# Differential Geometry III
# Gauge Theory
# Problem Set 7

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(1) Let $(p\colon P \to B, R)$ be a $\mathrm{GL}_n(\mathbf{C})$–principal bundle. Work out explicit formulae for the Chern classes $c_0(p,R)$, $c_1(p,R)$ and $c_2(p,R)$ in terms of a connection on $(p,R)$.x

(2) Compute the the first Chern class $c_1$ of the Hopf bundle $(q\colon S^3 \to \mathbf{C}P^1, S)$.

(3) Prove that if $E$ is a complex rank $r$ vector bundle, then $c_k(E) = 0$ for $k > r$.

(4) Prove that $E_1$ and $E_2$ are complex vector bundles, then

$$c(E_1 \oplus E_2) = c(E_1) \cup c(E_2).$$

(5) Prove that $E_1$ and $E_2$ are complex vector bundles, then

$$\mathrm{ch}(E_1 \oplus E_2) = \mathrm{ch}(E_1) + \mathrm{ch}(E_2) \quad \text{and} \quad \mathrm{ch}(E_1 \otimes E_2) = \mathrm{ch}(E_1) \cup \mathrm{ch}(E_2).$$

(6) Prove that if $E$ is a complex vector bundle, then

$$\mathrm{ch}_3(E) = \frac{1}{6}(c_1(E)^3 - 3c_1(E)c_2(E) + 3c_3(E)).$$

(7) Let $E$ be a real vector bundle. Show that $c_{2k+1}(E \otimes_{\mathbf{R}} \mathbf{C}) = 0$.

(8) Suppose $B$ is a Riemannian closed 4–manifold. Let $G$ be a semi-simple Lie group and $P$ a principal $G$–bundle. Then minus the Killing form is a metric on $\mathfrak{g}_P \coloneqq P \times_{\mathrm{Ad}} \mathfrak{g}$. Show that there are constants $c_1 > 0$ and $c_2 \in \mathbf{R}$ such that for any $A \in \mathscr{A}(p,R)$

$$\mathrm{YM}(A) = c_1 \int_M |F_A + *F_A|^2 + c_2 \int_M p_1(\mathfrak{g}_P).$$

Since the second term on the right-hand side depends only on $P$, this shows, in particular, that anti-self-dual instantons are absolute minima of YM (and not just critical points).