x (v_1, \dots, v_k) = \int_{g \in G} \alpha_{(g,x)}(v_1, \dots, v_k, \dots)$$

for $(v_1, \dots, v_k) \in T_xM$.

Exercise 3.39. Verify that this is a chain map. ?

We can now write $i \circ \mathrm{av}$ as the composition

$$i \circ \mathrm{av} = \mathrm{av}_\nu \coloneqq \int_G \circ\, (\nu \wedge \cdot) \circ L^*.$$

$H^\bullet(\nu \wedge \cdot)$ depends only on $[\nu]$; hence, if $\eta \in \Omega^{\dim G}(G)$ with $\int_G \eta = 1$, i.e., $[\mu] = [\eta]$, then $H^\bullet(i) \circ H^\bullet(\mathrm{av}) = H^\bullet(\mathrm{av}_\eta)$.

Heuristically, if we could take η to be a δ volume form at $\mathbf{1} \in G$, then $\mathrm{av}_\delta = \mathrm{id}_{\Omega^\bullet(X)}$; and as the induced map on cohomology does not change when as η becomes closer and closer to δ the proof is complete. It is not terribly difficult to make the above heuristic rigorous, but we will follow the standard route and chose η supported in neighbourhood U of $\mathbf{1} \in G$ which is smoothly contractible.

If $j\colon U\times X\to G\times X$, then

$$\mathrm{av}_\eta = \int_U \circ (\eta\wedge\cdot)\circ j^*\circ L^*.$$

Since U is smoothly contractible, on cohomology $j^*\circ\rho^* = (\rho\circ j)^*$ is the same as pulling back via the projection $\mathrm{pr}_X\colon U\times X\to X$. However,

$$\int_U \circ (\eta\wedge\cdot)\circ \mathrm{pr}_X^* = \mathrm{id}_{\Omega^\bullet(X)}. \qquad\blacksquare$$

Remark 3.40. The advantage of not following the heuristic argument, is that one can (at least in principle) write down a chain homotopy h such that

$$i\circ\mathrm{av} - \mathrm{id} = \mathrm{d}\circ h + h\circ\mathrm{d}. \qquad\clubsuit$$

Let us now use Proposition 3.38 to compute the de Rham cohomology in a few simple cases.

Example 3.41. $G = \mathrm{SO}(n+1)$ acts transitively on S^n. The stabilizer of any $x\in S^n$ is $\mathrm{SO}(T_xS^n)\cong\mathrm{SO}(n)$. A moments thought shows that

$$\begin{aligned}\Omega^\bullet(S^n)^G &= (\Lambda^* T_xS^n)^{\mathrm{SO}(T_xS^n)}\\ &= (\Lambda^*(\mathbf{R}^n)^*)^{\mathrm{SO}(n)}\\ &= \mathbf{R}\cdot 1\oplus\mathbf{R}\cdot\mathrm{d}x_1\wedge\ldots\wedge\mathrm{d}x_n\\ &= \mathbf{R}[0]\oplus\mathbf{R}[n].\end{aligned}$$

The differential necessarily vanishes (for dimension reasons if $n>1$); hence, this already is $\mathrm{H}^\bullet_{\mathrm{dR}}(S^n)$. The last step in the above computation is a fact from the representation theory of $\mathrm{SO}(n)$. ♠

Example 3.42. $G = \mathrm{U}(n+1)$ acts transitively on $\mathbf{C}P^n$ with stabiliser of any $\mathbf{C}\cdot z\in\mathbf{C}P^n$ isomorphic to $\mathrm{U}(z^\perp) = \mathrm{U}(T_z\mathbf{C}P^n)\cong\mathrm{U}(n)$. We compute

$$\Omega^\bullet(\mathbf{C}P^n)^{U(n+1)}\otimes\mathbf{C} = (\Lambda^*(\mathbf{C}^n)^*)^{\mathrm{U}(n)}.$$

The latter is generated as a $\mathbf{C}$*–algebra* by the standard symplectic form

$$\omega := \sum_{i=1}^n \frac{i\mathrm{d}z_i\wedge\mathrm{d}\bar z_i}{2};$$

that is,

$$\begin{aligned}(\Lambda^*(\mathbf{C}^n)^*)^{\mathrm{U}(n)} &= \mathbf{C}\cdot 1\oplus\mathbf{C}\cdot\omega\oplus\cdots\oplus\mathbf{C}\cdot\omega^n\\ &= \mathbf{C}[\omega]/(\omega^{n+1}).\end{aligned}$$

Since this complex is supported in even degrees, the differential vanishes and this already is $H^\bullet_{\mathrm{dR}}(\mathbf{C}P^n)\otimes\mathbf{C}$. ♠

Example 3.43. Let G be a Lie group. Let $\mathfrak{g} := \mathrm{Lie}(G)$. If we consider the L action of G on itself, then

$$\Omega^\bullet(G)^L = \Lambda^*\mathfrak{g}^* = \mathrm{Hom}(\Lambda^*\mathfrak{g}, \mathbf{R}).$$

The differential, which is usually denoted by δ, does not vanish. It can be computed to be

$$\begin{aligned}(\delta\alpha)(\xi_1, \dots, x_{k+1}) = &\sum_{i=1}^{k+1}(-1)^{i+1}\xi_i \cdot \alpha(\xi_1, \dots, \widehat{\xi_i}, \dots, \xi_{k+1}) \\ &+ \sum_{i<j=1}^{k+1}(-1)^{i+j}\alpha([\xi_i, \xi_j], \xi_1, \dots, \widehat{\xi_i}, \dots, \widehat{\xi_j}, \dots, \xi_{k+1});\end{aligned}$$

in fact, since the Lie algebra acts trivially on $\mathbf{R}$ the first term vanishes. $(\mathrm{Hom}(\Lambda^*\mathfrak{g}, \mathbf{R}), \delta)$ is the **Chevalley–Eilenberg cochain complex** (although it was discovered decades before Chevalley–Eilenberg by Cartan). It is defines for every Lie algebra $\mathfrak{g}$. Its cohomology

$$\mathrm{H}^\bullet(\mathfrak{g}) := \mathrm{H}^\bullet(C^\bullet(\mathfrak{g}), \delta).$$

is the **Lie algebra cohomology** of $\mathfrak{g}$.

If V is any representation of $\mathfrak{g}$, then δ as defined above makes $\mathrm{Hom}(\Lambda^*\mathfrak{g}, M)$ into a cochain complex. $\mathrm{H}^\bullet(\mathfrak{g}; V) := \mathrm{H}^\bullet(\mathrm{Hom}(\Lambda^*\mathfrak{g}, V))$ is called the **Lie algebra cohomology** of $\mathfrak{g}$ with coefficients in V. Proposition 3.38 shows that $\mathrm{H}^\bullet_{\mathrm{dR}}(G) = \mathrm{H}^\bullet(\mathfrak{g}; \mathbf{R})$. The notion of Lie algebra cohomology goes back to Chevalley and Eilenberg [CE48]. ♠

Remark 3.44. Let $\rho\colon G \to \mathrm{GL}(V)$ be a Lie group representation. Consider the trivial vector bundle $\mathrm{pr}_G\colon \underline{V} = G \times V \to G$. G acts on $\underline{V}$ by $L \times \rho$. This turns $\underline{V}$ into a G–equivariant vector bundle. The formula $\mathrm{d}_\nabla s := \mathrm{d}s + (\mathrm{Lie}(\rho) \circ \mu)s$ defines a G–equivariant flat connection on $\underline{V}$. A moment's thought shows that $\mathrm{H}^\bullet_{\mathrm{dR}}(G, \underline{V}) = \mathrm{H}^\bullet(\mathfrak{g}, V)$. ♣

