Differential Geometry III: Gauge Theory
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Contents

1 Covering maps

The purpose of this section is to *review* the theory of covering maps. Here are some classical (popular?) l references: Jänich [Jän05, Kapitel 9], Fulton [Ful95, Parts VI and VII], May [May99, §1-§4], and Hatcher [Hat02, §1].

1.1 Introduction

Definition 1.1. A continuous map $p\colon X \to B$ is a **covering map** if for every $b \in B$ there are an open subset $U \subset B$ with $b \in U$, a discrete space D, and a homeomorphism $\tau\colon p^{-1}(U) \to U \times D$ such that

$$\mathrm{pr}_1 \circ \tau = p|_{p^{-1}(U)}. \qquad \bullet$$

Here are some examples (which might not be sufficiently convincing).

Example 1.2. Let B be a topological space. Let D be a discrete space. The projection map $\mathrm{pr}_1\colon B \times D \to B$ is a covering map: the **trivial covering map of** B **with fibre** D. ♠

Example 1.3. The exponential map $\exp\colon \mathbf{C} \to \mathbf{C}^\times$ is a covering map. ♠

Example 1.4. The cosine map $\cos\colon \mathbf{C}\backslash\pi\mathbf{Z} \to \mathbf{C}\backslash\{\pm 1\}$ is a covering map. ♠

Example 1.5. Consider $S^1 := [0,1]/\{0,1\}$.

(1) The map $p_\infty\colon \mathbf{R} \to S^1$ defined by $p_\infty(x) := [\lfloor x \rfloor]$ is a covering map.

(2) For every $k \in \mathbf{N}_0$, the map $p_k\colon S^1 \to S^1$ defined by $p_k([x]) := [\lfloor kx \rfloor]$ is a covering map. ♠

The degree of a covering map Here are a straight-forward observation and another example.

Definition 1.6. Let $p\colon X \to B$ be a proper covering map. The **degree of** p is the map $\deg_\cdot(p)\colon B \to \mathbf{N}_0$ defined by

$$\deg_b(p) := \#p^{-1}(b). \qquad \bullet$$

Proposition 1.7. *Let $p\colon X \to B$ be a proper covering map. The map $\deg_\cdot(p)$ is locally constant.* ■

Example 1.8.

(1) Denote by

$$\mathrm{Poly} \cong \coprod_{d\in\mathbf{N}_0} \mathbf{C}^\times \times \mathbf{C}^d$$

the space of complex polynomials. The topology on Poly is choosen so that map $\mathrm{pdeg}\colon \mathrm{Poly} \to \mathbf{N}_0$ which assigns to every polynomial its degree is continuous. Consider the **universal set of roots**

$$\mathrm{Roots} := \{(p,z) \in \mathrm{Poly} \times \mathbf{C} : p(z) = 0\}.$$

The projection map $q\colon \mathrm{Roots} \to \mathrm{Poly}$ is a *not a covering map*—it violates Proposition 1.7: $p_\varepsilon(z) := z^2 + \varepsilon$ has a unique root if $\varepsilon = 0$, but 2 distinct roots if $\varepsilon \neq 0$.

(2) The above issue is easily rectified (or rather ignored) as follows. Denote by $\mathrm{Poly}^\circ \subset \mathrm{Poly}$ the open subset of those $p \in \mathrm{Poly}$ with $\mathrm{pdeg}(p) = \#p^{-1}(0)$ roots or, equivalently, with non-zero discriminant. Set $\mathrm{Roots}^\circ := \mathrm{Roots} \cap (\mathrm{Poly}^\circ \times \mathbf{C})$. The restriction

$$q^\circ := q|_{\mathrm{Roots}^\circ}\colon \mathrm{Roots}^\circ \to \mathrm{Poly}^\circ$$

is a covering map. ♠

Covering maps and the regular value theorem In differential geometry, covering map are almost unavoidable because of the following observations.

Proposition 1.9. *If $p\colon X \to B$ is a proper local homeomorphism, then it is a covering map.* ■

Proposition 1.10. *Let $p\colon X \to B$ be a proper equi-dimensional smooth map. Set $B^\circ := B\backslash p(\mathrm{Crit}(p))$ and $X^\circ := p^{-1}(B^\circ)$. The restriction $p^\circ := p|_{X^\circ}\colon X^\circ \to B^\circ$ is a covering map.* ■

Proposition 1.11. *Let $p\colon X \to B$ be a covering map. If B is a smooth manifold, then X admits a unique smooth structure such that p is smooth (indeed: a local diffeomorphism).* ■

Here are two examples to illustrate the above.

Example 1.12. Denote by $\mathbf{H}$ the normed $\mathbf{R}$–algebra of the **quaternions**. Set

$$\mathrm{Sp}(1) := \{q \in \mathbf{H} : |q| = 1\}.$$

The map $\mathrm{Ad}\colon \mathrm{Sp}(1) \to \mathrm{SO}(\mathrm{Im}\,\mathbf{H}) = \mathrm{SO}(3)$ defined by

$$\mathrm{Ad}(q)x := qxq^*$$

is a covering map of degree 2; moreover: it is a Lie group homomorphism. ♠

Remark 1.13. $\mathrm{SO}(3)$ naturally is a submanifold of $\mathbf{R}^{3\times 3} = \mathbf{R}^9$. Since $\dim \mathrm{SO}(3) = 3$, it seems quite wasteful to encode a rotation as a 3×3–matrix and inefficient to compute the composition of two rotations via matrix multiplication. Example 1.12 offer a more parsimonious and efficient solution to this problem—at the expense of a slight over-parametrisation. ♣

Example 1.14.

(1) For a unit vector $v \in S^2 \subset \mathbf{R}^3$ and an angle $\alpha \in S^1 := \mathbf{R}/2\pi\mathbf{Z}$ denote by $R_v(\alpha) \in \mathrm{SO}(3)$ the rotation around v by α. Define $E\colon T^3 := (S^1)^3 \to \mathrm{SO}(3)$ by

$$E(\phi, \theta, \psi) := R_{e_3}(\phi) R_{e_2}(\theta) R_{e_3}(\psi).$$

This is the over-parametrization of $\mathrm{SO}(3)$ by **Euler angles**—in the intrinsic zyz convention. E is proper and surjective, but *not a covering map* because $\mathrm{Crit}(E) \neq \varnothing$.

(2) Proposition 1.10 produces a covering map $E^\circ\colon (T^3)^\circ \to \mathrm{SO}(3)^\circ$ of degree 2; but: not a Lie group homomorphism. ♠

Remark 1.15.

(1) Example 1.14 provides an even more parsimonious encoding of $\mathrm{SO}(3)$ than Example 1.12; however: computing compositions is not straight-forward.

(2) Here is how Example 1.14 arises in practice. T^3 describes the rotations of a **three-axis gimbal**; e.g., a robot arm. The map E encodes the rotation of object suspended in the gimbal effected by the rotations of the gimbal. $\mathrm{Crit}(E)$ is the set of Euler angles for which **gimbal lock** occurs. ♣

Covering maps and Riemann surfaces Let Σ, T be closed Riemann surfaces. If $f\colon \Sigma \to T$ is a non-constant holomorphic map, then $\mathrm{Crit}(f)$ is a finite set. As a consequence, passing to the covering map $f^\circ\colon \Sigma^\circ \to T^\circ$ obtained by Proposition 1.10 does not incur a substantial loss of information.

Proposition 1.16. *Let T be a Riemann surface. Let $B \subset T$ be a finite subset. Set $T^\circ := T\backslash B$. Let $f^\circ\colon \Sigma^\circ \to T^\circ$ be a covering map. There is a closed Riemann surface Σ, a finite subset $R \subset \Sigma$, and a holomorphic map $f\colon \Sigma \to T$ such that $\Sigma\backslash R = \Sigma^\circ$ and $f^\circ = f|_{\Sigma^\circ}$.* ■

It suffices to prove this for $T^\circ = D^\times$. In this case, the result follows from the classification of covering maps of $p\colon \Sigma^\circ \to D^\times$. If Σ° is connected, then every covering map is essentially of the form $D^\times \to D^\times, z \mapsto z^k$.

The above observation is particularly important because of the following foundational result.

Theorem 1.17 (Riemann's existence theorem). *Every closed Riemann surface Σ admits a non-constant holomorphic map $f\colon \Sigma \to \mathbf{C}P^1$.* ■

The proof of Theorem 1.17 is somewhat difficult (certainly by historic standards) and requires some (e.g., analytic) machinery.

Here is a more elementary observation.

Proposition 1.18. *Let Σ, T be closed Riemann surfaces. Let $f\colon \Sigma \to T$ be a holomorphic map. If f is non-constant and T is connected, then f is surjective.* ■

This implies the **fundamental theorem of calculus** as follows; cf. Milnor [Mil97, p.8?]. Let

$$p(z) = \sum_{k=0}^{d} a_k z^k \in \mathbf{C}[z]$$

be a polynomial of degree $d \geqslant 1$. Since $P\colon \mathbf{C}P^1 \to \mathbf{C}P^1$ defined by

$$P([z:w]) := \left[\sum_{k=0}^{d} a_k z^k w^{d-k} : w^d\right]$$

is surjective, p must have a root.

Riemann surfaces rather naturally appear in complex analysis; in particular, through the following concept.

Definition 1.19. A **multi-valued holomorphic function** on $U \subset \mathbf{C}$ is connected Riemann surface $\Sigma \subset U \times \mathbf{C}$ such that $\mathrm{pr}_1\colon \Sigma \to U$ is surjective. ●

Example 1.20. The **square-root** is

$$\mathrm{Sqrt} := \{(z,w) \in \mathbf{C}^\times \times \mathbf{C} : z = w^2\}. \quad ♠$$

Example 1.21. The **logarithm** is

$$\mathrm{Log} := \{(z,w) \in \mathbf{C}^\times \times \mathbf{C} : z = e^w\}. \quad ♠$$

Covering maps and quotients Finally, covering maps arise from quotients by (exceptionally tame) group actions.

Definition 1.22. Let G be group. Let X be a topological space. A right action $X \circlearrowleft G$ is a **covering space action** if every $x \in X$ has an open neighborhood U such that $U \cdot g \cap U \neq \varnothing$ if and only if $g = \mathbf{1} \in G$. ●

Proposition 1.23. *Let $X \circlearrowleft G$ be a covering space action. Let $H < G$ The projection map $p\colon X/H \to X/G$ is a covering map.* ■

Two examples shall suffice to illustrate this.

Example 1.24.

(1) Consider S^n with the **round metric** g; that is: the metric induced by the Euclidean metric on $\mathbf{R}^{n+1}$. The isometry group $\mathrm{Isom}(S^n, g)$ is $\mathrm{O}(n+1)$. If $\Gamma^{\mathrm{op}} < \mathrm{O}(n+1)$ induces a covering space action, then g descends to a Riemannian metric $\check{g}$ on S^n/Γ with sectional curvature $\sec_{\check{g}} = 1$. In fact, by the **Riemann–Hopf–Killing theorem**, every **spherical space form** (that is: a complete Riemannian manifold (X,g) with $\sec_g = 1$) arises from this construction (up to isometry).

(2) It is know which subgroups $\Gamma^{\mathrm{op}} < \mathrm{O}(n+1)$ occur. This is quite difficult and discussed, e.g., in Wolf [Wol11].

(3) Let $n \in \mathbf{N}$. Let $p \in \mathbf{N}$ and $q_1, \dots, q_n \in \mathbf{Z}$ such that

$$\gcd(p, q_i) = 1.$$

Identify $\mathbf{R}^{2n} = \mathbf{C}^n$ and define $\phi \in \mathrm{SO}(2n)$ by

$$\phi(z_1, \dots, z_n) := (e^{2\pi i q_1/p} z_1, \cdots, e^{2\pi i q_n/p} z_1).$$

By construction, the subgroup $\langle \phi \rangle \subset \mathrm{SO}(2n)$ is cyclic of order p and acts freely on S^{2n-1}. The **lens space** $L(p; q_1, \dots, q_n)$ is the quotient

$$L(p; q_1, \dots, q_n) := S^{2n-1}/\langle \phi \rangle.$$

In particular, $L(2, 1, 1) = \mathbf{R}P^3$. The projection map $S^{2n-1} \to L(p; q_1, \dots, q_n)$ is a covering map. ♠

Example 1.25. Let X be a topological space. Let $k \in \mathbf{N}$.

(1) $X^k \circlearrowleft S_k$ via

$$(x_1, \dots, x_k) \cdot \sigma := (x_{\sigma(1)}, \dots, x_{\sigma(k)}).$$

The **k–th symmetric power of** X is

$$\mathrm{Sym}^k(X) := X^k/S_k.$$

The projection map $p\colon\ X^k \to \mathrm{Sym}^k(X)$ is *not a covering map*—with the exception of a few edge cases. This is also called the **configuration space** of k points in X. $\mathrm{Sym}^k(\mathbf{R}^n)$ plays an important role in the study of multi-valued functions, as it appears, e.g., in Almgren [Almoo].

(2) The map p fails to be a covering map along the **fat diagonal** $\Delta \subset X^k$ defined by

$$\Delta := \left\{(x_1, \dots, x_k) \in X^k : \#\{x_1, \dots, x_k\} < k\right\}.$$

The **regular part of the k–th symmetric power of** X is

$$\mathrm{Sym}^k(X)^\circ := (X^k \backslash \Delta)/S_k.$$

The projection map $p^\circ\colon\ X^k \backslash \Delta \to \mathrm{Sym}^k(X)^\circ$ is a covering map. This is also called the **unordered configuration space**.

(3) If $G < S_k$ is a subgroup, then $q\colon\ (X^k \backslash \Delta)/G \to \mathrm{Sym}^k(X)^\circ$ is a covering map. $\mathrm{Sym}^k(X)^\circ$ is the space of subsets $S \subset X$ with $\#S = k$. If $S_{k-1} \cong G < S_k$ is subgroup fixing $1 \in \{1, \dots, k\}$, then $(X^k \backslash \Delta)/G$ is the space of subset $S \subset X$ with $\#S = k$ together with a choice of $x \in S$. For $X = \mathbf{C}$ this (essentially) recovers Example 1.8 (restricted to degree k polynomials). ♠

1.2 The category of covering maps of B

At this point the reader is (hopefully) convinced that the concept of covering map is sufficiently relevant to introduce a category of covering maps.

Definition 1.26. Let $\mathbf{C}$ be a category. Let b be an object of $\mathbf{C}$. The **slice category** is the category $\mathbf{C}/b$ whose objects are morphisms $p\colon x \to b$ in $\mathbf{C}$, and whose morphisms $\phi\colon (p\colon x \to b) \to (q\colon y \to b)$ are morphsisms $\phi\colon x \to y$ in $\mathbf{C}$ satisfying

$$p = q \circ \phi. \qquad \bullet$$

Definition 1.27. Let B be a topological space. The **category of covering maps of** B is the full subcategory $\mathbf{Cov}(B) \subset \mathbf{Top}/B$ whose objects are covering maps $p\colon X \to B$. $\bullet$

Proposition 1.28. $\mathbf{Cov}(B)$ *has products and coproducts.* ■

Example 1.29. $\mathbf{Cov}(\{*\}) \cong \mathbf{Set}$. ♠

Under rather mild connectivity assumptions on B, $\mathbf{Cov}(B)$ can be (essentially) determined algbraically from the fundamental group $\pi_1(B, b)$; see Section 1.10.

Here is an observation about $\mathbf{Cov}(B)$ which is so trivial that it usually is not even mentioned.

Definition 1.30. Let B be a topological space. Denote by $\mathrm{Op}(B)$ the category whose objects are open subset $U \subset B$ and whose morphisms $U \to V$ are inclusions $U \subset B$. Define the functor $\mathbf{Cov}_B\colon \mathrm{Op}(B)^{\mathrm{op}} \to \mathbf{Cat}$ as follows.

(1) For every open subset $U \subset B$ set $\mathbf{Cov}_B(U) \coloneqq \mathbf{Cov}(U)$

(2) For every inclusion of open subset $U \subset V \subset B$, $\mathbf{Cov}_B(U \subset V)\colon \mathbf{Cov}(V) \to \mathbf{Cov}(U)$ is the (obvious) restriction functor. $\bullet$

Proposition 1.31. $\mathbf{Cov}_B$ *is a sheaf; that is: for every open cover* $\{U_i : i \in I\}$ *of* $U \subset B$ *the diagram*

$$\mathbf{Cov}_B(U) \to \coprod_{i\in I} \mathbf{Cov}_B(U_i) \rightrightarrows \coprod_{(i,j)\in I^2} \mathbf{Cov}_B(U_i \cap U_j)$$

is an equaliser. ■

1.3 The fibred category of covering maps

See Vistoli's notes `http://homepage.sns.it/vistoli/descent.pdf` for more on fibred categories.

Definition 1.32. Let $\mathbf{C}$ be a category. The **arrow category** is the category $\mathbf{Arr}(\mathbf{C})$ whose objects are morphisms $p\colon x \to a$ in $\mathbf{C}$, and whose morphisms $(p\colon x \to a) \to (q\colon y \to b)$ are pairs of morphisms $\phi\colon x \to y$ and $f\colon a \to b$ in $\mathbf{C}$ satisfying

$$f \circ p = q \circ \phi. \qquad \bullet$$

Definition 1.33. The **category of covering maps** is the full subcategory $\mathbf{Cov} \subset \mathbf{Arr}(\mathbf{Top})$ whose objects are covering maps $p\colon X \to B$. $\bullet$

The main reason to introduce **Cov** is to talk about pull-packs. This is a good excuse to introduce fibred categories.

Definition 1.34. Let $P\colon \mathbf{X} \to \mathbf{B}$ be a functor. A morphism $\phi\colon x \to y$ is **cartesian** if for every object z of $\mathbf{X}$, every morphism $g\colon P(z) \to P(x)$, and every morphism $\psi\colon z \to x$ with $P(\psi) = P(\phi) \circ g$ there is a unique morphism $\zeta\colon z \to y$ such that

$$\psi = \phi \circ \zeta.$$

that is:

$$\begin{tikzcd} z \arrow[dr, dashed, "\exists!\zeta"] \arrow[drr, bend left, "\psi"] & & \\ P(z) \arrow[dr, "g"] & x \arrow[r, "\phi"] & y \\ & P(x) \arrow[r, "P(\phi)"] & P(y) \end{tikzcd}$$ •

Definition 1.35. A **fibred category** is a functor $P\colon \mathbf{X} \to \mathbf{B}$ such that for every object x of $\mathbf{X}$ and every morphism $f\colon a \to P(x)$ there is a **cartesian lift**; that is: a cartesian morphism $\phi\colon y \to x$ with

$$P(\phi) = f.$$ •

Proposition 1.36. *The codomain functor $U\colon$ **Cov** $\to$ **Top** is a fibred category.*

Proof. Let $p\colon X \to B$ be a be a covering map. Let $f\colon A \to B$ be a smooth map. Set

$$f^*X := \{(a,x) \in A \times X : f(a) = p(x)\} \subset A \times X.$$

Define $\phi\colon f^*X \to X$ by $\phi(a,x) := x$ and $f^*p\colon f^*X \to A$ by $f^*p(a,x) := a$. A moment's thought shows that f^*p is a covering map. Evidently, $(\phi, f)\colon q \to p$ is cartesian. ■

Definition 1.37. For every covering map $p\colon X \to B$ and every continuous map $f\colon A \to B$ *choose* a cartesian lift

$$\begin{tikzcd} f^*X \arrow[r, "p^*f"] \arrow[d, "f^*p"] & X \arrow[d, "p"] \\ A \arrow[r, "f"] & B. \end{tikzcd}$$

Denote this as *the* **pullback of p via f**. •

Remark 1.38. Let $p\colon X \to C$ be a covering map and $f\colon A \to B$, $g\colon B \to C$ be continuous maps. Typically, $(g \circ f)^*p$ is not *equal* to f^*g^*p but there is a *canonical* isomorphism

$$I_{f,g}\colon (g \circ f)^* \cong f^* \circ g^*$$

by the definition of cartesian morphism. In the implementation of pull-backs given above,

$$I_{f,g}(p)\colon (g \circ f)^*X \to f^*g^*X$$

is given by $I_{f,g}(p)(a,x) = (a, (f(a), x))$. ♣

A key fact about $\mathbf{Cov}(B)$ is that it is an invariant of the homotopy-type of B. This can be proved as follows.

