Differential Geometry III: Gauge Theory
Winter Semester 2023/24

Prof. Dr. Thomas Walpuski
Humboldt-Universität zu Berlin

2023-11-24

Contents

1 Covering maps **2**
- 1.1 Introduction . . . 2
- 1.2 The category of covering maps of B . . . 7
- 1.3 The fibred category of covering maps . . . 8
- 1.4 The lifting problem . . . 11
- 1.5 The unique homotopy lifting property . . . 12
- 1.6 The fundamental group(oid) . . . 14
- 1.7 Fibre transport and monodromy . . . 15
- 1.8 Lifting along covering maps . . . 18
- 1.9 Deck transformations . . . 19
- 1.10 Classification of covering maps . . . 21
- 1.11 Universal covering maps . . . 23
- 1.12 The Nielsen–Schreier Theorem . . . 25

2 Fibre bundles **27**
- 2.1 Introduction . . . 27
- 2.2 The category of fibre bundles . . . 28
- 2.3 Ehresmann connections . . . 29
- 2.4 Parallel transport . . . 31
- 2.5 The curvature of an Ehresmann connection . . . 33
- 2.6 Decomposition of the de Rham complex on fibre bundles . . . 36
- 2.7 Digression: The Fröhlicher–Nijenhuis bracket . . . 37
- 2.8 The spectral sequence of a filtered complex . . . 42
- 2.9 The Leray–Serre spectral sequence . . . 45
- 2.10 Fibre integration . . . 47
- 2.11 The Gysin sequence . . . 49
- 2.12 Symplectic fibre bundles . . . 50

3 Lie groups 52
3.1 Definition . . . 52
3.2 Lie group actions . . . 52
3.3 The slice theorem . . . 53
3.4 Lie algebra . . . 55
3.5 Exponential map . . . 57
3.6 Haar volume form . . . 59
3.7 The Killing form . . . 60
3.8 de Rham cohomology of manifolds with G–actions . . . 60

Index 67

1 Covering maps

The purpose of this section is to *review* the theory of covering maps. Here are some classical (popular?) l references: Jänich [Jän05, Kapitel 9], Fulton [Ful95, Parts VI and VII], May [May99, §1-§4], and Hatcher [Hat02, §1].

1.1 Introduction

Definition 1.1. A continuous map $p\colon X \to B$ is a **covering map** if for every $b \in B$ there are an open subset $U \subset B$ with $b \in U$, a discrete space D, and a homeomorphism $\tau\colon p^{-1}(U) \to U \times D$ such that

$$\mathrm{pr}_1 \circ \tau = p|_{p^{-1}(U)}.$$

•

Here are some examples (which might not be sufficiently convincing).

Example 1.2. Let B be a topological space. Let D be a discrete space. The projection map $\mathrm{pr}_1\colon B \times D \to B$ is a covering map: the **trivial covering map of B with fibre D**. ♠

Example 1.3. The exponential map $\exp\colon \mathbf{C} \to \mathbf{C}^\times$ is a covering map. ♠

Example 1.4. The cosine map $\cos\colon \mathbf{C}\backslash\pi\mathbf{Z} \to \mathbf{C}\backslash\{\pm 1\}$ is a covering map. ♠

Example 1.5. Consider $S^1 := [0,1]/\{0,1\}$.

(1) The map $p_\infty\colon \mathbf{R} \to S^1$ defined by $p_\infty(x) := [\lfloor x \rfloor]$ is a covering map.

(2) For every $k \in \mathbf{N}_0$, the map $p_k\colon S^1 \to S^1$ defined by $p_k([x]) := [\lfloor kx \rfloor]$ is a covering map. ♠

The degree of a covering map Here are a straight-forward observation and another example.

Definition 1.6. Let $p\colon X \longrightarrow B$ be a proper covering map. The **degree of** p is the map $\deg_\cdot(p)\colon B \longrightarrow \mathbf{N}_0$ defined by

$$\deg_b(p) := \#p^{-1}(b). \qquad \bullet$$

Proposition 1.7. *Let $p\colon X \longrightarrow B$ be a proper covering map. The map $\deg_\cdot(p)$ is locally constant.* ■

Example 1.8.

(1) Denote by

$$\mathrm{Poly} \cong \coprod_{d\in\mathbf{N}_0} \mathbf{C}^\times \times \mathbf{C}^d$$

the space of complex polynomials. The topology on Poly is choosen so that map $\mathrm{pdeg}\colon \mathrm{Poly} \to \mathbf{N}_0$ which assigns to every polynomial its degree is continuous. Consider the **universal set of roots**

$$\mathrm{Roots} := \{(p, z) \in \mathrm{Poly} \times \mathbf{C} : p(z) = 0\}.$$

The projection map $q\colon \mathrm{Roots} \longrightarrow \mathrm{Poly}$ is a *not a covering map*—it violates Proposition 1.7: $p_\varepsilon(z) := z^2 + \varepsilon$ has a unique root if $\varepsilon = 0$, but 2 distinct roots if $\varepsilon \neq 0$.

(2) The above issue is easily rectified (or rather ignored) as follows. Denote by $\mathrm{Poly}^\circ \subset \mathrm{Poly}$ the open subset of those $p \in \mathrm{Poly}$ with $\mathrm{pdeg}(p) = \#p^{-1}(0)$ roots or, equivalently, with non-zero discriminant. Set $\mathrm{Roots}^\circ := \mathrm{Roots} \cap (\mathrm{Poly}^\circ \times \mathbf{C})$. The restriction

$$q^\circ := q|_{\mathrm{Roots}^\circ}\colon \mathrm{Roots}^\circ \longrightarrow \mathrm{Poly}^\circ$$

is a covering map. ♠

Covering maps and the regular value theorem In differential geometry, covering map are almost unavoidable because of the following observations.

Proposition 1.9. *If $p\colon X \longrightarrow B$ is a proper local homeomorphism, then it is a covering map.* ■

Proposition 1.10. *Let $p\colon X \longrightarrow B$ be a proper equi-dimensional smooth map. Set $B^\circ := B\backslash p(\mathrm{Crit}(p))$ and $X^\circ := p^{-1}(B^\circ)$. The restriction $p^\circ := p|_{X^\circ}\colon X^\circ \longrightarrow B^\circ$ is a covering map.* ■

Proposition 1.11. *Let $p\colon X \longrightarrow B$ be a covering map. If B is a smooth manifold, then X admits a unique smooth structure such that p is smooth (indeed: a local diffeomorphism).* ■

Here are two examples to illustrate the above.

Example 1.12. Denote by $\mathbf{H}$ the normed $\mathbf{R}$–algebra of the **quaternions**. Set

$$\mathrm{Sp}(1) \coloneqq \{q \in \mathbf{H} : |q| = 1\}.$$

The map $\mathrm{Ad}\colon \mathrm{Sp}(1) \longrightarrow \mathrm{SO}(\mathrm{Im}\,\mathbf{H}) = \mathrm{SO}(3)$ defined by

$$\mathrm{Ad}(q)x \coloneqq qxq^*$$

is a covering map of degree 2; moreover: it is a Lie group homomorphism. ♠

Remark 1.13. $\mathrm{SO}(3)$ naturally is a submanifold of $\mathbf{R}^{3\times 3} = \mathbf{R}^9$. Since $\dim \mathrm{SO}(3) = 3$, it seems quite wasteful to encode a rotation as a 3×3–matrix and inefficient to compute the composition of two rotations via matrix multiplication. Example 1.12 offer a more parsimonious and efficient solution to this problem—at the expense of a slight over-parametrisation. ♣

Example 1.14.

(1) For a unit vector $v \in S^2 \subset \mathbf{R}^3$ and an angle $\alpha \in S^1 \coloneqq \mathbf{R}/2\pi\mathbf{Z}$ denote by $R_v(\alpha) \in \mathrm{SO}(3)$ the rotation around v by α. Define $E\colon T^3 \coloneqq (S^1)^3 \longrightarrow \mathrm{SO}(3)$ by

$$E(\phi, \theta, \psi) \coloneqq R_{e_3}(\phi)R_{e_2}(\theta)R_{e_3}(\psi).$$

This is the over-parametrization of $\mathrm{SO}(3)$ by **Euler angles**—in the intrinsic *zyz* convention. E is proper and surjective, but *not a covering map* because $\mathrm{Crit}(E) \neq \varnothing$.

(2) Proposition 1.10 produces a covering map $E^\circ\colon (T^3)^\circ \longrightarrow \mathrm{SO}(3)^\circ$ of degree 2; but: not a Lie group homomorphism. ♠

Remark 1.15.

(1) Example 1.14 provides an even more parsimonious encoding of $\mathrm{SO}(3)$ than Example 1.12; however: computing compositions is not straight-forward.

(2) Here is how Example 1.14 arises in practice. T^3 describes the rotations of a **three-axis gimbal**; e.g., a robot arm. The map E encodes the rotation of object suspended in the gimbal effected by the rotations of the gimbal. $\mathrm{Crit}(E)$ is the set of Euler angles for which

gimbal lock occurs. ♣

Covering maps and Riemann surfaces Let Σ, T be closed Riemann surfaces. If $f\colon \Sigma \to T$ is a non-constant holomorphic map, then $\mathrm{Crit}(f)$ is a finite set. As a consequence, passing to the covering map $f^\circ\colon \Sigma^\circ \to T^\circ$ obtained by Proposition 1.10 does not incur a substantial loss of information.

Proposition 1.16. *Let T be a Riemann surface. Let $B \subset T$ be a finite subset. Set $T^\circ := T \backslash B$. Let $f^\circ\colon \Sigma^\circ \to T^\circ$ be a covering map. There is a closed Riemann surface Σ, a finite subset $R \subset \Sigma$, and a holomorphic map $f\colon \Sigma \to T$ such that $\Sigma \backslash R = \Sigma^\circ$ and $f^\circ = f|_{\Sigma^\circ}$.* ■

It suffices to prove this for $T^\circ = D^\times$. In this case, the result follows from the classification of covering maps of $p\colon \Sigma^\circ \to D^\times$. If Σ° is connected, then every covering map is essentially of the form $D^\times \to D^\times, z \mapsto z^k$.

The above observation is particularly important because of the following foundational result.

Theorem 1.17 (Riemann's existence theorem). *Every closed Riemann surface Σ admits a non-constant holomorphic map $f\colon \Sigma \to \mathbf{C}P^1$.* ■

The proof of Theorem 1.17 is somewhat difficult (certainly by historic standards) and requires some (e.g., analytic) machinery.

Here is a more elementary observation.

Proposition 1.18. *Let Σ, T be closed Riemann surfaces. Let $f\colon \Sigma \to T$ be a holomorphic map. If f is non-constant and T is connected, then f is surjective.* ■

This implies the **fundamental theorem of calculus** as follows; cf. Milnor [Mil97, p.8?]. Let

$$p(z) = \sum_{k=0}^{d} a_k z^k \in \mathbf{C}[z]$$

be a polynomial of degree $d \geqslant 1$. Since $P\colon \mathbf{C}P^1 \to \mathbf{C}P^1$ defined by

$$P([z : w]) := \left[\sum_{k=0}^{d} a_k z^k w^{d-k} : w^d\right]$$

is surjective, p must have a root.

Riemann surfaces rather naturally appear in complex analysis; in particular, through the following concept.

Definition 1.19. A **multi-valued holomorphic function** on $U \subset \mathbf{C}$ is connected Riemann surface $\Sigma \subset U \times \mathbf{C}$ such that $\mathrm{pr}_1 \colon \Sigma \to U$ is surjective. •

Example 1.20. The **square-root** is

$$\mathrm{Sqrt} \coloneqq \{(z, w) \in \mathbf{C}^\times \times \mathbf{C} : z = w^2\}.$$ ♠

Example 1.21. The **logarithm** is

$$\mathrm{Log} \coloneqq \{(z, w) \in \mathbf{C}^\times \times \mathbf{C} : z = e^w\}.$$ ♠

Covering maps and quotients Finally, covering maps arise from quotients by (exceptionally tame) group actions.

Definition 1.22. Let G be group. Let X be a topological space. A right action $X \circlearrowleft G$ is a **covering space action** if every $x \in X$ has an open neighborhood U such that $U \cdot g \cap U \neq \varnothing$ if and only if $g = \mathbf{1} \in G$. •

Proposition 1.23. *Let $X \circlearrowleft G$ be a covering space action. Let $H < G$ The projection map $p \colon X/H \to X/G$ is a covering map.* ■

Two examples shall suffice to illustrate this.

Example 1.24.

(1) Consider S^n with the **round metric** g; that is: the metric induced by the Euclidean metric on $\mathbf{R}^{n+1}$. The isometry group $\mathrm{Isom}(S^n, g)$ is $\mathrm{O}(n+1)$. If $\Gamma^{\mathrm{op}} < \mathrm{O}(n+1)$ induces a covering space action, then g descends to a Riemannian metric $\check{g}$ on S^n/Γ with sectional curvature $\sec_{\check{g}} = 1$. In fact, by the **Riemann–Hopf–Killing theorem**, every **spherical space form** (that is: a complete Riemannian manifold (X, g) with $\sec_g = 1$) arises from this construction (up to isometry).

(2) It is know which subgroups $\Gamma^{\mathrm{op}} < \mathrm{O}(n+1)$ occur. This is quite difficult and discussed, e.g., in Wolf [Wol11].

(3) Let $n \in \mathbf{N}$. Let $p \in \mathbf{N}$ and $q_1, \dots, q_n \in \mathbf{Z}$ such that

$$\gcd(p, q_i) = 1.$$

Identify $\mathbf{R}^{2n} = \mathbf{C}^n$ and define $\phi \in \mathrm{SO}(2n)$ by

$$\phi(z_1, \dots, z_n) \coloneqq (e^{2\pi i q_1/p} z_1, \cdots, e^{2\pi i q_n/p} z_1).$$

By construction, the subgroup $\langle \phi \rangle \subset \mathrm{SO}(2n)$ is cyclic of order p and acts freely on S^{2n-1}. The **lens space** $L(p; q_1, \ldots, q_n)$ is the quotient

$$L(p; q_1, \ldots, q_n) := S^{2n-1}/\langle \phi \rangle.$$

In particular, $L(2, 1, 1) = \mathbf{R}P^3$. The projection map $S^{2n-1} \to L(p; q_1, \ldots, q_n)$ is a covering map. ♠

Example 1.25. Let X be a topological space. Let $k \in \mathbf{N}$.

(1) $X^k \circlearrowleft S_k$ via

$$(x_1, \ldots, x_k) \cdot \sigma := (x_{\sigma(1)}, \ldots, x_{\sigma(k)}).$$

The k–**th symmetric power of** X is

$$\mathrm{Sym}^k(X) := X^k/S_k.$$

The projection map $p\colon X^k \to \mathrm{Sym}^k(X)$ is *not a covering map*—with the exception of a few edge cases. This is also called the **configuration space** of k points in X. $\mathrm{Sym}^k(\mathbf{R}^n)$ plays an important role in the study of multi-valued functions, as it appears, e.g., in Almgren [Almoo].

(2) The map p fails to be a covering map along the **fat diagonal** $\Delta \subset X^k$ defined by

$$\Delta := \big\{(x_1, \ldots, x_k) \in X^k : \#\{x_1, \ldots, x_k\} < k\big\}.$$

The **regular part of the** k–**th symmetric power of** X is

$$\mathrm{Sym}^k(X)^\circ := (X^k \backslash \Delta)/S_k.$$

The projection map $p^\circ\colon X^k\backslash\Delta \to \mathrm{Sym}^k(X)^\circ$ is a covering map. This is also called the **unordered configuration space**.

(3) If $G < S_k$ is a subgroup, then $q\colon (X^k\backslash\Delta)/G \to \mathrm{Sym}^k(X)^\circ$ is a covering map. $\mathrm{Sym}^k(X)^\circ$ is the space of subsets $S \subset X$ with $\#S = k$. If $S_{k-1} \cong G < S_k$ is subgroup fixing $1 \in \{1, \ldots, k\}$, then $(X^k\backslash\Delta)/G$ is the space of subset $S \subset X$ with $\#S = k$ together with a choice of $x \in S$. For $X = \mathbf{C}$ this (essentially) recovers Example 1.8 (restricted to degree k polynomials). ♠

1.2 The category of covering maps of B

At this point the reader is (hopefully) convinced that the concept of covering map is sufficiently relevant to introduce a category of covering maps.

Definition 1.26. Let $\mathbf{C}$ be a category. Let b be an object of $\mathbf{C}$. The **slice category** is the category $\mathbf{C}/b$ whose objects are morphisms $p\colon x \to b$ in $\mathbf{C}$, and whose morphisms $\phi\colon (p\colon x \to b) \to$

$(q\colon y \to b)$ are morphsisms $\phi\colon x \to y$ in **C** satisfying

$$p = q \circ \phi.$$

•

Definition 1.27. Let B be a topological space. The **category of covering maps of** B is the full subcategory $\mathbf{Cov}(B) \subset \mathbf{Top}/B$ whose objects are covering maps $p\colon X \to B$. •

Proposition 1.28. $\mathbf{Cov}(B)$ *has products and coproducts.* ■

Example 1.29. $\mathbf{Cov}(\{*\}) \cong \mathbf{Set}$. ♠

Under rather mild connectivity assumptions on B, $\mathbf{Cov}(B)$ can be (essentially) determined algbraically from the fundamental group $\pi_1(B, b)$; see Section 1.10.

Here is an observation about $\mathbf{Cov}(B)$ which is so trivial that it usually is not even mentioned.

Definition 1.30. Let B be a topological space. Denote by $\mathrm{Op}(B)$ the category whose objects are open subset $U \subset B$ and whose morphisms $U \to V$ are inclusions $U \subset B$. Define the functor $\mathbf{Cov}_B\colon \mathrm{Op}(B)^{\mathrm{op}} \to \mathbf{Cat}$ as follows.

(1) For every open subset $U \subset B$ set $\mathbf{Cov}_B(U) \coloneqq \mathbf{Cov}(U)$

(2) For every inclusion of open subset $U \subset V \subset B$, $\mathbf{Cov}_B(U \subset V)\colon \mathbf{Cov}(V) \to \mathbf{Cov}(U)$ is the (obvious) restriction functor. •

Proposition 1.31. $\mathbf{Cov}_B$ *is a sheaf; that is: for every open cover* $\{U_i : i \in I\}$ *of* $U \subset B$ *the diagram*

$$\mathbf{Cov}_B(U) \to \coprod_{i \in I} \mathbf{Cov}_B(U_i) \rightrightarrows \coprod_{(i,j) \in I^2} \mathbf{Cov}_B(U_i \cap U_j)$$

is an equaliser. ■

1.3 The fibred category of covering maps

See Vistoli's notes `http://homepage.sns.it/vistoli/descent.pdf` for more on fibred categories.

Definition 1.32. Let **C** be a category. The **arrow category** is the category $\mathbf{Arr}(\mathbf{C})$ whose objects are morphisms $p\colon x \to a$ in **C**, and whose morphisms $(p\colon x \to a) \to (q\colon y \to b)$ are pairs of morphisms $\phi\colon x \to y$ and $f\colon a \to b$ in **C** satisfying

$$f \circ p = q \circ \phi.$$

•

Definition 1.33. The **category of covering maps** is the full subcategory $\mathbf{Cov} \subset \mathbf{Arr}(\mathbf{Top})$ whose objects are covering maps $p\colon X \to B$. •

The main reason to introduce **Cov** is to talk about pull-packs. This is a good excuse to introduce fibred categories.

Definition 1.34. Let $P\colon \mathbf{X} \to \mathbf{B}$ be a functor. A morphism $\phi\colon x \to y$ is **cartesian** if for every object z of $\mathbf{X}$, every morphism $g\colon P(z) \to P(x)$, and every morphism $\psi\colon z \to x$ with $P(\psi) = P(\phi) \circ g$ there is a unique morphism $\zeta\colon z \to y$ such that

$$\psi = \phi \circ \zeta.$$

that is:

$$\begin{tikzcd}
z \arrow[drr, dashed, "\exists!\zeta"] \arrow[drrrr, bend left, "\psi"] & & & & \\
P(z) \arrow[drr, "g"] & & x \arrow[rr, "\phi"] & & y \\
& & P(x) \arrow[rr, "P(\phi)"] & & P(y)
\end{tikzcd}$$

•

Definition 1.35. A **fibred category** is a functor $P\colon \mathbf{X} \to \mathbf{B}$ such that for every object x of $\mathbf{X}$ and every morphism $f\colon a \to P(x)$ there is a **cartesian lift**; that is: a cartesian morphism $\phi\colon y \to x$ with

$$P(\phi) = f.$$

•

Proposition 1.36. *The codomain functor* $U\colon \mathbf{Cov} \to \mathbf{Top}$ *is a fibred category.*

Proof. Let $p\colon X \to B$ be a be a covering map. Let $f\colon A \to B$ be a smooth map. Set

$$f^*X := \{(a, x) \in A \times X : f(a) = p(x)\} \subset A \times X.$$

Define $\phi\colon f^*X \to X$ by $\phi(a, x) := x$ and $f^*p\colon f^*X \to A$ by $f^*p(a, x) := a$. A moment's thought shows that f^*p is a covering map. Evidently, $(\phi, f)\colon q \to p$ is cartesian. ∎

Definition 1.37. For every covering map $p\colon X \to B$ and every continuous map $f\colon A \to B$ *choose* a cartesian lift

$$\begin{tikzcd} f^*X \arrow[r, "p^*f"] \arrow[d, "f^*p"] & X \arrow[d, "p"] \\ A \arrow[r, "f"] & B. \end{tikzcd}$$

Denote this as *the* **pullback of** p **via** f. •

Remark 1.38. Let $p\colon X \to C$ be a covering map and $f\colon A \to B$, $g\colon B \to C$ be continuous maps. Typically, $(g \circ f)^*p$ is not *equal* to f^*g^*p but there is a *canonical* isomorphism

$$I_{f,g}\colon (g \circ f)^* \cong f^* \circ g^*$$

by the definition of cartesian morphism. In the implementation of pull-backs given above,

$$I_{f,g}(p)\colon (g \circ f)^*X \to f^*g^*X$$

is given by $I_{f,g}(p)(a, x) = (a, (f(a), x))$. ♣

A key fact about $\mathbf{Cov}(B)$ is that it is an invariant of the homotopy-type of B. This can be proved as follows.

Proposition 1.39. *Let $p\colon X \to [0, 1] \times B$ be a covering map. For every $b \in B$ there are an open subset $V \subset B$ with $b \in V$, a discrete space D, and a homeomorphism $\tau\colon p^{-1}([0, 1] \times V) \to [0, 1] \times V \times D$ such that* $(\mathrm{pr}_1, \mathrm{pr}_2) \circ \tau = p|_{p^{-1}([0,1]\times V)}$.

Proof. By Definition 1.1 and since $[0, 1]$ is compact, for every $b \in B$ there are an open subset $V \subset B$ with $b \in V$, $0 = t_0 < t_1 < \cdots < t_n = 1$, and for every $i \in \{0, \ldots, n - 1\}$ a discrete space D_i and a homeomorphism $\tau_i\colon p^{-1}([t_i, t_{i+1}] \times V) \to [t_i, t_{i+1}] \times V \times D_i$ such that $(\mathrm{pr}_1, \mathrm{pr}_2) \circ \tau_i = p|_{p^{-1}([t_i,t_{i+1}]\times V)}$.