Example 3.45. Let G be a connected compact Lie group. Let $H < G$ be a connected closed Lie subgroup. Set $\mathfrak{g} := \mathrm{Lie}(G)$ and $\mathfrak{h} := \mathrm{Lie}(H)$. Set $C^\bullet(\mathfrak{g}) := \mathrm{Hom}(\Lambda^\bullet\mathfrak{g}, \mathbf{R})$ and define δ as above. Denote by $C^\bullet(\mathfrak{g}, \mathfrak{h})$ the subcomplex of those $\alpha \in C^\bullet(\mathfrak{g})$ with

$$i_\xi\alpha = 0 \quad \text{and} \quad i_\xi\delta\alpha = 0 \quad \text{for every} \quad \xi \in \mathfrak{h}.$$

The **relative Lie algebra cohomology** of $\mathfrak{g} \supset \mathfrak{h}$ is

$$\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{h}) := \mathrm{H}^\bullet(C^\bullet(\mathfrak{g}, \mathfrak{h}), \delta).$$

The adjoint action of the Lie algebra $\mathfrak{h}$ on $\mathfrak{g}$ descends to $\mathfrak{g}/\mathfrak{h}$. Denote by $\mathrm{Hom}(\Lambda^\bullet\mathfrak{g}/\mathfrak{h}, \mathbf{R})^\mathfrak{h}$ the corresponding invariant subspace of $\mathrm{Hom}(\Lambda^\bullet\mathfrak{g}/\mathfrak{h}, \mathbf{R})$. $\mathrm{Hom}(\Lambda^\bullet\mathfrak{g}/\mathfrak{h}, \mathbf{R})^\mathfrak{h}$ can be regarded as a subspace $C^\bullet(\mathfrak{g})$. A moment's thought identifies it as $C^\bullet(\mathfrak{g}, \mathfrak{h})$. Moreover, $\mathrm{Hom}(\Lambda^\bullet\mathfrak{g}/\mathfrak{h}, \mathbf{R})^\mathfrak{h} \cong \Omega^\bullet(G/H)^H$ and the differentials δ and d agree. Therefore,

$$\mathrm{H}^\bullet_{\mathrm{dR}}(G/H) \cong \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{h}).$$ ♠

Exercise 3.46. Show that $\mathrm{H}^1(\mathfrak{g}, \mathbf{R}) = (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*$. ?

Example 3.47. Set $\tilde{R}(g, h) := hg^{-1}$. If we consider the action $L \times \tilde{R}$ of $G \times G$ on G, then

$$\Omega^\bullet(G)^{L\times\tilde{R}} = (\Lambda^\bullet \mathfrak{g}^*)^{\mathrm{Ad}}.$$

Here Ad denotes the coadjoint action. Suppose $\alpha \in \Omega^k(G)^{L\times\tilde{R}}$, that is, α is left invariant and right invariant. Since derivative of the map $i\colon G \to G, g \mapsto g^{-1}$ is

$$\mathrm{d}_g i = -\mathrm{d}L_{g^{-1}} \circ \mathrm{d}R_{g^{-1}},$$

we have

$$i^*\alpha = (-1)^k \alpha.$$

It follows that

$$\mathrm{d}\alpha = (-1)^k \mathrm{d}i^*\alpha = (-1)^k i^* \mathrm{d}\alpha = -\mathrm{d}\alpha;$$

hence, the differential vanishes on $\Omega^\bullet(G)^{L\times\tilde{R}}$ and

$$\mathrm{H}^\bullet_{\mathrm{dR}}(G) = (\Lambda^* \mathfrak{g}^*)^{\mathrm{Ad}}.$$

The formula

$$\gamma(\xi, \eta, \zeta) := B([\xi, \eta], \zeta)$$

defines an element $\gamma \in (\Lambda^3 \mathfrak{g}^*)^{\mathrm{Ad}}$. If G is semisimple, then $\gamma \neq 0$; hence $b_3(G) \geqslant 1$. ♠

Example 3.48. Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation. $\mathrm{Lie}(\rho)\colon \mathfrak{g} \to \mathfrak{gl}(V)$ can be regarded as an element of $\theta_\rho \in \mathfrak{g}^* \otimes \mathfrak{gl}(V)$. Evidently, θ_ρ is invariant under the action induces by Ad and ρ and so is $\theta_\rho^{\wedge k} \in \Lambda^k \mathfrak{g}^* \otimes \mathfrak{gl}(V)$. Therefore, $\mathrm{tr}(\theta_\rho^{\wedge k}) \in \Lambda^k \mathfrak{g}^*$. ♠

Remark 3.49. The multiplication map $m\colon G \times G \to G$ induces a map $\Delta\colon \mathrm{H}^\bullet(G) \to \mathrm{H}^\bullet(G) \otimes \mathrm{H}^\bullet(G)$. This turns $\mathrm{H}^\bullet(G)$ into a **Hopf algebra**. ♣

4 Principal bundles

4.1 Definition and examples

Definition 4.1. Let G be a Lie group. A G–**principal fibre bundle** is smooth map $p\colon P \to B$ together with a right action $R\colon P \times G \to P$ such that for every $b \in B$ there are an open subset $b \in U \subset B$, and a G–equivariant diffeomorphism $\tau\colon p^{-1}(U) \to U \times G$ such that

$$\mathrm{pr}_U \circ \tau = p;$$

that is: the diagram

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\ \tau\ } & U \times G \\ & {\scriptstyle p}\searrow & \downarrow{\scriptstyle \mathrm{pr}_U} \\ & & U \end{array}$$

commutes. G is the **structure group** of (p, R). •

Remark 4.2. In the situation of Definition 4.1, $p\colon X \to B$ is a quotient $X \to X/G$, R is free, fibre-preserving, and its restriction to any fibre $p^{-1}(b)$ is transitive. ♣

Example 4.3. The **trivial G–principal bundle** over B is $\mathrm{pr}_B\colon B \times G \to B$ with $R((b,g),h) := (b, gh)$. ♠

Example 4.4. Let B be a smooth manifold. Let $V \to B$ be a vector bundle of rank r. Denote by

$$\mathrm{Fr}(V) := \{(b,\phi) : b \in B, \phi\colon \mathbf{R}^r \to V_b \text{ isomorphism}\}$$

the **frame bundle of** V. Denote by $p\colon \mathrm{Fr}(V) \to B$ the projection map. $\mathrm{GL}_r(\mathbf{R})$ acts on the right of $\mathrm{Fr}(V)$ via

$$R((b,\phi),\tau) := (b, \phi \circ \tau).$$

There is a unique smooth structure on $\mathrm{Fr}(V)$ such that (p, R) is a $\mathrm{GL}_r(\mathbf{R})$–principal bundle. ♠

Example 4.5. Let $k, r \in \mathbf{N}$ with $k < r$. The **Stiefel manifold** $\mathrm{St}_k^*(\mathbf{R}^r)$ is the submanifold

$$\mathrm{St}_k^*(\mathbf{R}^r) := \{(v_1, \ldots, v_k) \in (\mathbf{R}^r)^k : v_1, \ldots, v_k \text{ linearly independent}\}$$

or, equivalently,

$$\mathrm{St}_k^*(\mathbf{R}^r) := \{A \in \mathrm{Hom}(\mathbf{R}^k, \mathbf{R}^r) : A \text{ is injective}\}.$$

$\mathrm{GL}_k(\mathbf{R})$ acts on the right of $\mathrm{St}_k^*(\mathbf{R}^r)$ via $R(A,\tau) := A \circ \tau$. The map $p\colon \mathrm{St}_k^*(\mathbf{R}^r) \to \mathrm{Gr}_k(\mathbf{R}^r)$ defined by

$$p(A) := \operatorname{im} A$$

together with R is a $\mathrm{GL}_k(\mathbf{R})$–principal bundle. Of course, $\mathrm{St}_k^*(\mathbf{R}^r)$ is the frame bundle of the tautological bundle over $\mathrm{Gr}_k(\mathbf{R}^r)$. ♠

Proposition 4.6. *Let G be a compact Lie group. Let P be smooth manifold. If $R\colon P \times G \to P$ is a free right action, then $p\colon P \to B := P/G$ together with R is a G–principal fibre bundle.*

Proof. Exercise. ■

Exercise 4.7. The Hopf bundle $p\colon S^{2n+1} \to \mathbf{C}P^n$ together with the right action $R\colon S^{2n+1} \times \mathrm{U}(1) \to S^{2n+1}$ defined by $R(z, e^{i\alpha}) := ze^{i\alpha}$ is a $\mathrm{U}(1)$–principal bundle. ?