Proposition 1.39. *Let $p\colon X \to [0,1]\times B$ be a covering map. For every $b \in B$ there are an open subset $V \subset B$ with $b \in V$, a discrete space D, and a homeomorphism $\tau\colon p^{-1}([0,1]\times V) \to [0,1]\times V\times D$ such that $(\mathrm{pr}_1,\mathrm{pr}_2)\circ\tau = p|_{p^{-1}([0,1]\times V)}$.*

Proof. By Definition 1.1 and since $[0,1]$ is compact, for every $b \in B$ there are an open subset $V \subset B$ with $b \in V$, $0 = t_0 < t_1 < \cdots < t_n = 1$, and for every $i \in \{0,\dots,n-1\}$ a discrete space D_i and a homeomorphism $\tau_i\colon p^{-1}([t_i,t_{i+1}]\times V) \to [t_i,t_{i+1}]\times V\times D_i$ such that $(\mathrm{pr}_1,\mathrm{pr}_2)\circ\tau_i = p|_{p^{-1}([t_i,t_{i+1}]\times V)}$.

Set $\sigma_i := (\mathrm{pr}_2,\mathrm{pr}_3)\circ\tau_{i-1}^{-1}\circ\tau_i(t_i,\cdot)\colon V\times D_i \to V\times D_{i-1}$. Evidently, σ_i is a homeomorphism satisfying $\mathrm{pr}_1 = \mathrm{pr}_1\circ\sigma_i$, and the homeomorphisms $\tilde\tau_i := \sigma_0\circ\cdots\circ\sigma_i\circ\tau_i$ glue to the desired homeomorphism τ. ■

Proposition 1.40. *For every covering map $p\colon X \to [0,1]\times B$ there is a unique isomorphism*

$$\phi\colon \mathrm{pr}_B^* i_0^* p \cong p$$

in $\mathbf{Cov}([0,1]\times B)$ such that $i_0^*\phi\colon i_0^*\mathrm{pr}_2^* i_0^* p \cong i_0^* p$ is the canonical isomorphism in $\mathbf{Cov}(B)$. Here $i_0\colon B \to [0,1]\times B$ is defined by $i_0(b) := (0,b)$.

Proof. If ϕ,ψ are two such isomorphisms, then $(\phi,\psi)^{-1}(\Delta)$ is open and closed in $\mathrm{pr}_2^* i_0^* X \cong [0,1]\times p^{-1}(\{0\}\times B)$ and contains $\{0\}\times p^{-1}(\{0\}\times B)$. Therefore, $\phi = \psi$.

The isomorphisms ϕ exists for $(\mathrm{pr}_1,\mathrm{pr}_2)\colon [0,1]\times B\times D \to [0,1]\times B$. Therefore, by uniqueness and Proposition 1.39, it exists for every covering map $p\colon X \to [0,1]\times B$. ■

Corollary 1.41. *If f_0, f_1 are homotopy-equivalent, then $\mathbf{Cov}(f_0)$, $\mathbf{Cov}(f_1)$ are naturally isomorphic. Moreover, a homotopy equivalence induces an natural isomorphism.* ■

1.4 The lifting problem

Definition 1.42. Let C be a category. Let $p\colon x \to b$ and $f\colon a \to b$ be morphisms in C. A morphism $\tilde f\colon a \to x$ is a **lift of f along p** if

$$f = p\circ\tilde f;$$

that is: the diagram

$$\begin{tikzcd} & x \arrow[d, "p"] \\ a \arrow[ur, dashed, "\tilde f"] \arrow[r, "f"'] & b \end{tikzcd}$$

commutes or, equivalently,

$$\tilde f \in \mathrm{Hom}_{\mathrm{C}/b}(f,p).$$ •

The **lifting problem** is to determine $\mathrm{Hom}_{\mathrm{C}/b}(f,p)$, the set of lifts of f along p. Many questions in geometry and topology are lifting problems. Here are some examples.

Example 1.43. Which open subsets $U \subset \mathbf{C}^\times$ admit a **logarithm**; that is: a lift of $U \subset \mathbf{C}^\times$ along exp? ♠

Example 1.44. Let $P\colon A \to \mathrm{Poly}$ be a continous map. Is is possible to continuously choose roots of P; that is: a lift of P along p in Example 1.8? ♠

Example 1.45. Imagine a robot arm holding a banana. A movement in time of the banana is encoded by a path $\gamma\colon [0,1] \to \mathrm{SO}(3)$. The question of how to operate the robot arm to achieve this path is the lifting problem along E. ♠

Example 1.46. Let $f\colon A \to \mathrm{Sym}^k(X)$ be a k–valued function. Is it possible to lift it to k single-valued functions? ♠

Example 1.47. Via pull-backs the lifting problem can always be reduced to id_A: Indeed, there is a canonical bijection $\mathrm{Hom}_{\mathbf{Top}/B}(f,p) \cong \mathrm{Hom}_{\mathbf{Top}/A}(\mathrm{id}_A, f^*p)$. ♠

Example 1.48. Pullbacks allow trivial solutions of lifting problems. Let $f\colon B \to \mathrm{Sym}^k(X)^\circ$ be a k–valued map. Consider the covering map $p^\circ\colon X^k\backslash\Delta \to \mathrm{Sym}^k(X)^\circ$. The map f might not lift, but $f \circ f^*p^\circ$ canonically lifts: the lift is p^*f. ♠

1.5 The unique homotopy lifting property

This section lays the foundation for solving the lifting problem in Section 1.8. The following asserts that the lifting problem along covering maps is very rigid.

Definition 1.49. Let $p\colon X \to B$ and $f\colon A \to B$ be continous maps. Let $a \in A$. Set $b \coloneqq f(a)$. Define the **evaluation map**

$$\mathrm{ev}_a\colon \mathrm{Hom}_{\mathbf{Top}/B}(f,p) \to p^{-1}(b)$$

by

$$\mathrm{ev}_a(\tilde{f}) \coloneqq \tilde{f}(a).$$ •

Proposition 1.50. *Let $p\colon X \to B$ and $f\colon A \to B$ be continous maps. Let $a \in A$. If p is a covering map and A is connected, then ev_a is injective.*

This is an immediate consequence of the following.

Lemma 1.51. *Let $p\colon X \to B$ be a covering map. Set*

$$X \times_B X \coloneqq \{(x,y) \in X \times X : p(x) = p(y)\} \quad \text{and} \quad \Delta \coloneqq \{(x,x) \in X \times_B X\}.$$

The subset $\Delta \subset X \times_B X$ is open and closed.

Proof. For $x \in X$ denote by V_x an open neighborhood of $x \in X$ such that $p(V_x)$ is open and $p|_{V_x}\colon V_x \to p(V_x)$ is a homeomorphism. $(V \times V) \cap (X \times_B X)$ is an open neighborhood of $(x,x) \in X \times_B X$ and contained in Δ. Therefore, Δ is open.

Let $(x,y) \in (X \times_B X)\backslash\Delta$. Choose V_x, V_y as above with $V_x \cap V_y = \varnothing$. $(V_x \times V_y) \cap X \times_B X$ is an open neighborhood of $(x,y) \in X \times_B X$ and does not intersect Δ. Therefore, Δ is closed. ■

Proof of Proposition 1.50. Suppose $\tilde{f}_1, \tilde{f}_2\colon A \to X$ are lifts of f along p with $\tilde{f}_i(a) = x$. By Lemma 1.51, $S \coloneqq (\tilde{f}_1, \tilde{f}_2)^{-1}(\Delta)$ is open and closed. Since $a \in S$ and A is connected, $S = A$; hence: $\tilde{f}_1 = \tilde{f}_2$. ■

There are obstructions to the lifting problem along covering maps; that is: ev_a need not be a bijection. Here is an example to illustrate this.

Example 1.52. There is *no lift* of id_{S^1} along p_∞ from Example 1.5:

$$\mathrm{Hom}_{\mathbf{Top}/S^1}(\mathrm{id}_{S^1}, p_\infty) = \varnothing.$$

Indeed, if $\tilde{f}\colon S^1 \to \mathbf{R}$ were a lift, then $\tilde{f}(S^1)$ would be compact interval $[a,b]$ and $p_\infty|_{[a,b]}$ would be a bijection. However, $p_\infty|_{[a,b]}$ is injective if and only if $b-a<1$ and surjective if and only if $b-a \geqslant 1$. ♠

However, there are no obstructions to lifting continuous paths (even in families). Here is a precise formulation of this observation.

Definition 1.53. Let A be a topological space. A continuous map $p\colon X \to B$ has the **homotopy lifting property (HLP) with respect to** A if for every homotopy $h\colon [0,1]\times A \to B$ and lift $\tilde{h}_0\colon A \to X$ of $h_0 \coloneqq h(0,\cdot)\colon A \to B$ there is a homotopy $\tilde{h}\colon [0,1]\times A \to X$ which is a lift $\tilde{h}$ of h with $\tilde{h}(0,\cdot) = \tilde{h}_0$; that is: the diagram

$$\begin{tikzcd} A \arrow[r, "\tilde{h}_0"] \arrow[d, hook] & X \arrow[d, "p"] \\ {[0,1]\times A} \arrow[r, "h"'] \arrow[ur, dashed, "\tilde{h}"] & B \end{tikzcd}$$

commutes. •

Definition 1.54. A continuous map $p\colon X \to B$ is a **Hurewicz fibration** if it has the HLP with respect to every topological space. •

Theorem 1.55. *Every covering map $p\colon X \to B$ is a Hurewicz fibration.*

Proof. If D is a discrete space, then $\mathrm{pr}_1\colon B\times D \to B$ is a Hurewicz fibration. Consequently, every $b \in B$ has a neighborhood U such that $p|_{p^{-1}(U)}\colon p^{-1}(U) \to U$ is a Hurewicz fibration.

Let A be a topological space. Let $h\colon [0,1]\times A \to B$ be a homotopy. For every $(t,a) \in [0,1]\times A$ choose a neighborhood $U_{t,a}$ of $h(t,a)$ as above. Since $[0,1]$ is compact, there are $0 = t_0 < t_1 < \cdots < t_n = 1$ and an open neighborhood V_a of $a \in A$ with $[t_i,t_{i+1}]\times V_a \subset h^{-1}(U_{t,a})$ for some $t \in [0,1]$. Let $\tilde{h}_0$ be a lift of $h(0,\cdot)$. Since $p|_{p^{-1}(h([t_i,t_{i+1}]\times V_a))}$ is a Hurewicz fibration, a finite induction argument constructs a lift $\tilde{h}_{V_a}$ of $h|_{[0,1]\times V_a}$ with $\tilde{h}_{V_a}(0,\cdot) = \tilde{h}_0|_{V_a}$.

By Proposition 1.50 and because $[0,1]$ is connected, $\tilde{h}_{V_a}$ and $\tilde{h}_{V_b}$ agree on $[0,1]\times(V_a\cap V_b)$. Therefore, they assemble into a lift $\tilde{h}$ of h with $\tilde{h}(0,\cdot) = \tilde{h}_0$. ■

1.6 The fundamental group(oid)

It is useful to encode the unique homotopy lifting property (Proposition 1.50 and Theorem 1.55) algebraically. This requires the following terminology from algebraic topology.

Definition 1.56. A **groupoid** is a category $\mathbf{G}$ in which every morphism is an isomorphism. The **category of groupoids** is the full subcategory $\mathbf{Gpd} \subset \mathbf{Cat}$ whose objects are groupoids. •

Definition 1.57. The **fundamental groupoid (functor)** is the functor $\Pi_1\colon \mathbf{Top} \to \mathbf{Gpd}$ defined as follows:

(Ob) Let X be a topological space. The **fundamental groupoid of** X is the groupoid $\Pi_1(X)$ whose objects are the elements of X, and whose morphisms $[\gamma]\colon x \to y$ are homotopy classes rel $\{0,1\}$ of continuous paths $\gamma\colon [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$, composed by concatenation.

(Hom) Let $f\colon X \to Y$ be a continuous map. $\Pi_1(f)\colon \Pi_1(X) \to \Pi_1(Y)$ is defined by

$$\Pi_1(f)(x) \coloneqq f(x) \quad \text{and} \quad \Pi_1(f)([\gamma]) \coloneqq [f \circ \gamma]. \qquad \bullet$$

The fundamental groupoid $\Pi_1(X)$ is useful in constructions, e.g., in Section 1.10; however: it is also rather unwiedly. Fortunately, it is possible to drastically compress $\Pi_1(X)$—*essentially* without loss of information.

Definition 1.58. The **fundamental group (functor)** is the functor $\pi_1\colon \mathbf{pTop} \to \mathbf{Grp}$ defined as follows:

$$\pi_1(X,x) \coloneqq \mathrm{Aut}_{\Pi_1(X)}(x) \quad \text{and} \quad \pi_1(f) \coloneqq \Pi_1(f)|_{\pi_1(X,x)}. \qquad \bullet$$

Proposition 1.59. *Let (X,x) be a pointed topological space. X is path-connected if and only if the inclusion $\mathrm{B}\pi_1(X,x) = \Pi_1(X)_x \subset \Pi_1(X)$ is an equivalence of categories.* ■

Example 1.60. Let $[\gamma] \in \pi_1(S^1,[0])$. By Proposition 1.50 and Theorem 1.55, there $[\gamma]$ has a unique lift $[\tilde{\gamma}]$ to a homotopy class rel $\{0,1\}$ of continuous paths $\tilde{\gamma}\colon [0,1] \to \mathbf{R}$ with $\tilde{\gamma}(0) = 0$. Define the **winding number map** $w\colon \pi_1(S^1,[0]) \to \mathbf{Z}$ by

$$w([\gamma]) \coloneqq [\tilde{\gamma}](1).$$

A few moments' thought show that this is a group isomorphism. ♠

Example 1.61. Let $k \in \mathbf{N}$. Consider $\mathrm{Sym}^k(\mathbf{C})^\circ$ from Example 1.25. Set $* \coloneqq [1,2,\dots,k] \in \mathrm{Sym}^k(\mathbf{C})^\circ$. The fundamental group

$$B_k \coloneqq \pi_1(\mathrm{Sym}^k(\mathbf{C})^\circ, *)$$

is the **braid group on k–strands**. It is superficially obvious (but quite cumbersome to prove) that B_k has the **Artin presentation** in terms of generators and relations:

$$B_k \cong \langle \sigma_1,\dots,\sigma_{k-1} : \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ \sigma_i\sigma_j = \sigma_j\sigma_i\ (i-j \geqslant 2)\rangle.$$

Here $\sigma_i \in B_k$ is the obvious element swapping i and $i+1$ in anti-clockwise fashion. ♠

The (obvious) homotopy-invariance of $\pi_1(X,x)$ and the following result are useful for computations.

Theorem 1.62 (Seifert–van Kampen for π_1). *Let (X,x) be a topological space. Let $\mathcal{U}$ be a subcategory of* **pTop** *whose objects are open subsets of X containing x and whose morphisms are inclusions. If the objects of $\mathcal{U}$ cover X and are closed under finite intersections, then*

$$\pi_1(X,x) \cong \operatorname{colim} \pi_1|_{\mathcal{U}}$$

in **Grp**. ■

Example 1.63. The fundamental group of bouquet of k circles is isomorphic to the free group F_k; cf. Section 1.12. ♠

1.7 Fibre transport and monodromy

Here is the desired algebraic formulation of the unique homotopy lifting property (Proposition 1.50 and Theorem 1.55).

Corollary 1.64. *Let $p\colon X \to B$ is a covering map. Let $x \in X$ and $b \in B$. The map*

$$\coprod_{y \in p^{-1}(b)} \mathrm{Hom}_{\Pi_1(X)}(x, y) \to \mathrm{Hom}_{\Pi_1(B)}(p(x), b)$$

induced by $\Pi_1(p)$ is bijective. ■

Corollary 1.65. *If $p\colon X \to B$ is a covering map, then for every $x \in X$ the homomorphism $\pi_1(p)\colon \pi_1(X, x) \to \pi_1(B, p(x))$ is injective.* ■

As a consequence of Corollary 1.64 the following construction is well-defined.

Definition 1.66. The **fibre transport functor**

$$\mathrm{Fib}\colon \mathbf{Cov}(B) \to \Pi_1(B)\text{–}\mathbf{Set} := \mathbf{Fun}(\Pi_1(B), \mathbf{Set})$$

is the functor defined as follows:

(Ob) Let $p\colon X \to B$ be a covering map. The **fibre transport of** p is the functor $\mathrm{Fib}(p)\colon \Pi_1(B) \to \mathbf{Set}$ defined by

$$\mathrm{Fib}(p)(b) := p^{-1}(b) \quad \text{and} \quad \mathrm{Fib}(p)([\gamma])(x) := [\tilde{\gamma}](1)$$

with $[\tilde{\gamma}]$ denoting the unique lift of $[\gamma]$ with $[\tilde{\gamma}](0) = x$.

(Hom) Let $\phi\colon p \to q$ be a morphism of covering map of B. The natural transformation $\mathrm{Fib}(\phi)\colon \mathrm{Fib}(p) \to \mathrm{Fib}(p)$ is defined by

$$\mathrm{Fib}(\phi)_b := \phi|_{p^{-1}(b)}\colon \mathrm{Fib}(p)(b) = p^{-1}(b) \to \mathrm{Fib}(q)(b) = q^{-1}(b). \qquad \bullet$$

Example 1.67. Consider $q^\circ\colon \mathrm{Roots}^\circ \to \mathrm{Poly}^\circ$ from Example 1.8. Consider the continuous loop $\gamma(t)\colon [0,1] \to \mathrm{Poly}^\circ$ defined by $\gamma(t) := z^2 - e^{2\pi i t}$. A moment's thought shows that $\mathrm{Fib}(q)(z^2 - 1) = \{z^2 - 1\} \times \{\pm 1\}$ and $\mathrm{Fib}(q)([\gamma])$ acts by exchanges the two roots. ♠

The fibre transport functor is quite remarkable. It transforms geometric data into algebraic data, and in the process does not lose any information. Indeed, under mild connectivity assumptions, it is an equivalence of categories; see Section 1.10.

Although it is usually not terribly difficult to compute $\mathrm{Fib}(p)([\gamma])$ for concrete p and $[\gamma]$, completely determining $\mathrm{Fib}(p)$ is, at least, very cumbersome. It is useful to compress Fib as follows.