Set $\sigma_i := (\mathrm{pr}_2, \mathrm{pr}_3) \circ \tau_{i-1}^{-1} \circ \tau_i(t_i, \cdot)\colon V \times D_i \to V \times D_{i-1}$. Evidently, σ_i is a homeomorphism satisfying $\mathrm{pr}_1 = \mathrm{pr}_1 \circ \sigma_i$, and the homeomorphisms $\tilde{\tau}_i := \sigma_0 \circ \cdots \circ \sigma_i \circ \tau_i$ glue to the desired homeomorphism τ. ■

Proposition 1.40. *For every covering map $p\colon X \to [0, 1] \times B$ there is a unique isomorphism*

$$\phi\colon \mathrm{pr}_B^* i_0^* p \cong p$$

in $\mathbf{Cov}([0, 1] \times B)$ such that $i_0^*\phi\colon i_0^*\mathrm{pr}_2^* i_0^* p \cong i_0^* p$ is the canonical isomorphism in $\mathbf{Cov}(B)$. Here $i_0\colon B \to [0, 1] \times B$ is defined by $i_0(b) := (0, b)$.

Proof. If ϕ, ψ are two such isomorphisms, then $(\phi,\psi)^{-1}(\Delta)$ is open and closed in $\mathrm{pr}_2^* i_0^* X \cong [0,1] \times p^{-1}(\{0\} \times B)$ and contains $\{0\} \times p^{-1}(\{0\} \times B)$. Therefore, $\phi = \psi$.

The isomorphisms ϕ exists for $(\mathrm{pr}_1, \mathrm{pr}_2)\colon [0,1] \times B \times D \to [0,1] \times B$. Therefore, by uniqueness and Proposition 1.39, it exists for every covering map $p\colon X \to [0,1] \times B$. ∎

Corollary 1.41. *If f_0, f_1 are homotopy-equivalent, then $\mathbf{Cov}(f_0), \mathbf{Cov}(f_1)$ are naturally isomorphic. Moreover, a homotopy equivalence induces an natural isomorphism.* ∎

1.4 The lifting problem

Definition 1.42. Let C be a category. Let $p\colon x \to b$ and $f\colon a \to b$ be morphisms in C. A morphism $\tilde{f}\colon a \to x$ is a **lift of f along p** if

$$f = p \circ \tilde{f};$$

that is: the diagram

$$\begin{tikzcd} & x \arrow[d, "p"] \\ a \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & b \end{tikzcd}$$

commutes or, equivalently,

$$\tilde{f} \in \mathrm{Hom}_{\mathrm{C}/b}(f, p).$$ •

The **lifting problem** is to determine $\mathrm{Hom}_{\mathrm{C}/b}(f,p)$, the set of lifts of f along p. Many questions in geometry and topology are lifting problems. Here are some examples.

Example 1.43. Which open subsets $U \subset \mathbf{C}^\times$ admit a **logarithm**; that is: a lift of $U \subset \mathbf{C}^\times$ along exp? ♠

Example 1.44. Let $P\colon A \to \mathrm{Poly}$ be a continous map. Is is possible to continuously choose roots of P; that is: a lift of P along p in Example 1.8? ♠

Example 1.45. Imagine a robot arm holding a banana. A movement in time of the banana is encoded by a path $\gamma\colon [0,1] \to \mathrm{SO}(3)$. The question of how to operate the robot arm to achieve this path is the lifting problem along E. ♠

Example 1.46. Let $f\colon A \to \mathrm{Sym}^k(X)$ be a k−valued function. Is it possible to lift it to k single-valued functions? ♠

Example 1.47. Via pull-backs the lifting problem can always be reduced to id_A: Indeed, there is a canonical bijection $\mathrm{Hom}_{\mathbf{Top}/B}(f,p) \cong \mathrm{Hom}_{\mathbf{Top}/A}(\mathrm{id}_A, f^*p)$. ♠

Example 1.48. Pullbacks allow trivial solutions of lifting problems. Let $f\colon B \to \mathrm{Sym}^k(X)^\circ$ be a k−valued map. Consider the covering map $p^\circ\colon X^k\backslash\Delta \to \mathrm{Sym}^k(X)^\circ$. The map f might not lift, but $f \circ f^*p^\circ$ canonically lifts: the lift is p^*f. ♠

1.5 The unique homotopy lifting property

This section lays the foundation for solving the lifting problem in Section 1.8. The following asserts that the lifting problem along covering maps is very rigid.

Definition 1.49. Let $p\colon X \to B$ and $f\colon A \to B$ be continous maps. Let $a \in A$. Set $b := f(a)$. Define the **evaluation map**

$$\mathrm{ev}_a\colon \mathrm{Hom}_{\mathbf{Top}/B}(f,p) \to p^{-1}(b)$$

by

$$\mathrm{ev}_a(\tilde{f}) := \tilde{f}(a). \qquad \bullet$$

Proposition 1.50. *Let $p\colon X \to B$ and $f\colon A \to B$ be continous maps. Let $a \in A$. If p is a covering map and A is connected, then ev_a is injective.*

This is an immediate consequence of the following.

Lemma 1.51. *Let $p\colon X \to B$ be a covering map. Set*

$$X \times_B X := \{(x,y) \in X \times X : p(x) = p(y)\} \quad \textit{and} \quad \Delta := \{(x,x) \in X \times_B X\}.$$

The subset $\Delta \subset X \times_B X$ is open and closed.

Proof. For $x \in X$ denote by V_x an open neighborhood of $x \in X$ such that $p(V_x)$ is open and $p|_{V_x}\colon V_x \to p(V_x)$ is a homeomorphism. $(V \times V) \cap (X \times_B X)$ is an open neighborhood of $(x,x) \in X \times_B X$ and contained in Δ. Therefore, Δ is open.

Let $(x,y) \in (X \times_B X)\backslash\Delta$. Choose V_x, V_y as above with $V_x \cap V_y = \varnothing$. $(V_x \times V_y) \cap X \times_B X$ is an open neighborhood of $(x,y) \in X \times_B X$ and does not intersect Δ. Therefore, Δ is closed. ■

Proof of Proposition 1.50. Suppose $\tilde{f}_1, \tilde{f}_2 \colon A \to X$ are lifts of f along p with $\tilde{f}_i(a) = x$. By Lemma 1.51, $S \coloneqq (\tilde{f}_1, \tilde{f}_2)^{-1}(\Delta)$ is open and closed. Since $a \in S$ and A is connected, $S = A$; hence: $\tilde{f}_1 = \tilde{f}_2$. ■

There are obstructions to the lifting problem along covering maps; that is: ev_a need not be a bijection. Here is an example to illustrate this.

Example 1.52. There is *no lift* of id_{S^1} along p_∞ from Example 1.5:

$$\mathrm{Hom}_{\mathbf{Top}/S^1}(\mathrm{id}_{S^1}, p_\infty) = \varnothing.$$

Indeed, if $\tilde{f} \colon S^1 \to \mathbf{R}$ were a lift, then $\tilde{f}(S^1)$ would be compact interval $[a, b]$ and $p_\infty|_{[a,b]}$ would be a bijection. However, $p_\infty|_{[a,b]}$ is injective if and only if $b - a < 1$ and surjective if and only if $b - a \geqslant 1$. ♠

However, there are no obstructions to lifting continuous paths (even in families). Here is a precise formulation of this observation.

Definition 1.53. Let A be a topological space. A continuous map $p \colon X \to B$ has the **homotopy lifting property (HLP) with respect to** A if for every homotopy $h \colon [0,1] \times A \to B$ and lift $\tilde{h}_0 \colon A \to X$ of $h_0 \coloneqq h(0, \cdot) \colon A \to B$ there is a homotopy $\tilde{h} \colon [0,1] \times A \to X$ which is a lift $\tilde{h}$ of h with $\tilde{h}(0, \cdot) = \tilde{h}_0$; that is: the diagram

$$\begin{tikzcd} A \arrow[r, "\tilde{h}_0"] \arrow[d, hook] & X \arrow[d, "p"] \\ {[0,1] \times A} \arrow[r, "h"'] \arrow[ur, dashed, "\tilde{h}"] & B \end{tikzcd}$$

commutes. •

Definition 1.54. A continuous map $p \colon X \to B$ is a **Hurewicz fibration** if it has the HLP with respect to every topological space. •

Theorem 1.55. *Every covering map $p \colon X \to B$ is a Hurewicz fibration.*

Proof. If D is a discrete space, then $\mathrm{pr}_1 \colon B \times D \to B$ is a Hurewicz fibration. Consequently, every $b \in B$ has a neighborhood U such that $p|_{p^{-1}(U)} \colon p^{-1}(U) \to U$ is a Hurewicz fibration.

Let A be a topological space. Let $h \colon [0,1] \times A \to B$ be a homotopy. For every $(t, a) \in [0,1] \times A$ choose a neighborhood $U_{t,a}$ of $h(t,a)$ as above. Since $[0,1]$ is compact, there are $0 = t_0 < t_1 < \cdots < t_n = 1$ and an open neighborhood V_a of $a \in A$ with $[t_i, t_{i+1}] \times V_a \subset h^{-1}(U_{t,a})$ for some $t \in [0,1]$. Let $\tilde{h}_0$ be a lift of $h(0, \cdot)$. Since $p|_{p^{-1}(h([t_i,t_{i+1}] \times V_a))}$ is a Hurewicz fibration, a finite induction argument constructs a lift $\tilde{h}_{V_a}$ of $h|_{[0,1] \times V_a}$ with $\tilde{h}_{V_a}(0, \cdot) = \tilde{h}_0|_{V_a}$.

By Proposition 1.50 and because $[0,1]$ is connected, $\tilde{h}_{V_a}$ and $\tilde{h}_{V_b}$ agree on $[0,1] \times (V_a \cap V_b)$. Therefore, they assemble into a lift $\tilde{h}$ of h with $\tilde{h}(0, \cdot) = \tilde{h}_0$. ■

1.6 The fundamental group(oid)

It is useful to encode the unique homotopy lifting property (Proposition 1.50 and Theorem 1.55) algebraically. This requires the following terminology from algebraic topology.

Definition 1.56. A **groupoid** is a category $\mathbf{G}$ in which every morphism is an isomorphism. The **category of groupoids** is the full subcategory $\mathbf{Gpd} \subset \mathbf{Cat}$ whose objects are groupoids. •

Definition 1.57. The **fundamental groupoid (functor)** is the functor $\Pi_1 \colon \mathbf{Top} \to \mathbf{Gpd}$ defined as follows:

(Ob) Let X be a topological space. The **fundamental groupoid of** X is the groupoid $\Pi_1(X)$ whose objects are the elements of X, and whose morphisms $[\gamma] \colon x \to y$ are homotopy classes rel $\{0,1\}$ of continuous paths $\gamma \colon [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$, composed by concatenation.

(Hom) Let $f \colon X \to Y$ be a continuous map. $\Pi_1(f) \colon \Pi_1(X) \to \Pi_1(Y)$ is defined by

$$\Pi_1(f)(x) \coloneqq f(x) \quad \text{and} \quad \Pi_1(f)([\gamma]) \coloneqq [f \circ \gamma].$$ •

The fundamental groupoid $\Pi_1(X)$ is useful in constructions, e.g., in Section 1.10; however: it is also rather unwiedly. Fortunately, it is possible to drastically compress $\Pi_1(X)$—*essentially* without loss of information.

Definition 1.58. The **fundamental group (functor)** is the functor $\pi_1 \colon \mathbf{pTop} \to \mathbf{Grp}$ defined as follows:

$$\pi_1(X,x) \coloneqq \mathrm{Aut}_{\Pi_1(X)}(x) \quad \text{and} \quad \pi_1(f) \coloneqq \Pi_1(f)|_{\pi_1(X,x)}.$$ •

Proposition 1.59. *Let (X,x) be a pointed topological space. X is path-connected if and only if the inclusion $\mathbf{B}\pi_1(X,x) = \Pi_1(X)_x \subset \Pi_1(X)$ is an equivalence of categories.* ■

Example 1.60. Let $[\gamma] \in \pi_1(S^1,[0])$. By Proposition 1.50 and Theorem 1.55, there $[\gamma]$ has a unique lift $[\tilde{\gamma}]$ to a homotopy class rel $\{0,1\}$ of continuous paths $\tilde{\gamma} \colon [0,1] \to \mathbf{R}$ with $\tilde{\gamma}(0) = 0$. Define the **winding number map** $w \colon \pi_1(S^1,[0]) \to \mathbf{Z}$ by

$$w([\gamma]) \coloneqq [\tilde{\gamma}](1).$$

A few moments' thought show that this is a group isomorphism. ♠

Example 1.61. Let $k \in \mathbf{N}$. Consider $\mathrm{Sym}^k(\mathbf{C})^\circ$ from Example 1.25. Set $* := [1, 2, \dots, k] \in \mathrm{Sym}^k(\mathbf{C})^\circ$. The fundamental group

$$B_k := \pi_1(\mathrm{Sym}^k(\mathbf{C})^\circ, *)$$

is the **braid group on k–strands**. It is superficially obvious (but quite cumbersome to prove) that B_k has the **Artin presentation** in terms of generators and relations:

$$B_k \cong \langle \sigma_1, \dots, \sigma_{k-1} : \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ \sigma_i\sigma_j = \sigma_j\sigma_i\ (i - j \geqslant 2)\rangle.$$

Here $\sigma_i \in B_k$ is the obvious element swapping i and $i + 1$ in anti-clockwise fashion. ♠

The (obvious) homotopy-invariance of $\pi_1(X, x)$ and the following result are useful for computations.

Theorem 1.62 (Seifert–van Kampen for π_1). *Let (X, x) be a topological space. Let $\mathcal{U}$ be a subcategory of* **pTop** *whose objects are open subsets of X containing x and whose morphisms are inclusions. If the objects of $\mathcal{U}$ cover X and are closed under finite intersections, then*

$$\pi_1(X, x) \cong \operatorname{colim} \pi_1|_{\mathcal{U}}$$

in **Grp**. ■

Example 1.63. The fundamental group of bouquet of k circles is isomorphic to the free group F_k; cf. Section 1.12. ♠

1.7 Fibre transport and monodromy

Here is the desired algebraic formulation of the unique homotopy lifting property (Proposition 1.50 and Theorem 1.55).

Corollary 1.64. *Let $p\colon X \to B$ is a covering map. Let $x \in X$ and $b \in B$. The map*

$$\coprod_{y \in p^{-1}(b)} \mathrm{Hom}_{\Pi_1(X)}(x, y) \to \mathrm{Hom}_{\Pi_1(B)}(p(x), b)$$

induced by $\Pi_1(p)$ is bijective. ■

Corollary 1.65. *If $p\colon X \to B$ is a covering map, then for every $x \in X$ the homomorphism $\pi_1(p)\colon \pi_1(X, x) \to \pi_1(B, p(x))$ is injective.* ■

As a consequence of Corollary 1.64 the following construction is well-defined.

Definition 1.66. The **fibre transport functor**

$$\mathrm{Fib}\colon\ \mathbf{Cov}(B) \to \Pi_1(B)\text{–}\mathbf{Set} \coloneqq \mathbf{Fun}(\Pi_1(B), \mathbf{Set})$$

is the functor defined as follows:

(Ob) Let $p\colon\ X \to B$ be a covering map. The **fibre transport of** p is the functor $\mathrm{Fib}(p)\colon\ \Pi_1(B) \to \mathbf{Set}$ defined by

$$\mathrm{Fib}(p)(b) \coloneqq p^{-1}(b) \quad \text{and} \quad \mathrm{Fib}(p)([\gamma])(x) \coloneqq [\tilde{\gamma}](1)$$

with $[\tilde{\gamma}]$ denoting the unique lift of $[\gamma]$ with $[\tilde{\gamma}](0) = x$.

(Hom) Let $\phi\colon\ p \to q$ be a morphism of covering map of B. The natural transformation $\mathrm{Fib}(\phi)\colon\ \mathrm{Fib}(p) \to \mathrm{Fib}(p)$ is defined by

$$\mathrm{Fib}(\phi)_b \coloneqq \phi|_{p^{-1}(b)}\colon\ \mathrm{Fib}(p)(b) = p^{-1}(b) \to \mathrm{Fib}(q)(b) = q^{-1}(b). \qquad \bullet$$

Example 1.67. Consider $q^\circ\colon\ \mathrm{Roots}^\circ \to \mathrm{Poly}^\circ$ from Example 1.8. Consider the continuous loop $\gamma(t)\colon\ [0,1] \to \mathrm{Poly}^\circ$ defined by $\gamma(t) \coloneqq z^2 - e^{2\pi i t}$. A moment's thought shows that $\mathrm{Fib}(q)(z^2-1) = \{z^2-1\} \times \{\pm 1\}$ and $\mathrm{Fib}(q)([\gamma])$ acts by exchanges the two roots. ♠

The fibre transport functor is quite remarkable. It transforms geometric data into algebraic data, and in the process does not lose any information. Indeed, under mild connectivity assumptions, it is an equivalence of categories; see Section 1.10.

Although it is usually not terribly difficult to compute $\mathrm{Fib}(p)([\gamma])$ for concrete p and $[\gamma]$, completely determining $\mathrm{Fib}(p)$ is, at least, very cumbersome. It is useful to compress Fib as follows.

Definition 1.68. Let $b \in B$. The **evaluation functor**

$$\mathrm{ev}_b\colon\ \Pi_1(B)\text{–}\mathbf{Set} \to \pi_1(B,b)\text{–}\mathbf{Set}$$

is obtained by composition with the inclusion $\mathbf{B}\pi_1(B,b) \hookrightarrow \Pi_1(B)$. $\bullet$

Corollary 1.69. *Let $b \in B$. If B is path-connected, then ev_b is an equivalence of categories.* ■

Definition 1.70. Let $b \in B$. The **monodromy representation (functor) at** b is

$$\mu_b \coloneqq \mathrm{ev}_b \circ \mathrm{Fib}\colon\ \mathbf{Cov}(B) \to \pi_1(B,b)\text{–}\mathbf{Set}.$$

The **monodromy representation** of a covering map $p\colon X \to B$ is the group homomorphism $\mu_b(p)\colon \pi_1(B,b) \to p^{-1}(B)$. •

It is important to understand that the monodromy representation is quite computable in practice and can often be determined very explicitly. Here are two examples.

Example 1.71. Consider $\exp\colon \mathbf{C} \to \mathbf{C}^\times$ from Example 1.3. The fundamental group $\pi_1(\mathbf{C}^\times, 1)$ is generated by $[\gamma]$ with $\gamma(t) \coloneqq e^{2\pi i t}$. Evidently, $\mu_1(\exp) = 2\pi i\mathbf{Z}$ with $[\gamma]$ acting as a shift by $2\pi i$. ♠

Example 1.72. Consider $p^\circ\colon \mathbf{C}^k\backslash\Delta \to \mathrm{Sym}^k(\mathbf{C})^\circ$ from Example 1.25. Set $* \coloneqq [1,2,\dots,k] \in \mathrm{Sym}^k(\mathbf{C})^\circ$. The monodromy representation $\mu_*(p^\circ)$ is a group homomorphism $\pi_1(\mathrm{Sym}^k(\mathbf{C})^\circ, *) \to S_k = \mathrm{Bij}(\{1,\dots,k\})$. This is the obvious homomorphism $B_k \to S_k$ in the Artin presentation. ♠

Exercise 1.73. Compute the monodromy of $\cos\colon \mathbf{C}\backslash\pi\mathbf{Z} \to \mathbf{C}\backslash\{\pm 1\}$ from Example 1.4.

The monodromy representation can further be understood as follows.

Proposition 1.74. *Let $b \in B$. Let $p\colon X \to B$ be a covering map. If X is path-connected, then $\mu_b(p)\colon \pi_1(B,b) \to \mathrm{Bij}(p^{-1}(b))$ is transitive.* ■

If is a *transitive* left action $G \circlearrowright S$ and $s \in S$, then $G \circlearrowright S$ is isomorphic to $G \circlearrowright G/\mathrm{Stab}_G(s)$. Therefore,

Definition 1.75. Let $p\colon X \to B$ be a continuous map. Let $x \in X$. Set $b \coloneqq p(x)$. The **characteristic subgroup of** (p,x) is

$$C(p,x) \coloneqq \mathrm{im}(\pi_1(p)\colon \pi_1(X,x) \to \pi_1(B,b)) < \pi_1(B,b).$$ •

Proposition 1.76. *Let $p\colon X \to B$ be a covering map. Let $b \in B$. The stabiliser of $x \in p^{-1}(b)$ with respect to $\mu_b(p)$ is*

$$\mathrm{Stab}_{\pi_1(B,b)}(x) = C(p,x).$$ ■

❉

Here is a needlessly formal way to say what is going on with the characteristic subgroup.

Corollary 1.77. *Let (B,b) be a pointed topological space. Denote by* **pTop** *the **category of pointed topological spaces**. Denote by* $\mathbf{pCov}^\circ(B,b)$ *the full subcategory whose objects are pointed covering map map $p\colon (X,x) \to (B,b)$ with X path-connected. Denote by* $\mathbf{SubGrp}(\pi_1(B,b))$ *the category whose objects are subgroups of $\pi_1(B,b)$ and whose morphisms are inclusions. Denote by*

$$C\colon \mathbf{pTop}/(B,b) \to \mathbf{SubGrp}(\pi_1(B,b))$$

the ***characteristic subgroup functor*** *and by*

$$\mathrm{Quot}\colon \mathbf{SubGrp}(\pi_1(B,b)) \to \pi_1(B,b)\text{-}\mathbf{Orb}$$

the quotient functor. The diagram

$$\begin{tikzcd} \mathbf{pCov}^\circ(B,b) \arrow[r, "C"] \arrow[d, "U"] & \mathbf{SubGrp}(\pi_1(B,b)) \arrow[d, "\mathrm{Quot}"] \\ \mathbf{Cov}^\circ(B) \arrow[r, "\mu_b"] & \pi_1(B,b)\text{-}\mathbf{Orb} \end{tikzcd}$$

commutes upto natural isomorphism. ■

1.8 Lifting along covering maps

The following result is a rather satisfactory answer to the lifting problem raised in Section 1.4. It shows that *monodromy is the only obstruction to lifting* under mild connectivity assumptions..

Theorem 1.78. *Let $p\colon X \to B$ and $f\colon A \to B$ be continous maps. Let $a \in A$. Set $b := f(a)$. If A is path-connected and locally path-connected, then ev_a induces a bijection*

$$\mathrm{ev}_a\colon \mathrm{Hom}_{\mathbf{Top}/B}(f,p) \to p^{-1}(b)^{\pi_1(A,a)} = \{x \in p^{-1}(b) : C(p,x) \supset C(f,a)\}$$

with $\pi_1(A,a) \circlearrowright p^{-1}(b)$ via $\mu_b(p) \circ \pi_1(f)$.

The proof of Theorem 1.78 requires the following preparation.