Exercise 4.8. Let $n \in \mathbf{N}$. $\mathrm{Sp}(1) := \{q \in \mathbf{H} : |q| = 1\}$ acts on $S^{4n+3} \subset \mathbf{H}^{n+1}$ by $R\colon S^{4n+3} \times \mathrm{Sp}(1) \to S^{4n+3}$ with

$$R_+(x,q) := q^{-1}x.$$

The quotient of R is the $\mathbf{H}P^n$, the space of $\mathbf{H}$–left modules $L \subset \mathbf{H}^{n+1}$ of dimension 1. The projection map $q\colon S^{4n+3} \to \mathbf{H}P^n$ together with R is an $\mathrm{Sp}(1)$–principal bundle—the **quaternionic Hopf bundle**. ?

∼

Definition 4.9. Let $(p\colon P \to B, R)$ and $(q\colon Q \to B, S)$ be G–principal fibre bundles. A **morphism** $(p,R) \to (q,S)$ is a G–equivariant smooth map $\phi\colon P \to Q$ satisfying $q \circ \phi = p$. A **gauge transformation** of (p,R) is a isomorphism $(p,R) \to (p,R)$. The **gauge group** of (p,R) is the group of all gauge transformations of (p,R) and denoted by

$$\mathscr{G}(p,R).$$

•

Exercise 4.10. Prove that every morphism of G–principal bundles is an isomomorphism. ?

Exercise 4.11. Let (p,R) be a G–principal bundle. Prove that (p,R) is isomorphic to the trivial G–principal bundle if and only if p admits a section. ?

The gauge group plays a very important role. The following concrete description of the gauge group is quite useful.

Proposition 4.12. *Let $(p\colon P \to B, R)$ be G–principal fibre bundles. Denote by $C^\infty(P,G)^C$ the subspace of $u \in C^\infty(P,G)$ satisfying*

$$R_g^* u = C_{g^{-1}} u, \quad \text{i.e.,} \quad u(xg) = g^{-1}u(x)g$$

for every $g \in G$

(1) *For every $u \in C^\infty(P,G)^C$ the map $\tilde{u} \in C^\infty(P,P)$ defined by*

$$\tilde{u}(x) = x \cdot u(x)$$

is a gauge transformation.

(2) *The map $\tilde{\cdot}\colon C^\infty(P,G)^C \to \mathscr{G}(p,R)$ defined by the above is a bijection; in fact, a group isomorphism.*

Proof. To prove (1), it suffices to verify that $\tilde{u}$ is G–equivariant:

$$\tilde{u}(xg) = xg \cdot u(xg) = xg \cdot g^{-1}u(x)g = xu(x)g = \tilde{u}(x)g.$$

Evidently, the map $\tilde{\cdot}\colon C^\infty(P,G)^C \to \mathscr{G}(p,R)$ is injective. To prove that it is surjective, observe that if $\tilde{u} \in \mathscr{G}(p,R)$ then for every $x \in P$ there is a unique $u(x) \in G$ such that $\tilde{u}(x) = x \cdot u(x)$. The map $u \in \mathrm{Map}(P,G)$ thus defined is smooth. The G–equivariance of $\tilde{u}$ follows from the G–equivariance of u.

It remains to prove that $\tilde{\cdot}$ is a group homomorphism. To see this observe that

$$\tilde{v}(\tilde{u}(x)) = x \cdot u(x) \cdot v(x \cdot u(x)) = x \cdot v(x) \cdot u(x).$$

■

Definition 4.13. Denote by $\gamma_\cdot\colon \mathscr{G}(p,R) \to C^\infty(P,G)^C$, $u \mapsto \gamma_u$ the inverse of $\tilde{\cdot}\colon C^\infty(P,G)^C \to \mathscr{G}(p,R)$. (This notation is not standard and probably a bad choice. I'm not aware of any standard notation for this map.) •

Example 4.14. If G is an abelian group, then $C^\infty(P,G)^C$ consists precisely of the G–invariant maps $C^\infty(P,G)^G \cong C^\infty(B,G)$. Therefore, $\mathscr{G}(p,R) \cong C^\infty(B,G)$.

Let G be an arbitrary Lie group. Denote by $Z(G) := \{g \in G : gh = hg \text{ for every } h \in G\}$ the **center of** G. Evidently,

$$C^\infty(B, Z(G)) \cong C^\infty(P, Z(G))^C \hookrightarrow \mathscr{G}(p,R). \qquad \spadesuit$$

Example 4.15. Let $V \to B$ be a vector bundle of rank r. Consider the frame bundle $(p\colon \operatorname{Fr}(V) \to B, R)$. For every $\lambda \in \mathbf{R}^*$ the map $\varepsilon\colon \operatorname{Fr}(V) \to \operatorname{Fr}(V)$ defined by

$$\varepsilon(b,(v_1,\dots,v_r)) := \varepsilon(b,(\lambda v_1,\dots,\lambda v_r)). \qquad \spadesuit$$

Example 4.16. For the trivial G–principal bundle $(p\colon B\times G \to B, R)$ the map $C^\infty(B,G) \to \mathscr{G}(p,R)$ defined by $C^\infty(B,G) \ni \gamma \mapsto u_\gamma$ defined by $u_\gamma(b,g) := (b,\gamma(b)g)$ is a group isomorphism. (Observe that for general G–principal bundle the left-multiplication is not available.) ♠

Proposition 4.17. *Let $p\colon P \to B$ with R be a G–principal bundle. Let $f\colon A \to B$ be a smooth map. Denote by $f^*p\colon f^*P \to A$ the pullback of $p\colon P \to B$. Define $f^*R\colon f^*P \times G \to f^*P$ by*

$$f^*R((a,p),g) = (a, R(p,g)).$$

(f^*p, f^*R) is a G–principal fibre bundle.

Definition 4.18. The G–principal bundle (f^*p, f^*R) constructed above is the **pullback** of (p,R) via f. •

∼

Example 4.19. Let G be a Lie group. Here is how to construct a fiber bundle $p\colon X \to B$ with fibres diffeomorphic to G but which cannot be turned into a G–principal fibre bundle. Let $\phi \in \operatorname{Diff}(G)$. Denote by $X_\phi := ([0,1]\times G)/\sim$ with $\sim$ generated by $(1,x) \sim (0, f(x))$ the mapping torus of ϕ. The projection $p\colon X_\phi \to S^1$ is a fibre bundle whose fibres are diffeomorphic to G.

The right action of G on $[0,1]\times G$ descends to X_ϕ if and only if $\phi(g) = hg$ for some $h \in G$. Therefore, usually, p cannot be turned into a G–principal fibre in the obvious way. In fact, often, p cannot be turned into a G–principal at all. To see this, observe that if there is a $g \in G$ such that g and $\phi(g)$ lie in the same connected component of G, then p admits a section. Therefore, p is isomorphic to $\operatorname{pr}_{S^1}\colon S^1 \times G \to S^1$. However, this implies that ϕ is isotopic to id_G.