Definition 1.68. Let $b \in B$. The **evaluation functor**

$$\mathrm{ev}_b\colon \Pi_1(B)\text{–}\mathbf{Set} \to \pi_1(B, b)\text{–}\mathbf{Set}$$

is obtained by composition with the inclusion $\mathbf{B}\pi_1(B, b) \hookrightarrow \Pi_1(B)$. •

Corollary 1.69. *Let $b \in B$. If B is path-connected, then ev_b is an equivalence of categories.* ■

Definition 1.70. Let $b \in B$. The **monodromy representation (functor) at b** is

$$\mu_b := \mathrm{ev}_b \circ \mathrm{Fib}\colon\ \mathbf{Cov}(B) \to \pi_1(B,b)\text{–}\mathbf{Set}.$$

The **monodromy representation** of a covering map $p\colon\ X \to B$ is the group homomorphism $\mu_b(p)\colon\ \pi_1(B,b) \to p^{-1}(B)$. •

It is important to understand that the monodromy representation is quite computable in practice and can often be determined very explicitly. Here are two examples.

Example 1.71. Consider $\exp\colon\ \mathbf{C} \to \mathbf{C}^\times$ from Example 1.3. The fundamental group $\pi_1(\mathbf{C}^\times, 1)$ is generated by $[\gamma]$ with $\gamma(t) := e^{2\pi i t}$. Evidently, $\mu_1(\exp) = 2\pi i\mathbf{Z}$ with $[\gamma]$ acting as a shift by $2\pi i$. ♠

Example 1.72. Consider $p^\circ\colon\ \mathbf{C}^k\backslash\Delta \to \mathrm{Sym}^k(\mathbf{C})^\circ$ from Example 1.25. Set $* := [1,2,\ldots,k] \in \mathrm{Sym}^k(\mathbf{C})^\circ$. The monodromy representation $\mu_*(p^\circ)$ is a group homomorphism $\pi_1(\mathrm{Sym}^k(\mathbf{C})^\circ, *) \to S_k = \mathrm{Bij}(\{1,\ldots,k\})$. This is the obvious homomorphism $B_k \to S_k$ in the Artin presentation. ♠

Exercise 1.73. Compute the monodromy of $\cos\colon\ \mathbf{C}\backslash\pi\mathbf{Z} \to \mathbf{C}\backslash\{\pm 1\}$ from Example 1.4.

The monodromy representation can further be understood as follows.

Proposition 1.74. *Let $b \in B$. Let $p\colon\ X \to B$ be a covering map. If X is path-connected, then $\mu_b(p)\colon\ \pi_1(B,b) \to \mathrm{Bij}(p^{-1}(b))$ is transitive.* ■

If is a *transitive* left action $G \circlearrowleft S$ and $s \in S$, then $G \circlearrowleft S$ is isomorphic to $G \circlearrowleft G/\mathrm{Stab}_G(s)$. Therefore,

Definition 1.75. Let $p\colon\ X \to B$ be a continuous map. Let $x \in X$. Set $b := p(x)$. The **characteristic subgroup of** (p,x) is

$$C(p,x) := \mathrm{im}(\pi_1(p)\colon\ \pi_1(X,x) \to \pi_1(B,b)) < \pi_1(B,b). \quad \bullet$$

Proposition 1.76. *Let $p\colon\ X \to B$ be a covering map. Let $b \in B$. The stabiliser of $x \in p^{-1}(b)$ with respect to $\mu_b(p)$ is*

$$\mathrm{Stab}_{\pi_1(B,b)}(x) = C(p,x). \quad ■$$

❉

Here is a needlessly formal way to say what is going on with the characteristic subgroup.

Corollary 1.77. *Let (B,b) be a pointed topological space. Denote by* $\mathbf{pTop}$ *the* ***category of pointed topological spaces****. Denote by* $\mathbf{pCov}^\circ(B,b)$ *the full subcategory whose objects are pointed covering map map $p\colon\ (X,x) \to (B,b)$ with X path-connected. Denote by* $\mathbf{SubGrp}(\pi_1(B,b))$ *the category whose objects are subgroups of $\pi_1(B,b)$ and whose morphisms are inclusions. Denote by*

$$C\colon\ \mathbf{pTop}/(B,b) \to \mathbf{SubGrp}(\pi_1(B,b))$$

the ***characteristic subgroup functor*** *and by*

$$\mathrm{Quot}\colon\ \mathbf{SubGrp}(\pi_1(B,b)) \to \pi_1(B,b)\text{–}\mathbf{Orb}$$

the quotient functor. The diagram

$$\begin{tikzcd} \mathbf{pCov}^\circ(B,b) \arrow[r, "C"] \arrow[d, "U"] & \mathbf{SubGrp}(\pi_1(B,b)) \arrow[d, "\mathrm{Quot}"] \\ \mathbf{Cov}^\circ(B) \arrow[r, "\mu_b"] & \pi_1(B,b)\text{-}\mathbf{Orb} \end{tikzcd}$$

commutes upto natural isomorphism. ■

1.8 Lifting along covering maps

The following result is a rather satisfactory answer to the lifting problem raised in Section 1.4. It shows that *monodromy is the only obstruction to lifting* under mild connectivity assumptions..

Theorem 1.78. *Let $p\colon X \to B$ and $f\colon A \to B$ be continous maps. Let $a \in A$. Set $b := f(a)$. If A is path-connected and locally path-connected, then ev_a induces a bijection*

$$\mathrm{ev}_a\colon\ \mathrm{Hom}_{\mathbf{Top}/B}(f,p) \to p^{-1}(b)^{\pi_1(A,a)} = \{x \in p^{-1}(b) : C(p,x) \supset C(f,a)\}$$

with $\pi_1(A,a) \circlearrowright p^{-1}(b)$ via $\mu_b(p) \circ \pi_1(f)$.

The proof of Theorem 1.78 requires the following preparation.

Definition 1.79. Let X, Y be topological spaces. A (set-theoretic) map $f\colon X \to Y$ is **path-preserving** if for for every continuous path $\gamma\colon [0,1] \to X$ the composition $f \circ \gamma\colon [0,1] \to Y$ is a continuous path. •

Lemma 1.80. *Let $p\colon X \to B$ be a covering map. Let $\tilde{f}\colon A \to X$ be path-preserving. If A is locally path-connected and $p \circ \tilde{f}$ is continuous, then $\tilde{f}$ is continuous.*

Proof. Let $a \in A$. Choose an open neighborhood $V \ni b := p \circ \tilde{f}(a)$, a discrete space D, a homeomorphism $\tau\colon p^{-1}(V) \to V \times D$ such that $\mathrm{pr}_1 \circ \tau = p|_{p^{-1}(V)}$. Choose a path-connected open neighborhood $U \ni a$ with $p \circ \tilde{f}(U) \subset V$. It remains to prove that $\mathrm{pr}_2 \circ \tau \circ \tilde{f}|_U\colon U \to D$ is continuous. Since this map is path-preserving, U is path-connected, and D is discrete, it must be constant. ■

Proof of Theorem 1.78. The asserted equality is a consequence of Proposition 1.76.

If $\tilde{f} \in \mathrm{Hom}_{\mathbf{Top}/B}(f,p)$ and $x := \mathrm{ev}_a(\tilde{f}) = \tilde{f}(a)$, then $C(f,a) \subset C(p,x)$ because $\pi_1(f) = \pi_1(p) \circ \pi_1(\tilde{f})$.

Conversely, let $x \in p^{-1}(f(a))^{\pi_1(A,a)}$. Define $\tilde{f}\colon A \to X$ by

$$\tilde{f}(b) := \mathrm{Fib}(p)(\Pi_1(f)[\gamma])(x)$$

for $[\gamma] \in \mathrm{Hom}_{\Pi_1(A)}(a,b)$. By hypothesis, this is independent of the choice of $[\gamma]$. By construction, $\tilde{f}$ is path-preserving and $p \circ \tilde{f} = f$. Therefore, by Lemma 1.80, it is continous. ■

Example 1.81. $U \subset \mathbf{C}^\times$ admits a logarithm if and only if for every $x \in U$ the map $\pi_1(U,x) \to \pi_1(\mathbf{C}^\times, x)$ is trivial. ♠

1.9 Deck transformations

Definition 1.82. The **deck transformation group** of a covering map $p\colon X \to B$ is its automorphism group
$$\mathrm{Deck}(p) := \mathrm{Aut}_{\mathbf{Cov}(B)}(p). \qquad \bullet$$

Proposition 1.83. *Let $p\colon X \to B$ be a covering map. If X is connected and locally connected, then the action $\mathrm{Deck}(p) \circlearrowleft X$ is a covering space action.*

Proof. By Proposition 1.50, the action is free.

Let $b \in B$. Since X is locally connected, there are U, D, and τ be as in Definition 1.1 with U connected. Let $\phi \in \mathrm{Deck}(p)\backslash\{\mathrm{id}_X\}$. Since U is connected, there is a bijection $f_\sharp \in \mathrm{Bij}(D)$ such that
$$\tau \circ \phi \circ \tau^{-1} = \mathrm{id}_U \times \phi_\sharp.$$
Since ϕ has no fixed-points, $\phi_\sharp$ has no fixed-points. Therefore,
$$\phi(V_d) \cap V_d = \varnothing \quad \text{for} \quad V_d := \tau^{-1}(U \times \{d\})$$
for every $d \in D$. ■

The deck transformation group can be computed as follows.

Proposition 1.84. *Let $p\colon X \to B$ be a covering map. Let $x \in X$. Set $b := p(x)$. Assume that X is path-connected and locally path-connected. There is a unique isomorphism*
$$\iota_x\colon \mathrm{Deck}(p)^{\mathrm{op}} \to W_{\pi_1(B,b)}(C(p,x)) := N_{\pi_1(B,b)}(C(p,x))/C(p,x)$$
such that for every $\phi \in \mathrm{Deck}(p)$
$$\phi(x) = \mu_b(p)(\iota_x(\phi))(x).$$

Proof. By Theorem 1.78, the evaluation map
$$\mathrm{ev}_x\colon \mathrm{Deck}(p) \to p^{-1}(b)^{\pi_1(X,x)} = \{y \in p^{-1}(b) : C(p,y) = C(p,x)\}$$
is a bijection. Since X is path-connected and by Proposition 1.76, the map
$$\mu_b(p)(\cdot)(x)\colon \pi_1(B,b)/C(p,x) \to p^{-1}(b)$$
is a bijection. Since
$$C(p,y) = [\gamma]C(p,x)[\gamma]^{-1} \quad \text{with} \quad y := \mu_b(p)([\gamma])(x),$$
the above map induces a bijection
$$\mu_b(p)(\cdot)(x)\colon W_{\pi_1(B,b)}(C(p,x)) \to p^{-1}(b)^{\pi_1(X,x)}.$$
Therefore, ι_x is uniquely determined as a (set-theoretic) map.

It remains to veryify that ι_x is a group homomorphism. This is an immediate consequence of the fact that deck transformations $f \in \mathrm{Deck}(p)$ commute with the monodromy representation $\mu_b(p)$; indeed:
$$\begin{aligned}\phi(\psi(x)) &= \phi(\mu_b(p)(\iota_x(\psi))(x)) = \mu_b(p)(\iota_x(\psi))(\phi(x)) \\ &= \mu_b(p)(\iota_x(\psi))(\mu_b(p)(\iota_x(\phi))(x)) = \mu_b(p)(\iota_x(\psi)\iota_x(\phi))(x).\end{aligned} \qquad ■$$

Example 1.85. For $p_\infty\colon \mathbf{R} \to S^1$ from Example 1.5, $\mathrm{Deck}(p_\infty) = \mathbf{Z} \circlearrowright \mathbf{R}$ and ι_0 is the inverse of the winding number map from Example 1.60. ♠

The following definition is possibly inescapable.

Definition 1.86. A covering map $p\colon X \to B$ with X path-connected and locally path-connected is **principal** (or **Galois** or **normal**) if p induces descends to a homeomorphism $X/\mathrm{Deck}(p) \cong B$. •

Corollary 1.87. *A covering map $p\colon X \to B$ with X path-connected and locally path-connected is principal if and only if for $C(p, x) < \pi_1(B, p(x))$ is a normal subgroup.* ■

Remark 1.88. Since subgroups are rarely normal, covering maps are rarely principal. ♣

Example 1.89. The normaliser of $\langle(12)\rangle < S_3$ is $\langle(12)\rangle$. ♠

The following observation is sometimes useful to compute fundamental groups.

Corollary 1.90. *If p is principal, then exact sequence*

$$\pi_1(X, x) \hookrightarrow \pi_1(B, b) \twoheadrightarrow \mathrm{Deck}(p)^{\mathrm{op}}. \qquad ■$$

1.10 Classification of covering maps

Proposition 1.91. *If B is locally path-connected, then* $\mathrm{Fib}\colon \mathbf{Cov}(B) \to \mathbf{Tra}(B)$ *is full and faithful.*

Proof. Evidently, Fib is faithful.

Let $p\colon X \to B$ and $q\colon Y \to B$ be covering maps. Let $\phi\colon \mathrm{Fib}(p) \to \mathrm{Fib}(p)$ be a morphism in $\mathbf{Tra}(B)$. Define the (set-theoretic) map $\phi\colon X \to Y$ by

$$\phi|_{p^{-1}(b)} \coloneqq \phi_b.$$

By construction, $p = q \circ f$ and f is path-perserving. By Lemma 1.80, ϕ is continuous. Therefore, Fib is full. ■

Proposition 1.91 can be strengthened as follows.

Definition 1.92. A topological space X is **semi-locally simply-connected** if every $x \in X$ has a neighborhood U such that for every $y, z \in U$

$$\#\,\mathrm{im}(\mathrm{Hom}_{\Pi_1(U)}(y, z) \to \mathrm{Hom}_{\Pi_1(X)}(y, z)) = 1. \qquad •$$

Theorem 1.93 (Classification of covering maps, I). *If B is locally path-connected and semi-locally simply-connected, then*

$$\mathrm{Fib}\colon \mathbf{Cov}(B) \to \Pi_1(B)\text{–}\mathbf{Set}$$

is an equivalence of categories.

The proof of Theorem 1.93 relies the following construction.

Definition 1.94. The **set-theoretic reconstruction functor** $\mathrm{rec}\colon \Pi_1(B)\text{–}\mathbf{Set} \to \mathbf{Set}/B$ is defined by:

(Ob) For every object T of $\Pi_1(B)$–**Set** define

$$\mathrm{rec}(T)\colon\ X_T \coloneqq \coprod_{b\in B} T(b) \to B$$

to be the canonical projection.

(Hom) For every morphism $f\colon\ T \to S$ in $\Pi_1(B)$–**Set** define

$$\mathrm{rec}(f) \coloneqq \coprod_{b\in B} f(b)\colon\ X_T \to X_S. \qquad \bullet$$

Definition 1.95. Let B be a topological space. Let T be an object of $\Pi_1(B)$–**Set**.

(1) Denote by $\mathscr{U}$ the set of open subsets $U \subset B$ be open such that for every $b, c \in U$

$$\#\,\mathrm{im}\big(\mathrm{Hom}_{\Pi_1(U)}(b,c) \to \mathrm{Hom}_{\Pi_1(B)}(b,c)\big) = 1.$$

(2) Let $b \in U \in \mathscr{U}$. Define the bijection $\tau_{b,U}\colon\ X_T \to U \times T(b)$ by

$$\tau_{b,U}(c,x) \coloneqq (c, T([\gamma^c_{b,U}])x) \quad \text{with} \quad \{[\gamma^c_{b,U}]\} \coloneqq \mathrm{im}\big(\mathrm{Hom}_{\Pi_1(U)}(c,b) \to \mathrm{Hom}_{\Pi_1(B)}(c,b)\big).$$

(3) The **transport topology** is the coarsest topology on X_T with respect to which the maps $\tau_{b,U}$ ($b \in U \in \mathscr{U}$, $b \in U$) are continuous. $\bullet$

The following construction lifts rec along $\mathbf{Cov}(B) \to \mathbf{Set}/B$.

Proposition 1.96. *Let B be a locally path-connected and semi-locally simply-connected topological space.*

(1) *For every object T of $\Pi_1(B)$–**Set**, $\mathrm{rec}(T)\colon\ X_T \to B$ is a covering map of B with respect to the transport topology on X_T.*

(2) *For every morphism $f\colon\ T \to S$ in $\Pi_1(B)$–**Set**, $\mathrm{rec}(f)\colon\ X_T \to X_S$ is a morphism covering maps of B with respect to the transport topology on X_T and X_S.*

Proof. The assumptions guarantee that $\mathscr{U}$ is an open cover of B. Therefore, it suffices to prove that for every $U \in \mathscr{U}$ and $b \in U$ the bijection $\tau_{b,U}$ is a homeomorphism with respect to the transport topology. In fact, it suffices to prove that

$$\tau_{b,U} \circ \tau^{-1}_{c,V}\colon\ (U \cap V) \times T(c) \to (U \cap V) \times T(b)$$

is continuous for every $b \in U \in \mathscr{U}$ and $c \in V \in \mathscr{U}$.

Let $d \in U \cap V$. Since B is locally path-connected, d has a path-connected open neighborhood $W \subset U \cap V$. Evidently, $W \in \mathscr{U}$. For every $e \in W$

$$T([\gamma^e_{b,U}][\gamma^c_{e,V}]) = T([\gamma^e_{b,U}][\gamma^d_{e,W}][\gamma^e_{d,W}][\gamma^c_{e,V}]) = T([\gamma^d_{b,U}][\gamma^c_{d,V}]) \in \mathrm{Bij}(T(c), T(b));$$

in particular, it does not depend on e. Therefore, $\tau_{b,U} \circ \tau^{-1}_{c,V}$ is continuous. ■

Definition 1.97. The **reconstruction functor** $\mathrm{Rec}\colon \Pi_1(B)\text{–}\mathbf{Set} \to \mathbf{Cov}(B)$ is the lift of $\mathrm{rec}\colon \Pi_1(B)\text{–}\mathbf{Set} \to \mathbf{Set}/B$ along the forgetful functor $\mathbf{Cov}(B) \to \mathbf{Set}/B$ obtained from Proposition 1.96. •

Proof of Theorem 1.93. Because of Proposition 1.91, it remains to verify that Fib is essentially surjective. This is an immediate consequence of the (nearly obvious) fact that $\mathrm{Fib} \circ \mathrm{Rec}$ is naturally isomorphic to the identity. ■

❉

Theorem 1.93 and Corollary 1.69 imply the following.

Corollary 1.98 (Classification of covering maps, II). *Let $b \in B$. If B is path-connected, locally path-connected and semi-locally simply-connected, then*

$$\mu_b\colon \mathbf{Cov}(B) \to \pi_1(B,b)\text{–}\mathbf{Set}$$

is an equivalence of categories. ■

❉

Definition 1.99. Denote by $\mathbf{Cov}^\circ(B) \subset \mathbf{Cov}(B)$ the full subcategory whose objects are covering maps $p\colon X \to B$ with X path-connected. •

Definition 1.100. The **category of G–orbits** is the full subcategory $G\text{–}\mathbf{Orb} \subset G\text{–}\mathbf{Set}$ whose objects are *transitive* G–actions. •

Proposition 1.101. $\mathbf{Cov}(B)$ *is the free coproduct completion of* $\mathbf{Cov}^\circ(B)$. ■

Proposition 1.102. $G\text{–}\mathbf{Set}$ *is the free coproduct completion of* $G\text{–}\mathbf{Orb}$. ■

Corollary 1.98 immediately implies the following.