Definition 1.79. Let X, Y be topological spaces. A (set-theoretic) map $f\colon X \to Y$ is **path-preserving** if for for every continuous path $\gamma\colon [0,1] \to X$ the composition $f \circ \gamma\colon [0,1] \to Y$ is a continuous path. •

Lemma 1.80. *Let $p\colon X \to B$ be a covering map. Let $\tilde{f}\colon A \to X$ be path-preserving. If A is locally path-connected and $p \circ \tilde{f}$ is continuous, then $\tilde{f}$ is continuous.*

Proof. Let $a \in A$. Choose an open neighborhood $V \ni b := p \circ \tilde{f}(a)$, a discrete space D, a homeomorphism $\tau\colon p^{-1}(V) \to V \times D$ such that $\mathrm{pr}_1 \circ \tau = p|_{p^{-1}(V)}$. Choose a path-connected open neighborhood $U \ni a$ with $p \circ \tilde{f}(U) \subset V$. It remains to prove that $\mathrm{pr}_2 \circ \tau \circ \tilde{f}|_U\colon U \to D$ is continuous. Since this map is path-preserving, U is path-connected, and D is discrete, it must be constant. ■

Proof of Theorem 1.78. The asserted equality is a consequence of Proposition 1.76.

If $\tilde{f} \in \mathrm{Hom}_{\mathbf{Top}/B}(f,p)$ and $x := \mathrm{ev}_a(\tilde{f}) = \tilde{f}(a)$, then $C(f,a) \subset C(p,x)$ because $\pi_1(f) = \pi_1(p) \circ \pi_1(\tilde{f})$.

Conversely, let $x \in p^{-1}(f(a))^{\pi_1(A,a)}$. Define $\tilde{f}\colon A \to X$ by

$$\tilde{f}(b) := \mathrm{Fib}(p)(\Pi_1(f)[\gamma])(x)$$

for $[\gamma] \in \mathrm{Hom}_{\Pi_1(A)}(a,b)$. By hypothesis, this is independent of the choice of $[\gamma]$. By construction, $\tilde{f}$ is path-preserving and $p \circ \tilde{f} = f$. Therefore, by Lemma 1.80, it is continous. ∎

Example 1.81. $U \subset \mathbf{C}^\times$ admits a logarithm if and only if for every $x \in U$ the map $\pi_1(U,x) \to \pi_1(\mathbf{C}^\times, x)$ is trivial. ♠

1.9 Deck transformations

Definition 1.82. The **deck transformation group** of a covering map $p\colon X \to B$ is its automorphism group

$$\mathrm{Deck}(p) := \mathrm{Aut}_{\mathbf{Cov}(B)}(p).$$

•

Proposition 1.83. *Let $p\colon X \to B$ be a covering map. If X is connected and locally connected, then the action $\mathrm{Deck}(p) \circlearrowright X$ is a covering space action.*

Proof. By Proposition 1.50, the action is free.

Let $b \in B$. Since X is locally connected, there are U, D, and τ be as in Definition 1.1 with U connected. Let $\phi \in \mathrm{Deck}(p)\backslash\{\mathrm{id}_X\}$. Since U is connected, there is a bijection $f_\sharp \in \mathrm{Bij}(D)$ such that

$$\tau \circ \phi \circ \tau^{-1} = \mathrm{id}_U \times \phi_\sharp.$$

Since ϕ has no fixed-points, $\phi_\sharp$ has no fixed-points. Therefore,

$$\phi(V_d) \cap V_d = \varnothing \quad \text{for} \quad V_d := \tau^{-1}(U \times \{d\})$$

for every $d \in D$. ∎

The deck transformation group can be computed as follows.

Proposition 1.84. *Let $p\colon X \to B$ be a covering map. Let $x \in X$. Set $b := p(x)$. Assume that X is path-connected and locally path-connected. There is a unique isomorphism*

$$\iota_x\colon\ \mathrm{Deck}(p)^{\mathrm{op}} \to W_{\pi_1(B,b)}(C(p,x)) := N_{\pi_1(B,b)}(C(p,x))/C(p,x)$$

such that for every $\phi \in \mathrm{Deck}(p)$

$$\phi(x) = \mu_b(p)(\iota_x(\phi))(x).$$

Proof. By Theorem 1.78, the evaluation map

$$\mathrm{ev}_x\colon\ \mathrm{Deck}(p) \to p^{-1}(b)^{\pi_1(X,x)} = \{y \in p^{-1}(b) : C(p,y) = C(p,x)\}$$

is a bijection. Since X is path-connected and by Proposition 1.76, the map

$$\mu_b(p)(\cdot)(x)\colon\ \pi_1(B,b)/C(p,x) \to p^{-1}(b)$$

is a bijection. Since

$$C(p,y) = [\gamma]C(p,x)[\gamma]^{-1} \quad \text{with} \quad y \coloneqq \mu_b(p)([\gamma])(x),$$

the above map induces a bijection

$$\mu_b(p)(\cdot)(x)\colon\ W_{\pi_1(B,b)}(C(p,x)) \to p^{-1}(b)^{\pi_1(X,x)}.$$

Therefore, ι_x is uniquely determined as a (set-theoretic) map.

It remains to veryify that ι_x is a group homomorphism. This is an immediate consequence of the fact that deck transformations $f \in \mathrm{Deck}(p)$ commute with the monodromy representation $\mu_b(p)$; indeed:

$$\begin{aligned}\phi(\psi(x)) &= \phi(\mu_b(p)(\iota_x(\psi))(x)) = \mu_b(p)(\iota_x(\psi))(\phi(x)) \\ &= \mu_b(p)(\iota_x(\psi))(\mu_b(p)(\iota_x(\phi))(x)) = \mu_b(p)(\iota_x(\psi)\iota_x(\phi))(x).\end{aligned}$$ ■

Example 1.85. For $p_\infty\colon\ \mathbf{R} \to S^1$ from Example 1.5, $\mathrm{Deck}(p_\infty) = \mathbf{Z} \circlearrowright \mathbf{R}$ and ι_0 is the inverse of the winding number map from Example 1.60. ♠

The following definition is possibly inescapable.

Definition 1.86. A covering map $p\colon\ X \to B$ with X path-connected and locally path-connected is **principal** (or **Galois** or **normal**) if p induces descends to a homeomorphism $X/\mathrm{Deck}(p) \cong B$. •

Corollary 1.87. *A covering map $p\colon\ X \to B$ with X path-connected and locally path-connected is principal if and only if for $C(p,x) < \pi_1(B,p(x))$ is a normal subgroup.* ■

Remark 1.88. Since subgroups are rarely normal, covering maps are rarely principal. ♣

Example 1.89. The normaliser of $\langle(12)\rangle < S_3$ is $\langle(12)\rangle$. ♠

The following observation is sometimes useful to compute fundamental groups.

Corollary 1.90. *If p is principal, then exact sequence*

$$\pi_1(X,x) \hookrightarrow \pi_1(B,b) \twoheadrightarrow \mathrm{Deck}(p)^{\mathrm{op}}. \qquad \blacksquare$$

1.10 Classification of covering maps

Proposition 1.91. *If B is locally path-connected, then* $\mathrm{Fib}\colon \mathbf{Cov}(B) \to \mathbf{Tra}(B)$ *is full and faithful.*

Proof. Evidently, Fib is faithful.

Let $p\colon X \to B$ and $q\colon Y \to B$ be covering maps. Let $\phi\colon \mathrm{Fib}(p) \to \mathrm{Fib}(p)$ be a morphism in $\mathbf{Tra}(B)$. Define the (set-theoretic) map $\phi\colon X \to Y$ by

$$\phi|_{p^{-1}(b)} \coloneqq \phi_b.$$

By construction, $p = q \circ f$ and f is path-perserving. By Lemma 1.80, ϕ is continuous. Therefore, Fib is full. ■

Proposition 1.91 can be strengthened as follows.

Definition 1.92. A topological space X is **semi-locally simply-connected** if every $x \in X$ has a neighborhood U such that for every $y, z \in U$

$$\#\,\mathrm{im}(\mathrm{Hom}_{\Pi_1(U)}(y,z) \to \mathrm{Hom}_{\Pi_1(X)}(y,z)) = 1. \qquad \bullet$$

Theorem 1.93 (Classification of covering maps, I). *If B is locally path-connected and semi-locally simply-connected, then*

$$\mathrm{Fib}\colon \mathbf{Cov}(B) \to \Pi_1(B)\text{–}\mathbf{Set}$$

is an equivalence of categories.

The proof of Theorem 1.93 relies the following construction.

Definition 1.94. The **set-theoretic reconstruction functor** $\mathrm{rec}\colon \Pi_1(B)\text{–}\mathbf{Set} \to \mathbf{Set}/B$ is defined by:

(Ob) For every object T of $\Pi_1(B)\text{–}\mathbf{Set}$ define

$$\mathrm{rec}(T)\colon X_T \coloneqq \coprod_{b \in B} T(b) \to B$$

to be the canonical projection.

(Hom) For every morphism $f\colon T \to S$ in $\Pi_1(B)$–**Set** define

$$\operatorname{rec}(f) \coloneqq \coprod_{b\in B} f(b)\colon X_T \to X_S. \qquad \bullet$$

Definition 1.95. Let B be a topological space. Let T be an object of $\Pi_1(B)$–**Set**.

(1) Denote by $\mathscr{U}$ the set of open subsets $U \subset B$ be open such that for every $b, c \in U$

$$\#\operatorname{im}\big(\operatorname{Hom}_{\Pi_1(U)}(b,c) \to \operatorname{Hom}_{\Pi_1(B)}(b,c)\big) = 1.$$

(2) Let $b \in U \in \mathscr{U}$. Define the bijection $\tau_{b,U}\colon X_T \to U \times T(b)$ by

$$\tau_{b,U}(c,x) \coloneqq (c, T([\gamma^c_{b,U}])x) \quad \text{with} \quad \{[\gamma^c_{b,U}]\} \coloneqq \operatorname{im}\big(\operatorname{Hom}_{\Pi_1(U)}(c,b) \to \operatorname{Hom}_{\Pi_1(B)}(c,b)\big).$$

(3) The **transport topology** is the coarsest topology on X_T with respect to which the maps $\tau_{b,U}$ ($b \in U \in \mathscr{U}$, $b \in U$) are continuous. $\bullet$

The following construction lifts rec along $\mathbf{Cov}(B) \to \mathbf{Set}/B$.

Proposition 1.96. *Let B be a locally path-connected and semi-locally simply-connected topological space.*

(1) *For every object T of $\Pi_1(B)$–**Set**, $\operatorname{rec}(T)\colon X_T \to B$ is a covering map of B with respect to the transport topology on X_T.*

(2) *For every morphism $f\colon T \to S$ in $\Pi_1(B)$–**Set**, $\operatorname{rec}(f)\colon X_T \to X_S$ is a morphism covering maps of B with respect to the transport topology on X_T and X_S.*

Proof. The assumptions guarantee that $\mathscr{U}$ is an open cover of B. Therefore, it suffices to prove that for every $U \in \mathscr{U}$ and $b \in U$ the bijection $\tau_{b,U}$ is a homeomorphism with respect to the transport topology. In fact, it suffices to prove that

$$\tau_{b,U} \circ \tau_{c,V}^{-1}\colon (U \cap V) \times T(c) \to (U \cap V) \times T(b)$$

is continuous for every $b \in U \in \mathscr{U}$ and $c \in V \in \mathscr{U}$.

Let $d \in U \cap V$. Since B is locally path-connected, d has a path-connected open neighborhood $W \subset U \cap V$. Evidently, $W \in \mathscr{U}$. For every $e \in W$

$$T([\gamma^e_{b,U}][\gamma^c_{e,V}]) = T([\gamma^e_{b,U}][\gamma^d_{e,W}][\gamma^e_{d,W}][\gamma^c_{e,V}]) = T([\gamma^d_{b,U}][\gamma^c_{d,V}]) \in \operatorname{Bij}(T(c), T(b));$$

in particular, it does not depend on e. Therefore, $\tau_{b,U} \circ \tau_{c,V}^{-1}$ is continuous. ∎

Definition 1.97. The **reconstruction functor** $\mathrm{Rec}\colon \Pi_1(B){-}\mathbf{Set} \to \mathbf{Cov}(B)$ is the lift of $\mathrm{rec}\colon \Pi_1(B){-}\mathbf{S}$
$\mathbf{Set}/B$ along the forgetful functor $\mathbf{Cov}(B) \to \mathbf{Set}/B$ obtained from Proposition 1.96. •

Proof of Theorem 1.93. Because of Proposition 1.91, it remains to verify that Fib is essentially surjective. This is an immediate consequence of the (nearly obvious) fact that $\mathrm{Fib} \circ \mathrm{Rec}$ is naturally isomorphic to the identity. ■

❉

Theorem 1.93 and Corollary 1.69 imply the following.

Corollary 1.98 (Classification of covering maps, II). *Let $b \in B$. If B is path-connected, locally path-connected and semi-locally simply-connected, then*

$$\mu_b\colon \mathbf{Cov}(B) \to \pi_1(B,b){-}\mathbf{Set}$$

is an equivalence of categories. ■

❉

Definition 1.99. Denote by $\mathbf{Cov}^\circ(B) \subset \mathbf{Cov}(B)$ the full subcategory whose objects are covering maps $p\colon X \to B$ with X path-connected. •

Definition 1.100. The **category of G–orbits** is the full subcategory $G{-}\mathbf{Orb} \subset G{-}\mathbf{Set}$ whose objects are *transitive* G–actions. •

Proposition 1.101. $\mathbf{Cov}(B)$ *is the free coproduct completion of* $\mathbf{Cov}^\circ(B)$. ■

Proposition 1.102. $G{-}\mathbf{Set}$ *is the free coproduct completion of* $G{-}\mathbf{Orb}$. ■

Corollary 1.98 immediately implies the following.

Corollary 1.103 (Classification of covering maps, III). *Let $b \in B$. If B is path-connected, locally path-connected and semi-locally simply-connected, then*

$$\mu_b\colon \mathbf{Cov}^\circ(B) \to \pi_1(B,b){-}\mathbf{Orb}$$

is an equivalence of categories. ■

The functor μ_b from Corollary 1.103 is imminently computable because of Proposition 1.76.

1.11 Universal covering maps

Definition 1.104. A covering map $p\colon\ X \longrightarrow B$ is **universal** if and only if p is surjective and X is simply-connected. •

Proposition 1.105. *Let B be path-connected and locally path-connected. B is semi-locally simply-connected if and only if it admits a universal covering map.*

Proof. If B admits a universal cover, then it is semi-locally simply-connected. Conversely, if B is semi-locally simply-connected, then a universal cover exists by Corollary 1.98. ∎

Remark 1.106. Proposition 1.105 can be proved directly. However, the usual construction boils down to a special case of the construction of the reconstruction functor Rec. Sometimes, however, a universal covering map is known to exists a priori. ♣

Corollary 1.107. *Every universal covering map is principal.* ∎

Corollary 1.108. *If $p\colon\ X \longrightarrow B$ is a universal covering map, then $\iota_x\colon\ \mathrm{Aut}(p)^{\mathrm{op}} \cong \pi_1(B, b)$.* ∎

Definition 1.109. Let $p\colon\ X \longrightarrow B$ be a covering map such that $\mathrm{Aut}(p) \circlearrowright X$ is a covering space action. The **associated covering map functor**

$$\mathrm{A}_p\colon\ \mathrm{Aut}(p)^{\mathrm{op}}\text{–}\mathbf{Set} \longrightarrow \mathbf{Cov}(B)$$

is defined as follows:

(Ob) Let S be a $\mathrm{Aut}(p)^{\mathrm{op}}$–set. $X \times S \circlearrowleft \mathrm{Aut}(p)^{\mathrm{op}}$ via

$$(x, s) \cdot f \coloneqq (f(x), f^{-1} \cdot s).$$

The **associated covering map** is the canonical projection

$$\mathrm{A}_p(S)\colon\ X_S \coloneqq (X \times S)/\mathrm{Aut}(p)^{\mathrm{op}} \longrightarrow B.$$

(Hom) Let $f\colon\ S \longrightarrow G$ be a morphism of $\mathrm{Aut}(p)^{\mathrm{op}}$–sets. The continuous map $\mathrm{id}_X \times f\colon\ X \times S \longrightarrow X \times T$ is $\mathrm{Aut}(p)^{\mathrm{op}}$–equivariant and descends to

$$\mathrm{A}_p(f)\colon\ \mathrm{A}_p(S) \longrightarrow \mathrm{A}_p(T).$$ •

Proposition 1.110. *If $p\colon X \to B$ is a universal covering map, then the diagram*

$$\begin{tikzcd} \mathrm{Aut}(p)^{\mathrm{op}}\text{–}\mathbf{Set} \arrow[r, "A_p"] \arrow[dr, "\iota_x"'] & \mathbf{Cov}(B) \arrow[d, "\mu_b"] \\ & \pi_1(B,b)\text{–}\mathbf{Set} \end{tikzcd}$$

commutes upto natural isomorphism. ■

Corollary 1.111. *Let B be path-connected and locally path-connected. B admits a universal cover if and only if μ_b is an equivalence of categories.* ■

1.12 The Nielsen–Schreier Theorem

Here is an application of the theory of covering maps to algebra.

Definition 1.112. Let S be a set. The **free group on** S is the group $F(S)$ generated by S. A group G is **free** if it is isomorphic to $F(S)$ for some S. The **rank** of G is $\mathrm{rk}(G) \coloneqq \#S$. •

Theorem 1.113 (Nielsen–Schreier Theorem). *If G is a free group, then every subgroup $H < G$ is free. If $\mathrm{rk}(G) = r \in \mathbf{N}_0$ and $|H : G| = i \in \mathbf{N}$, then $\mathrm{rk}(H) = i(r-1)+1$.*

The proof relies on realising G as a fundamental group and the theory of covering maps.

Definition 1.114.

(1) A **graph** is a triple $\Gamma = (V, E, \alpha)$ with V a set, E a set of unordered pairs, and a map $\alpha\colon \bigcup E \to V$. The **vertices** and **edges** of Γ are the elements of V and E respectively. An edge e **connects** $x, y \in V$ if $\alpha(e) = \{x, y\}$.

(2) For every unordered pair $e = \{x, y\}$ set

$$I_e \coloneqq (e \times [0,1])/\sim$$

with $\sim$ denoting the equivalence relation generated by $(x,t) \sim (y, 1-t)$.

(3) The **topological realisation** of Γ is

$$X(\Gamma) \coloneqq \left(V \amalg \coprod_{e\in E} I_e \right) \Big/ \sim$$

with $\sim$ denoting the equivalence relation generated by $[x, 0] \sim \alpha(x)$. •

Example 1.115. Let S be a set. Set $V := \{*\}$ and $E := \{0,1\} \times S$. There is a unique map $\alpha\colon E \to V$. The graph $\Gamma = (V, E, \alpha)$ has a unique vertex $*$ and an edge connecting $*$ to itself for every $s \in S$. The topological realisation $X(\Gamma)$ of Γ is homeomorphic to a **bouquet of circles** indexed by S:

$$X(\Gamma) \cong \bigvee_{s\in S} \{s\} \times S^1 := \left(\coprod_{s\in S} \{s\} \times S^1 \right) \Big/ \sim$$

with $\sim$ denoting the equivalence relation generated by $(s, [0]) \sim (t, [0])$. By the Seifert–van Kampen theorem,

$$\pi_1(X(\Gamma), *) \cong F(S).$$

♠

Definition 1.116. Let $\Gamma = (V, E, \alpha)$ be a graph.

(1) A **subgraph** of a graph $\Gamma = (V, E, \alpha)$ is a graph $\Delta = (W, F, \beta)$ with $W \subset V$, $F \subset E$, and $\beta = \alpha|_F$.

(2) A **path** in Γ is a is a sequence of vertices $v_0, \ldots, v_n$ together with a sequence of edges $e_0, \ldots, e_n$ such that e_i connects v_i and v_{i+1}. A **cycle** in Γ is a path with $n \geqslant 1$, $v_0 = v_n$, and $e_i \neq e_{i+1}$.

(3) Γ is **connected** if for every $v, w \in V$ there is a path with $v_0 = v$ and $v_n = w$.

(4) A **forest** is a graph without cycles. A **tree** is a connected forest. •

Proposition 1.117. *Let Γ be a connected graph. $X(\Gamma)$ is homotopy-equivalent to a bouquet of circles.*

Proof sketch. Denote by $\mathscr{T}$ the set of subgraphs of Γ which are trees. There is an obvious order on $\mathscr{T}$. Use Zorn's lemma to construct a maximal $T \in \mathscr{T}$. A moment's thought shows that T has the same vertices as Γ. The subspace $X(T) \subset X(\Gamma)$ is contractible. $X(\Gamma)/X(T)$ is homeomorphic to a bouquet of circles. Finally, the projection $X(\Gamma) \to X(\Gamma)/X(T)$ is a homotopy equivalence. ■

Lemma 1.118. *Let Γ be a graph. If $p\colon Y \to X(\Gamma)$ is a covering map, then Y is homeomorphic to $X(\Delta)$ for some graph Δ.* ■

Proof of Theorem 1.113. Let G be a free group. Construct a graph Γ with $\pi_1(X(\Gamma)) \cong G$. If $H < G$ is a subgroup, then there is a covering map $p\colon Y \to X(\Gamma)$ with characteristic subgroup isomorphic to H. By the above, $\pi_1(Y)$ is free.

If F has rank r and $|H : G| = i$, then $\deg(p) = i$; hence:

$$1 - \mathrm{rk}(H) = \chi(Y) = i\chi(X(\Gamma)) = i(1 - r).$$

This implies $\mathrm{rk}(H) = i(r - 1) + 1$. ■

2 Fibre bundles

The purpose of this section is to explain the salient points of theory of Ehresmann connections on (unstructured) fibre bundles. Most of gauge theory is concerned with the less general but slightly more complicated G–principal fibre bundles (or fibre bundles with structure groups). The current section can be understood as a warm-up.

[Ste51] is the classical reference of the topological theory of fibre bundles. [Hus94] is a more modern reference. The theory of connections of fibre bundles is due to Ehresmann [Ehr51]. Kolář, Michor, and Slovák [KMS93] have an extensive treatment.

2.1 Introduction

Definition 2.1. A **fibre bundle** is a smooth map $p\colon X \to B$ such that for every $b \in B$ there are an open subset $b \in U \subset B$, a smooth manifold F, and a diffeomorphism $\tau\colon p^{-1}(U) \to U \times F$ such that

$$\mathrm{pr}_1 \circ \tau = p;$$

that is: the diagram

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\tau"] \arrow[dr, "p"'] & U \times F \arrow[d, "\mathrm{pr}_1"] \\ & U \end{tikzcd}$$

commutes. The **total space** of p is X. The **base space** of p is B. For $b \in B$ the **fibre of p over b** is $X_b := p^{-1}(b)$. •

This concept formalises the concept of smoothly varying families of manifolds.

Proposition 2.2. *Let $p\colon X \to B$ be a fiber bundle. If B is connected and $b_0, b_1 \in B$, then X_{b_0} and X_{b_1} are diffeomorphic.* ■

Example 2.3. Let B, F be smooth manifolds. The **trivial fibre bundle** over B with fibre F is the projection map $\mathrm{pr}_1\colon B \times F \to B$. ♠

Example 2.4. Let $n \in \mathbf{N}$. The skew fields $\mathbf{R}, \mathbf{C}, \mathbf{H}$ give rise to the Hopf bundles.

(1) The projection $p\colon S^n \subset \mathbf{R}^{n+1}\backslash\{0\} \to \mathbf{R}P^n$ is a fibre bundle.