To make this concrete consider the orientation reversing diffeomorphism $\phi \in \operatorname{Diff}(\mathrm{U}(1))$ defined by $\phi(e^{i\alpha}) := e^{-i\alpha}$. The mapping torus T_ϕ is the Klein bottle; hence, not diffeomorphic to $S^1 \times \mathrm{U}(1)$. However, the projection $p\colon T_\phi \to S^1$ admits a section $s(b) := [b, \mathbf{1}]$. ♠

4.2 G–principal connections

Proposition 4.20. *Let $(p\colon P \to B, R\colon P\times G \to P)$ be a G–principal fibre bundle. The map $\kappa\colon P \times \mathfrak{g} \to V_p$ defined by*

$$\kappa(p,\xi) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} R(p, \exp(t\xi)) = T_pR_1 \circ \operatorname{ev}_1(\xi).$$

is an isomorphism. ■

The isomorphism κ simplifies the theory of connections (or at least it makes it more concrete).

Definition 4.21. Let $(p\colon P \to B, R\colon P \times G \to P)$ be a G–principal fibre bundle. For $\xi \in \mathfrak{g}$ define $v_\xi \in \Gamma(V_p)$ by

$$v_\xi(p) := \kappa(p, \xi).$$

•

Exercise 4.22. Prove that $\mathfrak{g} \to \mathrm{Vect}(P)$, $\xi \mapsto v_\xi$ is a Lie algebra homomorphism. ?

Exercise 4.23. Construct a fibre bundle $p\colon X \to B$ whose fibres are diffeomorphic to S^1 but which cannot be equipped with a S^1 action R making (p, R) into an S^1–principal bundle. ?

Definition 4.24. Let $(p\colon P \to B, R\colon P \times G \to P)$ be a G–principal fibre bundle. A **G–principal connection** is an Ehresmann connection $A\colon TP \to V_p \oplus p^*TB$ satisfying

$$A \circ TR_g = (TR_g \oplus \mathrm{id}_{p^*TB}) \circ A.$$

The G–**principal connection** 1–**form** of A is the 1–form $\theta_A \in \Omega^1(P, \mathfrak{g})$ defined by

$$\theta_A := \mathrm{pr}_{\mathfrak{g}} \circ \kappa^{-1} \circ \mathrm{pr}_{V_p} \circ A.$$

The space of connections on (p, R) is denoted by $\mathscr{A}(p, R)$. •

Proposition 4.25. *Let $(p\colon P \to B, R\colon P \times G \to P)$ be a G–principal fibre bundle.*

(1) *An Ehresmann connection A on p is a G–principal connection if and only if for every $g \in R$*

$$TR_g(H_A) = H_A.$$

(2) *An Ehresmann connection A on p is a G–principal connection if and only if the 1–form*

$$\theta_A := \mathrm{pr}_{\mathfrak{g}} \circ \kappa^{-1} \circ \mathrm{pr}_{V_p} \circ A.$$

satisfies

$$R_g^*\theta_A = \mathrm{Ad}(g^{-1}) \circ \theta_A$$

for every $g \in G$. ■

Proposition 4.26. *Every G–principal connection is complete.*

Proof. Exercise. ■

∼

Definition 4.27. Let $(p\colon P \to B, R\colon P \times G \to P)$ be a G–principal fibre bundle. Let V be a finite-dimensional vector space. Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation of G. G acts on $\Omega^\bullet(P, V)$ via

$$g \cdot \alpha := \rho(g) \circ R_g^*\alpha.$$

Set

$$\Omega^\bullet_{\mathrm{hor}}(P, V)^G = \Omega^\bullet_{\mathrm{hor}}(P, V)^\rho := \{\alpha \in \Omega^\bullet_{\mathrm{hor}}(P, V) : g \cdot \alpha = \alpha \text{ for every } g \in G\}.$$

•

Remark 4.28. The above construction is particularly important for the adjoint representation $\rho\colon G \to \mathrm{GL}(\mathrm{Lie}(G))$. ♣

Proposition 4.29. *Let $(p\colon P \to B, R\colon P \times G \to P)$ be a G–principal fibre bundle. $\mathscr{A}(p, R)$ is an affine space modelled on $\Omega^1_{\mathrm{hor}}(P, \mathrm{Lie}(G))^{\mathrm{Ad}}$.*

Proof. Exercise; cf. Proposition 2.28. ■

Proposition 4.30. *Let $(p\colon P \to B, R\colon P\times G \to P)$ be a G–principal fibre bundle. Let $A \in \mathscr{A}(p,R)$. Denote by $\sigma_\alpha\colon TP \to H_A$ the projection onto H_A. Let V be a finite-dimensional vector space. Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation of G. Define $\mathrm{d}_A\colon \Omega^\bullet_{\mathrm{hor}}(P,V) \to \Omega^{\bullet+1}_{\mathrm{hor}}(P,V)$ by*

$$(\mathrm{d}_A\alpha)(v_1,\dots,v_k) \coloneqq (\mathrm{d}\alpha)(\sigma_A(v_1),\dots,\sigma_A(v_k)).$$

(This is essentially the map $\mathrm{d}^{1,0}_A$ from Proposition 2.72.)

(1) *Let $\alpha \in \Omega^\bullet_{\mathrm{hor}}(P,V)$. If α is G–invariant, then so is $\mathrm{d}_A\alpha$. Therefore, d_A induces a map $\mathrm{d}_A\colon \Omega^\bullet_{\mathrm{hor}}(P,V)^\rho \to \Omega^{\bullet+1}_{\mathrm{hor}}(P,V)^\rho$.*

(2) *The map $\mathrm{d}_A\colon \Omega^\bullet_{\mathrm{hor}}(P,V)^\rho \to \Omega^{\bullet+1}_{\mathrm{hor}}(P,V)^\rho$ satisfies*

$$\mathrm{d}_A\alpha = \mathrm{d}\alpha + (\mathrm{Lie}(\rho)\theta_A)\wedge\alpha.$$

Proof. To prove (1), observe that $\sigma_A \circ TR_g = TR_g \circ \sigma_A$ and compute

$$\begin{aligned}\rho(g)R_g^*\mathrm{d}_A\alpha(v_1,\dots,v_k) &= \rho(g)\mathrm{d}\alpha(\sigma_A\circ TR_g(v_1),\dots,\sigma_A\circ TR_g(v_k))\\ &= \rho(g)\mathrm{d}\alpha(TR_g\circ\sigma_A(v_1),\dots,TR_g\circ\sigma_A(v_k))\\ &= (\rho(g)R_g^*\mathrm{d}\alpha)(\sigma_A(v_1),\dots,TR_g\circ\sigma_A(v_k)).\end{aligned}$$

The assertion follows since

$$R_g^*\mathrm{d}\alpha = \mathrm{d}R_g^*\alpha = \rho(g^{-1})\mathrm{d}\alpha.$$