Corollary 1.103 (Classification of covering maps, III). *Let $b \in B$. If B is path-connected, locally path-connected and semi-locally simply-connected, then*

$$\mu_b\colon \mathbf{Cov}^\circ(B) \to \pi_1(B,b)\text{–}\mathbf{Orb}$$

is an equivalence of categories. ■

The functor μ_b from Corollary 1.103 is imminently computable because of Proposition 1.76.

1.11 Universal covering maps

Definition 1.104. A covering map $p\colon X \to B$ is **universal** if and only if p is surjective and X is simply-connected. •

Proposition 1.105. *Let B be path-connected and locally path-connected. B is semi-locally simply-connected if and only if it admits a universal covering map.*

Proof. If B admits a universal cover, then it is semi-locally simply-connected. Conversely, if B is semi-locally simply-connected, then a universal cover exists by Corollary 1.98. ■

Remark 1.106. Proposition 1.105 can be proved directly. However, the usual construction boils down to a special case of the construction of the reconstruction functor Rec. Sometimes, however, a universal covering map is known to exists a priori. ♣

Corollary 1.107. *Every universal covering map is principal.* ■

Corollary 1.108. *If $p\colon X \to B$ is a universal covering map, then $\iota_x\colon \operatorname{Aut}(p)^{\mathrm{op}} \cong \pi_1(B, b)$.* ■

Definition 1.109. Let $p\colon X \to B$ be a covering map such that $\operatorname{Aut}(p) \circlearrowright X$ is a covering space action. The **associated covering map functor**

$$\mathrm{A}_p\colon \operatorname{Aut}(p)^{\mathrm{op}}\text{–}\mathbf{Set} \to \mathbf{Cov}(B)$$

is defined as follows:

(Ob) Let S be a $\operatorname{Aut}(p)^{\mathrm{op}}$–set. $X \times S \circlearrowleft \operatorname{Aut}(p)^{\mathrm{op}}$ via

$$(x, s) \cdot f := (f(x), f^{-1} \cdot s).$$

The **associated covering map** is the canonical projection

$$\mathrm{A}_p(S)\colon X_S := (X \times S)/\operatorname{Aut}(p)^{\mathrm{op}} \to B.$$

(Hom) Let $f\colon S \to G$ be a morphism of $\operatorname{Aut}(p)^{\mathrm{op}}$–sets. The continuous map $\mathrm{id}_X \times f\colon X \times S \to X \times T$ is $\operatorname{Aut}(p)^{\mathrm{op}}$–equivariant and descends to

$$\mathrm{A}_p(f)\colon \mathrm{A}_p(S) \to \mathrm{A}_p(T).$$ •

Proposition 1.110. *If $p\colon X \to B$ is a universal covering map, then the diagram*

$$\begin{tikzcd}
\operatorname{Aut}(p)^{\mathrm{op}}\text{–}\mathbf{Set} \arrow[r, "\mathrm{A}_p"] \arrow[dr, "\iota_x"'] & \mathbf{Cov}(B) \arrow[d, "\mu_b"] \\
& \pi_1(B, b)\text{–}\mathbf{Set}
\end{tikzcd}$$

commutes upto natural isomorphism. ■

Corollary 1.111. *Let B be path-connected and locally path-connected. B admits a universal cover if and only if μ_b is an equivalence of categories.* ■

1.12 The Nielsen–Schreier Theorem

Here is an application of the theory of covering maps to algebra.

Definition 1.112. Let S be a set. The **free group on** S is the group $F(S)$ generated by S. A group G is **free** if it is isomorphic to $F(S)$ for some S. The **rank** of G is $\operatorname{rk}(G) := \#S$. •

Theorem 1.113 (Nielsen–Schreier Theorem). *If G is a free group, then every subgroup $H < G$ is free. If $\operatorname{rk}(G) = r \in \mathbf{N}_0$ and $|H : G| = i \in \mathbf{N}$, then $\operatorname{rk}(H) = i(r-1) + 1$.*

The proof relies on realising G as a fundamental group and the theory of covering maps.

Definition 1.114.

(1) A **graph** is a triple $\Gamma = (V, E, \alpha)$ with V a set, E a set of unordered pairs, and a map $\alpha\colon \bigcup E \to V$. The **vertices** and **edges** of Γ are the elements of V and E respectively. An edge e **connects** $x, y \in V$ if $\alpha(e) = \{x, y\}$.

(2) For every unordered pair $e = \{x, y\}$ set

$$I_e \coloneqq (e \times [0,1])/\sim$$

with $\sim$ denoting the equivalence relation generated by $(x, t) \sim (y, 1-t)$.

(3) The **topological realisation** of Γ is

$$X(\Gamma) \coloneqq \left(V \amalg \coprod_{e \in E} I_e\right) \Big/ \sim$$

with $\sim$ denoting the equivalence relation generated by $[x, 0] \sim \alpha(x)$. •

Example 1.115. Let S be a set. Set $V \coloneqq \{*\}$ and $E \coloneqq \{0,1\} \times S$. There is a unique map $\alpha\colon E \to V$. The graph $\Gamma = (V, E, \alpha)$ has a unique vertex $*$ and an edge connecting $*$ to itself for every $s \in S$. The topological realisation $X(\Gamma)$ of Γ is homeomorphic to a **bouquet of circles** indexed by S:

$$X(\Gamma) \cong \bigvee_{s \in S} \{s\} \times S^1 \coloneqq \left(\coprod_{s \in S} \{s\} \times S^1\right) \Big/ \sim$$

with $\sim$ denoting the equivalence relation generated by $(s, [0]) \sim (t, [0])$. By the Seifert–van Kampen theorem,

$$\pi_1(X(\Gamma), *) \cong F(S).$$ ♠

Definition 1.116. Let $\Gamma = (V, E, \alpha)$ be a graph.

(1) A **subgraph** of a graph $\Gamma = (V, E, \alpha)$ is a graph $\Delta = (W, F, \beta)$ with $W \subset V$, $F \subset E$, and $\beta = \alpha|_F$.

(2) A **path** in Γ is a is a sequence of vertices $v_0, \dots, v_n$ together with a sequence of edges $e_0, \dots, e_n$ such that e_i connects v_i and v_{i+1}. A **cycle** in Γ is a path with $n \geqslant 1$, $v_0 = v_n$, and $e_i \neq e_{i+1}$.

(3) Γ is **connected** if for every $v, w \in V$ there is a path with $v_0 = v$ and $v_n = w$.

(4) A **forest** is a graph without cycles. A **tree** is a connected forest. •

Proposition 1.117. *Let Γ be a connected graph. $X(\Gamma)$ is homotopy-equivalent to a bouquet of circles.*

Proof sketch. Denote by $\mathscr{T}$ the set of subgraphs of Γ which are trees. There is an obvious order on $\mathscr{T}$. Use Zorn's lemma to construct a maximal $T \in \mathscr{T}$. A moment's thought shows that T has the same vertices as Γ. The subspace $X(T) \subset X(\Gamma)$ is contractible. $X(\Gamma)/X(T)$ is homeomorphic to a bouquet of circles. Finally, the projection $X(\Gamma) \to X(\Gamma)/X(T)$ is a homotopy equivalence. ■

Lemma 1.118. *Let Γ be a graph. If $p\colon Y \to X(\Gamma)$ is a covering map, then Y is homeomorphic to $X(\Delta)$ for some graph Δ.* ■

Proof of Theorem 1.113. Let G be a free group. Construct a graph Γ with $\pi_1(X(\Gamma)) \cong G$. If $H < G$ is a subgroup, then there is a covering map $p\colon Y \to X(\Gamma)$ with characteristic subgroup isomorphic to H. By the above, $\pi_1(Y)$ is free.

If F has rank r and $|H : G| = i$, then $\deg(p) = i$; hence:

$$1 - \mathrm{rk}(H) = \chi(Y) = i\chi(X(\Gamma)) = i(1 - r).$$

This implies $\mathrm{rk}(H) = i(r - 1) + 1$. ■

2 Fibre bundles

The purpose of this section is to explain the salient points of theory of Ehresmann connections on (unstructured) fibre bundles. Most of gauge theory is concerned with the less general but slightly more complicated G–principal fibre bundles (or fibre bundles with structure groups). The current section can be understood as a warm-up.

[Ste51] is the classical reference of the topological theory of fibre bundles. [Hus94] is a more modern reference. The theory of connections of fibre bundles is due to Ehresmann [Ehr51]. Kolář, Michor, and Slovák [KMS93] have an extensive treatment.

2.1 Introduction

Definition 2.1. A **fibre bundle** is a smooth map $p\colon X \to B$ such that for every $b \in B$ there are an open subset $b \in U \subset B$ and a **local trivialisation** of $p|_{p^{-1}(U)}$; that is: a smooth manifold F, and a diffeomorphism $\tau\colon p^{-1}(U) \to U \times F$ such that

$$\mathrm{pr}_B \circ \tau = p|_{p^{-1}(U)};$$

The **total space** of p is X. The **base space** of p is B. For $b \in B$ the **fibre of** p **over** b is $X_b := p^{-1}(b)$. •

This concept formalises the concept of smoothly varying families of manifolds.

Proposition 2.2. *Let $p\colon X \to B$ be a fiber bundle. If B is connected and $b_0, b_1 \in B$, then X_{b_0} and X_{b_1} are diffeomorphic.* ■

Example 2.3. Let B, F be smooth manifolds. The **trivial fibre bundle** over B with fibre F is the projection map $\mathrm{pr}_1\colon B \times F \to B$. ♠

Example 2.4. Let $n \in \mathbf{N}$. The skew fields $\mathbf{R}, \mathbf{C}, \mathbf{H}$ give rise to the Hopf bundles.

(1) The projection $p\colon S^n \subset \mathbf{R}^{n+1}\backslash\{0\} \to \mathbf{R}P^n$ is a fibre bundle.

(2) The **Hopf bundle** $p\colon S^{2n+1} \subset \mathbf{C}^{n+1}\backslash\{0\} \to \mathbf{C}P^n$ is a fibre bundle.

(3) The **quaternionic Hopf bundle** $p\colon S^{4n+3} \subset \mathbf{H}^{n+1}\backslash\{0\} \to \mathbf{H}P^n$ is a fibre bundle.

(4) There is a fibre bundle $p\colon S^{15} \to S^8$ but a naive construction of a family of octonionic Hopf bundles does not work. ♠

Example 2.5. Let X, B be smooth manifolds. If $p\colon X \to B$ is a covering map, then it is a fiber bundle. ♠

Example 2.6. Every smooth vector bundle $p\colon E \to B$ is a fibre bundle. ♠

Example 2.7. Let $E \to B$ be a Euclidean vector bundle. The **sphere bundle**

$$p\colon S(V) \to B \quad \text{with} \quad S(V) := \{v \in V : |v| = 1\}.$$

is a fibre bundle. ♠

Example 2.8. Let $p\colon E \to B$ be a vector bundle. For $r \in \mathbf{N}_0$ denote by

$$\mathrm{Gr}_r(V) := \{(b, \Pi) : b \in B, \Pi \subset V_b \text{ with } \dim \Pi = r\}$$

the **Grassmannian of r–planes in** V. $\mathrm{Gr}_r(V)$ admits the structure of a smooth manifold such that the map $q\colon \mathrm{Gr}_r(V) \to B$ obtained by restriction of pr_1 is a fibre bundle. ♠

Example 2.9. Let $p\colon E \to B$ be a vector bundle. Denote by

$$\mathrm{Fr}(V) := \{(b, \phi) : b \in B, \phi\colon \mathbf{R}^{\mathrm{rk}_b V} \to V_b \text{ isomomorphism}\}$$

the **frame bundle of** E. $\mathrm{Fr}(E)$ admits the structure of a smooth manifold such that the map $q\colon \mathrm{Fr}(E) \to B$ obtained by restriction of pr_1 is a fibre bundle. ♠

Theorem 2.10 (Ehresmann fibration theorem). *Every proper submersion $p\colon X \to B$ is a fibre bundle.* ■

Example 2.11. Let $f\colon X \to \mathbf{R}$ be a proper smooth function. If $[a, b] \cap f(\mathrm{Crit}(f)) = \varnothing$, then $f^{-1}(a)$ and $f^{-1}(b)$ are diffeomorphic. ♠

Corollary 2.12. *Deformation equivalent closed complex manifolds are diffeomorphic.* ■

2.2 The category of fibre bundles

Definition 2.13. The **category of fibre bundles** is the full subcategory **FibBun** of **Arr**(**Sm**) whose objects are fibre bundles. •

Proposition 2.14. **FibBun** *has finite products and arbitrary coproducts.* ■

Proposition 2.15. *The codomain functor $U\colon$ **FibBun** $\to$ **Sm** is a fibred category.*

Proof. Let $p\colon X \to B$ be a fibre bundle. Let $f\colon A \to B$ be a smooth map. Since p is a submersion, $f \times p$ is transverse to the diagonal $\Delta \subset B \times B$. Therefore,

$$f^*X := \{(a, x) \in A \times X : f(a) = p(x)\} \subset A \times X$$

is a submanifold. Set $p^*f := \mathrm{pr}_1\colon f^*X \to X$ and $f^*p := \mathrm{pr}_2\colon f^*X \to A$. Evidently, the morphism $(p^*f, f)\colon f^*p \to p$ is cartesian. ■

Definition 2.16. For every fibre bundle $p\colon X \to B$ and every smooth map $f\colon A \to B$ *choose* a cartesian lift

$$\begin{tikzcd} f^*X \arrow[r, "p^*f"] \arrow[d, "f^*p"] & X \arrow[d, "p"] \\ A \arrow[r, "f"] & B. \end{tikzcd}$$

This is the **pullback via** f •

As in Section 1.3, $\mathbf{FibBun}_B$ is a sheaf.

2.3 Ehresmann connections

It is not terribly difficult to prove the following.

Theorem 2.17. *If $p\colon X \to B$ is a fibre bundle, then it is a Hurewicz fibration.* ■

However, the lifting problem is quite flabby. The following definition helps to rigidify the lifting problem.

Definition 2.18. Let $p\colon X \to B$ be a fibre bundle. The **vertical tangent bundle** of p is the vector bundle

$$V_p \coloneqq \ker(Tp\colon TX \to p^*TB) \to X.$$ •

Definition 2.19. Let $p\colon X \to B$ be a fibre bundle. An **Ehresmann connection** on p is a left splitting of the short exact sequence of vector bundles

$$V_p \overset{\iota}{\hookrightarrow} TX \overset{Tp}{\twoheadrightarrow} p^*TB;$$

that is: an $A \in \Omega^1(X, V_p) = \Gamma(X, \mathrm{Hom}(TX, V_p))$ such that

$$A \circ \iota = \mathrm{id}_{V_p}.$$

Denote by $\mathscr{A}(p) \subset \Omega^1(X, V_p)$ the subset of Ehresmann connections on p. •

Example 2.20. Consider the trivial fibre bundle $\mathrm{pr}_B\colon B \times F \to B$. There is a canonical isomorphism $T(B\times F) \cong \mathrm{pr}_B^*TB \oplus \mathrm{pr}_F^*TF$ with respect to which $V_{\mathrm{pr}_B} = \mathrm{pr}_F^*TF$. The **product connection** A on pr_B is the obvious projection. ♠

Example 2.21. Let $p\colon X \to B$ be a fibre bundle. Let g be a Riemannian metric on X. The orthogonal projection $A_g\colon TX = V_p \oplus V_p^\perp \to V_p$ is an Ehresmann connection. ♠

Remark 2.22. Let $p\colon X \to B$ be a fibre bundle. The construction in Example 2.21 induces a map $\mathscr{M}et(X) \to \mathscr{A}(p)$. This map is surjective, but very far from injective. ♣

The structure of the space $\mathscr{A}(p)$ is very simple.

Definition 2.23. Let $p\colon X \to B$ be a fibre bundle. Let E be a vector bundle over X. A differential form $\alpha \in \Omega^\bullet(X, E)$ is **horizontal** if for every $v \in V_p$

$$i_v\alpha = 0.$$

The subspace of horizontal differential forms is denoted by $\Omega^\bullet_{\mathrm{hor}}(X, E)$. •

Proposition 2.24. *$\mathscr{A}(p) \subset \Omega^1(X, V_p)$ is an affine subspace modelled on $\Omega^1_{\mathrm{hor}}(X, V_p)$.*

Proof. By Example 2.21, there exists an $A_0 \in \mathscr{A}(p)$. Let $A \in \Omega^1(X, V_p)$. Evidently, $A \in \mathscr{A}(p)$ if and only if $A - A_0 \in \Omega^1_{\mathrm{hor}}(X, V_p)$. ■

Remark 2.25. Informally, the fact that $\mathscr{A}(p)$ is contractible, means that *choosing* an $A \in \mathscr{A}(p)$ is mostly harmless. ♣

Ehresmann connections can be pulled-back as follows.

Definition 2.26. Let $p\colon X \to B$ be a fibre bundle. Let $f\colon C \to B$ be a smooth map. Denote by

$$V_f\colon V_{f^*p} \to (p^*f)^*V_p.$$

the isomorphism induced by $Tp^*f\colon Tf^*X \to (p^*f)TX$. Define the **pull-back** map $f^\sharp\colon \mathscr{A}(p) \to \mathscr{A}(f^*p)$ by

$$f^\sharp A \coloneqq V_f^{-1}(p^*f)^*A. \qquad \bullet$$

2.4 Parallel transport

Definition 2.27. Let $p\colon X \to B$ be a fibre bundle together with an Ehresmann connection $A \in \mathscr{A}(p)$. A smooth map $f\colon C \to X$ is A–**horizontal** if

$$f^*A = 0.$$

Let $f\colon A \to B$ be a smooth map. Denote by $\mathrm{Hom}_{\mathbf{Sm}/B}(f, p)^A \subset \mathrm{Hom}_{\mathbf{Sm}/B}$ the subset of A–horizontal lifts of f along p. •

The **horizontal lifting problem** is to determine $\mathrm{Hom}_{\mathbf{Sm}/B}(f, p)^A$. The horizontal lifting problem is rather similar to the lifting problem along covering maps.