(2) The **Hopf bundle** $p\colon S^{2n+1} \subset \mathbf{C}^{n+1}\backslash\{0\} \to \mathbf{C}P^n$ is a fibre bundle.

(3) The **quaternionic Hopf bundle** $p\colon S^{4n+3} \subset \mathbf{H}^{n+1}\backslash\{0\} \to \mathbf{H}P^n$ is a fibre bundle.

(4) There is a fibre bundle $p\colon S^{15} \to S^8$ but a naive construction of a family of octonionic

Hopf bundles does not work. ♠

Example 2.5. Let X, B be smooth manifolds. If $p\colon X \to B$ is a covering map, then it is a fiber bundle. ♠

Example 2.6. Every smooth vector bundle $p\colon E \to B$ is a fibre bundle. ♠

Example 2.7. Let $E \to B$ be a Euclidean vector bundle. The **sphere bundle**

$$p\colon S(V) \to B \quad \text{with} \quad S(V) \coloneqq \{v \in V : |v| = 1\}.$$

is a fibre bundle. ♠

Example 2.8. Let $p\colon E \to B$ be a vector bundle. For $r \in \mathbf{N}_0$ denote by

$$\mathrm{Gr}_r(V) \coloneqq \{(b, \Pi) : b \in B, \Pi \subset V_b \text{ with } \dim \Pi = r\}$$

the **Grassmannian of r–planes in** V. $\mathrm{Gr}_r(V)$ admits the structure of a smooth manifold such that the map $q\colon \mathrm{Gr}_r(V) \to B$ obtained by restriction of pr_1 is a fibre bundle. ♠

Example 2.9. Let $p\colon E \to B$ be a vector bundle. Denote by

$$\mathrm{Fr}(V) \coloneqq \{(b, \phi) : b \in B, \phi\colon \mathbf{R}^{\mathrm{rk}_b V} \to V_b \text{ isomomorphism}\}$$

the **frame bundle of** E. $\mathrm{Fr}(E)$ admits the structure of a smooth manifold such that the map $q\colon \mathrm{Fr}(E) \to B$ obtained by restriction of pr_1 is a fibre bundle. ♠

Theorem 2.10 (Ehresmann fibration theorem). *Every proper submersion $p\colon X \to B$ is a fibre bundle.* ■

Example 2.11. Let $f\colon X \to \mathbf{R}$ be a proper smooth function. If $[a, b] \cap f(\mathrm{Crit}(f)) = \varnothing$, then $f^{-1}(a)$ and $f^{-1}(b)$ are diffeomorphic. ♠

Corollary 2.12. *Deformation equivalent closed complex manifolds are diffeomorphic.* ■

2.2 The category of fibre bundles

Definition 2.13. The **category of fibre bundles** is the full subcategory **FibBun** of **Arr**(**Sm**) whose objects are fibre bundles. •

Proposition 2.14. **FibBun** *has finite products and arbitrary coproducts.* ■

Proposition 2.15. *The codomain functor* $U\colon$ **FibBun** $\to$ **Sm** *is a fibred category.*

Proof. Let $p\colon X \to B$ be a fibre bundle. Let $f\colon A \to B$ be a smooth map. Since p is a submersion, $f \times p$ is transverse to the diagonal $\Delta \subset B \times B$. Therefore,

$$f^*X := \{(a,x) \in A \times X : f(a) = p(x)\} \subset A \times X$$

is a submanifold. Set $p^*f := \mathrm{pr}_1\colon f^*X \to X$ and $f^*p := \mathrm{pr}_2\colon f^*X \to A$. Evidently, the morphism $(p^*f, f)\colon f^*p \to p$ is cartesian. ■

Definition 2.16. For every fibre bundle $p\colon X \to B$ and every smooth map $f\colon A \to B$ *choose* a cartesian lift

$$\begin{tikzcd} f^*X \arrow[r, "p^*f"] \arrow[d, "f^*p"] & X \arrow[d, "p"] \\ A \arrow[r, "f"] & B. \end{tikzcd}$$

This is the **pullback via** f •

As in Section 1.3, $\mathbf{FibBun}_B$ is a sheaf.

2.3 Ehresmann connections

It is not terribly difficult to prove the following.

Theorem 2.17. *If* $p\colon X \to B$ *is a fibre bundle, then it is a Hurewicz fibration.* ■

However, the lifting problem is quite flabby. The following definition helps to rigidify the lifting problem.

Definition 2.18. Let $p\colon X \to B$ be a fibre bundle. The **vertical tangent bundle** of p is the vector bundle

$$V_p := \ker(Tp\colon TX \to p^*TB) \to X.$$ •

Definition 2.19. Let $p\colon X \to B$ be a fibre bundle. An **Ehresmann connection** on p is a left splitting of the short exact sequence

$$V_p \xhookrightarrow{\iota} TX \xrightarrow{Tp} p^*TB;$$

that is: an $A \in \Omega^1(X, V_p) = \Gamma(X, \mathrm{Hom}(TX, V_p))$ such that

$$A \circ \iota = \mathrm{id}_{V_p}.$$

Denote by $\mathscr{A}(p) \subset \Omega^1(X, V_p)$ the subset of Ehresmann connections on p. •

Example 2.20. Consider the trivial fibre bundle $\mathrm{pr}_B\colon B \times F \to B$. There is a canonical isomorphism $T(B\times F) \cong \mathrm{pr}_B^*TB \oplus \mathrm{pr}_F^*TF$ with respect to which $V_{\mathrm{pr}_B} = \mathrm{pr}_F^*TF$. The **product connection** A on pr_B is the obvious projection. ♠

Example 2.21. Let $p\colon X \to B$ be a fibre bundle. Let g be a Riemannian metric on X. The orthogonal projection $A_g\colon TX = V_p \oplus V_p^\perp \to V_p$ is an Ehresmann connection. ♠

Remark 2.22. Let $p\colon X \to B$ be a fibre bundle. The construction in Example 2.21 induces a map $\mathscr{Met}(X) \to \mathscr{A}(p)$. This map is surjective, but very far from injective. ♣

The structure of the space $\mathscr{A}(p)$ is very simple.

Definition 2.23. Let $p\colon X \to B$ be a fibre bundle. Let E be a vector bundle over X. A differential form $\alpha \in \Omega^\bullet(X, E)$ is **horizontal** if for every $v \in V_p$

$$i_v\alpha = 0.$$

The subspace of horizontal differential forms is denoted by $\Omega^\bullet_{\mathrm{hor}}(X, E)$. •

Proposition 2.24. *$\mathscr{A}(p) \subset \Omega^1(X, V_p)$ is an affine subspace modelled on $\Omega^1_{\mathrm{hor}}(X, V_p)$.*

Proof. By Example 2.21, there exists an $A_0 \in \mathscr{A}(p)$. Let $A \in \Omega^1(X, V_p)$. Evidently, $A \in \mathscr{A}(p)$ if and only if $A - A_0 \in \Omega^1_{\mathrm{hor}}(X, V_p)$. ■

Remark 2.25. Informally, the fact that $\mathscr{A}(p)$ is contractible, means that *choosing* an $A \in \mathscr{A}(p)$ is mostly harmless. ♣

Ehresmann connections can be pulled-back as follows.

Definition 2.26. Let $p\colon X \to B$ be a fibre bundle. Let $f\colon C \to B$ be a smooth map. Denote by

$$V_f\colon V_{f^*p} \to (p^*f)^*V_p.$$

the isomorphism induced by $Tp^*f\colon Tf^*X \to (p^*f)TX$. Define the **pull-back** map $f^\sharp\colon \mathscr{A}(p) \to \mathscr{A}(f^*p)$ by

$$f^\sharp A \coloneqq V_f^{-1}(p^*f)^*A.$$

•

2.4 Parallel transport

Definition 2.27. Let $p\colon X \to B$ be a fibre bundle together with an Ehresmann connection $A \in \mathscr{A}(p)$. A smooth map $f\colon C \to X$ is A–**horizontal** if

$$f^*A = 0.$$

Let $f\colon A \to B$ be a smooth map. Denote by $\mathrm{Hom}_{\mathbf{Sm}/B}(f,p)^A \subset \mathrm{Hom}_{\mathbf{Sm}/B}$ the subset of A–horizontal lifts of f along p. •

The **horizontal lifting problem** is to determine $\mathrm{Hom}_{\mathbf{Sm}/B}(f,p)^A$. The horizontal lifting problem is rather similar to the lifting problem along covering maps.

Proposition 2.28. *Let $p\colon X \to B$ be a fibre bundle together with an Ehresmann connection $A \in \mathscr{A}(p)$. Let $f\colon C \to B$ be a smooth map. The map $p^*f \circ \cdot\colon \mathrm{Hom}_{\mathbf{Sm}/B}(\mathrm{id}_C, f^*p) \to \mathrm{Hom}_{\mathbf{Sm}/B}(f,p)$ induces a bijection $\mathrm{Hom}_{\mathbf{Sm}/B}(f,p)^A \cong \mathrm{Hom}_{\mathbf{Sm}/B}(\mathrm{id}, f^*p)^{f^*A}$.* ■

Definition 2.29. Let $p\colon X \to B$ be a fibre bundle together with an Ehresmann connection $A \in \mathscr{A}(p)$. Let $f\colon C \to B$ be a smooth map. Let $c \in C$. Set $b \coloneqq f(c)$. Define the **evaluation map**

$$\mathrm{ev}_c\colon \mathrm{Hom}_{\mathbf{Sm}/B}(f,p)^A \to p^{-1}(b)$$

by

$$\mathrm{ev}_c(\tilde f) \coloneqq \tilde f(c).$$

•

Proposition 2.30. *Let $p\colon X \to B$ be a fibre bundle together with an Ehresmann connection $A \in \mathscr{A}(p)$. Let $f\colon C \to B$ be a smooth map. Let $c \in C$. If C is connected, then ev_c is injective.*

Proof. Since C is path-connected, it suffices to prove this for $C = [0,1]$. By the above observation, it suffices to consider $B = C = [0,1]$ and $f = \mathrm{id}_B$. There is a unique vector field $\tilde v \in \mathrm{Vect}(X)$ which is p–related to ∂_t on $[0,1]$. A lift $\tilde f$ is A–horizontal if and only if it is a integral curve of $\tilde v$. The assertion therefore follows from the Picard–Lindelöf Theorem. ■

Definition 2.31. Let $p\colon X \to B$ be a fibre bundle. An Ehresmann connection $A \in \mathscr{A}(p)$ is **complete** if for every smooth path $\gamma\colon [0,1] \to B$ and every $x \in p^{-1}(\gamma(0))$ there is an A–horizontal lift $\tilde{\gamma}$ with $\tilde{\gamma}(0) = x_0$. •

Theorem 2.32 (del Hoyo [dHoy16]). *Every fibre bundle admits a complete Ehresmann connection.* ■

Proposition 2.33. *If p is a proper fibre bundle, then every $A \in \mathscr{A}(p)$ is complete.* ■

Remark 2.34. The attentive reader will have observed that the theory of Ehresmann connections, discussed so far, does not make use of p being a fibre bundle. It would have sufficed to assume that p is a submersion. What singles out fibre bundles is the existence of complete Ehresmann connections. ♣

Definition 2.35. The **path groupoid functor** is the functor $P_1\colon \mathbf{Sm} \to \mathbf{Gpd}$ defined as follows:

(Ob) Let X be a smooth manifold. The **path groupoid of** X is the groupoid $P_1(X)$ whose objects are the elements of X, and whose morphism $[\gamma]\colon x \to y$ are smooth paths $\gamma\colon [0,1] \to X$ which are constant in a neighborhood of $\{0,1\}$ up to thin homotopy (i.e.: a homotopy $h\colon [0,1]^2 \to X$ with $\operatorname{rk} Th \leqslant 1$), composed by concatenation.

(Hom) Let $f\colon X \to Y$ be a smooth map. The natural transformation $P_1(f)$ is given by composition
$$P_1(f)[\gamma] \coloneqq [f \circ \gamma].$$ •

Definition 2.36. Let $p\colon X \to B$ be a fibre bundle together with an Ehresmann connection $A \in \mathscr{A}(p)$. The **horizontal path groupoid** $P_1(X, A)$ is the subgroupoid of $P_1(X)$ whose morphisms are equivalence classes of A–horizontal paths. •

Proposition 2.37. *Let $p\colon X \to B$ be a fibre bundle together with a complete Ehresmann connection $A \in \mathscr{A}(p)$. For every $x \in X$ and $b \in B$ the map*
$$\coprod_{y \in p^{-1}(b)} \operatorname{Hom}_{P_1(X,A)}(x, y) \to \operatorname{Hom}_{P_1(B)}(p(x), b)$$
induced by $P_1(p)$ is bijective. ■

Definition 2.38. Let $p\colon X \to B$ be a fibre bundle together with a complete Ehresmann connection $A \in \mathscr{A}(p)$. The **parallel transport** is the functor

$$\mathrm{tra}^A\colon P_1(B) \to \mathbf{Sm}$$

defined by

$$\mathrm{tra}^A(b) := p^{-1}(b) \quad \text{and} \quad \mathrm{tra}^A([\gamma])(x) := [\tilde{\gamma}(1)]$$

with $\tilde{\gamma}$ denoting the unique A–horizontal lift of γ with $\gamma(0) = x$. •

Remark 2.39. To determine $\mathrm{tra}^A([\gamma])$ is to solve an ODE. ♣

Remark 2.40. The fact that is functor goes to **Sm** instead of **Set** is the smooth dependence on initial conditions for solutions of ODE. ♣

Definition 2.41. Let $p\colon X \to B$ be a fibre bundle together with a complete Ehresmann connection $A \in \mathscr{A}(p)$. Let $b \in B$. The **holonomy group of** A **based at** b is the subgroup $\mathrm{Hol}_b(A) < \mathrm{Diff}(p^{-1}(b))$ defined by

$$\mathrm{Hol}_b(A) := \left\{\mathrm{tra}^A(\gamma) : \gamma \in \mathrm{Aut}_{P_1(B)}(b)\right\}. \qquad \bullet$$

Remark 2.42. In practice, it is not feasible to compute $\mathrm{Hol}_b(A)$ from the definition. ♣

Corollary 2.43 (holonomy principle). *Let $p\colon X \to B$ be a fibre bundle together with an Ehresmann connection $A \in \mathscr{A}(p)$. Let $f\colon C \to B$ be a smooth map. Let $c \in C$. If C is connected, then the evaluation map induces a bijection*

$$\mathrm{ev}_c\colon \mathrm{Hom}_{\mathbf{Sm}/B}(f, p)^A \to p^{-1}(b)^{\mathrm{Aut}_{P_1(C)}(c)}. \qquad \blacksquare$$

2.5 The curvature of an Ehresmann connection

Definition 2.44. Let $p\colon X \to B$ be a fibre bundle. Let $A \in \mathscr{A}(p)$. The **curvature of** A unique horizontal 2–form $F_A \in \Omega^2_{\mathrm{hor}}(X, V_p)$ such that for every $v, w \in \mathrm{Vect}(X)$

$$F_A(v, w) = -A([v - A(v), w - A(w)]). \qquad \bullet$$

Of course, F_A is the obstruction to the integrability of the **horizontal distribution** $H_A := \ker A \subset TX$.

Example 2.45. Consider the Hopf bundle $p\colon\ S^{2n+1} \subset \mathbf{C}^{n+1}\backslash\{0\} \to \mathbf{C}P^n$. The vertical tangent bundle is spanned by the vector field $\partial_\alpha \in \mathrm{Vect}(S^{n+1})$ defined by

$$\partial_\alpha(z) = iz.$$

The connection A defines by the metric on S^{2n+1} satisfies

$$H_A = \{v \in TS^{2n+1} : v \perp \partial_\alpha\}.$$

Let $v, w \in \mathrm{Vect}(S^{2n+1})$ The curvature of A is

$$\begin{aligned} F_A(v,w) &= -\langle [v,w], iz\rangle \otimes iz \\ &= -\langle \nabla_v w - \nabla_w v, iz\rangle \otimes iz \\ &= -2\langle v, iw\rangle \otimes iz \\ &= -2\pi \cdot p^*\omega_{\mathrm{FS}} \otimes \partial_\alpha. \end{aligned}$$

Here $\omega_{\mathrm{FS}} \in \Omega^2(\mathbf{C}P^n)$ is the Fubini–Study form on $\mathbf{C}P^n$. ♠

Exercise 2.46. Let $U_a \coloneqq \{[z_0,\dots,z_n] \in \mathbf{C}P^n : z_a \neq 0\}$ and define $\phi_a\colon\ U_a \to \mathbf{C}^n$ by

$$\phi([z_0,\dots,z_n]) \coloneqq [z_0/z_a : \cdots : \widehat{z_a/z_a} : \cdots : z_n/z_a].$$

Prove that there is a unique 2–form ω_{FS} on $\mathbf{C}P^n$ satisfying

$$(\phi_a)_*\omega_{\mathrm{FS}} = \frac{i}{2\pi}\left(\sum_{b=1}^n \frac{\mathrm{d}z_b \wedge \mathrm{d}\bar z_b}{1+|z|^2} - \sum_{b,c=1}^n \frac{\bar z_c \mathrm{d}z_c \wedge z_b \mathrm{d}\bar z_b}{(1+|z|^2)^2}\right).$$

Prove that the above formula for F_A indeed holds.

Proposition 2.47. *Let $p\colon\ X \to B$ be fibre bundle. Let $A \in \mathscr{A}(p)$. Let $f\colon\ C \to B$ be a smooth map. The curvature of A and f^*A satisfy*

$$F_{f^*A} = V_f^{-1}(p^*f)^*F_A.$$

Definition 2.48. Let $p\colon\ X \to B$ be a fibre bundle. An Ehresmann connection $A \in \mathscr{A}(p)$ is **flat** if $F_A = 0$. •

Example 2.49. Let $p\colon\ X \to B$ be a smooth covering map. The unique Ehresmann connection on p is flat. ♠

Example 2.50 ([MS74, Appendix C]). Let Σ be a closed, connected Riemann surface of genus $g \geqslant 2$. The fibre bundle $p\colon ST\Sigma \to \Sigma$ admits a flat connection.

By the uniformization theorem, there is a $\Gamma^{\mathrm{op}} < \mathbf{PSL}(2,\mathbf{R}) = \mathrm{Isom}(H)$ with $\Sigma = H/\Gamma$. In particular, $ST\Sigma = STH/\Gamma$. Define $P\colon STH \to H \times (\mathbf{R} \cup \{\infty\})$ by

$$P(z,v) := \lim_{t\to\infty} \exp_x(tv).$$

Here $\exp_x$ is computed with respect to the hyperbolic metric g_{-1} on H. $\mathbf{PSL}(2,\mathbf{R})$ acts on $\mathbf{R} \cup \{\infty\} \cong S^1$. A moment's thought shows that P is $\mathbf{PSL}_2(\mathbf{R})$–equivariant. Therefore,

$$ST\Sigma = STH/\Gamma \cong (H \times (\mathbf{R} \cup \{\infty\}))/\Gamma.$$

The flat Ehresmann connection on $\mathrm{pr}_1\colon H \times (\mathbf{R} \cup \{\infty\}) \to \mathbf{H}$ is $\mathbf{PSL}_2(\mathbf{R})$–invariant and descends to $p\colon ST\Sigma \to \Sigma$. ♠

Proposition 2.51 (Flat connections and covering maps). *Let $p\colon X \to B$ be a fibre bundle. If $A \in \mathscr{A}(p)$ is complete and flat, then there is a covering map $q\colon S \to B$ and a bijective immersion $\iota\colon S \looparrowright X$ such that $q = p \circ \iota$ and $T\iota\colon TS \hookrightarrow \iota^*TX$ induces an isomorphism $TS \cong \iota^* H_A$; in particular:* tra^A *depends only on the homotopy class rel $\{0,1\}$ of γ; that is* tra^A *factors through $P_1(B) \to \Pi_1(B)$.*

Proof. A is flat if and only if the distribution H_A is involutive. Frobenius's theorem guarantees the existence of a bijective immersion $\iota\colon S \looparrowright X$ such that $T\iota\colon TS \hookrightarrow \iota^*TX$ induces an isomorphism $TS \cong \iota^* H_A$.

To prove that $q := p \circ \iota$ is a covering map, let $b_0 \in B$ and let U be a connected, simply-connected, open neighborhood of b_0. Let V be a connected component of $S \cap \iota^{-1}(U)$. It remains to prove that $q|_V\colon V \to U$ is a diffeomorphism. By construction, $q|_V$ is a local diffeomorphism. Since U is path-connected, $q|_V$ is surjective. To prove that $q|_V$ is injective, let $s_0, s_1 \in q^{-1}(b_0) \cap V$. Since V is path-connected, there is a smooth path $\tilde\gamma\colon [0,1] \to S$ with $\tilde\gamma(0) = s_0$ and $\tilde\gamma(1) = s_1$. Since U is simply-connected, there is a smooth homotopy $\Gamma\colon [0,1] \times [0,1] \to B$ rel $\{0,1\}$ with $\Gamma(0,\cdot) = q \circ \tilde\gamma$ and $\Gamma(1,\cdot) = b_0$. The task at hand is to find an A–horizontal lift $\tilde\Gamma\colon [0,1] \times [0,1] \to X$ of Γ along p with $\tilde\Gamma(0,0) = x_0 := \iota(s_0)$.

It suffices to consider $B = [0,1] \times [0,1]$ and $\gamma = \mathrm{id}_B$. Denote by v_1, v_2 the A–horizontal lifts of ∂_1, ∂_2. The lift

$$\tilde\Gamma(t_1,t_2) := \mathrm{flow}_{v_1}^{t_1} \circ \mathrm{flow}_{v_2}^{t_2}(x_0)$$

maps into the maximal integral submanifold through x_0; hence, it is A–parallel. ■

Proposition 2.52. *Let $p\colon X \to B$ be a fibre bundle. If there is a flat Ehresmann connection $A \in \mathscr{A}(p)$ and B is simply-connected, then p is isomorphic to a trivial fibre bundle.* ■

2.6 Decomposition of the de Rham complex on fibre bundles

Definition 2.53. Let $p\colon X \to B$ be a fibre bundle together with an Ehresmann connection $A \in \mathscr{A}(p)$. The **bi-grading on** $\Omega(X)$ **induced by** A is defined by

$$\Omega_A^{p,q}(X) \coloneqq \Gamma(\Lambda_A^{p,q}T^*X). \quad \text{with} \quad \Lambda_A^{p,q}T^*X \coloneqq \Lambda^p H_A^* \otimes \Lambda^q V_p^*.$$

Denote by $\mathrm{d}_A^{p,q}$ the component of d of bi-degree (p,q). •

Proposition 2.54. *Let $p\colon X \to B$ be a fibre bundle together with an Ehresmann connection. The exterior derivative decomposes into three components of bidegree $(1,0)$, $(0,1)$, and $(2,-1)$:*

$$\mathrm{d} = \mathrm{d}_A^{1,0} + \mathrm{d}_A^{0,1} + \mathrm{d}_A^{2,-1};$$

moreover:

$$\mathrm{d}_A^{2,-1} = i_{F_A}.$$

where i_{F_A} is the graded derivation of degree 1 on $\Omega(X)$ defined by

$$i_{F_A} f = 0 \quad \text{and} \quad i_{F_A}\alpha = \alpha \circ F_A.$$

Proof. The exterior derivative d is a graded derivation of $\Omega^\bullet(X)$ of degree 1. Consequently, $\mathrm{d}_A^{p,q} = 0$ vanishes unless $p + q = 1$. Since a graded derivation of $\Omega^\bullet(X)$ is determined by its restriction to $\Omega^0(X) \oplus \Omega^1(X)$, $\mathrm{d}_A^{p,q} = 0$ unless $p, q \geqslant -1$.