One can deduce (2) more or less directly from Proposition 2.72 (2) (but we didn't prove that in class). Instead let us verify this directly. It suffice to verify the formula on $v_0,\dots,v_k$ which are each either vertical or horizontal. If they are all horizontal, then second term on the RHS vanishes and the formula hold. If at least two are vertical, then both sides vanish. Therefore, without loss of generality, v_0 is vertical and $v_1,\dots,v_k$ are horizontal. Extend the v_j to local G–invariant vector fields with $v_0 = v_\xi$ and $v_2,\dots,v_{k+1}$ horizontal. In this case,

$$\begin{aligned}\mathrm{d}\alpha(v_\xi, v_1\dots,v_k) &= \mathscr{L}_{v_\xi}(\alpha(v_1,\dots,v_k))\\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(R^*_{\exp(t\xi)}\alpha)(v_1,\dots,v_k))\\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\rho(\exp(-t\xi))\circ\alpha(v_1,\dots,v_k))\\ &= -\mathrm{Lie}(\rho)(\xi)\circ\alpha(v_1,\dots,v_k)\\ &= -(\mathrm{Lie}(\rho)(\theta_A)\wedge\alpha)(v_\xi,v_1,\dots,v_k).\end{aligned}$$ ■

Remark 4.31. The maps $\mathrm{d}_A \colon \Omega^\bullet_{\mathrm{hor}}(P,V)^\rho \to \Omega^{\bullet+1}_{\mathrm{hor}}(P,V)^\rho$ are compatible with the usual operations on representations, in particular, $\otimes$ and $\oplus$. One consequence of this is that $\Omega^\bullet_{\mathrm{hor}}(P,V)$ is a (left-)module over $\Omega^\bullet_{\mathrm{hor}}(P)^G \cong \Omega^\bullet(B)$ and d_A is a the corresponding Leibniz rule holds. ♣

~

Proposition 4.32. *Let $(p\colon P \to B, R\colon P\times G \to P)$ be a G–principal fibre bundle. Let $A \in \mathscr{A}(p,R)$.*

(1) *There is a unique G–invariant horizontal 2–form $F_A \in \Omega^2_{\mathrm{hor}}(P,\mathfrak{g})^{\mathrm{Ad}}$ such that for every $v,w \in \Gamma(H_A)$*

$$F_A(v,w) = -\theta_A([v,w]).$$

*F_A is the **curvature** of A.*

(2) *F_A can be computed as*

$$F_A = \mathrm{d}\theta_A + \frac{1}{2}[\theta_A \wedge \theta_A]$$

(3) *F_A satisfies the **Bianchi identity***

$$\mathrm{d}_A F_A = 0.$$

(4) *If $\rho\colon G \to \mathrm{GL}(V)$ is a finite-dimensional representation of G, then $\mathrm{d}_A\colon \Omega^\bullet_{\mathrm{hor}}(P,V)^\rho \to \Omega^{\bullet+1}_{\mathrm{hor}}(P,V)^\rho$ satisfies*

$$\mathrm{d}_A \circ \mathrm{d}_A = (\mathrm{Lie}(\rho)\circ F_A)\wedge\cdot.$$

Remark 4.33. One sometimes sees the formula $F_A = \mathrm{d}_A\theta_A$. This is correct, but it easily leads to confusion. The issue is that on is tempted to forget the original definition of d_A and use Proposition 4.30 (2) instead; however: θ_A is (not at all) horizontal and this formula obviously does not hold for θ_A. ♣

Proof of Proposition 4.32. Define $F_A \coloneqq \mathrm{d}\theta_A + \frac{1}{2}[\theta_A \wedge \theta_A]$ and compute

$$F_A(v,w) = \mathscr{L}_v\theta_A(w) - \mathscr{L}_w\theta_A(v) - \theta_A([v,w]) + [\theta_A(v),\theta_A(w)].$$

This matches $-\theta_A(v,w)$ for $v,w \in \Gamma(H_A)$.

If $\xi \in \mathfrak{g}$ and $w \in \Gamma(H_A)$ is G–invariant, then

$$F_A(v_\xi, w) = \mathscr{L}_w\xi - \theta_A([v_\xi,w]) = 0.$$

The first term vanishes because ξ is constant. The second term vanishes because w is G–invariant.

For $\xi,\eta \in \mathfrak{g}$

$$F_A(v,w) = \mathscr{L}_{v_\xi}\eta - \mathscr{L}_{v_\eta}\xi - \theta_A([v_\xi,v_\eta]) + [\xi,\eta].$$

The first two term vanish because ξ,η are constant. The last two terms cancel because $\xi \mapsto v_\xi$ is a Lie algebra homomorphism. (This is, of course, essentially the proof of the Maurer–Cartan equation.)

The G–invariance of F_A follows from the G–invariance of θ_A. Thus (1) and (2) are proved.

$$\begin{aligned} d_A F_A &= (d + [\theta_A \wedge \cdot])(d\theta_A + \frac{1}{2}[\theta_A \wedge \theta_A]) \\ &= \frac{1}{2}([d\theta_A \wedge \theta_A] - [\theta_A \wedge d\theta_A]) + [\theta_A \wedge d\theta_A] + \frac{1}{2}[\theta_A \wedge [\theta_A \wedge \theta_A]] \\ &= 0 \end{aligned}$$

because the first three term cancel and the last term vanishes by the Jacobi identity.

(4) follows by direct computation. ∎

∼

Proposition 4.34. *Let $(p\colon P \to B, R\colon P \times G \to P)$ be a G–principal fibre bundle. Let $A \in \mathscr{A}(p,R)$. Let $\gamma\colon [0,1] \to B$ be a piecewise smooth path. The parallel transport* tra_γ^A *is G–equivariant; that is: for every $x \in p^{-1}(\gamma(0))$ and $g \in G$*

$$\mathrm{tra}_\gamma^A(x) \cdot g = \mathrm{tra}_\gamma^A(x \cdot g).$$

Proof. This is a consequence of A being G–invariant. ∎

Remark 4.35. Let $x_0 \in p^{-1}(b_0)$. There is a $g \in G$ such that $\mathrm{tra}_\gamma^A(x_0) = x_0 \cdot g$. Every element of $p^{-1}(b_0)$ is of the form $x_0 \cdot h$ for some $h \in G$. Since tra_A^γ is G–invariant,

$$\mathrm{tra}_\gamma^A(x_0 h) = \mathrm{tra}_\gamma^A(x_0) h = x_0 \cdot gh. \quad ♣$$

Definition 4.36. Let $(p\colon P \to B, R\colon P \times G \to P)$ be a G–principal fibre bundle. Let $A \in \mathscr{A}(p,R)$. Let $b_0 \in B$ and $x_0 \in p^{-1}(b_0)$. The **holonomy group** of A based at x_0 is the subgroup $\mathrm{Hol}_{x_0}(A) < G$ defined by

$$\mathrm{Hol}_{x_0}(A) := \{g \in G : (\star)\}$$

with $(\star)$ meaning that there is a piecewise smooth loop $\gamma\colon [0,1] \to B$ based at b_0 with $\mathrm{tra}_\gamma^A(x_0) = x_0 \cdot g$. The **restricted holonomy group** of A based at x_0 is the subgroup $\mathrm{Hol}_{x_0}^0(A) < G$ defined by

$$\mathrm{Hol}_{x_0}^0(A) := \{g \in G : (\dagger)\}$$

with $(\dagger)$ meaning that there is a null-homotopic piecewise smooth loop $\gamma\colon [0,1] \to B$ based at b_0 with $\mathrm{tra}_\gamma^A(x_0) = x_0 \cdot g$. •

Proposition 4.37. *The holonomy group and the restricted holonomy group are Lie subgroups of G.*

Proof sketch. $\mathrm{Hol}_{x_0}^0(A)$ is path-connected and therefore a Lie subgroup of G. Parallel transport defines a group homomorphism $\pi_1(B, b_0) \to \mathrm{Hol}_{x_0}(A)/\mathrm{Hol}_{x_0}^0(A)$. With Γ denoting its image

$$\mathrm{Hol}_{x_0}(A) = \coprod_{\gamma \in \Gamma} g \cdot \mathrm{Hol}_{x_0}^0(A).$$