Proposition 2.28. *Let $p\colon X \to B$ be a fibre bundle together with an Ehresmann connection $A \in \mathscr{A}(p)$. Let $f\colon C \to B$ be a smooth map. The map $p^*f \circ \cdot\colon \mathrm{Hom}_{\mathbf{Sm}/B}(\mathrm{id}_C, f^*p) \to \mathrm{Hom}_{\mathbf{Sm}/B}(f, p)$ induces a bijection $\mathrm{Hom}_{\mathbf{Sm}/B}(f, p)^A \cong \mathrm{Hom}_{\mathbf{Sm}/B}(\mathrm{id}, f^*p)^{f^*A}$.* ■

Definition 2.29. Let $p\colon X \to B$ be a fibre bundle together with an Ehresmann connection $A \in \mathscr{A}(p)$. Let $f\colon C \to B$ be a smooth map. Let $c \in C$. Set $b \coloneqq f(c)$. Define the **evaluation map**

$$\mathrm{ev}_c\colon \mathrm{Hom}_{\mathbf{Sm}/B}(f, p)^A \to p^{-1}(b)$$

by

$$\mathrm{ev}_c(\tilde{f}) \coloneqq \tilde{f}(c). \qquad \bullet$$

Proposition 2.30. *Let $p\colon X \to B$ be a fibre bundle together with an Ehresmann connection $A \in \mathscr{A}(p)$. Let $f\colon C \to B$ be a smooth map. Let $c \in C$. If C is connected, then ev_c is injective.*

Proof. Since C is path-connected, it suffices to prove this for $C = [0, 1]$. By the above observation, it suffices to consider $B = C = [0, 1]$ and $f = \mathrm{id}_B$. There is a unique vector field $\tilde{v} \in \mathrm{Vect}(X)$ which is p–related to ∂_t on $[0, 1]$. A lift $\tilde{f}$ is A–horizontal if and only if it is a integral curve of $\tilde{v}$. The assertion therefore follows from the Picard–Lindelöf Theorem. ■

Definition 2.31. Let $p\colon X \to B$ be a fibre bundle. An Ehresmann connection $A \in \mathscr{A}(p)$ is **complete** if for every smooth path $\gamma\colon [0,1] \to B$ and every $x \in p^{-1}(\gamma(0))$ there is an A–horizontal lift $\tilde{\gamma}$ with $\tilde{\gamma}(0) = x_0$. •

Theorem 2.32 (del Hoyo [dHoy16]). *Every fibre bundle admits a complete Ehresmann connection.* ■

Proposition 2.33. *If p is a proper fibre bundle, then every $A \in \mathscr{A}(p)$ is complete.* ■

Remark 2.34. The attentive reader will have observed that the theory of Ehresmann connections, discussed so far, does not make use of p being a fibre bundle. It would have sufficed to assume that p is a submersion. What singles out fibre bundles is the existence of complete Ehresmann connections. ♣

Definition 2.35. The **path groupoid functor** is the functor $P_1\colon \mathbf{Sm} \to \mathbf{Gpd}$ defined as follows:

(Ob) Let X be a smooth manifold. The **path groupoid of** X is the groupoid $P_1(X)$ whose objects are the elements of X, and whose morphism $[\gamma]\colon x \to y$ are smooth paths $\gamma\colon [0,1] \to X$ which are constant in a neighborhood of $\{0,1\}$ up to thin homotopy (i.e.: a homotopy $h\colon [0,1]^2 \to X$ with $\operatorname{rk} Th \leqslant 1$), composed by concatenation.

(Hom) Let $f\colon X \to Y$ be a smooth map. The natural transformation $P_1(f)$ is given by composition
$$P_1(f)[\gamma] \coloneqq [f \circ \gamma].$$ •

Definition 2.36. Let $p\colon X \to B$ be a fibre bundle together with an Ehresmann connection $A \in \mathscr{A}(p)$. The **horizontal path groupoid** $P_1(X, A)$ is the subgroupoid of $P_1(X)$ whose morphisms are equivalence classes of A–horizontal paths. •

Proposition 2.37. *Let $p\colon X \to B$ be a fibre bundle together with a complete Ehresmann connection $A \in \mathscr{A}(p)$. For every $x \in X$ and $b \in B$ the map*
$$\coprod_{y \in p^{-1}(b)} \mathrm{Hom}_{P_1(X,A)}(x, y) \to \mathrm{Hom}_{P_1(B)}(p(x), b)$$
induced by $P_1(p)$ is bijective. ■

Definition 2.38. Let $p\colon X \to B$ be a fibre bundle together with a complete Ehresmann connection $A \in \mathscr{A}(p)$. The **parallel transport** is the functor
$$\mathrm{tra}^A\colon P_1(B) \to \mathbf{Sm}$$
defined by
$$\mathrm{tra}^A(b) \coloneqq p^{-1}(b) \quad \text{and} \quad \mathrm{tra}^A([\gamma])(x) \coloneqq [\tilde{\gamma}(1)]$$
with $\tilde{\gamma}$ denoting the unique A–horizontal lift of γ with $\gamma(0) = x$. •

Remark 2.39. To determine $\mathrm{tra}^A([\gamma])$ is to solve an ODE. ♣

Remark 2.40. The fact that is functor goes to **Sm** instead of **Set** is the smooth dependence on initial conditions for solutions of ODE. ♣

Definition 2.41. Let $p\colon X \to B$ be a fibre bundle together with a complete Ehresmann connection $A \in \mathscr{A}(p)$. Let $b \in B$. The **holonomy group of** A **based at** b is the subgroup $\mathrm{Hol}_b(A) < \mathrm{Diff}(p^{-1}(b))$ defined by

$$\mathrm{Hol}_b(A) := \left\{ \mathrm{tra}^A(\gamma) : \gamma \in \mathrm{Aut}_{P_1(B)}(b) \right\}. \qquad \bullet$$

Remark 2.42. In practice, it is not feasible to compute $\mathrm{Hol}_b(A)$ from the definition. ♣

Corollary 2.43 (holonomy principle). *Let $p\colon X \to B$ be a fibre bundle together with an Ehresmann connection $A \in \mathscr{A}(p)$. Let $f\colon C \to B$ be a smooth map. Let $c \in C$. If C is connected, then the evaluation map induces a bijection*

$$\mathrm{ev}_c\colon \mathrm{Hom}_{\mathbf{Sm}/B}(f,p)^A \to p^{-1}(b)^{\mathrm{Aut}_{P_1(C)}(c)}. \qquad \blacksquare$$

2.5 The curvature of an Ehresmann connection

Definition 2.44. Let $p\colon X \to B$ be a fibre bundle. Let $A \in \mathscr{A}(p)$. The **curvature of** A unique horizontal 2–form $F_A \in \Omega^2_{\mathrm{hor}}(X, V_p)$ such that for every $v, w \in \mathrm{Vect}(X)$

$$F_A(v,w) = -A([v - A(v), w - A(w)]). \qquad \bullet$$

Of course, F_A is the obstruction to the integrability of the **horizontal distribution** $H_A := \ker A \subset TX$.

Example 2.45. Consider the Hopf bundle $p\colon S^{2n+1} \subset \mathbf{C}^{n+1}\backslash\{0\} \to \mathbf{C}P^n$. The vertical tangent bundle is spanned by the vector field $\partial_\alpha \in \mathrm{Vect}(S^{n+1})$ defined by

$$\partial_\alpha(z) = iz.$$

The connection A defines by the metric on S^{2n+1} satisfies

$$H_A = \{v \in TS^{2n+1} : v \perp \partial_\alpha\}.$$

Let $v, w \in \mathrm{Vect}(S^{2n+1})$ The curvature of A is

$$\begin{aligned} F_A(v,w) &= -\langle [v,w], iz\rangle \otimes iz \\ &= -\langle \nabla_v w - \nabla_w v, iz\rangle \otimes iz \\ &= -2\langle v, iw\rangle \otimes iz \\ &= -2\pi \cdot p^*\omega_{\mathrm{FS}} \otimes \partial_\alpha. \end{aligned}$$

Here $\omega_{\mathrm{FS}} \in \Omega^2(\mathbf{C}P^n)$ is the Fubini–Study form on $\mathbf{C}P^n$. ♠

Exercise 2.46. Let $U_a := \{[z_0, \dots, z_n] \in \mathbf{C}P^n : z_a \neq 0\}$ and define $\phi_a\colon U_a \to \mathbf{C}^n$ by

$$\phi([z_0, \dots, z_n]) := [z_0/z_a : \cdots : \widehat{z_a/z_a} : \cdots : z_n/z_a].$$

Prove that there is a unique 2–form ω_{FS} on $\mathbf{C}P^n$ satisfying

$$(\phi_a)_*\omega_{\mathrm{FS}} = \frac{i}{2\pi}\left(\sum_{b=1}^n \frac{\mathrm{d}z_b \wedge \mathrm{d}\bar{z}_b}{1+|z|^2} - \sum_{b,c=1}^n \frac{\bar{z}_c \mathrm{d}z_c \wedge z_b \mathrm{d}\bar{z}_b}{(1+|z|^2)^2} \right).$$

Prove that the above formula for F_A indeed holds.

Proposition 2.47. *Let $p\colon X \to B$ be fibre bundle. Let $A \in \mathscr{A}(p)$. Let $f\colon C \to B$ be a smooth map. The curvature of A and f^*A satisfy*

$$F_{f^*A} = V_f^{-1}(p^*f)^*F_A.$$

Definition 2.48. Let $p\colon X \to B$ be a fibre bundle. An Ehresmann connection $A \in \mathscr{A}(p)$ is **flat** if $F_A = 0$. •

Example 2.49. Let $p\colon X \to B$ be a smooth covering map. The unique Ehresmann connection on p is flat. ♠

Example 2.50 ([MS74, Appendix C]). Let Σ be a closed, connected Riemann surface of genus $g \geqslant 2$. The fibre bundle $p\colon ST\Sigma \to \Sigma$ admits a flat connection.

By the uniformization theorem, there is a $\Gamma^{\mathrm{op}} < \mathbf{PSL}(2,\mathbf{R}) = \mathrm{Isom}(H)$ with $\Sigma = H/\Gamma$. In particular, $ST\Sigma = STH/\Gamma$. Define $P\colon STH \to H \times (\mathbf{R} \cup \{\infty\})$ by

$$P(z,v) \coloneqq \lim_{t\to\infty} \exp_x(tv).$$

Here $\exp_x$ is computed with respect to the hyperbolic metric g_{-1} on H. $\mathbf{PSL}(2,\mathbf{R})$ acts on $\mathbf{R} \cup \{\infty\} \cong S^1$. A moment's thought shows that P is $\mathbf{PSL}_2(\mathbf{R})$–equivariant. Therefore,

$$ST\Sigma = STH/\Gamma \cong (H \times (\mathbf{R} \cup \{\infty\}))/\Gamma.$$

The flat Ehresmann connection on $\mathrm{pr}_1\colon H \times (\mathbf{R} \cup \{\infty\}) \to \mathbf{H}$ is $\mathbf{PSL}_2(\mathbf{R})$–invariant and descends to $p\colon ST\Sigma \to \Sigma$. ♠

Proposition 2.51 (Flat connections and covering maps). *Let $p\colon X \to B$ be a fibre bundle. If $A \in \mathscr{A}(p)$ is complete and flat, then there is a covering map $q\colon S \to B$ and a bijective immersion $\iota\colon S \looparrowright X$ such that $q = p \circ \iota$ and $T\iota\colon TS \hookrightarrow \iota^*TX$ induces an isomorphism $TS \cong \iota^*H_A$; in particular:* tra^A *depends only on the homotopy class rel $\{0,1\}$ of γ; that is* tra^A *factors through $P_1(B) \to \Pi_1(B)$.*

Proof. A is flat if and only if the distribution H_A is involutive. Frobenius's theorem guarantees the existence of a bijective immersion $\iota\colon S \looparrowright X$ such that $T\iota\colon TS \hookrightarrow \iota^*TX$ induces an isomorphism $TS \cong \iota^*H_A$.

To prove that $q \coloneqq p \circ \iota$ is a covering map, let $b_0 \in B$ and let U be a connected, simply-connected, open neighborhood of b_0. Let V be a connected component of $S \cap \iota^{-1}(U)$. It remains to prove that $q|_V\colon V \to U$ is a diffeomorphism. By construction, $q|_V$ is a local diffeomorphism. Since U is path-connected, $q|_V$ is surjective. To prove that $q|_V$ is injective, let $s_0, s_1 \in q^{-1}(b_0) \cap V$. Since V is path-connected, there is a smooth path $\tilde\gamma\colon [0,1] \to S$ with $\tilde\gamma(0) = s_0$ and $\tilde\gamma(1) = s_1$. Since U is simply-connected, there is a smooth homotopy $\Gamma\colon [0,1]\times[0,1] \to B$ rel $\{0,1\}$ with $\Gamma(0,\cdot) = q \circ \tilde\gamma$ and $\Gamma(1,\cdot) = b_0$. The task at hand is to find an A–horizontal lift $\tilde\Gamma\colon [0,1]\times[0,1] \to X$ of Γ along p with $\tilde\Gamma(0,0) = x_0 \coloneqq \iota(s_0)$.

It suffices to consider $B = [0,1]\times[0,1]$ and $\gamma = \mathrm{id}_B$. Denote by v_1, v_2 the A–horizontal lifts of ∂_1, ∂_2. The lift

$$\tilde\Gamma(t_1,t_2) \coloneqq \mathrm{flow}_{v_1}^{t_1} \circ \mathrm{flow}_{v_2}^{t_2}(x_0)$$

maps into the maximal integral submanifold through x_0; hence, it is A–parallel. ∎

Proposition 2.52. *Let $p\colon X \to B$ be a fibre bundle. If there is a flat Ehresmann connection $A \in \mathscr{A}(p)$ and B is simply-connected, then p is isomorphic to a trivial fibre bundle.* ∎

2.6 Decomposition of the de Rham complex on fibre bundles

Definition 2.53. Let $p\colon X \to B$ be a fibre bundle together with an Ehresmann connection $A \in \mathscr{A}(p)$. The **bi-grading on** $\Omega(X)$ **induced by** A is defined by

$$\Omega_A^{p,q}(X) := \Gamma(\Lambda_A^{p,q}T^*X). \quad \text{with} \quad \Lambda_A^{p,q}T^*X := \Lambda^p H_A^* \otimes \Lambda^q V_p^*.$$

Denote by $\mathrm{d}_A^{p,q}$ the component of d of bi-degree (p,q). •

Proposition 2.54. *Let $p\colon X \to B$ be a fibre bundle together with an Ehresmann connection. The exterior derivative decomposes into three components of bidegree $(1,0)$, $(0,1)$, and $(2,-1)$:*

$$\mathrm{d} = \mathrm{d}_A^{1,0} + \mathrm{d}_A^{0,1} + \mathrm{d}_A^{2,-1};$$

moreover:

$$\mathrm{d}_A^{2,-1} = i_{F_A}.$$

where i_{F_A} is the graded derivation of degree 1 on $\Omega(X)$ defined by

$$i_{F_A} f = 0 \quad \textit{and} \quad i_{F_A}\alpha = \alpha \circ F_A.$$

Proof. The exterior derivative d is a graded derivation of $\Omega^\bullet(X)$ of degree 1. Consequently, $\mathrm{d}_A^{p,q} = 0$ vanishes unless $p + q = 1$. Since a graded derivation of $\Omega^\bullet(X)$ is determined by its restriction to $\Omega^0(X) \oplus \Omega^1(X)$, $\mathrm{d}_A^{p,q} = 0$ unless $p, q \geqslant -1$.

Evidently, $\mathrm{d}_A^{-1,2}$ vanishes on $\Omega^0(X) \oplus \Omega_A^{0,1}(X)$. Moreover, for every $\alpha \in \Omega_A^{1,0}$ and $v, w \in \mathrm{Vect}(X)$

$$\begin{aligned}(\mathrm{d}_A^{-1,2}\alpha)(v,w) &= (\mathrm{d}\alpha)(A(v), A(w)) \\ &= -\alpha([A(v), A(w)]) = 0.\end{aligned}$$

Therefore, $\mathrm{d}_A^{-1,2} = 0$.

Evidently, $\mathrm{d}_A^{2,-1}$ vanishes on $\Omega^0(X) \oplus \Omega_A^{1,0}(X)$. Moreover, for every $\alpha \in \Omega_A^{0,1}$ and $v, w \in \mathrm{Vect}(X)$,

$$\begin{aligned}(\mathrm{d}_A^{-1,2}\alpha)(v,w) &= (\mathrm{d}\alpha)(v - A(v), v - A(w)) \\ &= -\alpha(A[v - A(v), w - A(w)]) \\ &= (i_{F_A}\alpha)(v,w).\end{aligned}$$

■

Remark 2.55. Proposition 2.54 is a justification for the sign appearing in the definition of F_A. ♣

The following is an immediate consequence of $\mathrm{d}^2 = 0$.

Proposition 2.56. *The operators $\mathrm{d}_A^{1,0}$, $\mathrm{d}_A^{0,1}$, $\mathrm{d}_A^{2,-1}$ satisfy*

$$\begin{aligned}\left(\mathrm{d}_A^{1,0}\right)^2 + \mathrm{d}_A^{0,1}\mathrm{d}_A^{2,-1} + \mathrm{d}_A^{2,-1}\mathrm{d}_A^{0,1} &= 0, \\ \left(\mathrm{d}_A^{0,1}\right)^2 &= 0, \\ \left(\mathrm{d}_A^{2,-1}\right)^2 &= 0, \\ \mathrm{d}_A^{1,0}\mathrm{d}_A^{0,1} + \mathrm{d}_A^{0,1}\mathrm{d}_A^{1,0} &= 0, \quad \textit{and} \\ \mathrm{d}_A^{1,0}\mathrm{d}_A^{2,-1} + \mathrm{d}_A^{2,-1}\mathrm{d}_A^{1,0} &= 0.\end{aligned}$$

■

Remark 2.57.

(1) $d_A^{1,0}d_A^{2,-1} + d_A^{2,-1}d_A^{1,0}$ is the **Bianchi identity**.

(2) $d_A^{0,1}$ and $d_A^{2,-1}$ are differentials, but $d_A^{1,0}$ is not. However, it descends to a differential on $H(\Omega(X), d_A^{0,1})$ and on $H(\Omega(X), d_A^{2,-1})$. ♣

2.7 Digression: The Fröhlicher–Nijenhuis bracket

This section is a slight digression on the Fröhlicher–Nijenhuis bracket. This provides some context but is not necessary. [KMS93, §8] contains a more detailed treatment of the material discussed below.

Definition 2.58. Let $k \in \mathbf{Z}$. A **graded derivation of degree** k on $\Omega^\bullet(X)$ is an $\mathbf{R}$–linear map $\delta\colon \Omega^\bullet(X) \to \Omega^{\bullet+k}(X)$ satisfying the **graded Leibniz rule**

$$\delta(\alpha \wedge \beta) = (\delta\alpha) \wedge \beta + (-1)^{k\cdot\ell}\alpha \wedge (\delta\beta)$$

for every $\alpha \in \Omega^\ell(X), \beta \in \Omega^\bullet(X)$. The graded derivations of $\Omega^\bullet(X)$ form a graded Lie algebra $\mathrm{Der}_\bullet(\Omega^\bullet(X))$. •

Exercise 2.59. Verify the last sentence in the above definition.

Example 2.60. The exterior derivative d is a graded derivation of degree 1 of $\Omega^\bullet(X)$. If $v \in \mathrm{Vect}(X)$, then i_v is a graded derivation of degree -1 of $\Omega^\bullet$. By Cartan's formula, their graded commutator is the Lie derivative

$$\mathscr{L}_v = \mathrm{d}i_v + i_v\mathrm{d} = [i_v, \mathrm{d}];$$

itself a graded derivation of degree 0. ♠

The derivations of $C^\infty(X)$ are precisely the vector fields on X: $\mathrm{Der}(C^\infty(X)) \cong \mathrm{Vect}(X)$. Is there an analogous result for $\mathrm{Der}_\bullet(\Omega^\bullet(X))$?

Definition 2.61. Let $k \in \mathbf{N}_0$. Denote by $i_\cdot\colon\ \Omega^{k+1}(X, TX) \to \mathrm{Der}_k(\Omega^\bullet(X))$ the unique linear map satisfying

$$i_{\xi\otimes v}\alpha = \xi \wedge i_v\alpha \quad \text{for} \quad \xi \in \Omega^{k+1}(X) \quad \text{and} \quad v \in \mathrm{Vect}(X).$$

Define $\mathscr{L}_\cdot\colon\ \Omega^k(X, TX) \to \mathrm{Der}_k(\Omega^\bullet(X))$ by

$$\mathscr{L}_\Xi \coloneqq [i_\Xi, \mathrm{d}].$$ •

Exercise 2.62. Prove that $\imath$ and $\mathscr{L}$ are injective.