Evidently, $\mathrm{d}_A^{-1,2}$ vanishes on $\Omega^0(X) \oplus \Omega_A^{0,1}(X)$. Moreover, for every $\alpha \in \Omega_A^{1,0}$ and $v, w \in \mathrm{Vect}(X)$

$$\begin{aligned}(\mathrm{d}_A^{-1,2}\alpha)(v,w) &= (\mathrm{d}\alpha)(A(v), A(w)) \\ &= -\alpha([A(v), A(w)]) = 0.\end{aligned}$$

Therefore, $\mathrm{d}_A^{-1,2} = 0$.

Evidently, $\mathrm{d}_A^{2,-1}$ vanishes on $\Omega^0(X) \oplus \Omega_A^{1,0}(X)$. Moreover, for every $\alpha \in \Omega_A^{0,1}$ and $v, w \in \mathrm{Vect}(X)$,

$$\begin{aligned}(\mathrm{d}_A^{-1,2}\alpha)(v,w) &= (\mathrm{d}\alpha)(v - A(v), v - A(w)) \\ &= -\alpha(A[v - A(v), w - A(w)]) \\ &= (i_{F_A}\alpha)(v,w).\end{aligned}$$

■

Remark 2.55. Proposition 2.54 is a justification for the sign appearing in the definition of F_A. ♣

The following is an immediate consequence of $\mathrm{d}^2 = 0$.

Proposition 2.56. *The operators* $\mathrm{d}_A^{1,0}$, $\mathrm{d}_A^{0,1}$, $\mathrm{d}_A^{2,-1}$ *satisfy*

$$\begin{aligned}
\left(\mathrm{d}_A^{1,0}\right)^2 + \mathrm{d}_A^{0,1}\mathrm{d}_A^{2,-1} + \mathrm{d}_A^{2,-1}\mathrm{d}_A^{0,1} &= 0,\\
\left(\mathrm{d}_A^{0,1}\right)^2 &= 0,\\
\left(\mathrm{d}_A^{2,-1}\right)^2 &= 0,\\
\mathrm{d}_A^{1,0}\mathrm{d}_A^{0,1} + \mathrm{d}_A^{0,1}\mathrm{d}_A^{1,0} &= 0, \quad \textit{and}\\
\mathrm{d}_A^{1,0}\mathrm{d}_A^{2,-1} + \mathrm{d}_A^{2,-1}\mathrm{d}_A^{1,0} &= 0.
\end{aligned}$$

■

Remark 2.57.

(1) $\mathrm{d}_A^{1,0}\mathrm{d}_A^{2,-1} + \mathrm{d}_A^{2,-1}\mathrm{d}_A^{1,0}$ is the **Bianchi identity**.

(2) $\mathrm{d}_A^{0,1}$ and $\mathrm{d}_A^{2,-1}$ are differentials, but $\mathrm{d}_A^{1,0}$ is not. However, it descends to a differential on $\mathrm{H}(\Omega(X), \mathrm{d}_A^{0,1})$ and on $\mathrm{H}(\Omega(X), \mathrm{d}_A^{2,-1})$. ♣

2.7 Digression: The Fröhlicher–Nijenhuis bracket

This section is a slight digression on the Fröhlicher–Nijenhuis bracket. This provides some context but is not necessary. [KMS93, §8] contains a more detailed treatment of the material discussed below.

Definition 2.58. Let $k \in \mathbf{Z}$. A **graded derivation of degree** k on $\Omega^\bullet(X)$ is an $\mathbf{R}$–linear map $\delta\colon \Omega^\bullet(X) \to \Omega^{\bullet+k}(X)$ satisfying the **graded Leibniz rule**

$$\delta(\alpha \wedge \beta) = (\delta\alpha) \wedge \beta + (-1)^{k\cdot\ell}\alpha \wedge (\delta\beta)$$

for every $\alpha \in \Omega^\ell(X), \beta \in \Omega^\bullet(X)$. The graded derivations of $\Omega^\bullet(X)$ form a graded Lie algebra $\mathrm{Der}_\bullet(\Omega^\bullet(X))$. •

Exercise 2.59. Verify the last sentence in the above definition.

Example 2.60. The exterior derivative d is a graded derivation of degree 1 of $\Omega^\bullet(X)$. If $v \in \mathrm{Vect}(X)$, then i_v is a graded derivation of degree −1 of $\Omega^\bullet$. By Cartan's formula, their graded commutator is the Lie derivative

$$\mathscr{L}_v = \mathrm{d}i_v + i_v\mathrm{d} = [i_v, \mathrm{d}];$$

itself a graded derivation of degree 0. ♠

The derivations of $C^\infty(X)$ are precisely the vector fields on X: $\mathrm{Der}(C^\infty(X)) \cong \mathrm{Vect}(X)$. Is there an analogous result for $\mathrm{Der}_\bullet(\Omega^\bullet(X))$?

Definition 2.61. Let $k \in \mathbf{N}_0$. Denote by $i_\cdot \colon \Omega^{k+1}(X, TX) \to \mathrm{Der}_k(\Omega^\bullet(X))$ the unique linear map satisfying

$$i_{\xi\otimes v}\alpha = \xi \wedge i_v\alpha \quad \text{for} \quad \xi \in \Omega^{k+1}(X) \quad \text{and} \quad v \in \mathrm{Vect}(X).$$

Define $\mathscr{L}_\cdot \colon \Omega^k(X, TX) \to \mathrm{Der}_k(\Omega^\bullet(X))$ by

$$\mathscr{L}_\Xi := [i_\Xi, \mathrm{d}].$$

•

Exercise 2.62. Prove that ι and $\mathscr{L}$ are injective.

Exercise 2.63. For $\Xi = \sum_i \xi_i \otimes v^i \in \Omega^k(X, TX)$ prove that

$$\mathscr{L}_\Xi\alpha = \sum_i \xi_i \wedge \mathscr{L}_{v^i}\alpha + (-1)^k(\mathrm{d}\xi_i) \wedge i_{v^i}\alpha.$$

Proposition 2.64. *Let X be a smooth manifold. Let $k \in \mathbf{Z}$. The map $\mathscr{L} + \iota \colon \Omega^k(X, TX) \oplus \Omega^{k+1}(X, TX) \to \mathrm{Der}_k(\Omega^\bullet(X))$ is an isomorphism. Moreover, $\delta \in \mathrm{im}\,\mathscr{L}$ if and only if $[\delta, \mathrm{d}] = 0$; and $\varepsilon \in \mathrm{im}\,\mathscr{L}$ if and only if $\varepsilon(\Omega^0(X)) = 0$.*

Proof. Every $\delta \in \mathrm{Der}_k(\Omega^\bullet(X))$ is determined by its restriction to $\Omega^0(X) \oplus \Omega^1(X)$. If $v_1, \dots, v_k$, then the map

$$f \mapsto (\delta f)(v_1, \dots, v_k)$$

is a derivation of $\Omega^0(X) = C^\infty(X)$. Hence, there is a unique vector field $\Xi(v_1, \dots, v_k)$ such that

$$(\delta f)(v_1, \dots, v_k) = \mathscr{L}_{\Xi(v_1,\dots,v_k)} f.$$

A moment's thought shows that $(v_1, \dots, v_k) \mapsto \Xi(v_1, \dots, v_k)$ is tensorial. Therefore, it defines a $\Xi \in \Omega^k(X, TX)$. The derivation $\varepsilon := \delta - \mathscr{L}_\Xi$ vanishes on $\Omega^0(X)$.

If $f \in C^\infty(X)$ and $\alpha \in \Omega^1(X)$, then

$$\varepsilon(f\alpha) = f \cdot \varepsilon\alpha;$$

that is: $\varepsilon \colon \Omega^1(X) \to \Omega^k(X)$ is tensorial. Therefore, there is a $\Theta \in \Omega^k(X, TX)$ with $\varepsilon = i_\Theta$.

By construction, $\delta = i_\Theta + \mathscr{L}_\Xi$ on $\Omega^0(X) \oplus \Omega^1(X)$. This proves the first assertion. The vanishing criterion for Θ is obvious. A brief computation shows that $[\mathscr{L}_\Xi, \mathrm{d}] = 0$. Since

$$[i_\Theta, \mathrm{d}] = \mathscr{L}_\Theta,$$

the final assertion follows. ■

Exercise 2.65. What are Θ and Ξ for $\delta = \mathrm{d}$?

Exercise 2.66. Use Proposition 2.64 to prove Cartan's formula $\mathscr{L}_v = \mathrm{d}i_v + i_v\mathrm{d}$.

The identification can be used to define the Lie bracket $[\cdot,\cdot]$ on $\mathrm{Vect}(X)$. Since

$$[[\mathscr{L}_\Theta, \mathscr{L}_\Xi], \mathrm{d}] = 0,$$

one obtains a graded Lie bracket on $\Omega^\bullet(X, TX)$.

Definition 2.67. The **Fröhlicher–Nijenhuis bracket** is the map

$$[\cdot,\cdot]\colon\ \Omega^\bullet(X, TX) \otimes \Omega^\bullet(X, TX) \to \Omega^\bullet(X, TX)$$

characterised by

$$[\mathscr{L}_\Theta, \mathscr{L}_\Xi] = \mathscr{L}_{[\Theta,\Xi]}.$$

•

It turns out (somewhat miraculously in my opinion) that Fröhlicher–Nijenhuis bracket consistently shows up as an obstruction to integrability.

Proposition 2.68. *If $\theta \in \Omega^1(X, TX)$, then the* ***Nijenhuis tensor***

$$N_\theta \coloneqq -\frac{1}{2}[\theta, \theta] \in \Omega^2(X, TX)$$

satisfies

$$N_\theta(v, w) = -\theta(\theta([v, w])) - [\theta(v), \theta(w)] + \theta([\theta(v), w] + [v, \theta(w)]).$$

The proof relies on the following.

Proposition 2.69. *For $\theta, \alpha \in \Omega^1(X, TX)$ and $v, w \in \mathrm{Vect}(X)$*

$$(\mathscr{L}_\theta\alpha)(v, w) = \mathscr{L}_{\theta(v)}(\alpha(w)) - \mathscr{L}_{\theta(w)}(\alpha(v)) - \alpha([\theta(v), w]) - \alpha([v, \theta(w)]) + \alpha(\theta([v, w])).$$

Proof. Let $\theta, \alpha \in \Omega^1(X, TX)$ and $v, w \in \mathrm{Vect}(X)$. Since

$$(\mathrm{d}\alpha)(v, w) = \mathscr{L}_v(\alpha(w)) - \mathscr{L}_w(\alpha(v)) - \alpha([v, w]),$$

by definition of $\mathscr{L}_\theta$,

$$\begin{aligned}
(\mathscr{L}_\theta\alpha)(v, w) &= (i_\theta \mathrm{d}\alpha - \mathrm{d}i_\theta a)(v, w) \\
&= \mathscr{L}_{\theta(v)}(\alpha(w)) - \mathscr{L}_w(\alpha(\theta(v))) - \alpha([\theta(v), w]) \\
&\quad + \mathscr{L}_v(\alpha(\theta(w))) - \mathscr{L}_{\theta(w)}(\alpha(v)) - \alpha([v, \theta(w)]) \\
&\quad - \mathscr{L}_v(\alpha(\theta(w))) + \mathscr{L}_w(\alpha(\theta(v))) + \alpha(\theta([v, w])) \\
&= \mathscr{L}_{\theta(v)}(\alpha(w)) - \alpha([\theta(v), w]) \\
&\quad - \mathscr{L}_{\theta(w)}(\alpha(v)) - \alpha([v, \theta(w)]) \\
&\quad + \alpha(\theta([v, w])).
\end{aligned}$$

■

Proof of Proposition 2.68. $N_\theta \in \Omega^2(X, TX)$ is determined by the action on $\Omega^0(X) = C^\infty(X)$. For $\theta \in \Omega^k(TX, X)$

$$(\mathscr{L}_\theta f)(v_1, \dots, v_k) = \mathscr{L}_{\theta(v_1,\dots,v_k)} f;$$

in particular, $(\mathscr{L}_\theta f)(v) = \mathscr{L}_{\theta(v)} f$. Therefore, using Proposition 2.69,

$$\begin{aligned}(\mathscr{L}_\theta \mathscr{L}_\theta f)(v, w) &= \mathscr{L}_{\theta(v)} \mathscr{L}_{\theta(w)} f - \mathscr{L}_{\theta([\theta(v),w])} f \\ &\quad - \mathscr{L}_{\theta(w)} \mathscr{L}_{\theta(v)} f - \mathscr{L}_{\theta([v,\theta(w)])} f \\ &\quad + \mathscr{L}_{\theta(\theta([v,w]))} f \\ &= \mathscr{L}_{[\theta(v),\theta(w)]} f - \mathscr{L}_{\theta([\theta(v),w])} f \\ &\quad - \mathscr{L}_{\theta([v,\theta(w)])} f + \mathscr{L}_{\theta(\theta([v,w]))} f.\end{aligned}$$

This implies the assertion. ■

Remark 2.70. If $J \in \mathrm{End}(TX)$ is an almost complex structure (that is: $J^2 = -\mathbf{1}$), then the vanishing of N_J characterises the integrability of J. Indeed, the Newlander–Nirenberg theorem asserts that $N_J = 0$ if and only if X admits a holomorphic structure which induces the almost complex structure J. ♣

Exercise 2.71. Let $p\colon X \to B$ be fibre bundle. Let $A \in \mathscr{A}(p)$. Regard the connection 1–form $A\colon \Omega^1(X, V_p)$ as TX–valued 1–form. Prove that

$$F_A = N_A$$

Remark 2.72. The graded Jacobi identity implies the **Bianchi identity**

$$[A, F_A] = 0.$$ ♣

Definition 2.73. Let X, Y be smooth manifolds Let $f\colon X \to Y$ be a smooth map. $\Theta \in \Omega^k(X, TX)$ and $\Xi \in \Omega^k(Y, TY)$ are f**–related** if for every $x \in X$, $v_1, \dots, v_k \in T_x X$

$$T_x f(\Theta(v_1, \dots, v_k)) = \Xi(T_x f(v_1), \dots, T_x f(v_k)).$$ •

Proposition 2.74. *Let X, Y be smooth manifolds Let $f\colon X \to Y$ be a smooth map. Let $\Theta_1, \Theta_2 \in \Omega^\bullet(X, TX)$ and $\Xi_1, \Xi_2 \in \Omega^\bullet(Y, TY)$. If Θ_i and Ξ_i are f–related, then $[\Theta_1, \Theta_2]$ and $[\Xi_1, \Xi_2]$ are f–related.*

Proof. Exercise. ■

Proposition 2.75. *Let $p\colon X \to B$ be a fibre bundle. Let $A \in \mathscr{A}(p)$. The operators $\mathrm{d}_A^{1,0}$, $\mathrm{d}_A^{0,1}$, $\mathrm{d}_A^{2,-1}$ satisfy*

$$\mathrm{d}_A^{1,0} = \mathscr{L}_{\mathrm{id}_{TX}-A} - 2i_{F_A}, \quad \mathrm{d}_A^{0,1} = \mathscr{L}_A + i_{F_A}, \quad \textit{and} \quad \mathrm{d}_A^{2,-1} = i_{F_A}.$$

Proof. The next two steps determine explicit formulae for $\mathrm{d}_A^{1,0}$ and $\mathrm{d}_A^{0,1}$. The computations are longish and not particularly illuminating.

Step 1. $\mathrm{d}_A^{1,0} = \mathscr{L}_{\mathrm{id}_{TX}-A} - 2i_{F_A}$.

It suffices to verify the identity on $C^\infty(X)$ and $\Omega^1(X)$.
For $f \in C^\infty(X)$

$$\begin{aligned}\mathrm{d}_A^{1,0} f &= \mathrm{d}f \circ (\mathrm{id}_{TX} - A)\\ &= (\mathscr{L}_{\mathrm{id}_{TX}-A} - 2i_{F_A})f.\end{aligned}$$

For $\alpha \in \Omega^1(X)$ and $v, w \in \mathrm{Vect}(X)$

$$\begin{aligned}(\mathscr{L}_{\mathrm{id}_{TX}-A}\alpha)(v,w) &= \mathscr{L}_{v-A(v)}(\alpha(w)) - \mathscr{L}_{w-A(w)}(\alpha(v))\\ &\quad - \alpha([v-A(v),w]) - \alpha([v,w-A(w)]) + \alpha([v,w]-A([v,w])).\end{aligned}$$

For $\alpha \in \Omega_A^{1,0}(X)$ and $v, w \in \mathrm{Vect}(X)$, since $\alpha \circ \theta = 0$,

$$\begin{aligned}(\mathrm{d}_A^{1,0}\alpha)(v,w) &= \mathrm{d}\alpha(v-A(v), w-A(w))\\ &= \mathscr{L}_{v-A(v)}(\alpha(w)) - \mathscr{L}_{w-\theta(w)}(\alpha(v)) - \alpha([v-\theta_a(v), w-\theta_a(w)])\\ &= (\mathscr{L}_{\mathrm{id}_{TX}-A}\alpha - 2i_{F_A}\alpha)(v,w)\\ &\quad + \alpha([v-A(v),w]) + \alpha([v,w-A(w)])\\ &\quad - \alpha([v,w]) - \alpha([v-\theta_a(v), w-\theta_a(w)]).\end{aligned}$$

The sum of the last four term vanishes because

$$[v-A(v),w] + [v,w-A(w)] - [v,w] - [v-\theta_a(v), w-\theta_a(w)] = -[A(v),A(w)]$$

is a vertical vector field.
For $\alpha \in \Omega_A^{0,1}(X)$ and $v, w \in \mathrm{Vect}(X)$, since $\alpha \circ \theta = \alpha$,

$$\begin{aligned}\mathrm{d}_A^{1,0}\alpha(v,w) &= \mathrm{d}\alpha(A(v), w-A(w)) + \mathrm{d}\alpha(v-A(v), A(w))\\ &= \mathscr{L}_{v-A(v)}(\alpha(w)) - \mathscr{L}_{w-A(w)}(\alpha(v))\\ &\quad - \alpha([\theta_a(v), w-\theta_a(w)]) - \alpha([v-\theta_a(v), \theta_a(w)])\\ &= \big(\mathscr{L}_{\mathrm{id}_{TX}-A}\alpha - 2i_{F_A}\alpha\big)(v,w)\\ &\quad + \alpha([v-A(v),w]) + \alpha([v,w-A(w)])\\ &\quad - 2\alpha([v-A(v), w-\theta(w)]\\ &\quad - \alpha([\theta_a(v), w-\theta_a(w)]) - \alpha([v-\theta_a(v), \theta_a(w)]).\end{aligned}$$

The sum of the last five terms vanishes.

Step 2. $\mathrm{d}_A^{0,1} = \mathscr{L}_A + i_{F_A}$.

For $f \in C^\infty(X)$

$$\begin{aligned}\mathrm{d}_A^{0,1} f &= \mathrm{d}f \circ A \\ &= (\mathscr{L}_A + i_{F_A}) f.\end{aligned}$$

For $\alpha \in \Omega^1(X)$ and $v, w \in \mathrm{Vect}(X)$

$$\begin{aligned}(\mathscr{L}_A \alpha)(v, w) = \mathscr{L}_{A(v)}(\alpha(w)) - \mathscr{L}_{A(w)}(\alpha(v)) \\ - \alpha([A(v), w]) - \alpha([v, A(w)]) + \alpha(A([v, w])).\end{aligned}$$

For $\alpha \in \Omega_A^{1,0}(X)$ and $v, w \in \mathrm{Vect}(X)$, since $\alpha \circ \theta = 0$,

$$\begin{aligned}(\mathrm{d}_A^{0,1}\alpha)(v, w) &= \mathrm{d}\alpha(A(v), w - A(w)) + \mathrm{d}\alpha(v - A(v), A(w)) \\ &= \mathscr{L}_{A(v)}(\alpha(w)) - \mathscr{L}_{A(w)}(\alpha(v)) \\ &\quad - \alpha([A(v), w - A(w)]) - \alpha([v - A(v), A(w)]) \\ &= (\mathscr{L}_A \alpha + i_{F_A}\alpha)(v, w) \\ &\quad + \alpha([A(v), w]) + \alpha([v, A(w)]) \\ &\quad - \alpha([A(v), w - A(w)]) - \alpha([v - A(v), A(w)]).\end{aligned}$$

The sum of the last four term vanishes because

$$[A(v), w] + [v, A(w)] - [A(v), w - A(w)] - [v - A(v), A(w)] = 2[A(v), A(w)]$$

is a vertical vector field.

For $\alpha \in \Omega_A^{0,1}(X)$ and $v, w \in \mathrm{Vect}(X)$, since $\alpha \circ \theta = \alpha$,

$$\begin{aligned}(\mathrm{d}_A^{0,1}\alpha)(v, w) &= \mathrm{d}\alpha(A(v), A(w)) \\ &= \mathscr{L}_{A(v)}(\alpha(w)) - \mathscr{L}_{\theta(w)}(\alpha(v)) - \alpha([A(v), A(w)]) \\ &= (\mathscr{L}_A \alpha + i_{F_A}\alpha)(v, w) \\ &\quad + \alpha([A(v), w]) + \alpha([v, A(w)]) - \alpha([v, w]) \\ &\quad + \alpha([v - A(v), w - A(w)]) - \alpha([A(v), A(w)]).\end{aligned}$$

The sum of the last five term vanishes. ■

2.8 The spectral sequence of a filtered complex

Here is some homological algebra in preparation of the Leray–Serre spectral sequence; see, e.g., [Wei94, §5.4].

