Differential Geometry III
Gauge Theory
Problem Set *

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(1) Let $n \in \mathbf{N}$. Let G be a Lie group. Set $\mathfrak{g} := \mathrm{Lie}(G)$. Consider the trivial bundle $(p\colon G \times \mathbf{R}^n, R)$. Let $\xi_1, \cdots, \xi_n \in \mathfrak{g}$. A connection on (p, R) is equivalent to a 1–form $A \in \Omega^1(\mathbf{R}^n, \mathfrak{g})$. For

$$A = \sum_{a=1}^{n} \xi_a \mathrm{d}x_a$$

compute the Bianchi identity $\mathrm{d}_A F_A = 0$ and the Yang–Mills equation $\mathrm{d}_A^* F_A = 0$ in terms of $(\xi_1, \ldots, x_n)$.

(2) For the BPST instanton

$$A = \frac{\mathrm{Im}(\bar{q}\mathrm{d}q)}{|q|^2 + 1}$$

on $\mathbf{H}$ compute $\mathrm{YM}(A)$.

(3) Set $S^7 := \{(q_1, q_2) \in \mathbf{H}^2 : |q_1|^2 + |q_2|^2 = 1\}$. $\mathrm{Sp}(1) := \{q \in \mathbf{H} : |q|^2 = 1\}$ acts on the right of S^7 by $R((q_1, q_2), q) = (q_1 q, q_2 q)$. The quotient $\mathbf{H}P^1 := S^7/\mathrm{Sp}(1)$ parametrizes rank 1 left $\mathbf{H}$–submodules $\ell \subset \mathbf{H}^2$. Define $\theta \in \Omega^1(S^7, \mathfrak{sp}(1))$ by

$$\theta := \mathrm{Im}(\bar{q}_1 \mathrm{d}q_1 + \bar{q}_2 \mathrm{d}q_2).$$

Prove that θ is a $\mathrm{Sp}(1)$–principal connection 1–form on $(p\colon S^7 \to \mathbf{H}P^1, R)$.

(4) Define $i\colon \mathbf{H} \to \mathbf{H}P^1$ by $i(q) := [q, 1]$. Find a trivialisation of $i^*(p, R)$ such that i^*A corresponds to the BPST instanton.

Remark. $\mathrm{Sp}(1)$ also acts on the right of S^7 via $L((q_1, q_2), q) := (\bar{q} q_1, \bar{q} q_2)$. The quotient $\overline{\mathbf{H}P}^1 := \mathrm{Sp}(7)/\mathrm{Sp}(1)$ parametrizes rank 1 right $\mathbf{H}$–submodules $\ell \subset \mathbf{H}^2$. Of course, $\overline{\mathbf{H}P}^1 \cong S^4 \cong \mathbf{H}P^1$. Therefore, there are two $\mathrm{Sp}(1)$–principal bundles over S^4 with total space S^7. These bundles are not isomorphic as $\mathrm{Sp}(1)$–principal bundle. They are, however, isomorphic after pulling back one of them via an orientation reversing diffeomorphism of S^4. ♣