Exercise 2.63. For $\Xi = \sum_i \xi_i \otimes v^i \in \Omega^k(X, TX)$ prove that

$$\mathscr{L}_\Xi\alpha = \sum_i \xi_i \wedge \mathscr{L}_{v^i}\alpha + (-1)^k(\mathrm{d}\xi_i) \wedge i_{v^i}\alpha.$$

Proposition 2.64. *Let X be a smooth manifold. Let $k \in \mathbf{Z}$. The map $\mathscr{L} + \imath\colon\ \Omega^k(X, TX) \oplus \Omega^{k+1}(X, TX) \to \mathrm{Der}_k(\Omega^\bullet(X))$ is an isomorphism. Moreover, $\delta \in \mathrm{im}\,\mathscr{L}$ if and only if $[\delta, \mathrm{d}] = 0$; and $\varepsilon \in \mathrm{im}\,\mathscr{L}$ if and only if $\varepsilon(\Omega^0(X)) = 0$.*

Proof. Every $\delta \in \mathrm{Der}_k(\Omega^\bullet(X))$ is determined by its restriction to $\Omega^0(X) \oplus \Omega^1(X)$. If $v_1, \ldots, v_k$, then the map

$$f \mapsto (\delta f)(v_1, \ldots, v_k)$$

is a derivation of $\Omega^0(X) = C^\infty(X)$. Hence, there is a unique vector field $\Xi(v_1, \ldots, v_k)$ such that

$$(\delta f)(v_1, \ldots, v_k) = \mathscr{L}_{\Xi(v_1,\ldots,v_k)} f.$$

A moment's thought shows that $(v_1, \ldots, v_k) \mapsto \Xi(v_1, \ldots, v_k)$ is tensorial. Therefore, it defines a $\Xi \in \Omega^k(X, TX)$. The derivation $\varepsilon := \delta - \mathscr{L}_\Xi$ vanishes on $\Omega^0(X)$.

If $f \in C^\infty(X)$ and $\alpha \in \Omega^1(X)$, then

$$\varepsilon(f\alpha) = f \cdot \varepsilon\alpha;$$

that is: $\varepsilon\colon\ \Omega^1(X) \to \Omega^k(X)$ is tensorial. Therefore, there is a $\Theta \in \Omega^k(X, TX)$ with $\varepsilon = i_\Theta$.

By construction, $\delta = i_\Theta + \mathscr{L}_\Xi$ on $\Omega^0(X) \oplus \Omega^1(X)$. This proves the first assertion. The vanishing criterion for Θ is obvious. A brief computation shows that $[\mathscr{L}_\Xi, \mathrm{d}] = 0$. Since

$$[i_\Theta, \mathrm{d}] = \mathscr{L}_\Theta,$$

the final assertion follows. ■

Exercise 2.65. What are Θ and Ξ for $\delta = \mathrm{d}$?

Exercise 2.66. Use Proposition 2.64 to prove Cartan's formula $\mathscr{L}_v = \mathrm{d}i_v + i_v\mathrm{d}$.

The identification can be used to define the Lie bracket $[\cdot,\cdot]$ on $\mathrm{Vect}(X)$. Since

$$[[\mathscr{L}_\Theta, \mathscr{L}_\Xi], \mathrm{d}] = 0,$$

one obtains a graded Lie bracket on $\Omega^\bullet(X, TX)$.

Definition 2.67. The **Fröhlicher–Nijenhuis bracket** is the map

$$[\cdot,\cdot]\colon\ \Omega^\bullet(X, TX) \otimes \Omega^\bullet(X, TX) \to \Omega^\bullet(X, TX)$$

characterised by

$$[\mathscr{L}_\Theta, \mathscr{L}_\Xi] = \mathscr{L}_{[\Theta,\Xi]}.$$ •

It turns out (somewhat miraculously in my opinion) that Fröhlicher–Nijenhuis bracket consistently shows up as an obstruction to integrability.

Proposition 2.68. *If $\theta \in \Omega^1(X, TX)$, then the **Nijenhuis tensor***

$$N_\theta := -\frac{1}{2}[\theta, \theta] \in \Omega^2(X, TX)$$

satisfies

$$N_\theta(v, w) = -\theta(\theta([v, w])) - [\theta(v), \theta(w)] + \theta([\theta(v), w] + [v, \theta(w)]).$$

The proof relies on the following.

Proposition 2.69. *For $\theta, \alpha \in \Omega^1(X, TX)$ and $v, w \in \mathrm{Vect}(X)$*

$$(\mathscr{L}_\theta \alpha)(v,w) = \mathscr{L}_{\theta(v)}(\alpha(w)) - \mathscr{L}_{\theta(w)}(\alpha(v)) - \alpha([\theta(v), w]) - \alpha([v, \theta(w)]) + \alpha(\theta([v,w])).$$

Proof. Let $\theta, \alpha \in \Omega^1(X, TX)$ and $v, w \in \mathrm{Vect}(X)$. Since

$$(\mathrm{d}\alpha)(v,w) = \mathscr{L}_v(\alpha(w)) - \mathscr{L}_w(\alpha(v)) - \alpha([v,w]),$$

by definition of $\mathscr{L}_\theta$,

$$\begin{aligned}
(\mathscr{L}_\theta \alpha)(v,w) &= (i_\theta \mathrm{d}\alpha - \mathrm{d}i_\theta a)(v,w) \\
&= \mathscr{L}_{\theta(v)}(\alpha(w)) - \mathscr{L}_w(\alpha(\theta(v))) - \alpha([\theta(v), w]) \\
&\quad + \mathscr{L}_v(\alpha(\theta(w))) - \mathscr{L}_{\theta(w)}(\alpha(v)) - \alpha([v, \theta(w)]) \\
&\quad - \mathscr{L}_v(\alpha(\theta(w))) + \mathscr{L}_w(\alpha(\theta(v))) + \alpha(\theta([v,w])) \\
&= \mathscr{L}_{\theta(v)}(\alpha(w)) - \alpha([\theta(v), w]) \\
&\quad - \mathscr{L}_{\theta(w)}(\alpha(v)) - \alpha([v, \theta(w)]) \\
&\quad + \alpha(\theta([v,w])).
\end{aligned}$$

■

Proof of Proposition 2.68. $N_\theta \in \Omega^2(X, TX)$ is determined by the action on $\Omega^0(X) = C^\infty(X)$. For $\theta \in \Omega^k(TX, X)$

$$(\mathscr{L}_\theta f)(v_1, \ldots, v_k) = \mathscr{L}_{\theta(v_1, \ldots, v_k)} f;$$

in particular, $(\mathscr{L}_\theta f)(v) = \mathscr{L}_{\theta(v)} f$. Therefore, using Proposition 2.69,

$$\begin{aligned}
(\mathscr{L}_\theta \mathscr{L}_\theta f)(v,w) &= \mathscr{L}_{\theta(v)} \mathscr{L}_{\theta(w)} f - \mathscr{L}_{\theta([\theta(v), w])} f \\
&\quad - \mathscr{L}_{\theta(w)} \mathscr{L}_{\theta(v)} f - \mathscr{L}_{\theta([v, \theta(w)])} f \\
&\quad + \mathscr{L}_{\theta(\theta([v,w]))} f \\
&= \mathscr{L}_{[\theta(v), \theta(w)]} f - \mathscr{L}_{\theta([\theta(v), w])} f \\
&\quad - \mathscr{L}_{\theta([v, \theta(w)])} f + \mathscr{L}_{\theta(\theta([v,w]))} f.
\end{aligned}$$

This implies the assertion. ■

Remark 2.70. If $J \in \mathrm{End}(TX)$ is an almost complex structure (that is: $J^2 = -\mathbf{1}$), then the vanishing of N_J characterises the integrability of J. Indeed, the Newlander–Nirenberg theorem asserts that $N_J = 0$ if and only if X admits a holomorphic structure which induces the almost complex structure J. ♣

Exercise 2.71. Let $p \colon X \to B$ be fibre bundle. Let $A \in \mathscr{A}(p)$. Regard the connection 1–form $A \colon \Omega^1(X, V_p)$ as TX–valued 1–form. Prove that

$$F_A = N_A$$

Remark 2.72. The graded Jacobi identity implies the **Bianchi identity**

$$[A, F_A] = 0.$$

♣

Definition 2.73. Let X, Y be smooth manifolds Let $f\colon X \to Y$ be a smooth map. $\Theta \in \Omega^k(X, TX)$ and $\Xi \in \Omega^k(Y, TY)$ are f**–related** if for every $x \in X$, $v_1, \dots, v_k \in T_xX$

$$T_xf(\Theta(v_1, \dots, v_k)) = \Xi(T_xf(v_1), \dots, T_xf(v_k)). \qquad \bullet$$

Proposition 2.74. *Let X, Y be smooth manifolds Let $f\colon X \to Y$ be a smooth map. Let $\Theta_1, \Theta_2 \in \Omega^\bullet(X, TX)$ and $\Xi_1, \Xi_2 \in \Omega^\bullet(Y, TY)$. If Θ_i and Ξ_i are f–related, then $[\Theta_1, \Theta_2]$ and $[\Xi_1, \Xi_2]$ are f–related.*

Proof. Exercise. ∎

Proposition 2.75. *Let $p\colon X \to B$ be a fibre bundle. Let $A \in \mathscr{A}(p)$. The operators $\mathrm{d}_A^{1,0}$, $\mathrm{d}_A^{0,1}$, $\mathrm{d}_A^{2,-1}$ satisfy*

$$\mathrm{d}_A^{1,0} = \mathscr{L}_{\mathrm{id}_{TX} - A} - 2i_{F_A}, \quad \mathrm{d}_A^{0,1} = \mathscr{L}_A + i_{F_A}, \quad \textit{and} \quad \mathrm{d}_A^{2,-1} = i_{F_A}.$$

Proof. The next two steps determine explicit formulae for $\mathrm{d}_A^{1,0}$ and $\mathrm{d}_A^{0,1}$. The computations are longish and not particularly illuminating.

Step 1. $\mathrm{d}_A^{1,0} = \mathscr{L}_{\mathrm{id}_{TX} - A} - 2i_{F_A}$.

It suffices to verify the identity on $C^\infty(X)$ and $\Omega^1(X)$.
For $f \in C^\infty(X)$

$$\begin{aligned}\mathrm{d}_A^{1,0} f &= \mathrm{d}f \circ (\mathrm{id}_{TX} - A)\\ &= (\mathscr{L}_{\mathrm{id}_{TX} - A} - 2i_{F_A})f.\end{aligned}$$

For $\alpha \in \Omega^1(X)$ and $v, w \in \mathrm{Vect}(X)$

$$\begin{aligned}(\mathscr{L}_{\mathrm{id}_{TX} - A}\alpha)(v, w) &= \mathscr{L}_{v - A(v)}(\alpha(w)) - \mathscr{L}_{w - A(w)}(\alpha(v))\\ &\quad - \alpha([v - A(v), w]) - \alpha([v, w - A(w)]) + \alpha([v, w] - A([v, w])).\end{aligned}$$

For $\alpha \in \Omega_A^{1,0}(X)$ and $v, w \in \mathrm{Vect}(X)$, since $\alpha \circ \theta = 0$,

$$\begin{aligned}(\mathrm{d}_A^{1,0}\alpha)(v, w) &= \mathrm{d}\alpha(v - A(v), w - A(w))\\ &= \mathscr{L}_{v - A(v)}(\alpha(w)) - \mathscr{L}_{w - \theta(w)}(\alpha(v)) - \alpha([v - \theta_a(v), w - \theta_a(w)])\\ &= (\mathscr{L}_{\mathrm{id}_{TX} - A}\alpha - 2i_{F_A}\alpha)(v, w)\\ &\quad + \alpha([v - A(v), w]) + \alpha([v, w - A(w)])\\ &\quad - \alpha([v, w]) - \alpha([v - \theta_a(v), w - \theta_a(w)]).\end{aligned}$$

The sum of the last four term vanishes because

$$[v - A(v), w] + [v, w - A(w)] - [v, w] - [v - \theta_a(v), w - \theta_a(w)] = -[A(v), A(w)]$$

is a vertical vector field.

For $\alpha \in \Omega^{0,1}_A(X)$ and $v, w \in \mathrm{Vect}(X)$, since $\alpha \circ \theta = \alpha$,

$$
\begin{aligned}
\mathrm{d}^{1,0}_A \alpha(v,w) &= \mathrm{d}\alpha(A(v), w - A(w)) + \mathrm{d}\alpha(v - A(v), A(w)) \\
&= \mathscr{L}_{v-A(v)}(\alpha(w)) - \mathscr{L}_{w-A(w)}(\alpha(v)) \\
&\quad - \alpha([\theta_a(v), w - \theta_a(w)]) - \alpha([v - \theta_a(v), \theta_a(w)]) \\
&= \big(\mathscr{L}_{\mathrm{id}_{TX}-A}\alpha - 2 i_{F_A}\alpha\big)(v,w) \\
&\quad + \alpha([v - A(v), w]) + \alpha([v, w - A(w)]) \\
&\quad - 2\alpha([v - A(v), w - \theta(w)] \\
&\quad - \alpha([\theta_a(v), w - \theta_a(w)]) - \alpha([v - \theta_a(v), \theta_a(w)]).
\end{aligned}
$$

The sum of the last five terms vanishes.

Step 2. $\mathrm{d}^{0,1}_A = \mathscr{L}_A + i_{F_A}$.

For $f \in C^\infty(X)$

$$
\begin{aligned}
\mathrm{d}^{0,1}_A f &= \mathrm{d}f \circ A \\
&= (\mathscr{L}_A + i_{F_A}) f.
\end{aligned}
$$

For $\alpha \in \Omega^1(X)$ and $v, w \in \mathrm{Vect}(X)$

$$
\begin{aligned}
(\mathscr{L}_A \alpha)(v,w) &= \mathscr{L}_{A(v)}(\alpha(w)) - \mathscr{L}_{A(w)}(\alpha(v)) \\
&\quad - \alpha([A(v), w]) - \alpha([v, A(w)]) + \alpha(A([v,w])).
\end{aligned}
$$

For $\alpha \in \Omega^{1,0}_A(X)$ and $v, w \in \mathrm{Vect}(X)$, since $\alpha \circ \theta = 0$,

$$
\begin{aligned}
(\mathrm{d}^{0,1}_A \alpha)(v,w) &= \mathrm{d}\alpha(A(v), w - A(w)) + \mathrm{d}\alpha(v - A(v), A(w)) \\
&= \mathscr{L}_{A(v)}(\alpha(w)) - \mathscr{L}_{A(w)}(\alpha(v)) \\
&\quad - \alpha([A(v), w - A(w)]) - \alpha([v - A(v), A(w)]) \\
&= (\mathscr{L}_A \alpha + i_{F_A}\alpha)(v,w) \\
&\quad + \alpha([A(v), w]) + \alpha([v, A(w)]) \\
&\quad - \alpha([A(v), w - A(w)]) - \alpha([v - A(v), A(w)]).
\end{aligned}
$$

The sum of the last four term vanishes because

$$
[A(v), w] + [v, A(w)] - [A(v), w - A(w)] - [v - A(v), A(w)] = 2[A(v), A(w)]
$$

is a vertical vector field.

For $\alpha \in \Omega^{0,1}_A(X)$ and $v, w \in \mathrm{Vect}(X)$, since $\alpha \circ \theta = \alpha$,

$$
\begin{aligned}
(\mathrm{d}^{0,1}_A \alpha)(v,w) &= \mathrm{d}\alpha(A(v), A(w)) \\
&= \mathscr{L}_{A(v)}(\alpha(w)) - \mathscr{L}_{\theta(w)}(\alpha(v)) - \alpha([A(v), A(w)]) \\
&= (\mathscr{L}_A \alpha + i_{F_A}\alpha)(v,w) \\
&\quad + \alpha([A(v), w]) + \alpha([v, A(w)]) - \alpha([v,w]) \\
&\quad + \alpha([v - A(v), w - A(w)]) - \alpha([A(v), A(w)]).
\end{aligned}
$$

The sum of the last five term vanishes. ∎

2.8 The spectral sequence of a filtered complex

Here is some homological algebra in preparation of the Leray–Serre spectral sequence; see, e.g., [Wei94, §5.4].

Definition 2.76. A **spectral sequence** is a sequence

$$(E_r, \mathrm{d}_r)_{r \in \mathbf{N}_0}$$

of $\mathbf{Z}^2$–graded complexes such that, for every $r \in \mathbf{N}_0$, d_r has bidegree $(r, -r+1)$ and

$$\mathrm{H}(E_r) \cong E_{r+1}.$$

The spectral sequence **degenerates** if $\mathrm{d}_r = 0$ for $r \gg 1$. In this case, E_∞ denotes a bigraded vector space with $E_\infty \cong E_r$ for $r \gg 1$. •

Definition 2.77. Let (C, d) be a $\mathbf{Z}$–graded complex. A **descending filtration** of (C, d) is a descending sequence of subcomplexes $(F^j C)_{j \in \mathbf{Z}}$:

$$C \supset \cdots \supset F^j C \supset F^{j+1} C \supset \cdots .$$

The filtration FC is **bounded** if $F^j C = C$ for $j \ll 1$ and $F^j C = 0$ for $j \gg 1$. The **associated $\mathbf{Z}^2$–graded complex** of (FC, d) is

$$\operatorname{Gr} C = \bigoplus_{p,q \in \mathbf{Z}} \operatorname{Gr}^p C^q \quad \text{with} \quad \operatorname{Gr}^p C^q := F^p C^q / F^{p+1} C^q$$

equipped with the differential of bidegree $(0, 1)$ induced by d. •

Lemma 2.78 (Spectral sequence of a filtered differential $\mathbf{Z}$–graded module). *Let (C, d) be a $\mathbf{Z}$–graded differential module equipped with a descending filtration FC. For $p, q, r \in \mathbf{Z}$ set*

$$\begin{aligned} Z_r^{p,q} &:= F^p C^{p+q} \cap \mathrm{d}^{-1}(F^{p+r} C^{p+q+1}) \quad \text{and} \\ B_r^{p,q} &:= \mathrm{d} Z_{r-1}^{p-r+1, q+r-2} = F^p C^{p+q} \cap \mathrm{d}(F^{p-r+1} C^{p+q-1}). \end{aligned}$$

There is a spectral sequence (E_r, d_r) with

$$E_r^{p,q} = \frac{Z_r^{p,q}}{B_r^{p,q} + Z_{r-1}^{p+1,q-1}} \quad \text{and} \quad \mathrm{d}_r[x] = [\mathrm{d}x];$$

in particular,

$$E_1^{p,q} = \mathrm{H}^{p,q}(\operatorname{Gr}^p C, \mathrm{d}).$$

If FC is bounded, then (E_r, d_r) degenerates and

$$E_\infty^{p,q} \cong \operatorname{Gr}^p \mathrm{H}^{p+q}(C, \mathrm{d}).$$

Proof. Since $\mathrm{d}Z_r^{p,q} \subset Z_r^{p+r, q-r+1}$, $\mathrm{d}B_r^{p,q} = 0$, and $\mathrm{d}Z_{r-1}^{p+1,q-1} \subset \mathrm{d}Z_{r-1}^{p+r+1,q-r}$, d descends to $\mathrm{d}_r^{p,q} \colon E_r^{p,q} \to E_r^{p+r, q-r+q}$.