Definition 2.76. A **spectral sequence** is a sequence

$$(E_r, \mathrm{d}_r)_{r \in \mathbf{N}_0}$$

of $\mathbf{Z}^2$–graded complexes such that, for every $r \in \mathbf{N}_0$, d_r has bidegree $(r, -r+1)$ and

$$\mathrm{H}(E_r) \cong E_{r+1}.$$

The spectral sequence **degenerates** if $\mathrm{d}_r = 0$ for $r \gg 1$. In this case, E_∞ denotes a bigraded vector space with $E_\infty \cong E_r$ for $r \gg 1$. •

Definition 2.77. Let (C, d) be a $\mathbf{Z}$–graded complex. A **descending filtration** of (C, d) is a descending sequence of subcomplexes $(F^jC)_{j\in\mathbf{Z}}$:

$$C \supset \cdots \supset F^jC \supset F^{j+1}C \supset \cdots.$$

The filtration FC is **bounded** if $F^jC = C$ for $j \ll 1$ and $F^jC = 0$ for $j \gg 1$. The **associated $\mathbf{Z}^2$–graded complex** of (FC, d) is

$$\mathrm{Gr}\, C = \bigoplus_{p,q\in\mathbf{Z}} \mathrm{Gr}^p\, C^q \quad \text{with} \quad \mathrm{Gr}^p\, C^q := F^pC^q/F^{p+1}C^q$$

equipped with the differential of bidegree $(0,1)$ induced by d. •

Lemma 2.78 (Spectral sequence of a filtered differential $\mathbf{Z}$–graded module). *Let (C, d) be a $\mathbf{Z}$–graded differential module equipped with a descending filtration FC. For $p, q, r \in \mathbf{Z}$ set*

$$\begin{aligned} Z_r^{p,q} &:= F^pC^{p+q} \cap \mathrm{d}^{-1}(F^{p+r}C^{p+q+1}) \quad \textit{and} \\ B_r^{p,q} &:= \mathrm{d}Z_{r-1}^{p-r+1,q+r-2} = F^pC^{p+q} \cap \mathrm{d}(F^{p-r+1}C^{p+q-1}). \end{aligned}$$

There is a spectral sequence (E_r, d_r) with

$$E_r^{p,q} = \frac{Z_r^{p,q}}{B_r^{p,q} + Z_{r-1}^{p+1,q-1}} \quad \textit{and} \quad \mathrm{d}_r[x] = [\mathrm{d}x];$$

in particular,

$$E_1^{p,q} = \mathrm{H}^{p,q}(\mathrm{Gr}^p\, C, \mathrm{d}).$$

If FC is bounded, then (E_r, d_r) degenerates and

$$E_\infty^{p,q} \cong \mathrm{Gr}^p\, \mathrm{H}^{p+q}(C, \mathrm{d}).$$

Proof. Since $\mathrm{d}Z_r^{p,q} \subset Z_r^{p+r,q-r+1}$, $\mathrm{d}B_r^{p,q} = 0$, and $\mathrm{d}Z_{r-1}^{p+1,q-1} \subset \mathrm{d}Z_{r-1}^{p+r+1,q-r}$, d descends to $\mathrm{d}_r^{p,q}\colon E_r^{p,q} \to E_r^{p+r,q-r+q}$.

By construction,

$$\mathrm{im}\, \mathrm{d}_r^{p-r,q+r-1} = \frac{B_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1}}{B_r^{p,q} + Z_{r-1}^{p+1,q-1}}.$$

A moment's thought reveals that

$$\ker \mathrm{d}_r^{p,q} = \frac{Z_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1}}{B_r^{p,q} + Z_{r-1}^{p+1,q-1}}.$$

Moreover, since $Z_{r+1}^{p,q} \cap Z_{r-1}^{p+1,q-1} = Z_r^{p+1,q-1}$,

$$\frac{\ker \mathrm{d}_r^{p,q}}{\operatorname{im} \mathrm{d}_r^{p-r,q+r-1}} \cong \frac{Z_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1}}{B_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1}} \cong E_{r+1}^{p,q}.$$

This establishes the existence of the desired spectral sequence.

It remains to establish the convergence if FC is bounded. To this end it suffices to observe that, for $r \gg 1$, $Z_r^{p,q} = F^p \ker(\mathrm{d}\colon C^{p+q} \to C^{p+q+1})$ and $B_r^{p,q} = F^p \operatorname{im}(\mathrm{d}\colon C^{p+q} \to C^{p+q+1})$. ■

Remark 2.79. Although Lemma 2.78 appears rather unwieldy, it is a very powerful tool. It turns out that often sufficient information can be extracted without going through the grueling process of unravelling the details of the construction in Lemma 2.78. ♣

Example 2.80. A **double complex** is a $\mathbf{Z}^2$–graded vector space C together with two differentials d of bidegree $(1,0)$ and δ of bidegree $(0,1)$ such that $\mathrm{d}\delta + \delta\mathrm{d} = 0$. The associated **total complex** is the graded complex

$$\operatorname{Tot} C^k := \bigoplus_{p+q=k} C^{p,q}$$

equipped with the differential $\mathrm{d} + \delta$. $\operatorname{Tot} C$ has two filtrations

$$'F^p \operatorname{Tot} C^n := \bigoplus_{\substack{n=p'+q \\ p' \geqslant p}} C^{p,q}$$

$$''F^q \operatorname{Tot} C^n := \bigoplus_{\substack{n=p+q' \\ q' \geqslant q}} C^{p,q}.$$

Consequently, $(\operatorname{Tot} C, \mathrm{d})$ has two spectral sequences. The E_2 pages of these spectral sequences are

$$'E_2^{p,q} \cong \mathrm{H}^p(\mathrm{H}^q(C,\delta),\mathrm{d}) \quad \text{and} \quad ''E_2^{p,q} \cong \mathrm{H}^q(\mathrm{H}^p(C,\mathrm{d}),\delta). \qquad ♠$$

Example 2.81. It is an frivolous but enlightening exercise to derive the snake lemma from considerations of the spectral sequences of the obvious double complex inherent in its setting.

The setting of the snake lemma can be understood as a double complex:

$$\begin{tikzcd} B_0 \arrow[r] & B_1 \arrow[r] & B_2 \\ A_0 \arrow[u, "f_0"] \arrow[r] & A_1 \arrow[u, "f_1"] \arrow[r] & A_2 \arrow[u, "f_2"] \end{tikzcd}$$

Since the rows are exact, $''E_1 = 0$. On the other hand, $'E_1$ is

$$\operatorname{coker} f_0 \longrightarrow \operatorname{coker} f_1 \longrightarrow \operatorname{coker} f_2$$

$$\ker f_0 \longrightarrow \ker f_1 \longrightarrow \ker f_2.$$

The $'E_2$ page is the cohomology of $'E_1$. For degree reasons, every differential except possibly $\mathrm{d}_2^{0,1}$ vanishes. As a consequence, $'E_2^{p,q} = 0$ unless $(p, q) \in \{(0, 1), (2, 0)\}$ and $\mathrm{d}_2^{0,1}$ is an isomorphism. This yields exact sequences

$$\ker(\operatorname{coker} f_0 \to \operatorname{coker} f_1) \hookrightarrow \operatorname{coker} f_1 \twoheadrightarrow \operatorname{coker} f_2$$

and

$$\ker f_0 \hookrightarrow \ker f_1 \twoheadrightarrow \operatorname{coker}(\ker f_1 \to \ker f_2).$$

and an isomorphism $\operatorname{coker}(\ker f_1 \to \ker f_2) \cong \ker(\operatorname{coker} f_0 \to \operatorname{coker} f_1)$. Splicing these short exact sequences gives the familiar long exact sequence.

$$\ker f_0 \hookrightarrow \ker f_1 \twoheadrightarrow \ker f_2 \to \operatorname{coker} f_0 \to \operatorname{coker} f_1 \twoheadrightarrow \operatorname{coker} f_2. \qquad \spadesuit$$

Lemma 2.82 (Spectral sequence of a filtered differential $\mathbf{Z}$–graded algebra). *Let (A, d) be a differential $\mathbf{Z}$–graded algebra equipped with a descending filtration $F^\bullet A^\bullet$. There is a product $\cdot = \cdot_r \colon E_r^{p,q} \otimes E_r^{p',q'} \to E_r^{p+p',q+q'}$ such that $[x][y] = [xy]$, d_r is derivation, and the isomorphism $\mathrm{H}(E_r) \cong E_{r+1}$ preserves the product.*

If $F^\bullet A$ is bounded, then $E_\infty \cong \mathrm{Gr}^p\, \mathrm{H}(A, \mathrm{d})$ is compatible with the product.

Proof. By direct inspection,

$$Z_r^{p,q} \cdot Z_r^{p',q'} \subset Z_r^{p+p',q+q'} \quad \text{and} \quad Z_r^{p,q} \cdot (B_r^{p',q'} + Z_{r-1}^{p'+1,q'-1}) \subset B_r^{p+p',q+q'} + Z_{r-1}^{p+p'+1,q+q'-1}.$$

This proves that $\cdot_r$ is defined. That d_r is a derivation and the rest are either obvious or an exercise. ■

2.9 The Leray–Serre spectral sequence

The following discusses the Leray–Spectral sequence in de Rham cohomology.

Definition 2.83. Let $p\colon X \to B$ be a fibre bundle. Define the **horizontal filtration** $F\Omega(X)$ by

$$\begin{aligned} F^\ell\Omega^k(X) &\coloneqq \{\alpha \in \Omega^\bullet(X) : i_{v_1} \dots i_{v_{k+1-\ell}}\alpha = 0 \text{ for every } v_1, \dots, v_{k+1-\ell} \in V_p\} \\ &= \Gamma(X, F^\ell\Lambda^k T^*X) \end{aligned}$$

with

$$F^\ell\Lambda^k T^*X \coloneqq \Lambda^\ell p^*T^*B \otimes \Lambda^{k-\ell}T^*X.$$

(There $p^*T^*B \subset T^*X$ is the annihilator of V_p.) The **Leray–Serre spectral sequence** is the spectral sequence associated with this filtration. •

Remark 2.84. $F^k\Omega^k(X) = \Omega^k_{\mathrm{hor}}(X)$; this is the penultimate step of the filtration. ♣

The E_2 page of the Leray–Serre spectral sequence can be described as follows.

Proposition 2.85. *Let $p\colon X \to B$ be a proper fibre bundle. There is a unique graded local system $(\mathscr{H}^\bullet_{\mathrm{dR}}(p), \nabla)$ over B such that:*

(1) *The fibre over $b \in B$ is $\mathrm{H}^k_{\mathrm{dR}}(p^{-1}(b))$.*

(2) *If $U \subset B$ is open, $[\alpha] \in \mathrm{H}_{\mathrm{dR}}(p^{-1}(U))$, then $s_{[\alpha]} \in \Gamma(U, \mathscr{H}^k_{\mathrm{dR}}(p))$ defined by*

$$s_{[\alpha]}(b) \coloneqq i_b^*[\alpha]$$

is parallel.

Definition 2.86. $(\mathscr{H}^\bullet_{\mathrm{dR}}(p), \nabla)$ is the **Gauß–Manin local system**. •

Proof of Proposition 2.85. If U is as in Definition 2.1, then (2) defines a bijection

$$\tau_U\colon U \times \mathrm{H}^k_{\mathrm{dR}}(p^{-1}(U)) \to \bigcup_{b\in U} \mathrm{H}^k_{\mathrm{dR}}(p^{-1}(b)).$$

It suffices to prove that if V is as in Definition 2.1, then the map

$$U \cap V \to \mathrm{Hom}(\mathrm{H}^k_{\mathrm{dR}}(p^{-1}(V)), \mathrm{H}^k_{\mathrm{dR}}(p^{-1}(U)))$$

defined by $b \mapsto \mathrm{pr}_2 \circ \tau_U^{-1} \circ \tau_V(b, \cdot)$ is locally constant. This is an immediate consequence of the homotopy-invariance of de Rham cohomology. ■

Proposition 2.87. *Let $p\colon X \to B$ be a fibre bundle. The Leray–Serre spectral sequence associated with p satisfies*

$$E_2^{p,q} \cong \mathrm{H}^p_{\mathrm{dR}}(B, \mathscr{H}^q_{\mathrm{dR}}(p)).$$

Proof. This is an exercise in tracing through the definitions; see [GH94, p. 464] for hints. ■

Corollary 2.88. *Let $p\colon X \to B$ be a fibre bundle. If B is simply-connected, then*

$$E_2^{p,q} \cong \mathrm{H}^p_{\mathrm{dR}}(B) \otimes \mathrm{H}^q_{\mathrm{dR}}(p^{-1}(b)).$$ ■

If B is not simply-connected, then $\mathscr{H}^q_{\mathrm{dR}}(p)$ might have monodromy.

Example 2.89. Let F be smooth manifold. Let $f \in \mathrm{Diff}(F)$. Denote X_f the **mapping torus** of f; that is:

$$X_f \coloneqq ([0,1] \times F)/\sim$$

with denoting the equivalence relation generated by $(0,x) \sim (1, f(x))$. X_f is a smooth manifold and the projection map $p\colon X \to S^1 = \mathbf{R}/\mathbf{Z}$ is a fibre bundle. The monodromy of the Gauß–Manin connection on $\mathscr{H}^\bullet_{\mathrm{dR}}(p)$ is precisely the action of $\mathbf{Z}$ on $\mathrm{H}^\bullet_{\mathrm{dR}}(X)$ generated by f^*. ♠

The Leray–Serre spectral sequence does not usually degenerate at E_2. This can be seen, e.g., for the Hopf bundle $S^{2n+1} \to \mathbf{C}P^n$. However, there are two notable exceptions.

Proposition 2.90. *If $p\colon X \to B$ is a proper smooth covering map, then*

$$\mathrm{H}^\bullet_{\mathrm{dR}}(X) \cong \mathrm{H}^\bullet_{\mathrm{dR}}(B, p_*\underline{\mathbf{R}}).$$

Here $p_*\underline{\mathbf{R}}$ is the (sheaf-theoretic) push-forward of the local system $\underline{\mathbf{R}}$ on X. ■

Theorem 2.91 (Deligne [ref?]). *If X, B are closed Kähler, then Leray–Serre spectral sequence degenerates at E_2.* ■

2.10 Fibre integration

Definition 2.92. Let $p\colon X \to B$ be a fibre bundle of relative dimension d. A **fibre orientation** on p is a an orientation on the line bundle $\det(V_p) \coloneqq \Lambda^d V_p \to X$. •

Proposition 2.93. *Let $d \in \mathbf{N}_0$. Let $p\colon X \to B$ be a proper fibre bundle of relative dimension d together with a fibre orientation.*

(1) *There is a unique linear map $p_*\colon \Omega^\bullet(X) \to \Omega^{\bullet-d}(B)$ such that for every $\alpha \in \Omega^{d+k}(X)$,*

$b \in B$, and $\tilde v_1, \dots, \tilde v_k \in \Gamma(TX|_{p^{-1}(b)})$ lifts of $v_1, \dots, v_k \in T_bB$

$$(p_*\alpha)(v_1, \dots, v_k) = \int_{p^{-1}(b)} i_{\tilde v_k} \cdots i_{\tilde v_1} \alpha|_{p^{-1}(b)}. \tag{2.94}$$

(2) *For every $\alpha \in \Omega^\bullet(X)$ and $\beta \in \Omega^\bullet(B)$*

$$p_*(\alpha \wedge p^*\beta) = p_*\alpha \wedge \beta.$$

(3) *Suppose that B is oriented. For every $\alpha \in \Omega^\bullet(X)$*

$$\int_X \alpha = \int_B p_*\alpha$$

(4) *Suppose that $\partial B = \varnothing$. Set $\partial p := p|_{\partial X} \colon \partial X \to B$. For every $\alpha \in \Omega^\bullet(X)$*

$$p_*\mathrm{d}\alpha - (-1)^d \mathrm{d}p_*\alpha = \partial p_*\alpha.$$

Proof. The right-hand side of (2.94) is independent of the lifts $\tilde v_1, \dots, \tilde v_k$. To verify that (2.94) does define the map p_* it suffices to it suffices the require smoothness. It is enough to verify this for $\mathrm{pr}_B \colon B \times F \to B$. This proves (1).

(2) is evident from the construction. (3) follows from Fubini's theorem.

(4) is a consequence of Stokes' theorem; indeed: for every $\alpha \in \Omega^{k+d}(X)$ and $\beta \in \Omega^\bullet(B)$

$$\begin{aligned}
\int_B (p_*\mathrm{d}\alpha) \wedge \beta &= \int_B p_*(\mathrm{d}\alpha \wedge p^*\beta) \\
&= \int_X \mathrm{d}\alpha \wedge p^*\beta \\
&= \int_{\partial X} \alpha \wedge (\partial p)^*\beta + (-1)^{k+d} \int_X \alpha \wedge p^*\mathrm{d}\beta \\
&= \int_B (\partial p)_*\alpha \wedge p^*\beta + (-1)^{k+d}(p_*\alpha) \wedge \mathrm{d}\beta \\
&= \int_B (\partial p)_*\alpha \wedge p^*\beta + (-1)^d \mathrm{d}(p_*\alpha) \wedge \mathrm{d}\beta.
\end{aligned}$$

■

Definition 2.95. In the situation of Proposition 2.93, the map p_* is the **fibre integration**. •

If $\partial X = \partial B = \varnothing$, then p_* descends to de Rham cohomology $p_* \colon \mathrm{H}^\bullet_{\mathrm{dR}}(X) \to \mathrm{H}^{\bullet-d}_{\mathrm{dR}}(B)$. Set $K^\bullet := \ker p_* \colon \Omega^\bullet(X) \to \Omega^{\bullet-d}(B)$. The short exact sequence

$$0 \to K^\bullet \to \Omega^\bullet(X) \to \Omega^{\bullet-d}(B) \to 0$$

induces a long exact sequence

$$\cdots \to \mathrm{H}^k(K^\bullet) \to \mathrm{H}^k_{\mathrm{dR}}(X) \to \mathrm{H}^{k-d}_{\mathrm{dR}}(B) \xrightarrow{\delta} \mathrm{H}^{k+1}(K^\bullet) \cdots.$$

Whether this is useful or not depends on whether one can compute $\mathrm{H}^k(K^\bullet)$. It might help to observe that $p^*\Omega(B) \subset K$.

2.11 The Gysin sequence

Definition 2.96. Let $p\colon X \to B$ be a proper fibre bundle of relative dimension $d > 0$. A **relative volume form** on p is a nowhere-vanishing section $\mathrm{vol}_{X/B} \in \Gamma(X, \Lambda^d V_p^*)$. A **relative probability form** is a relative volume form $\mathrm{vol}_{X/B}$ with $p_*\mathrm{vol}_{X/B} = 1 \in C^\infty(B)$. •

Given a relative probability form $\mathrm{vol}_{X/B}$, is there an $\eta \in \Omega^d(x)$ satisfying

(1) $\eta|_{p^{-1}(b)} = \mathrm{vol}_{X/B}|_{p^{-1}(b)}$ for every $b \in B$, and

(2) $\mathrm{d}\eta = 0$?

Certainly, it possible to choose η satisfying the first condition and this defines an element $[\eta] \in E_1^{0,d}$. In fact, this depends only on $\mathrm{vol}_{X/B}$. Whether the second condition can be satisfied is asking if η can be chosen so that $[\eta]$ lifts to $E_\infty^{0,d}$. $[\eta]$ lifts to $E_2^{0,k}$ if and only if $[\mathrm{vol}_{X/B}] \in \mathrm{H}^0(B, \mathscr{H}^k_{\mathrm{dR}}(p))$ if and only if $p_*\mathrm{vol}_{X/B}$ is locally constant. There Leray–Serre spectral sequence gives a sequence of obstructions. The Hopf bundle $p\colon S^3 \to S^2$ shows that these obstructions can be non-trivial.

If the fibres of p are rational homology spheres, i.e., $\mathrm{H}_{\mathrm{dR}}(p^{-1}(b)) \cong \mathrm{H}_{\mathrm{dR}}(S^d)$, then the above obstructions can be understood more concretely.

Definition 2.97. A **rational homology sphere bundle of relative dimension** d is a proper fibre bundle $p\colon X \to B$ such that, for every $b \in B$, $\mathrm{H}_{\mathrm{dR}}(p^{-1}(b)) \cong \mathrm{H}_{\mathrm{dR}}(S^d)$. •

Henceforth, assume the above.

The E_2 page ($\cong E_{d+1}$ page) of the Leray–Serre spectral sequence is $\mathrm{H}^\bullet_{\mathrm{dR}}(B) \otimes \mathrm{H}^\bullet_{\mathrm{dR}}(S^d)$; indeed, the choice of a relative probability form trivialises $\mathscr{H}^d_{\mathrm{dR}}(p)$ and $\mathscr{H}^0_{\mathrm{dR}}(p)$ is always trivial.

Definition 2.98. Consider a rational homology sphere bundle of relative dimension k, $p\colon X \to B$, together with a relative probability form $\mathrm{vol}_{X/B}$. Consider the Lerray–Serre spectral sequence of p. Evidently, By degree considerations, $[\mathrm{vol}_{X/B}]$ induces an $[\eta] \in E_{d+1}^{0,d}$. The **Euler class of** p is

$$e(p) \coloneqq \mathrm{d}_{d+1}[\eta] \in E_{d+1}^{d+1,0} \cong \mathrm{H}^{d+1}_{\mathrm{dR}}(B).$$

This is independent of the choice of $\mathrm{vol}_{X/B}$, but depends on the orientation. •

Example 2.99. Suppose $p\colon X \to B$ is a fibre bundle with S^1 fibres. Choose a relative probability form $\mathrm{vol}_{X/B}$. Let $A \in \mathscr{A}(p)$ be an Ehresmann connection. A induces a unique lift $\eta \in \Omega_A^{0,1}(X)$ of $\mathrm{vol}_{X/B}$. It satisfies

$$\mathrm{d}\eta = \mathrm{d}_A^{1,0}\eta + \mathrm{d}_A^{2,-1}\eta.$$

If A is volume-preserving, that is, $\mathrm{d}_A^{1,0}\mathrm{vol}_{X/B} = 0$, then

$$e(p) = \mathrm{d}_A^{2,-1}\mathrm{vol}_{X/B} = i_{F_A}\mathrm{vol}_{X/B}.$$

Note that $\mathrm{d}_A^{2,-1}\mathrm{vol}_{X/B}$ is horizontal and

$$\mathrm{d}_A^{0,1}\mathrm{d}_A^{2,-1}\mathrm{vol}_{X/B} = \mathrm{d}_A^{2,-1}\mathrm{d}_A^{0,1}\mathrm{vol}_{X/B} = 0.$$

Therefore, $\mathrm{d}_A^{2,-1}\mathrm{vol}_{X/B}$ is the pullback of a form on B (as expected.) This also shows that the flat connection on $ST\Sigma \to \Sigma$ discussed earlier is cannot be volume-preserving. ♠

Example 2.100. Let $V \to B$ be an Euclidean vector bundle of rank $d+1 = 2k+1$. Denote by $S \subset V$ the sphere bundle and by $p\colon S \to B$ the unit-sphere bundle. Consider the anti-podal map $a\colon p \to p$. Pulling-back by a flips is compatible with the filtration and therefore descends to a automorphism on the Leray–Serre spectral sequence. It flips sign of $\mathrm{vol}_{S/B}$ and consequently, $a^*\mathrm{d}_{d+1}[\eta] = \mathrm{d}_{d+1}[a^*\eta] = -\mathrm{d}_{k+1}[\eta]$. However, a^* acts trivially on $E_{k+1}^{0,k}$. ♠

Theorem 2.101 (The Gysin sequence). *The **Gysin sequence***

$$\cdots \to \mathrm{H}^\bullet_{\mathrm{dR}}(B) \xrightarrow{p^*} \mathrm{H}^\bullet_{\mathrm{dR}}(X) \xrightarrow{p_*} \mathrm{H}^{\bullet-d}_{\mathrm{dR}}(B) \xrightarrow{e(p)\wedge\cdot} \mathrm{H}^{\bullet+1}_{\mathrm{dR}}(B)\cdots$$

is exact.

