Differential Geometry III
Gauge Theory
Problem Set 2

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2021-11-07

(1) Prove that the Hopf bundle $p\colon S^{2n+1} \to \mathbf{C}P^n$ does not admit a flat Ehresmann connection.

(2) The following is from [MS74, Appendix C].

(a) Denote by $\mathrm{SL}_2(\mathbf{R})$ the 2×2 real matrices A with $\det A = 1$. Denote by $\mathbf{PSL}_2(\mathbf{R})$ the equivalence classes of the involution $A \mapsto -A$ on $\mathrm{SL}_2(\mathbf{R})$. $\mathbf{PSL}_2(\mathbf{R})$ acts on $H \coloneqq \{z \in \mathbf{C} : \operatorname{Im} z > 0\}$ by Möbius transformations.

$$\lambda_g(z) \coloneqq \frac{az+b}{cz+d} \quad \text{for} \quad g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Define $P\colon STH \to H \times (\mathbf{R} \cup \{\infty\})$ by

$$P(z,v) \coloneqq \lim_{t\to\infty} \exp_x(tv).$$

Here $\exp_x$ is computed with respect to the hyperbolic metric g_{-1} on H. (This is the projection to "the sphere at infinity".) $\mathbf{PSL}_2(\mathbf{R})$ acts on SH by

$$\Lambda_g(z,v) \coloneqq (\lambda_g(z), T_z\lambda_g(v))$$

and on $\mathbf{R} \cup \{\infty\}$ by Möbius transformations:

$$\lambda_g^\infty(x) \coloneqq \frac{ax+b}{cx+d} \quad \text{for} \quad g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Prove that P is $\mathbf{PSL}_2(\mathbf{R})$–equivariant; that is:

$$P \circ \Lambda_g = (\lambda_g \times \lambda_g^\infty) \circ P.$$

(b) Let Σ be a Riemann surface of genus $g \geqslant 2$. By the uniformization theorem there is a $\Gamma < \mathbf{PSL}(2,\mathbf{R})$ such that $\Sigma = \Gamma\backslash H$. Set $ST\Sigma \coloneqq \{(z,v) \in T\Sigma : |v| = 1\}$. Denote by $p\colon ST\Sigma \to \Sigma$ the canonical projection. Denote by $q\colon \Gamma\backslash STH \to \Gamma\backslash H = \Sigma$ the canonical projection. Prove that p and q are isomorphic.

(c) Prove that q admits a flat Ehresmann connection.

(d) Prove that the vector bundle $T\Sigma$ does not admit a flat covariant derivative. (Hint: Chern–Gauß–Bonnet.)