By construction,

$$\operatorname{im} \mathrm{d}_r^{p-r,q+r-1} = \frac{B_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1}}{B_r^{p,q} + Z_{r-1}^{p+1,q-1}}.$$

A moment's thought reveals that

$$\ker \mathrm{d}_r^{p,q} = \frac{Z_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1}}{B_r^{p,q} + Z_{r-1}^{p+1,q-1}}.$$

Moreover, since $Z_{r+1}^{p,q} \cap Z_{r-1}^{p+1,q-1} = Z_r^{p+1,q-1}$,

$$\frac{\ker \mathrm{d}_r^{p,q}}{\operatorname{im} \mathrm{d}_r^{p-r,q+r-1}} \cong \frac{Z_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1}}{B_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1}} \cong E_{r+1}^{p,q}.$$

This establishes the existence of the desired spectral sequence.

It remains to establish the convergence if FC is bounded. To this end it suffices to observe that, for $r \gg 1$, $Z_r^{p,q} = F^p \ker(\mathrm{d}\colon\, C^{p+q} \to C^{p+q+1})$ and $B_r^{p,q} = F^p \operatorname{im}(\mathrm{d}\colon\, C^{p+q} \to C^{p+q+1})$. ■

Remark 2.79. Although Lemma 2.78 appears rather unwieldy, it is a very powerful tool. It turns out that often sufficient information can be extracted without going through the grueling process of unravelling the details of the construction in Lemma 2.78. ♣

Example 2.80. A **double complex** is a $\mathbf{Z}^2$–graded vector space C together with two differentials d of bidegree $(1,0)$ and δ of bidegree $(0,1)$ such that $\mathrm{d}\delta + \delta\mathrm{d} = 0$. The associated **total complex** is the graded complex

$$\operatorname{Tot} C^k := \bigoplus_{p+q=k} C^{p,q}$$

equipped with the differential $\mathrm{d} + \delta$. $\operatorname{Tot} C$ has two filtrations

$$'F^p \operatorname{Tot} C^n := \bigoplus_{\substack{n=p'+q \\ p' \geqslant p}} C^{p,q}$$

$$''F^q \operatorname{Tot} C^n := \bigoplus_{\substack{n=p+q' \\ q' \geqslant q}} C^{p,q}.$$

Consequently, $(\operatorname{Tot} C, \mathrm{d})$ has two spectral sequences. The E_2 pages of these spectral sequences are

$$'E_2^{p,q} \cong \mathrm{H}^p(\mathrm{H}^q(C,\delta),\mathrm{d}) \quad \text{and} \quad ''E_2^{p,q} \cong \mathrm{H}^q(\mathrm{H}^p(C,\mathrm{d}),\delta). \qquad ♠$$

Example 2.81. It is an frivolous but enlightening exercise to derive the snake lemma from considerations of the spectral sequences of the obvious double complex inherent in its setting. The setting of the snake lemma can be understood as a double complex:

$$\begin{tikzcd}
B_0 \arrow[r] & B_1 \arrow[r] & B_2 \\
A_0 \arrow[r] \arrow[u, "f_0"] & A_1 \arrow[r] \arrow[u, "f_1"] & A_2 \arrow[u, "f_2"]
\end{tikzcd}$$

Since the rows are exact, $''E_1 = 0$. On the other hand, $'E_1$ is

$$\begin{array}{ccccc} \operatorname{coker} f_0 & \longrightarrow & \operatorname{coker} f_1 & \longrightarrow & \operatorname{coker} f_2 \\ \\ \ker f_0 & \longrightarrow & \ker f_1 & \longrightarrow & \ker f_2. \end{array}$$

The $'E_2$ page is the cohomology of $'E_1$. For degree reasons, every differential except possibly $\mathrm{d}_2^{0,1}$ vanishes. As a consequence, $'E_2^{p,q} = 0$ unless $(p,q) \in \{(0,1),(2,0)\}$ and $\mathrm{d}_2^{0,1}$ is an isomorphism. This yields exact sequences

$$\ker(\operatorname{coker} f_0 \to \operatorname{coker} f_1) \hookrightarrow \operatorname{coker} f_1 \twoheadrightarrow \operatorname{coker} f_2$$

and

$$\ker f_0 \hookrightarrow \ker f_1 \twoheadrightarrow \operatorname{coker}(\ker f_1 \to \ker f_2).$$

and an isomorphism $\operatorname{coker}(\ker f_1 \to \ker f_2) \cong \ker(\operatorname{coker} f_0 \to \operatorname{coker} f_1)$. Splicing these short exact sequences gives the familiar long exact sequence.

$$\ker f_0 \hookrightarrow \ker f_1 \twoheadrightarrow \ker f_2 \to \operatorname{coker} f_0 \to \operatorname{coker} f_1 \twoheadrightarrow \operatorname{coker} f_2.$$ ♠

Lemma 2.82 (Spectral sequence of a filtered differential $\mathbf{Z}$–graded algebra). *Let (A, d) be a differential $\mathbf{Z}$–graded algebra equipped with a descending filtration $F^\bullet A^\bullet$. There is a product $\cdot = \cdot_r \colon E_r^{p,q} \otimes E_r^{p',q'} \to E_r^{p+p',q+q'}$ such that $[x][y] = [xy]$, d_r is derivation, and the isomorphism $\mathrm{H}(E_r) \cong E_{r+1}$ preserves the product.*

If $F^\bullet A$ is bounded, then $E_\infty \cong \mathrm{Gr}^p \mathrm{H}(A, \mathrm{d})$ is compatible with the product.

Proof. By direct inspection,

$$Z_r^{p,q} \cdot Z_r^{p',q'} \subset Z_r^{p+p',q+q'} \quad \text{and} \quad Z_r^{p,q} \cdot (B_r^{p',q'} + Z_{r-1}^{p'+1,q'-1}) \subset B_r^{p+p',q+q'} + Z_{r-1}^{p+p'+1,q+q'-1}.$$

This proves that $\cdot_r$ is defined. That d_r is a derivation and the rest are either obvious or an exercise. ■

2.9 The Leray–Serre spectral sequence

The following discusses the Leray–Spectral sequence in de Rham cohomology.

Definition 2.83. Let $p\colon X \to B$ be a fibre bundle. Define the **horizontal filtration** $F\Omega(X)$ by

$$\begin{aligned} F^\ell\Omega^k(X) &\coloneqq \{\alpha \in \Omega^\bullet(X) : i_{v_1} \dots i_{v_{k+1-\ell}}\alpha = 0 \text{ for every } v_1, \dots, v_{k+1-\ell} \in V_p\} \\ &= \Gamma(X, F^\ell \Lambda^k T^* X) \end{aligned}$$

with

$$F^\ell \Lambda^k T^* X \coloneqq \Lambda^\ell p^* T^* B \otimes \Lambda^{k-\ell} T^* X.$$

(There $p^*T^*B \subset T^*X$ is the annihilator of V_p.) The **Leray–Serre spectral sequence** is the spectral sequence associated with this filtration. •

Remark 2.84. $F^k\Omega^k(X) = \Omega^k_{\mathrm{hor}}(X)$; this is the penultimate step of the filtration. ♣

The E_2 page of the Leray–Serre spectral sequence can be described as follows.

Proposition 2.85. *Let $p\colon X \to B$ be a proper fibre bundle. There is a unique graded local system $(\mathscr{H}^\bullet_{\mathrm{dR}}(p), \nabla)$ over B such that:*

(1) *The fibre over $b \in B$ is $\mathrm{H}^k_{\mathrm{dR}}(p^{-1}(b))$.*

(2) *If $U \subset B$ is open, $[\alpha] \in \mathrm{H}_{\mathrm{dR}}(p^{-1}(U))$, then $s_{[\alpha]} \in \Gamma(U, \mathscr{H}^k_{\mathrm{dR}}(p))$ defined by*

$$s_{[\alpha]}(b) := i_b^*[\alpha]$$

is parallel.

Definition 2.86. $(\mathscr{H}^\bullet_{\mathrm{dR}}(p), \nabla)$ is the **Gauß–Manin local system**. •

Proof of Proposition 2.85. If U is as in Definition 2.1, then (2) defines a bijection

$$\tau_U\colon U \times \mathrm{H}^k_{\mathrm{dR}}(p^{-1}(U)) \to \bigcup_{b\in U} \mathrm{H}^k_{\mathrm{dR}}(p^{-1}(b)).$$

It suffices to prove that if V is as in Definition 2.1, then the map

$$U \cap V \to \mathrm{Hom}(\mathrm{H}^k_{\mathrm{dR}}(p^{-1}(V)), \mathrm{H}^k_{\mathrm{dR}}(p^{-1}(U)))$$

defined by $b \mapsto \mathrm{pr}_2 \circ \tau_U^{-1} \circ \tau_V(b, \cdot)$ is locally constant. This is an immediate consequence of the homotopy-invariance of de Rham cohomology. ■

Proposition 2.87. *Let $p\colon X \to B$ be a fibre bundle. The Leray–Serre spectral sequence associated with p satisfies*

$$E_2^{p,q} \cong \mathrm{H}^p_{\mathrm{dR}}(B, \mathscr{H}^q_{\mathrm{dR}}(p)).$$

Proof. This is an exercise in tracing through the definitions; see [GH94, p. 464] for hints. ■

Corollary 2.88. *Let $p\colon X \to B$ be a fibre bundle. If B is simply-connected, then*

$$E_2^{p,q} \cong \mathrm{H}^p_{\mathrm{dR}}(B) \otimes \mathrm{H}^q_{\mathrm{dR}}(p^{-1}(b)).$$ ■

If B is not simply-connected, then $\mathscr{H}^q_{\mathrm{dR}}(p)$ might have monodromy.

Example 2.89. Let F be smooth manifold. Let $f \in \mathrm{Diff}(F)$. Denote X_f the **mapping torus** of f; that is:

$$X_f := ([0,1] \times F)/\sim$$

with denoting the equivalence relation generated by $(0, x) \sim (1, f(x))$. X_f is a smooth manifold and the projection map $p\colon X \to S^1 = \mathbf{R}/\mathbf{Z}$ is a fibre bundle. The monodromy of the Gauß–Manin connection on $\mathscr{H}^\bullet_{\mathrm{dR}}(p)$ is precisely the action of $\mathbf{Z}$ on $\mathrm{H}^\bullet_{\mathrm{dR}}(X)$ generated by f^*. ♠

The Leray–Serre spectral sequence does not usually degenerate at E_2. This can be seen, e.g., for the Hopf bundle $S^{2n+1} \to \mathbf{C}P^n$. However, there are two notable exceptions.

Proposition 2.90. *If $p\colon X \to B$ is a proper smooth covering map, then*

$$\mathrm{H}^\bullet_{\mathrm{dR}}(X) \cong \mathrm{H}^\bullet_{\mathrm{dR}}(B, p_*\underline{\mathbf{R}}).$$

Here $p_*\underline{\mathbf{R}}$ is the (sheaf-theoretic) push-forward of the local system $\underline{\mathbf{R}}$ on X. ■

Theorem 2.91 (Deligne [ref?]). *If X, B are closed Kähler, then Leray–Serre spectral sequence degenerates at E_2.* ■

2.10 Fibre integration

Definition 2.92. Let $p\colon X \to B$ be a fibre bundle of relative dimension d. A **fibre orientation** on p is a an orientation on the line bundle $\det(V_p) := \Lambda^d V_p \to X$. •

Proposition 2.93. *Let $d \in \mathbf{N}_0$. Let $p\colon X \to B$ be a proper fibre bundle of relative dimension d together with a fibre orientation.*

(1) *There is a unique linear map $p_*\colon \Omega^\bullet(X) \to \Omega^{\bullet-d}(B)$ such that for every $\alpha \in \Omega^{d+k}(X)$, $b \in B$, and $\tilde v_1, \ldots, \tilde v_k \in \Gamma(TX|_{p^{-1}(b)})$ lifts of $v_1, \ldots, v_k \in T_bB$*

$$(p_*\alpha)(v_1, \ldots, v_k) = \int_{p^{-1}(b)} i_{\tilde v_k} \cdots i_{\tilde v_1} \alpha|_{p^{-1}(b)}. \tag{2.94}$$

(2) *For every $\alpha \in \Omega^\bullet(X)$ and $\beta \in \Omega^\bullet(B)$*

$$p_*(\alpha \wedge p^*\beta) = p_*\alpha \wedge \beta.$$

(3) *Suppose that B is oriented. For every $\alpha \in \Omega^\bullet(X)$*

$$\int_X \alpha = \int_B p_*\alpha$$

(4) *Suppose that $\partial B = \varnothing$. Set $\partial p := p|_{\partial X}\colon \partial X \to B$. For every $\alpha \in \Omega^\bullet(X)$*

$$p_*\mathrm{d}\alpha - (-1)^d \mathrm{d}p_*\alpha = \partial p_*\alpha.$$

Proof. The right-hand side of (2.94) is independent of the lifts $\tilde v_1, \ldots, \tilde v_k$. To verify that (2.94) does define the map p_* it suffices to it suffices the require smoothness. It is enough to verify this for $\mathrm{pr}_B\colon B \times F \to B$. This proves (1).

(2) is evident from the construction. (3) follows from Fubini's theorem.

(4) is a consequence of Stokes' theorem; indeed: for every $\alpha \in \Omega^{k+d}(X)$ and $\beta \in \Omega^\bullet(B)$

$$\begin{aligned}
\int_B (p_*\mathrm{d}\alpha) \wedge \beta &= \int_B p_*(\mathrm{d}\alpha \wedge p^*\beta) \\
&= \int_X \mathrm{d}\alpha \wedge p^*\beta \\
&= \int_{\partial X} \alpha \wedge (\partial p)^*\beta + (-1)^{k+d} \int_X \alpha \wedge p^*\mathrm{d}\beta \\
&= \int_B (\partial p)_*\alpha \wedge p^*\beta + (-1)^{k+d}(p_*\alpha) \wedge \mathrm{d}\beta \\
&= \int_B (\partial p)_*\alpha \wedge p^*\beta + (-1)^d \mathrm{d}(p_*\alpha) \wedge \mathrm{d}\beta.
\end{aligned}$$ ■

Definition 2.95. In the situation of Proposition 2.93, the map p_* is the **fibre integration**. •

If $\partial X = \partial B = \varnothing$, then p_* descends to de Rham cohomology $p_*\colon \mathrm{H}^\bullet_{\mathrm{dR}}(X) \to \mathrm{H}^{\bullet-d}_{\mathrm{dR}}(B)$. Set $K^\bullet := \ker p_*\colon \Omega^\bullet(X) \to \Omega^{\bullet-d}(B)$. The short exact sequence

$$0 \to K^\bullet \to \Omega^\bullet(X) \to \Omega^{\bullet-d}(B) \to 0$$

induces a long exact sequence

$$\cdots \to \mathrm{H}^k(K^\bullet) \to \mathrm{H}^k_{\mathrm{dR}}(X) \to \mathrm{H}^{k-d}_{\mathrm{dR}}(B) \xrightarrow{\delta} \mathrm{H}^{k+1}(K^\bullet) \cdots .$$

Whether this is useful or not depends on whether one can compute $\mathrm{H}^k(K^\bullet)$. It might help to observe that $p^*\Omega(B) \subset K$.

2.11 The Gysin sequence

Definition 2.96. Let $p\colon X \to B$ be a proper fibre bundle of relative dimension $d > 0$. A **relative volume form** on p is a nowhere-vanishing section $\mathrm{vol}_{X/B} \in \Gamma(X, \Lambda^d V_p^*)$. A **relative probability form** is a relative volume form $\mathrm{vol}_{X/B}$ with $p_*\mathrm{vol}_{X/B} = 1 \in C^\infty(B)$. •

Given a relative probability form $\mathrm{vol}_{X/B}$, is there an $\eta \in \Omega^d(x)$ satisfying

(1) $\eta|_{p^{-1}(b)} = \mathrm{vol}_{X/B}|_{p^{-1}(b)}$ for every $b \in B$, and

(2) $\mathrm{d}\eta = 0$?

Certainly, it possible to choose η satisfying the first condition and this defines an element $[\eta] \in E_1^{0,d}$. In fact, this depends only on $\mathrm{vol}_{X/B}$. Whether the second condition can be satisfied is asking if η can be chosen so that $[\eta]$ lifts to $E_\infty^{0,d}$. $[\eta]$ lifts to $E_2^{0,k}$ if and only if $[\mathrm{vol}_{X/B}] \in \mathrm{H}^0(B, \mathscr{H}^k_{\mathrm{dR}}(p))$ if and only if $p_*\mathrm{vol}_{X/B}$ is locally constant. There Leray–Serre spectral sequence gives a sequence of obstructions. The Hopf bundle $p\colon S^3 \to S^2$ shows that these obstructions can be non-trivial.

If the fibres of p are rational homology spheres, i.e., $\mathrm{H}_{\mathrm{dR}}(p^{-1}(b)) \cong \mathrm{H}_{\mathrm{dR}}(S^d)$, then the above obstructions can be understood more concretely.

Definition 2.97. A **rational homology sphere bundle of relative dimension** d is a proper fibre bundle $p\colon X \to B$ such that, for every $b \in B$, $\mathrm{H}_{\mathrm{dR}}(p^{-1}(b)) \cong \mathrm{H}_{\mathrm{dR}}(S^d)$. •

Henceforth, assume the above.

The E_2 page ($\cong E_{d+1}$ page) of the Leray–Serre spectral sequence is $\mathrm{H}^\bullet_{\mathrm{dR}}(B) \otimes \mathrm{H}^\bullet_{\mathrm{dR}}(S^d)$; indeed, the choice of a relative probability form trivialises $\mathscr{H}^d_{\mathrm{dR}}(p)$ and $\mathscr{H}^0_{\mathrm{dR}}(p)$ is always trivial.