Proof of Theorem 2.101. By direct inspection, the sequence

$$E_\infty^{k,d} \hookrightarrow E_{d+1}^{k,d} \xrightarrow{\mathrm{d}_{d+1}} E_{d+1}^{k+d+1,0} \twoheadrightarrow E_\infty^{n+k+1,0}$$

is exact. The map

$$\mathrm{H}^k_{\mathrm{dR}}(B) \cong E_{d+1}^{k,d} \xrightarrow{\mathrm{d}_{d+1}} E_{d+1}^{k+d+1,0} \cong \mathrm{H}^{n+k+1}_{\mathrm{dR}}(B)$$

is $[\cdot \wedge \mathrm{e}(p)]$. Further inspection shows that the sequence

$$E_\infty^{k+d,0} \hookrightarrow \mathrm{H}^{k+d}_{\mathrm{dR}}(X) \twoheadrightarrow E_\infty^{k,d}$$

is exact; indeed, this is simply the convergence statement for the Leray–Serre spectral sequence. These combine into the Gysin sequence. ■

2.12 Symplectic fibre bundles

Definition 2.102. Let $p\colon X \to B$ be a fibre bundle of relative dimension $2k$. A **relative symplectic structure** is an $\omega \in \Gamma(X, \Lambda^2 V_p^*)$ such that ω^k is a relative volume form and $\mathrm{d}^V\omega = 0 \in \Gamma(X, \Lambda^3 V_p)$. A **symplectic fibre bundle** is a fibre bundle $p\colon X \to B$ together with a relative symplectic form

ω such that $\mathrm{d}_\nabla[\omega] = 0 \in \Omega^1(B, \mathscr{H}^2_{\mathrm{dR}}(p))$. •

Remark 2.103. This is not the usual definition of symplectic fibre bundle [MS98, §6.1] and it is a good exercise to prove that the two version of this concept are, in fact, equivalent. ♣

A relative straight-forward application shows that the above notion agrees with the usual notion in the symplectic literature.

The assumption we made says that $[\omega]$ lifts to $E_2^{0,2}$. Again, it is interesting to ask whether it possible to lift to $E_\infty^{0,2}$ or, equivalently, to find $\Omega \in \Omega^2(X)$ with

(1) $\Omega|_{p^{-1}(b)} = \omega|_{p^{-1}(b)}$ and

(2) $\mathrm{d}\Omega = 0$.

Additionally, one might want to require that Ω itself is symplectic.

If an Ω satisfying the first condition (but possibly not the second) exists it defines a connection $A = A_\Omega$ by the declaring

$$H_A := \ker(i.\Omega\colon\ TX \to V_p^*).$$

By construction $\Omega = \Omega^{0,2} + \Omega^{2,0}$. We have $\mathrm{d}\Omega^{0,2} = \mathrm{d}_A^{1,0}\Omega^{0,2} + \mathrm{d}_A^{2,-1}\Omega^{0,2}$. Observe that $\mathrm{d}_A^{1,0}\Omega^{0,2} = 0$ if and only if $\mathrm{d}\Omega \in F^2\Omega^3(X)$. This means that A is a **symplectic connection**. The term $\mathrm{d}_A^{2,-1}\Omega^{0,2}$ vanishes if and only if A is flat. In general, whether the error $\mathrm{d}_A^{2,-1}\Omega^{0,2}$ can be corrected is a question about the curvature F_A inducing Hamiltonian or just symplectic vector field on the fibres. Whether the further terms can be corrected (iteratively) is controlled by the Leray–Serre spectral sequence.

Example 2.104. Consider the fibre bundle $p\colon\ \mathbf{C}P^3 \to \mathbf{H}P^1 \cong S^4$. $\mathbf{C}P^3$ has a symplectic form, the Fubini–Study form ω_{FS}. The restriction to ω_{FS} to the fibres of p is symplectic. But note that S^4 does not carry a symplectic form. ♠

Example 2.105. Consider the **Hopf surface** $H := S^3 \times S^1 = (\mathbf{C}^2\backslash\{0\})/\mathbf{Z}$ with $k \in \mathbf{Z}$ acting by 2^k. H does not admits a symplectic structure, but the fibre bundle $p\colon\ H \to \mathbf{C}P^1$ admits a relative symplectic structure with fibres diffeomorphic to T^2. ♠

3 Lie groups

In this section I will introduce (review?) the concept of a Lie group, that is, a group in the category of manifolds. For the purpose of this course Lie groups will be a tool and a source of examples of manifolds. The theory of Lie groups is a vast subject and we will not even scrape the surface. A good reference is Bump [Bum13].

3.1 Definition

Definition 3.1. A **Lie group** is a smooth manifold G together with a group structure on G such that the maps $m\colon G \times G \to G$ defined by $m(g, h) := g \cdot h$, and $i\colon G \to G$ defined by $i(g) := g^{-1}$ are smooth. Let G and H be Lie groups. A **Lie group homomorphism** from G to H is a smooth group homomorphism $\rho\colon G \to H$. •

Example 3.2. $S^1 = \mathbf{R}/\mathbf{Z}$, $\mathrm{GL}_n(\mathbf{R})$, $\mathrm{GL}_n(\mathbf{C})$, $\mathrm{O}(n)$, $\mathrm{U}(n)$, $\mathrm{SO}(n)$, $\mathrm{SU}(n)$ are Lie groups. ♠

Example 3.3. Let V be a vector space. If $\omega \in \Lambda^2 V^*$ is a non-degenerate 2–form on V, then $H = H(V, \omega)$, the **Heisenberg group** of (V, ω), is defined by $H := \mathrm{U}(1) \times V$ with the group operation

$$(e^{i\alpha}, v) \cdot (e^{i\beta}, w) := (e^{i\alpha + i\beta + 2\pi i \omega(v,w)}, v + w).$$

H is a Lie group. ♠

Theorem 3.4 (`reference?`). *Let G be a Lie group. Let $H < G$ be a subgroup. If H is closed, then it is a submanifold; hence: H is a Lie group.*

Theorem 3.5 (Yamabe [Yam50]; see also Goto [Got69]). *Let G be a Lie group. Let $H < G$ be a subgroup. If H is path-connected, then H is an immersed submanifold; hence: H is a Lie group.*

3.2 Lie group actions

Definition 3.6. Let X be a smooth manifold. Let G be a Lie group.

(1) A **(left) action** of G on X is a smooth map $L\colon G \times X \to X$ satisfying

$$L(\mathbf{1}, \cdot) = \mathrm{id}_X \quad \text{and} \quad L(g, L(h, \cdot) = L(gh, \cdot).$$

Define $L_g \in \mathrm{Diff}(X)$ by $L_g := L(g, \cdot)$ and abbreviate $g \cdot x = L(g, x)$.

(2) The **orbit** of $x \in X$ is

$$\mathrm{Orb}_G(x) = G \cdot x := \{g \cdot x : g \in G\}.$$

(3) The **stabiliser** of $x \in X$ is

$$\mathrm{Stab}_G(x) = G_x := \{g \in G : g \cdot x = x\}.$$

(4) The action of G on X is **free** if $G_x = \mathbf{1}$ for every $x \in X$.

(5) The action of G on X is **proper** if the map $(L, \mathrm{pr}_X)\colon G \times X \to X \times X$ is proper. •

(6) A **right action** of G on X is a smooth map $R\colon X \times G \to G$ satisfying

$$R(\cdot, \mathbf{1}) = \mathrm{id}_X \quad \text{and} \quad R(R(\cdot, g), h) = R(\cdot, gh).$$

Set $R_g(\cdot) := R(\cdot, g)$ and abbreviate $x \cdot g := R(x, g)$. If R is a right action, then $L(g, x) := R(x, g^{-1})$ defines a left action. The notions orbit, stabiliser, free, proper carry over to right actions in the obvious way.

In this section, actions are assumed to be left actions unless explicitly stated otherwise.

Example 3.7. If G is a Lie group, then G acts on itself on the left by left multiplication $L\colon G \times G \to G$,

$$L(g, h) := g \cdot h.$$

The same formula also defines the action of G on itself on the right by right multiplication $R\colon G \times G$,

$$R(h, g) = h \cdot g.$$

These actions commute and G acts on itself on the left by conjugation $C\colon G \to G \to G$,

$$C(g, h) := ghg^{-1}.$$ ♠

Exercise 3.8. Let G be a Lie group. Let $H < G$ be a closed subgroup. Prove that the action of H on G is free and proper.

Example 3.9. $\mathrm{U}(1)$ acts on S^{2n+1} via $e^{i\alpha} \cdot z = e^{i\alpha} z$. ♠

Example 3.10. Let $\theta \in \mathbf{R}$. $\mathbf{R}$ acts on $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ via $L(t, [x, y]) := [x + t, y + \theta t]$. ♠

3.3 The slice theorem

Definition 3.11. Let X be a manifold. Let G be a Lie group acting on X. A **quotient** of X by G is a smooth manifold X/G together with a smooth G–invariant map $p\colon X \to X/G$ such that every G–invariant map $f\colon X \to Y$ uniquely factors though p. •

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"] & Y \\ X/G. \arrow[ur, dashed] & \end{tikzcd}$$

$$C^\infty(X/G, \cdot) \cong C^\infty(X, \cdot)^G$$

Which actions admit quotients?

Proposition 3.12. *Let X be a manifold. Let G be a Lie group. If G acts freely and properly on X, then it admits a quotient.*

Proof assuming that G is compact. Denote by X/G the topological quotient space and by $p\colon X \to X/G$ the projection map. X/G is paracompact and Hausdorff, and p is open. (Exercise!)

Let $x \in X$. The map $G \to X, g \mapsto gx$ is a proper injective immersion. Therefore, the orbit $G \cdot x \subset X$ is a submanifold. Choose a G–invariant metric g on X. (This is a red herring. The proof requires no Riemannian geometry, but it psychologically helpful.) Identify

$$N_x(G \cdot x) \cong T_x(G \cdot x)^\perp \subset T_xX$$

For $\varepsilon > 0$ set $V_x \coloneqq B_\varepsilon(0) \subset N_x(G \cdot x)$ and define $\jmath_x\colon G \times V_x \to X$ by

$$\jmath_x(g, v) \coloneqq g \exp_x(v).$$

Provided $\varepsilon \ll 1$, $\jmath$ is a G–equivariant embedding. Set $S_x \coloneqq \jmath_x(\{\mathbf{1}\} \times V_x)$ and $U_x \coloneqq p(\tilde{U}_x)$. The map $p|_{S_x}\colon S_x \to U_x$ is a homeomorphism. Define $\phi_x\colon U_x \to V_x$ by

$$\phi_x \coloneqq \mathrm{pr}_{V_x} \circ \jmath_x^{-1} \circ \big(p|_{S_x}\big)^{-1}.$$

The task at hand is to prove that the maps ϕ_x form a smooth atlas. Let $x, y \in X$. $U_x \cap U_y \neq \varnothing$ if and only if $(G \cdot S_x) \cap S_y \neq \varnothing$. By construction, $(g \cdot S_x) \cap S_x = \varnothing$ unless $g = 1$. Therefore, there is a unique map $\gamma_x^y\colon (G \cdot S_x) \cap S_y \to G$ satisfying $\gamma_x^y(z) \cdot z \in S_x$ or, equivalently, $\mathrm{pr}_G \circ \jmath_y^{-1}(\gamma_x^y(z) \cdot z) = \mathbf{1}$. By the implicit function theorem, γ_x^y is smooth. A moment's thought shows that the transition map $\phi_x \circ \phi_y^{-1}$ satisfies

$$\phi_x \circ \phi_y^{-1}(z) = \mathrm{pr}_{V_x} \circ \jmath_x\big(\gamma_x^y(\jmath_y^{-1}(\mathbf{1}, z)) \cdot \jmath_y^{-1}(\mathbf{1}, z)\big).$$

Therefore, it is smooth. This finishes the construction of the smooth atlas on X/G.

The universal property is evident from the construction. ■

Remark 3.13. For non-compact G one first proves that $G \cdot x$ is a submanifold and then produces an S_x in some (quite arbitrary way). ♣

Definition 3.14. A **homogeneous space** is a smooth manifold X together with a transitive G action. •

Proposition 3.15. *If X is a homogeneous space, then the map $G/G_{x_0} \to X$ induced by $g \mapsto g \cdot x_0$ is a diffeomorphism.* ■

Example 3.16. $\mathbf{C}P^n \cong S^{2n+1}/\mathrm{U}(1)$. ♠

Example 3.17. $\mathrm{Gr}_r(\mathbf{R}^n) \cong O(n)/(\mathrm{O}(r) \times \mathrm{O}(n-r))$. ♠

3.4 Lie algebra

Proposition 3.18. *Let G be a Lie group. Denote by*

$$\mathrm{Lie}(G) := \mathrm{Vect}(G)^L := \{\xi \in \mathrm{Vect}(G) : L_g^*\xi = \xi \text{ for every } g \in G\}.$$

the space of left-invariant vector fields on G.

(1) $\mathrm{Lie}(G) \subset \mathrm{Vect}(G)$ *is a Lie subalgebra.*

(2) *For $g \in G$ and $\xi \in \mathrm{Lie}(G)$, $R_g^*\xi \in \mathrm{Lie}(G)$.*

Proof. (1) is obvious. (2) holds because R_g and L_g commute. ■

Definition 3.19. Let G be a Lie group. The **Lie algebra of** G is the Lie algebra of left-invariant vector fields:

$$\mathfrak{g} = \mathrm{Lie}(G) := \mathrm{Vect}(G)^L.$$

The **adjoint representation** $\mathrm{Ad}\colon G \to \mathrm{End}(\mathrm{Lie}(G))$ is defined by

$$\mathrm{Ad}(g)\xi := R_g^*\xi.$$

The **adjoint representation** $\mathrm{ad}\colon \mathrm{Lie}(G) \to \mathrm{End}(\mathrm{Lie}(G))$ is defined by

$$\mathrm{ad}(\xi)\eta := [\xi, \eta].$$

Proposition 3.20. *Let G be a Lie group.*

(1) *The evaluation map* $\mathrm{ev}_{\mathbf{1}}\colon\ \mathrm{Vect}(G)^L \to T_{\mathbf{1}}G$ *is an isomorphism.*

(2) *For* $g \in G$ *and* $\xi \in \mathrm{Vect}(G)^L$

$$\mathrm{Ad}(g)\xi = \mathrm{ev}_{\mathbf{1}}^{-1} \circ T_{\mathbf{1}}C_g \circ \mathrm{ev}_{\mathbf{1}}(\xi).$$

(3) *For* $\xi, \eta \in \mathrm{Vect}(G)^L$

$$T_{\mathbf{1}}\,\mathrm{Ad}(\mathrm{ev}_{\mathbf{1}}(\xi))\eta = [\xi, \eta].$$

(4) *If* $\rho\colon\ G \to H$ *is a Lie group homomorphism, then* $\mathrm{Lie}(\rho)\colon\ \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ *defined by*

$$\mathrm{Lie}(\rho) = \mathrm{ev}_{\mathbf{1}}^{-1} \circ T_{\mathbf{1}}\rho \circ \mathrm{ev}_{\mathbf{1}}$$

is a Lie algebra homomorphism.

Proof. A left-invariant vector field v satisfies

$$v_g = T_{\mathbf{1}}L_g(v_{\mathbf{1}}).$$

Therefore, it is determined by $v_{\mathbf{1}}$. Conversely, the above formula defines a left-invariant vector field. This proves (1).

To prove (2), compute

$$\begin{aligned} \mathrm{ev}_{\mathbf{1}}(R_g^*\xi) &= T_gR_{g^{-1}}(\xi_g) \\ &= T_gR_{g^{-1}}T_{\mathbf{1}}L_g(\xi_{\mathbf{1}}) \\ &= T_{\mathbf{1}}C_g(\xi_{\mathbf{1}}). \end{aligned}$$

To prove (3), observe that

$$\begin{aligned} \mathrm{flow}_\xi^t(g) &= \mathrm{flow}_\xi^t(L_g(\mathbf{1})) \\ &= L_g(\mathrm{flow}_\xi^t(\mathbf{1})) \\ &= R_{\mathrm{flow}_\xi^t(\mathbf{1})}g. \end{aligned}$$

Therefore,

$$\begin{aligned} T_{\mathbf{1}}\,\mathrm{Ad}(\mathrm{ev}_{\mathbf{1}}(\xi))\eta &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} R^*_{\mathrm{flow}_\xi^t(\mathbf{1})}(\eta) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\mathrm{flow}_\xi^t)^*(\eta) \\ &= [\xi, \eta]. \end{aligned}$$

To prove (4), observe that by (2)

$$\mathrm{Ad}(\rho(g)) \circ \mathrm{Lie}(\rho)(\xi) = \mathrm{Lie}(\rho) \circ \mathrm{Ad}(g)(\xi).$$

By (3), this implies that $\mathrm{Lie}(\rho)$ is a Lie algebra homomorphism. ■

The following gadget turns out to be important for us later.

Definition 3.21. Let G be a Lie group. The **Maurer–Cartan form** $\mu \in \Omega^1(G, \mathrm{Lie}(G))$ is defined by

$$\mu_g(\xi) := \mathrm{ev}_{\mathbf{1}}^{-1} \circ T_g L_{g^{-1}}(\xi).$$

•

Proposition 3.22. *Let G be a Lie group.*

(1) *The Maurer–Cartan form μ satisfies $\mu(\xi) = \xi$ for every $\xi \in \mathrm{Lie}(G)$.*

(2) *For every $g \in G$*

$$R_g^* \mu = \mathrm{Ad}(g^{-1}) \circ \mu.$$

(3) *The Maurer–Cartan form μ satisfies the **Maurer–Cartan equation***

$$\mathrm{d}\mu + \tfrac{1}{2}[\mu \wedge \mu] = 0$$

Proof. (1) is obvious.

To prove (2), for $g \in G$ and $\xi \in \mathrm{Lie}(G)$ compute

$$(R_g^* \mu)(\xi) = \mu((R_g)_* \xi) = (R_{g^{-1}})^* \xi = \mathrm{Ad}(g^{-1})\xi.$$

To prove (3), compute

$$\begin{aligned}\big(\mathrm{d}\mu + \tfrac{1}{2}[\mu \wedge \mu]\big)(\xi, \eta) &= \mathscr{L}_\xi(\mu(\eta)) - \mathscr{L}_\eta(\mu(\xi)) - \mu([\xi, \eta]) \\ &\quad + \frac{1}{2}([\mu(\xi), \mu(\eta)] - [\mu(\xi), \mu(\eta)]) \\ &= 0.\end{aligned}$$

■

Exercise 3.23. Let $\rho\colon\ G \to H$ be a Lie group homomorphism. Prove that

$$\rho^* \mu_H = \mathrm{Lie}(\rho) \circ \mu_G.$$

3.5 Exponential map

Definition 3.24. Let G be a Lie group. The **exponential map** $\exp\colon \operatorname{Lie}(G) \to G$ is defined by

$$\exp(\xi) := \operatorname{flow}^1_\xi(\mathbb{1}).$$

•

Exercise 3.25. (1) Prove that exp is well-defined.

(2) Let $\rho\colon G \to H$ be a Lie group homomorphism. Prove that

$$\rho \circ \exp(\xi) = \exp \circ \operatorname{Lie}(\rho)(\xi).$$

(3) Prove that

$$C_g \circ \exp = \exp \circ \operatorname{Ad}_g .$$

Definition 3.26. Let X be smooth manifold. Let G be a Lie group. Let $L\colon G \times X \to X$ be a smooth left action. The **infinitesimal action** of $\operatorname{Lie}(G)$ on X is the map $v = v^L\colon \operatorname{Lie}(G) \to \operatorname{Vect}(X)$ defined by

$$v_\xi(x) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} L_{\exp(t\xi)}(x).$$

•

Proposition 3.27. *Let X be smooth manifold. Let G be a Lie group. Let $L\colon G \times X \to X$ be a smooth left action. Denote by $v\colon \operatorname{Lie}(G) \to \operatorname{Vect}(X)$ the corresponding infinitesimal action.*

(1) *For every $\xi \in \operatorname{Lie}(G)$*

$$L_{\exp(t\xi)} = \operatorname{flow}^t_{v_\xi} .$$

(2) *For every $g \in G$ and $\xi \in \operatorname{Lie}(G)$*

$$v_{\operatorname{Ad}(g)\xi} = L^*_{g^{-1}} v_\xi$$

(3) *The infinitesimal action v is an Lie algebra anti-isomorphism; that is: for every $\xi, \eta \in \operatorname{Lie}(G)$*

$$v_{[\xi,\eta]} = -[v_\xi, v_\eta].$$

Remark 3.28. If R is a right action and L is the corresponding left-action, then $v^R = -v^L$. In particular, v^R is a Lie algebra homomorphism. ♣

Proof. (1) is obvious.

To prove (2), compute

$$\begin{aligned} v_{\mathrm{Ad}(g)\xi}(x) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} L_{g\exp(t\xi)g^{-1}}(x) \\ &= T_{L_g(x)}L_g\left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} L_{\exp(t\xi)}L_{g^{-1}}(x)\right) \\ &= T_{L_g(x)}L_g\Big(v_\xi(L_{g^{-1}}(x))\Big) \\ &= (L_{g^{-1}}^* v_\xi)(x). \end{aligned}$$

To prove (3), differentiate

$$v_{\mathrm{Ad}(\exp(t\xi)\eta} = L_{\exp(-t\xi)}^* v_\eta = \left(\mathrm{flow}_{v_\xi}^t\right)^* v_\eta. \qquad \blacksquare$$

3.6 Haar volume form

Proposition 3.29. *Let G be a Lie group. Set $d := \dim G$. There is a unique left-invariant volume form up to multiplication by a non-zero constant:*

$$\dim \Omega^d(G)^L := \{\nu \in \Omega^d(G) : L_g^*\nu = \nu\} = 1.$$

Definition 3.30. Let G be a Lie group. A **Haar volume form** on G is a left-invariant volume form ν on G. ν is **normalised** if $\int_G \nu = 1$. •

Proof of Proposition 3.29. If $\nu \in \Omega^d(G)$ is left-invariant, then

$$\nu_g = \nu_1 \circ \Lambda^d T_g L_{g^{-1}}.$$

Therefore, ν is uniquely determined by $\nu_1 \in \Lambda^d T_1^* G$. Conversely, every $\nu_1 \in \Lambda^d T_1^G$ determines a left-invariant $\nu \in \Omega^d(G)$. ■

Exercise 3.31. Let G be a Lie group. Let ν be a Haar volume form on G. For every $g \in G$, $R_g^*\nu$ is a Haar volume form. The **modular function** of G is the function $\Delta \in C^\infty(G, \mathbf{R}^\times)$ defined by

$$\Delta(g) := \frac{R_g^*\nu}{\nu}.$$

(1) Prove that $\Delta = 1$ if and only if G admits a right-invariant Haar measure. These groups are **unimodular**.

(2) Prove that $\Delta\colon\ G \to \mathbf{R}^\times$ is a Lie group homomorphism.

(3) Prove that $\Delta = 1$ if G is compact.

(4) Define $i\colon\ G \to G$ by $i(g) := g^{-1}$. Prove that $i^*\nu = \Delta\nu$.

(5) Consider the Lie group

$$G := \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} : x > 0, y \in \mathbf{R} \right\}.$$

Compute modular function of G.