Use this to construct a smooth structure on $\mathrm{Hol}_{x_0}(A)$. ∎

Remark 4.38. The above underlines that G–principal connections are really much simpler than general Ehresmann connections. The holonomy group of an Ehresmann connection is a subgroup of $\mathrm{Diff}(p^{-1}(b_0))$, a possibly wild infinite-dimensional beast; while that of a G–principal connection sits in side a fixed finite dimensional Lie group. ♣

`One could discuss the relation between` F_A `and Hol here, but it is probably better to do this later in a section on Ambrose–Singer..`

∼

Proposition 4.39. *Let $(p\colon P \to B, R\colon P\times G \to P)$ be a G–principal fibre bundle. Let $A \in \mathscr{A}(p,R)$. Let $u \in \mathscr{G}(p,R)$. There is a unique $u^*A \in \mathscr{A}(p,R)$ such that*

$$\theta_{u^*A} = u^*\theta_A.$$

Moreover, the following hold:

(1) *The connection 1–form θ_{u^*A} can be written as*

$$\theta_{u^*A} = \mathrm{Ad}(\gamma_u^{-1}) \circ \theta_A + \gamma_u^*\mu$$

with $\mu \in \Omega^1(G,\mathfrak{g})$ denoting the Maurer–Cartan form.

(2) *The horizontal subspaces of A and u^*A are related by*

$$H_{u^*A} = Tu^{-1}(H_A).$$

(3) *The pullback via the gauge transformation u preserves $\Omega^\bullet(P,V)^\rho$ and*

$$\mathrm{d}_{u^*A} = u^* \circ \mathrm{d}_A \circ (u^{-1})^*.$$

(4) *The curvatures of A and u^*A are related by*

$$F_{u^*A} = \mathrm{Ad}(\gamma_u^{-1})F_A.$$

(5) *The parallel transports of A and u^*A are related by*

$$\mathrm{tra}_\gamma^{u^*A} = u^{-1} \circ \mathrm{tra}_\gamma^A \circ u.$$

The above defines a *right* action of $\mathscr{G}(p,R)$ on $\mathscr{A}(p,R)$.

Proposition 4.39. Since u is G–equivariant, $\theta_{u^*A} := u^*\theta_A$ is the connection 1–form of a G–principal connection u^*A. Evidently, $H_{u^*A} = Tu^{-1}(H_A)$.

Since

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} R_{g\exp(t\xi)}x &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} R_{\exp(t\xi)}R_g(x)\\ &= v_\xi(R_g(x)),\end{aligned}$$

the derivative of the map $x \mapsto R_{\gamma_u(x)}(y)$ is

$$v_{\gamma_u^*\mu}(R_{\gamma_u(x)}(y)).$$

Since $u(x) = R_{\gamma_u(x)}(x)$,

$$T_x u(\hat{x}) = T_x R_{\gamma_u(x)}(\hat{x}) + v_{(\gamma_u^*\mu)(\hat{x})}(R_{\gamma_u(x)}(x)).$$

Therefore,

$$\begin{aligned}\theta_{u^*A} &= u^*\theta_A \\ &= \mathrm{Ad}(\gamma_u^{-1}) \circ \theta_A + \gamma_u^*\mu.\end{aligned}$$

■

4.3 The tangent group

The following is a preparation for the study of associate fibre bundles, but it is also a interesting observation by itself.

Proposition 4.40. *Let G be a Lie group with multiplication $m\colon G \times G \to G$, inversion $i\colon G \to G$, and unit $\mathbf{1}$. Set $\mathfrak{g} := \mathrm{Lie}(G)$.*

(1) *The tangent bundle TG together with $Tm\colon TG \times TG \to TG$, $Ti\colon TG \to TG$, and $(\mathbf{1}, 0) \in TG$ is a Lie group.*

(2) *Denote by $G \ltimes \mathfrak{g} = G \ltimes_{\mathrm{Ad}} \mathfrak{g}$ the Lie group $G \times \mathfrak{g}$ with group multiplication*

$$(g, \xi)(h, \eta) := (gh, \xi + \mathrm{Ad}(g)\eta).$$

The map $G \times \mathfrak{g} \to TG$ defined by

$$(g, \xi) \mapsto (g, TR_g(\mathrm{ev}_{\mathbf{1}}(\xi))$$

is a Lie group isomorphism.

Proof. To prove (1) formulate the group conditions as commutative diagrams and apply the tangent functor. (2) follows from a computation. ■

Definition 4.41. TG is the **tangent group** associated with G. •

Proposition 4.42. *Let G be a Lie group. Let X be a smooth manifold.*

(1) *If $R\colon X \times G \to X$ is a right action, then $TR\colon TX \times TG \to TG$ is a right action.*

(2) *With respect to the isomorphism $G \ltimes \mathfrak{g} \cong TG$, TR is given by*

$$(x, v) \cdot (g, \xi) = (R_g x, TR_g(v + v_\xi(x))).$$

Proof. To prove (1), Write the condition for R to be a right action as a commutative diagram and apply the tangent functor.

To prove (2) compute. ■

Proposition 4.43. *Let G be a Lie group. If $(p\colon P \to B, R\colon P \times G \to P)$ is a G–principal fibre bundle, then $(Tp\colon TP \to TB, TR\colon TP \times TG \to TP)$ is a TG–principal fibre bundle.*

Proof. Exercise. *Hint:* apply the tangent functor to the local trivialisations. ∎

Remark 4.44. The above exhibits $Tp\colon TP \to TB$ as a quotient $TP \to TP/TG$. For every $b \in B$ and $x \in p^{-1}(b)$ the linear map

$$T_x P/\mathfrak{g} \to Tp^{-1}(b)/TG \to T_b B$$

is an isomorphism. ♣

4.4 Associated fibre bundles

The following construction gives away to "replace" the fibres of a principal bundle with a manifold F equipped with a G–action.

Proposition 4.45. *Let G be a Lie group. Let $(p\colon P \to B, R\colon P \times G \to P)$ be a G–principal fibre bundle. Let F be a smooth manifold. Let $L\colon G \times F \to F$ be a left action. Define a right action $S\colon (P \times F) \times G \to P \times F$ by*

$$S((x, f), g) := (xg, g^{-1}f).$$

1. *The action S admits a quotient*

$$q\colon P \times F \to (P \times F)/G =: P \times_L F = P \times_G F.$$

2. *The pair $(q\colon P \times F \to P \times_L F, S)$ is a G–principal fibre bundle.*
3. *The map $r\colon P \times_G F \to B$ induced by $p \circ \mathrm{pr}_P\colon P \times F \to B$ is a fibre bundle. This is the* ***fibre bundle associated with*** *(p, R)* ***and*** *L.*
4. *The map $\tau^{-1}\colon P \times F \to p^*(P \times_L F)$ defined by*

$$\tau^{-1}(x, f) := (p, [p, f]).$$

   is an isomorphism $\phi\colon \mathrm{pr}_P \to p^*r$ from the trivial bundle pr_P to p^*r. Denote the inverse of τ^{-1} by τ.

5. *Consider the G–equivariant maps*

$$C^\infty(P, F)^L := \{\hat{s} \in C^\infty(P, F) : \hat{s}(xg) = g\hat{s}(x)\}$$

   and

$$\Gamma(r) := \{s \in C^\infty(B, P \times_L F) : rs = \mathrm{id}_B\}.$$

   If $s \in \Gamma(r)$, then $\hat{s} := \mathrm{pr}_F \circ p^*s \in C^\infty(P, F)^L$. This defines a bijection

$$\hat{\cdot}\colon \Gamma(r) \to C^\infty(P, F)^L.$$

Here is a diagram summarising the above

$$\begin{array}{ccc} P \times F & \xrightarrow{q} & P \times_L F \\ \Big\downarrow{\scriptstyle \mathrm{pr}_P} & & \Big\downarrow{\scriptstyle r} \\ P & \xrightarrow{p} & B. \end{array}$$

Proof. For the trivial G–principal bundle $(\mathrm{pr}_B\colon P := B \times G, R)$ the quotient is $r = (\mathrm{pr}_B, L)\colon B \times G \times F \to B \times F$. (Verify this!)