3.7 The Killing form

Definition 3.32. Let $\mathfrak{g}$ be a Lie algebra. The **Killing form** $B \in S^2\mathfrak{g}^*$ is defined by

$$B(\xi, \eta) := \operatorname{tr}(\operatorname{ad}(\xi) \circ \operatorname{ad}(\eta)).$$ •

Exercise 3.33. Prove that

$$B([\xi, \eta], \zeta) = B(\eta, [\xi, \zeta]).$$

Definition 3.34. A Lie algebra is called **semisimple** if B is negative definite. G **semisimple** if $\operatorname{Lie}(G)$ is semisimple. •

Exercise 3.35. Prove that if $\mathfrak{g} = \mathfrak{gl}(n)$, then

$$B(\xi, \eta) = 2n \operatorname{tr}(\xi\eta) - 2 \operatorname{tr}(\xi) \operatorname{tr}(\eta).$$

Exercise 3.36. Prove that if $\mathfrak{g} = \mathfrak{su}(n)$, then

$$B(\xi, \eta) = 2n \operatorname{tr}(\xi\eta).$$

3.8 de Rham cohomology of manifolds with G–actions

Let X be a manifold. Let G be a Lie group. Let $L\colon G \times X \to X$ be an action. Such an action can tremendously simplify the computation of $\mathrm{H}^\bullet_{\mathrm{dR}}(X)$. To see this define

$$\Omega^\bullet(X)^L := \{\alpha \in \Omega^\bullet(X) : L_g^*\alpha = \alpha \text{ for every } g \in G\}.$$

Exercise 3.37. Prove that $\mathrm{d}\Omega^\bullet(X)^L \subset \Omega^\bullet(X)^L$.

We have an inclusion of cochain complexes $i\colon \Omega^\bullet(X)^L \to \Omega^\bullet(X)$.

Proposition 3.38. *If G is connected and compact, then i induces an isomorphism $H^\bullet(i)\colon \mathrm{H}^\bullet(\Omega^\bullet(X)^L) \cong \mathrm{H}^\bullet_{\mathrm{dR}}(X)$.*

Proof. Let ν be a normalized Haar volume form on G. Define $\mathrm{av}\colon \Omega^\bullet(X) \to \Omega^\bullet(X)^L$ by

$$\mathrm{av}(\alpha) \coloneqq \int_G L_g^*\alpha\, \nu(g)$$

This is a cochain map and

$$\mathrm{av} \circ i = \mathrm{id}_{\Omega^\bullet(X)^L};$$

hence, $H^\bullet(\mathrm{av})$ is a left inverse of $H^\bullet(i)$.

To show that $H^\bullet(\mathrm{av})$ also is a right inverse we proceed as follows. Denote by

$$\int_G\colon\ \Omega^\bullet(G \times X) \to \Omega^{\bullet - \dim G}(X)$$

the fibre integration map defined by the property that

$$\left(\int_G \alpha\right)_x (v_1, \ldots, v_k) = \int_{g \in G} \alpha_{(g,x)}(v_1, \ldots, v_k, \ldots)$$

for $(v_1, \ldots, v_k) \in T_x M$.

Exercise 3.39. Verify that this is a chain map.

We can now write $i \circ \mathrm{av}$ as the composition

$$i \circ \mathrm{av} = \mathrm{av}_\nu \coloneqq \int_G \circ (\nu \wedge \cdot) \circ L^*.$$

$H^\bullet(\nu \wedge \cdot)$ depends only on $[\nu]$; hence, if $\eta \in \Omega^{\dim G}(G)$ with $\int_G \eta = 1$, i.e., $[\mu] = [\eta]$, then $H^\bullet(i) \circ H^\bullet(\mathrm{av}) = H^\bullet(\mathrm{av}_\eta)$.

Heuristically, if we could take η to be a δ volume form at $\mathbf{1} \in G$, then $\mathrm{av}_\delta = \mathrm{id}_{\Omega^\bullet(X)}$; and as the induced map on cohomology does not change when as η becomes closer and closer to δ the proof is complete. It is not terribly difficult to make the above heuristic rigorous, but we will follow the standard route and chose η supported in neighbourhood U of $\mathbf{1} \in G$ which is smoothly contractible.

If $j\colon U \times X \to G \times X$, then

$$\mathrm{av}_\eta = \int_U \circ (\eta \wedge \cdot) \circ j^* \circ L^*.$$

Since U is smoothly contractible, on cohomology $j^* \circ \rho^* = (\rho \circ j)^*$ is the same as pulling back via the projection $\mathrm{pr}_X\colon U \times X \to X$. However,

$$\int_U \circ (\eta \wedge \cdot) \circ \mathrm{pr}_X^* = \mathrm{id}_{\Omega^\bullet(X)}. \qquad \blacksquare$$

Remark 3.40. The advantage of not following the heuristic argument, is that one can (at least in principle) write down a chain homotopy h such that

$$i \circ \mathrm{av} - \mathrm{id} = \mathrm{d} \circ h + h \circ \mathrm{d}.$$

♣

Let us now use Proposition 3.38 to compute the de Rham cohomology in a few simple cases.

Example 3.41. $G = \mathrm{SO}(n+1)$ acts transitively on S^n. The stabilizer of any $x \in S^n$ is $\mathrm{SO}(T_x S^n) \cong \mathrm{SO}(n)$. A moments thought shows that

$$\begin{aligned}\Omega^\bullet(S^n)^G &= (\Lambda^* T_x S^n)^{\mathrm{SO}(T_x S^n)} \\ &= (\Lambda^*(\mathbf{R}^n)^*)^{\mathrm{SO}(n)} \\ &= \mathbf{R} \cdot 1 \oplus \mathbf{R} \cdot \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n \\ &= \mathbf{R}[0] \oplus \mathbf{R}[n].\end{aligned}$$

The differential necessarily vanishes (for dimension reasons if $n > 1$); hence, this already is $\mathrm{H}^\bullet_{\mathrm{dR}}(S^n)$. The last step in the above computation is a fact from the representation theory of $\mathrm{SO}(n)$.

♠

Example 3.42. $G = \mathrm{U}(n+1)$ acts transitively on $\mathbf{C}P^n$ with stabiliser of any $\mathbf{C} \cdot z \in \mathbf{C}P^n$ isomorphic to $\mathrm{U}(z^\perp) = \mathrm{U}(T_z \mathbf{C}P^n) \cong \mathrm{U}(n)$. We compute

$$\Omega^\bullet(\mathbf{C}P^n)^{U(n+1)} \otimes \mathbf{C} = (\Lambda^*(\mathbf{C}^n)^*)^{\mathrm{U}(n)}.$$

The latter is generated as a $\mathbf{C}$−*algebra* by the standard symplectic form

$$\omega := \sum_{i=1}^n \frac{i \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i}{2};$$

that is,

$$\begin{aligned}(\Lambda^*(\mathbf{C}^n)^*)^{\mathrm{U}(n)} &= \mathbf{C} \cdot 1 \oplus \mathbf{C} \cdot \omega \oplus \cdots \oplus \mathbf{C} \cdot \omega^n \\ &= \mathbf{C}[\omega]/(\omega^{n+1}).\end{aligned}$$

Since this complex is supported in even degrees, the differential vanishes and this already is $H^\bullet_{\mathrm{dR}}(\mathbf{C}P^n) \otimes \mathbf{C}$.

♠

Example 3.43. Let G be a Lie group. Let $\mathfrak{g} := \mathrm{Lie}(G)$. If we consider the L action of G on itself, then

$$\Omega^\bullet(G)^L = \Lambda^* \mathfrak{g}^* = \mathrm{Hom}(\Lambda^* \mathfrak{g}, \mathbf{R}).$$

The differential, which is usually denoted by δ, does not vanish. It can be computed to be

$$\begin{aligned}(\delta\alpha)(\xi_1,\ldots,x_{k+1}) &= \sum_{i=1}^{k+1}(-1)^{i+1}\xi_i\cdot\alpha(\xi_1,\ldots,\widehat{\xi_i},\ldots,\xi_{k+1})\\ &\quad+\sum_{i<j=1}^{k+1}(-1)^{i+j}\alpha([\xi_i,\xi_j],\xi_1,\ldots,\widehat{\xi_i},\ldots,\widehat{\xi_j},\ldots,\xi_{k+1});\end{aligned}$$

in fact, since the Lie algebra acts trivially on $\mathbf{R}$ the first term vanishes. $(\mathrm{Hom}(\Lambda^*\mathfrak{g},\mathbf{R}),\delta)$ is the **Chevalley–Eilenberg cochain complex** (although it was discovered decades before Chevalley–Eilenberg by Cartan). It is defined for every Lie algebra $\mathfrak{g}$. Its cohomology

$$\mathrm{H}^\bullet(\mathfrak{g}) \coloneqq \mathrm{H}^\bullet(C^\bullet(\mathfrak{g}),\delta).$$

is the **Lie algebra cohomology** of $\mathfrak{g}$.

If V is any representation of $\mathfrak{g}$, then δ as defined above makes $\mathrm{Hom}(\Lambda^*\mathfrak{g},M)$ into a cochain complex. $\mathrm{H}^\bullet(\mathfrak{g};V) \coloneqq \mathrm{H}^\bullet(\mathrm{Hom}(\Lambda^*\mathfrak{g},V))$ is called the **Lie algebra cohomology** of $\mathfrak{g}$ with coefficients in V. Proposition 3.38 shows that $\mathrm{H}^\bullet_{\mathrm{dR}}(G) = \mathrm{H}^\bullet(\mathfrak{g};\mathbf{R})$. The notion of Lie algebra cohomology goes back to Chevalley and Eilenberg [CE48]. ♠

Remark 3.44. Let $\rho\colon G\to\mathrm{GL}(V)$ be a Lie group representation. Consider the trivial vector bundle $\mathrm{pr}_G\colon \underline{V} = G\times V\to G$. G acts on $\underline{V}$ by $L\times\rho$. This turns $\underline{V}$ into a G–equivariant vector bundle. The formula $\mathrm{d}_\nabla s \coloneqq \mathrm{d}s + (\mathrm{Lie}(\rho)\circ\mu)s$ defines a G–equivariant flat connection on $\underline{V}$. A moment's thought shows that $\mathrm{H}^\bullet_{\mathrm{dR}}(G,\underline{V}) = \mathrm{H}^\bullet(\mathfrak{g},V)$. ♣

Example 3.45. Let G be a connected compact Lie group. Let $H < G$ be a connected closed Lie subgroup. Set $\mathfrak{g} \coloneqq \mathrm{Lie}(G)$ and $\mathfrak{h} \coloneqq \mathrm{Lie}(H)$. Set $C^\bullet(\mathfrak{g}) \coloneqq \mathrm{Hom}(\Lambda^\bullet\mathfrak{g},\mathbf{R})$ and define δ as above. Denote by $C^\bullet(\mathfrak{g},\mathfrak{h})$ the subcomplex of those $\alpha\in C^\bullet(\mathfrak{g})$ with

$$i_\xi\alpha = 0 \quad\text{and}\quad i_\xi\delta\alpha = 0 \quad\text{for every}\quad \xi\in\mathfrak{h}.$$

The **relative Lie algebra cohomology** of $\mathfrak{g}\supset\mathfrak{h}$ is

$$\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{h}) \coloneqq \mathrm{H}^\bullet(C^\bullet(\mathfrak{g},\mathfrak{h}),\delta).$$

The adjoint action of the Lie algebra $\mathfrak{h}$ on $\mathfrak{g}$ descends to $\mathfrak{g}/\mathfrak{h}$. Denote by $\mathrm{Hom}(\Lambda^\bullet\mathfrak{g}/\mathfrak{h},\mathbf{R})^\mathfrak{h}$ the corresponding invariant subspace of $\mathrm{Hom}(\Lambda^\bullet\mathfrak{g}/\mathfrak{h},\mathbf{R})$. $\mathrm{Hom}(\Lambda^\bullet\mathfrak{g}/\mathfrak{h},\mathbf{R})^\mathfrak{h}$ can be regarded as a subspace $C^\bullet(\mathfrak{g})$. A moment's thought identifies it as $C^\bullet(\mathfrak{g},\mathfrak{h})$. Moreover, $\mathrm{Hom}(\Lambda^\bullet\mathfrak{g}/\mathfrak{h},\mathbf{R})^\mathfrak{h}\cong\Omega^\bullet(G/H)^H$ and the differentials δ and d agree. Therefore,

$$\mathrm{H}^\bullet_{\mathrm{dR}}(G/H)\cong\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{h}).$$

♠

Exercise 3.46. Show that $\mathrm{H}^1(\mathfrak{g}, \mathbf{R}) = (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*$.

Example 3.47. Set $\tilde{R}(g, h) := hg^{-1}$. If we consider the action $L \times \tilde{R}$ of $G \times G$ on G, then

$$\Omega^\bullet(G)^{L\times\tilde{R}} = (\Lambda^\bullet \mathfrak{g}^*)^{\mathrm{Ad}}.$$

Here Ad denotes the coadjoint action. Suppose $\alpha \in \Omega^k(G)^{L\times\tilde{R}}$, that is, α is left invariant and right invariant. Since derivative of the map $i\colon G \to G, g \mapsto g^{-1}$ is

$$\mathrm{d}_g i = -\mathrm{d}L_{g^{-1}} \circ \mathrm{d}R_{g^{-1}},$$

we have

$$i^*\alpha = (-1)^k\alpha.$$

It follows that

$$\mathrm{d}\alpha = (-1)^k \mathrm{d}i^*\alpha = (-1)^k i^*\mathrm{d}\alpha = -\mathrm{d}\alpha;$$

hence, the differential vanishes on $\Omega^\bullet(G)^{L\times\tilde{R}}$ and

$$\mathrm{H}^\bullet_{\mathrm{dR}}(G) = (\Lambda^*\mathfrak{g}^*)^{\mathrm{Ad}}.$$

The formula

$$\gamma(\xi, \eta, \zeta) := B([\xi, \eta], \zeta)$$

defines an element $\gamma \in (\Lambda^3\mathfrak{g}^*)^{\mathrm{Ad}}$. If G is semisimple, then $\gamma \neq 0$; hence $b_3(G) \geqslant 1$. ♠

Example 3.48. Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation. $\mathrm{Lie}(\rho)\colon \mathfrak{g} \to \mathfrak{gl}(V)$ can be regarded as an element of $\theta_\rho \in \mathfrak{g}^* \otimes \mathfrak{gl}(V)$. Evidently, θ_ρ is invariant under the action induces by Ad and ρ and so is $\theta_\rho^{\wedge k} \in \Lambda^k\mathfrak{g}^* \otimes \mathfrak{gl}(V)$. Therefore, $\mathrm{tr}(\theta_\rho^{\wedge k}) \in \Lambda^k\mathfrak{g}^*$. ♠

Remark 3.49. The multiplication map $m\colon G \times G \to G$ induces a map $\Delta\colon \mathrm{H}^\bullet(G) \to \mathrm{H}^\bullet(G) \otimes \mathrm{H}^\bullet(G)$. This turns $\mathrm{H}^\bullet(G)$ into a **Hopf algebra**. ♣

References

[Alm00] Frederick J. Almgren Jr. *Almgren's big regularity paper.* Vol. 1. World Scientific Monograph Series in Mathematics. *Q*-valued functions minimizing Dirichlet's integral and the regularity of area-minimizing rectifiable currents up to codimension 2, With a preface by Jean E. Taylor and Vladimir Scheffer. World Scientific Publishing Co., Inc., River Edge, NJ, 2000, pp. xvi+955 (cit. on p. 7)

[Bum13] D. Bump. *Lie groups.* Graduate Texts in Mathematics 225. Springer, 2013. DOI: 10.1007/978-1-4614-8024-2. MR: 3136522. Zbl: 1279.22001 (cit. on p. 52)

[CE48] C. Chevalley and S. Eilenberg. *Cohomology theory of Lie groups and Lie algebras. Transactions of the American Mathematical Society* 63 (1948), pp. 85–124. MR: 0024908 (cit. on p. 63)

[dHoy16] M. del Hoyo. *Complete connections on fiber bundles. Indagationes Mathematicae. New Series* 27.4 (2016), pp. 985–990. DOI: 10.1016/j.indag.2016.06.009. arXiv: 1512.03847. Zbl: 1346.53019 (cit. on p. 32)

[Ehr51] C. Ehresmann. *Les connexions infinitésimales dans un espace fibré différentiable. Colloque de topologie (espaces fibrés), Bruxelles, 1950.* Georges Thone, Liège; Masson & Cie, Paris, 1951, pp. 29–55 (cit. on p. 27)

[Ful95] W. Fulton. *Algebraic topology. A first course.* Graduate Texts in Mathematics 153. Springer, 1995. Zbl: 0852.55001 (cit. on p. 2)

[Got69] M. Goto. *On an arcwise connected subgroup of a Lie group. Proc. Amer. Math. Soc.* 20 (1969), pp. 157–162. MR: 0233923 (cit. on p. 52)

[GH94] P. Griffiths and J. Harris. *Principles of algebraic geometry.* Wiley Classics Library. Reprint of the 1978 original. New York: John Wiley & Sons Inc., 1994, pp. xiv+813. MR: 1288523 (cit. on p. 47)

[Hat02] A. Hatcher. *Algebraic topology.* Cambridge University Press, 2002. MR: 1867354. Zbl: 1044.55001 (cit. on p. 2)

[Hus94] D. H. Husemoller. *Fibre bundles.* Graduate Texts in Mathematics 20. Springer, 1994. DOI: 10.1007/978-1-4757-2261-1. MR: 1249482. Zbl: 0794.55001 (cit. on p. 27)

[Jän05] K. Jänich. *Topologie.* Berlin: Springer, 2005. Zbl: 1057.54001 (cit. on p. 2)

[KMS93] I. Kolář, P.W. Michor, and J. Slovák. *Natural operations in differential geometry.* Springer, 1993. DOI: 10.1007/978-3-662-02950-3. Zbl: 0782.53013. (cit. on pp. 27, 37)

[May99] J.P. May. *A concise course in algebraic topology.* University of Chicago Press, 1999. Zbl: 0923.55001. (cit. on p. 2)

[MS98] Dusa McDuff and Dietmar A. Salamon. *Introduction to symplectic topology.* Oxford Mathematical Monographs. Oxford University Press, 1998. MR: 1698616. Zbl: 1066.53137 (cit. on p. 51)

[Mil97] J. W. Milnor. *Topology from the differentiable viewpoint.* Princeton Landmarks in Mathematics. Based on notes by David W. Weaver, Revised reprint of the 1965 original. Princeton University Press, 1997, pp. xii+64. MR: 1487640 (cit. on p. 5)

[MS74] J. W. Milnor and J. D. Stasheff. *Characteristic classes.* Annals of Mathematics Studies, No. 76. Princeton University Press; University of Tokyo Press, 1974. MR: 0440554. Zbl: 0298.57008 (cit. on p. 35)

[Ste51] N. Steenrod. *The Topology of Fibre Bundles.* 14. Princeton University Press, 1951. MR: 0039258. Zbl: 0054.07103 (cit. on p. 27)

[Wei94] C. A. Weibel. *An introduction to homological algebra.* Vol. 38. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1994, pp. xiv+450. DOI: 10.1017/CBO9781139644136. MR: 1269324 (cit. on p. 42)

[Wol11] Joseph A. Wolf. *Spaces of constant curvature*. English. 6th ed. Providence, RI: AMS Chelsea Publishing, 2011. Zbl: 1216.53003 (cit. on p. 6)

[Yam50] H. Yamabe. *On an arcwise connected subgroup of a Lie group*. *Osaka Math. J.* 2 (1950), pp. 13–14. MR: 0036766 (cit. on p. 52)

Index

A–horizontal, 31
f–related, 40
k–th symmetric power of X, 7
(left) action, 52

adjoint representation, 55
arrow category, 8
Artin presentation, 15
associated $\mathbf{Z}^2$–graded complex, 43
associated covering map, 24
associated covering map functor, 24

bi-grading on $\Omega(X)$ induced by A, 36
Bianchi identity, 37, 40
bounded, 43
bouquet of circles, 26
braid group on k–strands, 15

cartesian, 9
cartesian lift, 9
category of G–orbits, 23
category of covering maps, 9
category of covering maps of B, 8
category of fibre bundles, 29
category of groupoids, 14
category of pointed topological spaces, 17
characteristic subgroup functor, 18
characteristic subgroup of (p, x), 17
Chevalley–Eilenberg cochain complex, 63
complete
    Ehresmann connection, 32
configuration space, 7
connected, 26
connects, 25
covering map, 2
covering space action, 6
curvature
    of an Ehresmann connection, 33
cycle, 26

deck transformation group, 19
degenerates, 43
degree
    of a covering map, 3
descending filtration, 43
double complex, 44

edges, 25
Ehresmann connection, 30
Euler angles, 4
Euler class of p, 49
evaluation functor, 16
evaluation map, 12, 31
exponential map, 58

fat diagonal, 7
fibre bundle, 27
fibre integration, 48
fibre of p over b, 27
fibre orientation, 47
fibre transport functor, 16
fibre transport of p, 16
fibred category, 9
flat, 34
forest, 26
frame bundle of E, 28
free, 25, 53
free group
    on S, 25
Fröhlicher–Nijenhuis bracket, 39
fundamental group (functor), 14
fundamental groupoid (functor), 14
fundamental groupoid of X, 14
fundamental theorem of calculus, 5

Galois, 20
Gauß–Manin local system, 46
gimbal lock, 5
graded derivation of degree k, 37
graded Leibniz rule, 37
graph, 25
Grassmannian of r–planes in V, 28
groupoid, 14
Gysin sequence, 50

Haar volume form, 59
Heisenberg group, 52
holonomy group of A based at b, 33
homogeneous space, 55
homotopy lifting property, 13
Hopf algebra, 64
Hopf bundle, 27
Hopf surface, 51
horizontal, 30
horizontal distribution, 33
horizontal filtration, 46
horizontal lifting problem, 31
horizontal path groupoid, 32
Hurewicz fibration, 13

infinitesimal action, 58

Killing form, 60

lens space, 7
Leray–Serre spectral sequence, 46
Lie algebra
  of a Lie group, 55
Lie algebra cohomology, 63
Lie group, 52
Lie group homomorphism, 52
lift
  of f along p, 11
lifting problem, 11
logarithm, 6, 11

mapping torus, 47
Maurer–Cartan equation, 57
Maurer–Cartan form, 57
modular function, 59
monodromy representation, 17
monodromy representation (functor) at b, 16
multi-valued holomorphic function, 6

Nijenhuis tensor, 39
normal, 20
normalised, 59

orbit, 52

parallel transport, 33
path, 26
path groupoid functor, 32
path groupoid of X, 32
path-preserving, 18
principal, 20
product connection, 30
proper, 53
pull-back
  of Ehresmann connections, 31
pullback of p via f, 10
pullback via f, 29

quaternionic Hopf bundle, 27
quaternions, 4
quotient, 53

rank
  of a free group, 25
rational homology sphere bundle of relative dimension d, 49
reconstruction functor, 23
regular part of the k–th symmetric power of X, 7
relative Lie algebra cohomology, 63
relative probability form, 49
relative symplectic structure, 50
relative volume form, 49
Riemann–Hopf–Killing theorem, 6
right action, 53
round metric, 6

semi-locally simply-connected, 21
semisimple, 60
set-theoretic reconstruction functor, 21
slice category, 7
spectral sequence, 42
sphere bundle, 28
spherical space form, 6
square-root, 6
stabiliser, 53
subgraph, 26
symplectic connection, 51
symplectic fibre bundle, 50

three-axis gimbal, 4
topological realisation, 25
total complex, 44
total space
  of a fiber bundle, 27
transport topology, 22
tree, 26
trivial covering map, 2
trivial fibre bundle, 27

unimodular, 59
universal, 24
universal set of roots, 3
unordered configuration space, 7

vertical tangent bundle
  of a fibre bundle, 29
vertices, 25

winding number map, 14