Denote by $r\colon P \times F \to (P \times F)/G =: P \times_G F$ the topological quotient. The map $p \circ \mathrm{pr}_P$ is G–invariant and thus descends to a continuous map $q\colon P \times_G G \to B$. For $b \in B$ let U_b and $\tau_b\colon p^{-1}(U_b) \to U_b \times G$ as in Definition 4.1. For $b_1, b_2 \in B$ there is a smooth map $\phi^{b_1}_{b_2}\colon U_{b_1} \cap U_{b_2} \to G$ such that

$$\tau_{b_2} \circ \tau^{-1}_{b_1}(b, g) = (b, \phi^{b_1}_{b_2}(b) \cdot g).$$

By the trivial case $q^{-1}(U_{b_i})$ is canonically homeomorphic to $U_{b_i} \times X$ and the transition between these homemorphisms is given by

$$(b, f) \mapsto (b, \phi^{b_1}_{b_2}(b) \cdot f).$$

This endows $P \times_G X$ with the structure of a smooth manifold and exhibits $q\colon P \times_G X \to B$ as a fibre bundle. This proves (1), (2), and (3).

(4) and (5) are exercises. ■

Example 4.46. Let X be a smooth manifold of dimension n. Denote by $p\colon \mathrm{Fr}(TX) \to X$ the frame bundle of TX. The orientation double cover of X is the fibre bundle associated with (p, R) and the action of $\mathrm{GL}_n(R)$ on $\{\pm 1\}$ induced by the group homomorphism $\mathrm{GL}_n(\mathbf{R}) \to \{\pm 1\}$ given by

$$\phi \mapsto \frac{\det \phi}{|\det \phi|}.$$

♠

Example 4.47. Let $s \geqslant 0$ and $n \in \mathbf{N}_0$. A s–**density** on $\mathbf{R}^n$ is a map $\mu\colon (\mathbf{R}^n)^{\times n} \to \mathbf{R}$ such that for every $A \in \mathrm{GL}_n(\mathbf{R})$ and $v_1, \dots, v_n \in \mathbf{R}^n$

$$\mu(\phi(v_1), \dots, \phi(v_n)) = |\det \phi| \mu(v_1, \dots, v_n).$$

The set of s–densities is a 1–dimensional vector space: $D^s(\mathbf{R}^n)$.

Let X be a smooth manifold of dimension n. Denote by $p\colon \mathrm{Fr}(TX) \to X$ the frame bundle of TX. The bundle of s–densities on X is

$$D^s(TX) := \mathrm{Fr}(TX) \times_{\mathrm{GL}_n(R)} D^s(\mathbf{R}^n).$$

A **density** is a 1–density and $D(TX) := D^1(TX)$. ♠

Example 4.48. Let $V \to B$ be a real (complex) vector bundle of rank r. Denote by $(p\colon \mathrm{Fr}(V) \to B, R)$ the frame bundle of $V \to B$. $\mathrm{GL}_r(\mathbf{R})$ ($\mathrm{GL}_r(\mathbf{C})$) acts on $\mathbf{R}P^n$ ($\mathbf{C}P^n$) The associated fibre bundle $\mathrm{Fr}(V) \times_{\mathrm{GL}_r(\mathbf{R})} \mathbf{R}P^n$ ($\mathrm{Fr}(V) \times_{\mathrm{GL}_r(\mathbf{R})} \mathbf{C}P^n$) is the **projectivisation** of V and denoted by $\mathbf{P}(V) \to B$. ♠

Example 4.49. Let $n \in \mathbf{Z}$. The **Hirzebruch surface** Σ_n is

$$\Sigma_n := \mathbf{P}(\mathscr{O}_{\mathbf{P}^1} \oplus \mathscr{O}_{\mathbf{P}^1}(n)). \qquad \spadesuit$$

Proposition 4.50. *Assume the situation of Proposition 4.45. Let $A \in \mathscr{A}(p, R)$.*

(1) *The V_r fits into the following diagram of exact sequences:*

$$\begin{array}{ccccc}
(P \times F) \times \mathfrak{g} & = & (P \times F) \times \mathfrak{g} & & \\
\downarrow & & \downarrow & & \\
V_p \times TF & \hookrightarrow & TP \times TF & \xrightarrow{T(p \circ \mathrm{pr}_P)} & p^*TB \\
\downarrow{\scriptstyle Tq} & & \downarrow{\scriptstyle Tq} & & \| \\
q^*V_r & \hookrightarrow & q^*T(P \times_L F) & \xrightarrow{q^*Tr} & q^*r^*TB.
\end{array}$$

Here $(P \times F) \times \mathfrak{g} \to TP \times TF$ is defined by

$$((x, f), \xi) \mapsto \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} S((x, f), \exp(t\xi)) = (v_\xi^R(x), -v_\xi^L(f)).$$

(2) *There is a unique Ehresmann connection $\tilde{A}$ on $r\colon P \times_L F \to B$ such that*

$$H_{\tilde{A}} = Tq(\mathrm{pr}_P^* H_A) \quad \text{and} \quad q^*\theta_{\tilde{A}} = \mathrm{pr}_P^*(\kappa \circ \theta_A) \oplus \mathrm{id}_{TF}.$$

This is the ***Ehresmann connection induced by*** *A.*

(3) *Let $s \in C^\infty(P, F)^L$. The corresponding section $\hat{s} \in \Gamma(r)$ is $\tilde{A}$–horizontal if and only if $Ts|_{V_p \times TF} = 0$.*

(4) *The parallel transports of A and $\tilde{A}$ are related by*

$$\mathrm{tra}_\gamma^{\tilde{A}}(q(x, f)) = q(\mathrm{tra}_\gamma^A(x), f).$$

(5) *The curvatures of A and $\tilde{A}$ are related by*

$$F_{\tilde{A}}(T_x q(v), T_x q(w)) = Tq(\kappa \circ F_A(v, w) \oplus 0) = Tq(0 \oplus v^L_{F_A(v,w)}).$$

Proof. (1) is an exercise.

Since H_A is R–invariant, $\mathrm{pr}_P^* H_A$ is S–invariant. Therefore, it descends to a distribution $H_{\tilde{A}} \subset T(P \times_L F)$. Evidently, $Tr\colon H_{\tilde{A}} \to r^*TB$ is an isomorphism. Therefore, $H_{\tilde{A}}$ defines an Ehresmann connection $\tilde{A}$ on $q\colon P \times_L F \to B$. Similarly, $\mathrm{pr}_p^*(\kappa \circ \theta_A) \oplus \mathrm{id}_{TF}$ is S–invariant; hence it descends to $\theta_{\tilde{A}}$. By construction, $\ker \theta_{\tilde{A}} = H_{\tilde{A}}$. This proves (2).

(3) holds by construction. This in turn implies (4).

To prove the fist formula in (5) choose lifts and compute on $P \times F$. The second formula follows from (1). ∎

Proposition 4.51. *Let G be a Lie group. Let $(p\colon P \to B, R\colon P \times G \to P)$ be a G–principal fibre bundle. Let $\rho\colon G \to \mathrm{GL}(V)$ be a finite-dimensional representation. $r\colon P \times_\rho V \to B$.*

(1) *There are unique vector space structure on the fibres of r such that r becomes a vector bundle and the map $\tau\colon p^*(P\times_\rho V) \to P \times V$ is an isomorphism of vector bundles. This is the* ***vector bundle associated with*** (p, R) ***and*** *ρ.*

(2) *If $\alpha \in \Omega^\bullet(B, P \times_\rho V)$, then $\hat{\alpha} \coloneqq \mathrm{pr}_V p^*\alpha \in \Omega^\bullet_{\mathrm{hor}}(P, V)^\rho$. This defines a bijection*

$$\hat{\cdot}\colon \Omega^\bullet(B, P \times_\rho V) \to \Omega^\bullet_{\mathrm{hor}}(P, V)^\rho.$$

(3) *There is a unique derivative corresponding d_A on $P \times_\rho V$ such that*

$$\begin{array}{ccc} \Omega^\bullet(B, P \times_\rho V) & \xrightarrow{\mathrm{d}_A} & \Omega^{\bullet+1}(B, P \times_\rho V) \\ \big\downarrow{\scriptstyle\hat{\cdot}} & & \big\downarrow{\scriptstyle\hat{\cdot}} \\ \Omega^\bullet_{\mathrm{hor}}(P, V)^\rho & \xrightarrow{\mathrm{d}_A} & \Omega^{\bullet+1}_{\mathrm{hor}}(P, V)^\rho. \end{array}$$

Proof. It suffices to prove (1) for the trivial bundle $\mathrm{pr}_B\colon B \times G \to B$. In this case the map $q\colon (B \times G) \times V \to B \times V$ defined by $((b, g), v) \mapsto (b, \rho(g)v)$ is G–invariant and every G–invariant map from $(B \times G) \times V$ factors through q. Therefore, $(B \times G) \times_\rho V$ is (isomorphic) to $B \times V$.

(3) is an exercise.

(3) follows from the fact that d_A is compatible with tensor products. ■

Exercise 4.52. Let X be a closed smooth manifold (possibly not oriented). Construct a linear map

$$\int_X\colon D(TX) \to \mathbf{R}$$

(worthy of its notation). ?

Example 4.53. Let $V \to B$ be a vector bundle rank r. Denote the corresponding frame bundle by $\mathrm{Fr}(V)$ of isomorphisms $\phi\colon \mathbf{R}^r \to V_x$. The map $\mathrm{ev}\colon \mathrm{Fr}(V) \times \mathbf{R}^r \to V$ defined by

$$((x, \phi), v) \coloneqq (x, \phi(v))$$

is G–invariant and exhibits V as (isomorphic to) $\mathrm{Fr}(V) \times_\rho \mathbf{R}^r$ with $\rho \coloneqq \mathrm{id}\colon \mathrm{GL}_r(\mathbf{R}) \to \mathrm{GL}_r(\mathbf{R})$. ♠

Remark 4.54. The frame bundle formalism is a convenient way to carry linear algebra constructions over to vector bundles:

(1) Denote by $\rho^*\colon \mathrm{GL}_r(\mathbf{R}) \to \mathrm{GL}((\mathbf{R}^r)^*)$ the contragredient representation defined by

$$\rho^*(g)\lambda \coloneqq \lambda \circ \rho(g^{-1}).$$

$\mathrm{Fr}(V) \times_{\rho^*} (\mathbf{R}^r)^* \cong V^*$.

(2) Denote by $\Lambda^k \rho\colon\ \mathrm{GL}_r\, r(\mathbf{R}) \to \mathrm{GL}(\Lambda^k(\mathbf{R}^r))$ the representation defined by

$$(\Lambda^k \rho)(g)\alpha \coloneqq \Lambda^k(\rho(g))\alpha.$$

$\mathrm{Fr}(V) \times_{\Lambda^k \rho} \Lambda^k \mathbf{R}^r \cong \Lambda^k V.$

♣

4.5 Extension and reduction of structure group

References

[Ban13] M. Banagl. *Isometric group actions and the cohomology of flat fiber bundles. Groups, Geometry, and Dynamics* 7.2 (2013), pp. 293–321. DOI: 10.4171/GGD/183. MR: 3054571. Zbl: 1275.55008 (cit. on pp. 37, 38)

[BT82] R. Bott and L. W. Tu. *Differential forms in algebraic topology.* Graduate Texts in Mathematics 82. Springer, 1982. MR: 658304 (cit. on pp. 29, 37)

[Bum13] D. Bump. *Lie groups.* Second. Graduate Texts in Mathematics 225. Springer, 2013. DOI: 10.1007/978-1-4614-8024-2. MR: 3136522 (cit. on p. 39)

[Bun] U. Bunke. *Vorlesung Algebraische Topologie 1.* (cit. on p. 37)

[CE48] C. Chevalley and S. Eilenberg. *Cohomology theory of Lie groups and Lie algebras. Transactions of the American Mathematical Society* 63 (1948), pp. 85–124. MR: 0024908 (cit. on p. 49)

[dHoy16] M. del Hoyo. *Complete connections on fiber bundles. Indagationes Mathematicae. New Series* 27.4 (2016), pp. 985–990. DOI: 10.1016/j.indag.2016.06.009. arXiv: 1512.03847. Zbl: 1346.53019 (cit. on p. 23)

[Ehr51] C. Ehresmann. *Les connexions infinitésimales dans un espace fibré différentiable. Colloque de topologie (espaces fibrés), Bruxelles, 1950.* Georges Thone, Liège; Masson & Cie, Paris, 1951, pp. 29–55 (cit. on p. 17)

[Ful95] W. Fulton. *Algebraic topology. A first course.* Graduate Texts in Mathematics 153. Springer, 1995. Zbl: 0852.55001 (cit. on p. 2)

[Got69] M. Goto. *On an arcwise connected subgroup of a Lie group. Proc. Amer. Math. Soc.* 20 (1969), pp. 157–162. MR: 0233923 (cit. on p. 40)

[Hat02] A. Hatcher. *Algebraic topology.* Cambridge: Cambridge University Press, 2002. Zbl: 1044.55001 (cit. on p. 2)

[Hus94] D. H. Husemoller. *Fibre bundles.* Graduate Texts in Mathematics 20. Springer, 1994. DOI: 10.1007/978-1-4757-2261-1. MR: 1249482. Zbl: 0794.55001 (cit. on p. 17)

[Jän05] K. Jänich. *Topologie.* Berlin: Springer, 2005. Zbl: 1057.54001 (cit. on p. 2)

[Kar99] H. Karcher. *Submersions via projections. Geometriae Dedicata* 74.3 (1999), pp. 249–260. DOI: 10.1023/A:1005083115426. MR: 1669359 (cit. on p. 28)

[KMS93] I. Kolář, P.W. Michor, and J. Slovák. *Natural operations in differential geometry.* Springer, 1993. DOI: 10.1007/978-3-662-02950-3. Zbl: 0782.53013. (cit. on pp. 17, 30)

[May99] J.P. May. *A concise course in algebraic topology.* University of Chicago Press, 1999. Zbl: 0923.55001. (cit. on p. 2)

[MS74] J. W. Milnor and J. D. Stasheff. *Characteristic Classes.* Annals of Mathematics Studies 76. Princeton University Press; University of Tokyo Press, 1974. MR: 0440554 (cit. on p. 37)

[Ste51] N. Steenrod. *The Topology of Fibre Bundles.* Princeton Landmarks in Mathematics 14. Princeton University Press, 1951. MR: 0039258. Zbl: 0054.07103 (cit. on p. 17)

[Yam50] H. Yamabe. *On an arcwise connected subgroup of a Lie group. Osaka Math. J.* 2 (1950), pp. 13–14. MR: 0036766 (cit. on p. 40)

Index