Definition 2.98. Consider a rational homology sphere bundle of relative dimension k, $p\colon X \to B$, together with a relative probability form $\mathrm{vol}_{X/B}$. Consider the Lerray–Serre spectral sequence of p. Evidently, By degree considerations, $[\mathrm{vol}_{X/B}]$ induces an $[\eta] \in E_{d+1}^{0,d}$. The **Euler class of** p is

$$e(p) := \mathrm{d}_{d+1}[\eta] \in E_{d+1}^{d+1,0} \cong \mathrm{H}^{d+1}_{\mathrm{dR}}(B).$$

This is independent of the choice of $\mathrm{vol}_{X/B}$, but depends on the orientation. •

Example 2.99. Suppose $p\colon X \to B$ is a fibre bundle with S^1 fibres. Choose a relative probability form $\mathrm{vol}_{X/B}$. Let $A \in \mathscr{A}(p)$ be an Ehresmann connection. A induces a unique lift $\eta \in \Omega^{0,1}_A(X)$ of $\mathrm{vol}_{X/B}$. It satisfies

$$\mathrm{d}\eta = \mathrm{d}^{1,0}_A\eta + \mathrm{d}^{2,-1}_A\eta.$$

If A is volume-preserving, that is, $\mathrm{d}^{1,0}_A\mathrm{vol}_{X/B} = 0$, then

$$e(p) = \mathrm{d}^{2,-1}_A\mathrm{vol}_{X/B} = i_{F_A}\mathrm{vol}_{X/B}.$$

Note that $\mathrm{d}^{2,-1}_A\mathrm{vol}_{X/B}$ is horizontal and

$$\mathrm{d}^{0,1}_A\mathrm{d}^{2,-1}_A\mathrm{vol}_{X/B} = \mathrm{d}^{2,-1}_A\mathrm{d}^{0,1}_A\mathrm{vol}_{X/B} = 0.$$

Therefore, $\mathrm{d}^{2,-1}_A\mathrm{vol}_{X/B}$ is the pullback of a form on B (as expected.) This also shows that the flat connection on $ST\Sigma \to \Sigma$ discussed earlier is cannot be volume-preserving. ♠

Example 2.100. Let $V \to B$ be an Euclidean vector bundle of rank $d + 1 = 2k + 1$. Denote by $S \subset V$ the sphere bundle and by $p\colon S \to B$ the unit-sphere bundle. Consider the anti-podal map $a\colon p \to p$. Pulling-back by a flips is compatible with the filtration and therefore descends to a automorphism on the Leray–Serre spectral sequence. It flips sign of $\mathrm{vol}_{S/B}$ and consequently, $a^*\mathrm{d}_{d+1}[\eta] = \mathrm{d}_{d+1}[a^*\eta] = -\mathrm{d}_{k+1}[\eta]$. However, a^* acts trivially on $E^{0,k}_{k+1}$. ♠

Theorem 2.101 (The Gysin sequence). *The **Gysin sequence***

$$\cdots \to \mathrm{H}^\bullet_{\mathrm{dR}}(B) \xrightarrow{p^*} \mathrm{H}^\bullet_{\mathrm{dR}}(X) \xrightarrow{p_*} \mathrm{H}^{\bullet-d}_{\mathrm{dR}}(B) \xrightarrow{e(p)\wedge\cdot} \mathrm{H}^{\bullet+1}_{\mathrm{dR}}(B)\cdots$$

is exact.

Proof of Theorem 2.101. By direct inspection, the sequence

$$E^{k,d}_\infty \hookrightarrow E^{k,d}_{d+1} \xrightarrow{\mathrm{d}_{d+1}} E^{k+d+1,0}_{d+1} \twoheadrightarrow E^{n+k+1,0}_\infty$$

is exact. The map

$$\mathrm{H}^k_{\mathrm{dR}}(B) \cong E^{k,d}_{d+1} \xrightarrow{\mathrm{d}_{d+1}} E^{k+d+1,0}_{d+1} \cong \mathrm{H}^{n+k+1}_{\mathrm{dR}}(B)$$

is $[\cdot \wedge \mathrm{e}(p)]$. Further inspection shows that the sequence

$$E^{k+d,0}_\infty \hookrightarrow \mathrm{H}^{k+d}_{\mathrm{dR}}(X) \twoheadrightarrow E^{k,d}_\infty$$

is exact; indeed, this is simply the convergence statement for the Leray–Serre spectral sequence. These combine into the Gysin sequence. ■

2.12 Symplectic fibre bundles

Definition 2.102. Let $p\colon X \to B$ be a fibre bundle of relative dimension $2k$. A **relative symplectic structure** is an $\omega \in \Gamma(X, \Lambda^2 V_p^*)$ such that ω^k is a relative volume form and $\mathrm{d}^V\omega = 0 \in \Gamma(X, \Lambda^3 V_p)$. A **symplectic fibre bundle** is a fibre bundle $p\colon X \to B$ together with a relative symplectic form ω such that $\mathrm{d}_\nabla[\omega] = 0 \in \Omega^1(B, \mathscr{H}^2_{\mathrm{dR}}(p))$. •

Remark 2.103. This is not the usual definition of symplectic fibre bundle [MS98, §6.1] and it is a good exercise to prove that the two version of this concept are, in fact, equivalent. ♣

A relative straight-forward application shows that the above notion agrees with the usual notion in the symplectic literature.

The assumption we made says that $[\omega]$ lifts to $E_2^{0,2}$. Again, it is interesting to ask whether it possible to lift to $E_\infty^{0,2}$ or, equivalently, to find $\Omega \in \Omega^2(X)$ with

(1) $\Omega|_{p^{-1}(b)} = \omega|_{p^{-1}(b)}$ and

(2) $\mathrm{d}\Omega = 0$.

Additionally, one might want to require that Ω itself is symplectic.

If an Ω satisfying the first condition (but possibly not the second) exists it defines a connection $A = A_\Omega$ by the declaring

$$H_A \coloneqq \ker(i_\cdot\Omega\colon TX \to V_p^*).$$

By construction $\Omega = \Omega^{0,2} + \Omega^{2,0}$. We have $\mathrm{d}\Omega^{0,2} = \mathrm{d}_A^{1,0}\Omega^{0,2} + \mathrm{d}_A^{2,-1}\Omega^{0,2}$. Observe that $\mathrm{d}_A^{1,0}\Omega^{0,2} = 0$ if and only if $\mathrm{d}\Omega \in F^2\Omega^3(X)$. This means that A is a **symplectic connection**. The term $\mathrm{d}_A^{2,-1}\Omega^{0,2}$ vanishes if and only if A is flat. In general, whether the error $\mathrm{d}_A^{2,-1}\Omega^{0,2}$ can be corrected is a question about the curvature F_A inducing Hamiltonian or just symplectic vector field on the fibres. Whether the further terms can be corrected (iteratively) is controlled by the Leray–Serre spectral sequence.

Example 2.104. Consider the fibre bundle $p\colon \mathbf{C}P^3 \to \mathbf{H}P^1 \cong S^4$. $\mathbf{C}P^3$ has a symplectic form, the Fubini–Study form ω_{FS}. The restriction to ω_{FS} to the fibres of p is symplectic. But note that S^4 does not carry a symplectic form. ♠

Example 2.105. Consider the **Hopf surface** $H \coloneqq S^3 \times S^1 = (\mathbf{C}^2\backslash\{0\})/\mathbf{Z}$ with $k \in \mathbf{Z}$ acting by 2^k. H does not admits a symplectic structure, but the fibre bundle $p\colon H \to \mathbf{C}P^1$ admits a relative symplectic structure with fibres diffeomorphic to T^2. ♠

3 Lie groups

In this section I will introduce (review?) the concept of a Lie group, that is, a group in the category of manifolds. For the purpose of this course Lie groups will be a tool and a source of examples of manifolds. The theory of Lie groups is a vast subject and we will not even scrape the surface. A good reference is Bump [Bum13].

3.1 Definition

Definition 3.1. A **Lie group** is a smooth manifold G together with a group structure on G such that the maps $m\colon G \times G \to G$ defined by $m(g,h) \coloneqq g \cdot h$, and $i\colon G \to G$ defined by $i(g) \coloneqq g^{-1}$ are smooth. Let G and H be Lie groups. A **Lie group homomorphism** from G to H is a smooth group homomorphism $\rho\colon G \to H$. •

Example 3.2. $S^1 = \mathbf{R}/\mathbf{Z}$, $\mathrm{GL}_n(\mathbf{R})$, $\mathrm{GL}_n(\mathbf{C})$, $\mathrm{O}(n)$, $\mathrm{U}(n)$, $\mathrm{SO}(n)$, $\mathrm{SU}(n)$ are Lie groups. ♠

Example 3.3. Let V be a vector space. If $\omega \in \Lambda^2 V^*$ is a non-degenerate 2–form on V, then $H = H(V, \omega)$, the **Heisenberg group** of (V, ω), is defined by $H := \mathrm{U}(1) \times V$ with the group operation

$$(e^{i\alpha}, v) \cdot (e^{i\beta}, w) := (e^{i\alpha + i\beta + 2\pi i \omega(v,w)}, v + w).$$

H is a Lie group. ♠

Theorem 3.4 (`reference?`). *Let G be a Lie group. Let $H < G$ be a subgroup. If H is closed, then it is a submanifold; hence: H is a Lie group.*

Theorem 3.5 (Yamabe [Yam50]; see also Goto [Got69]). *Let G be a Lie group. Let $H < G$ be a subgroup. If H is path-connected, then H is an immersed submanifold; hence: H is a Lie group.*

3.2 Lie group actions

Definition 3.6. Let X be a smooth manifold. Let G be a Lie group.

(1) A **(left) action** of G on X is a smooth map $L\colon G \times X \to X$ satisfying

$$L(\mathbf{1}, \cdot) = \mathrm{id}_X \quad \text{and} \quad L(g, L(h, \cdot) = L(gh, \cdot).$$

Define $L_g \in \mathrm{Diff}(X)$ by $L_g := L(g, \cdot)$ and abbreviate $g \cdot x = L(g, x)$.

(2) The **orbit** of $x \in X$ is

$$\mathrm{Orb}_G(x) = G \cdot x := \{g \cdot x : g \in G\}.$$

(3) The **stabiliser** of $x \in X$ is

$$\mathrm{Stab}_G(x) = G_x := \{g \in G : g \cdot x = x\}.$$

(4) The action of G on X is **free** if $G_x = \mathbf{1}$ for every $x \in X$.

(5) The action of G on X is **proper** if the map $(L, \mathrm{pr}_X)\colon G \times X \to X \times X$ is proper. •

(6) A **right action** of G on X is a smooth map $R\colon X \times G \to G$ satisfying

$$R(\cdot, \mathbf{1}) = \mathrm{id}_X \quad \text{and} \quad R(R(\cdot, g), h) = R(\cdot, gh).$$

Set $R_g(\cdot) := R(\cdot, g)$ and abbreviate $x \cdot g := R(x, g)$. If R is a right action, then $L(g, x) := R(x, g^{-1})$ defines a left action. The notions orbit, stabiliser, free, proper carry over to right actions in the obvious way.

In this section, actions are assumed to be left actions unless explicitly stated otherwise.

Example 3.7. If G is a Lie group, then G acts on itself on the left by left multiplication $L\colon G \times G \to G$,

$$L(g, h) := g \cdot h.$$

The same formula also defines the action of G on itself on the right by right multiplication $R\colon G \times G$,

$$R(h, g) = h \cdot g.$$

These actions commute and G acts on itself on the left by conjugation $C\colon G \to G \to G$,

$$C(g, h) := ghg^{-1}.$$

♠

Exercise 3.8. Let G be a Lie group. Let $H < G$ be a closed subgroup. Prove that the action of H on G is free and proper.

Example 3.9. $\mathrm{U}(1)$ acts on S^{2n+1} via $e^{i\alpha} \cdot z = e^{i\alpha} z$. ♠

Example 3.10. Let $\theta \in \mathbf{R}$. $\mathbf{R}$ acts on $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ via $L(t, [x, y]) := [x + t, y + \theta t]$. ♠

3.3 The slice theorem

Definition 3.11. Let X be a manifold. Let G be a Lie group acting on X. A **categorical quotient quotient** of X by G is a smooth manifold X/G together with a smooth G–invariant map $p\colon X \to X/G$ such that every G–invariant map $f\colon X \to Y$ uniquely factors though p. •

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"] & Y \\ X/G. \arrow[ur, dashed] & \end{tikzcd}$$

$$C^\infty(X/G, \cdot) \cong C^\infty(X, \cdot)^G$$

Which actions admit quotients?

Proposition 3.12. *Let X be a manifold. Let G be a Lie group. If G acts freely and properly on then, X it admits a categorical quotient $p\colon X \to X/G$; moreover: p is a surjective submersion.*

Remark 3.13. The significance of the existence of the s is that smoothness of a continuous map $Y \to X/G$ can, therefore, be characterised by the existence of local lifts to X. ♣

Proof assuming that G is compact. Denote by X/G the topological quotient space and by $p\colon X \to X/G$ the projection map. X/G is paracompact and Hausdorff, and p is open. (Exercise!)

Let $x \in X$. The map $G \to X, g \mapsto gx$ is a proper injective immersion. Therefore, the orbit $G \cdot x \subset X$ is a submanifold. Choose a G–invariant metric g on X. (This is a red herring. The proof requires no Riemannian geometry, but it psychologically helpful.) Identify

$$N_x(G \cdot x) \cong T_x(G \cdot x)^\perp \subset T_xX$$

For $\varepsilon > 0$ set $V_x := B_\varepsilon(0) \subset N_x(G \cdot x)$ and define $\jmath_x\colon G \times V_x \to X$ by

$$\jmath_x(g, v) := g \exp_x(v).$$

Provided $\varepsilon \ll 1$, $\jmath$ is a G–equivariant embedding. Set $S_x := \jmath_x(\{\mathbb{1}\} \times V_x)$ and $U_x := p(\tilde{U}_x)$. The map $p|_{S_x}\colon S_x \to U_x$ is a homeomorphism. Define $\phi_x\colon U_x \to V_x$ by

$$\phi_x := \mathrm{pr}_{V_x} \circ \jmath_x^{-1} \circ \left(p|_{S_x}\right)^{-1}.$$

The task at hand is to prove that the maps ϕ_x form a smooth atlas. Let $x, y \in X$. $U_x \cap U_y \neq \varnothing$ if and only if $(G \cdot S_x) \cap S_y \neq \varnothing$. By construction, $(g \cdot S_x) \cap S_x = \varnothing$ unless $g = \mathbb{1}$. Therefore, there is a unique map $\gamma_x^y\colon (G \cdot S_x) \cap S_y \to G$ satisfying $\gamma_x^y(z) \cdot z \in S_x$ or, equivalently,

$\mathrm{pr}_G \circ j_y^{-1}(\gamma_x^y(z) \cdot z) = \mathbf{1}$. By the implicit function theorem, γ_x^y is smooth. A moment's thought shows that the transition map $\phi_x \circ \phi_y^{-1}$ satisfies

$$\phi_x \circ \phi_y^{-1}(z) = \mathrm{pr}_{V_x} \circ j_x\big(\gamma_x^y(j_y^{-1}(\mathbf{1}, z)) \cdot j_y^{-1}(\mathbf{1}, z)\big).$$

Therefore, it is smooth. This finishes the construction of the smooth atlas on X/G.

The universal property is evident from the construction. ■

Remark 3.14. For non-compact G one first proves that $G \cdot x$ is a submanifold and then produces an S_x in some (quite arbitrary way). ♣

Definition 3.15. A **homogeneous space** is a smooth manifold X together with a transitive G action. •

Proposition 3.16. *If X is a homogeneous space, then the map $G/G_{x_0} \to X$ induced by $g \mapsto g \cdot x_0$ is a diffeomorphism.* ■

Example 3.17. $\mathbf{C}P^n \cong S^{2n+1}/\mathrm{U}(1)$. ♠

Example 3.18. $\mathrm{Gr}_r(\mathbf{R}^n) \cong O(n)/(\mathrm{O}(r) \times \mathrm{O}(n-r))$. ♠

3.4 Lie algebra

Proposition 3.19. *Let G be a Lie group. Denote by*

$$\mathrm{Lie}(G) := \mathrm{Vect}(G)^L := \{\xi \in \mathrm{Vect}(G) : L_g^* \xi = \xi \text{ for every } g \in G\}.$$

the space of left-invariant vector fields on G.

(1) $\mathrm{Lie}(G) \subset \mathrm{Vect}(G)$ *is a Lie subalgebra.*

(2) *For $g \in G$ and $\xi \in \mathrm{Lie}(G)$, $R_g^* \xi \in \mathrm{Lie}(G)$.*

Proof. (1) is obvious. (2) holds because R_g and L_g commute. ■

Definition 3.20. Let G be a Lie group. The **Lie algebra of** G is the Lie algebra of left-invariant vector fields:

$$\mathfrak{g} = \mathrm{Lie}(G) := \mathrm{Vect}(G)^L.$$

The **adjoint representation** $\mathrm{Ad}\colon G \to \mathrm{End}(\mathrm{Lie}(G))$ is defined by

$$\mathrm{Ad}(g)\xi := R_g^* \xi.$$

The **adjoint representation** $\mathrm{ad}\colon \mathrm{Lie}(G) \to \mathrm{End}(\mathrm{Lie}(G))$ is defined by

$$\mathrm{ad}(\xi)\eta := [\xi, \eta].$$

•

Proposition 3.21. *Let G be a Lie group.*

(1) *The evaluation map* $\mathrm{ev}_{\mathbf{1}}\colon \mathrm{Vect}(G)^L \to T_{\mathbf{1}}G$ *is an isomorphism.*

(2) *For $g \in G$ and $\xi \in \mathrm{Vect}(G)^L$*

$$\mathrm{Ad}(g)\xi = \mathrm{ev}_1^{-1} \circ T_1 C_g \circ \mathrm{ev}_1(\xi).$$

(3) *For $\xi, \eta \in \mathrm{Vect}(G)^L$*

$$T_1 \,\mathrm{Ad}(\mathrm{ev}_1(\xi))\eta = [\xi, \eta].$$

(4) *If $\rho\colon\ G \longrightarrow H$ is a Lie group homomorphism, then $\mathrm{Lie}(\rho)\colon\ \mathrm{Lie}(G) \longrightarrow \mathrm{Lie}(H)$ defined by*

$$\mathrm{Lie}(\rho) = \mathrm{ev}_1^{-1} \circ T_1\rho \circ \mathrm{ev}_1$$

is a Lie algebra homomorphism.

Proof. A left-invariant vector field v satisfies

$$v_g = T_1 L_g(v_1).$$

Therefore, it is determined by v_1. Conversely, the above formula defines a left-invariant vector field. This proves (1).

To prove (2), compute

$$\begin{aligned}
\mathrm{ev}_1(R_g^*\xi) &= T_g R_{g^{-1}}(\xi_g) \\
&= T_g R_{g^{-1}} T_1 L_g(\xi_1) \\
&= T_1 C_g(\xi_1).
\end{aligned}$$

To prove (3), observe that

$$\begin{aligned}
\mathrm{flow}_\xi^t(g) &= \mathrm{flow}_\xi^t(L_g(\mathbf{1})) \\
&= L_g(\mathrm{flow}_\xi^t(\mathbf{1})) \\
&= R_{\mathrm{flow}_\xi^t(\mathbf{1})} g.
\end{aligned}$$

Therefore,

$$\begin{aligned}
T_1 \,\mathrm{Ad}(\mathrm{ev}_1(\xi))\eta &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} R^*_{\mathrm{flow}_\xi^t(\mathbf{1})}(\eta) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\mathrm{flow}_\xi^t)^*(\eta) \\
&= [\xi, \eta].
\end{aligned}$$

To prove (4), observe that by (2)

$$\mathrm{Ad}(\rho(g)) \circ \mathrm{Lie}(\rho)(\xi) = \mathrm{Lie}(\rho) \circ \mathrm{Ad}(g)(\xi).$$

By (3), this implies that $\mathrm{Lie}(\rho)$ is a Lie algebra homomorphism. ∎

The following gadget turns out to be important for us later.

Definition 3.22. Let G be a Lie group. The **Maurer–Cartan form** $\mu \in \Omega^1(G, \mathrm{Lie}(G))$ is defined by

$$\mu_g(\xi) := \mathrm{ev}_1^{-1} \circ T_g L_{g^{-1}}(\xi).$$ •

Proposition 3.23. *Let G be a Lie group.*

(1) *The Maurer–Cartan form μ satisfies $\mu(\xi) = \xi$ for every $\xi \in \mathrm{Lie}(G)$.*

(2) *For every $g \in G$*

$$R_g^* \mu = \mathrm{Ad}(g^{-1}) \circ \mu.$$

(3) *The Maurer–Cartan form μ satisfies the **Maurer–Cartan equation***

$$\mathrm{d}\mu + \tfrac{1}{2}[\mu \wedge \mu] = 0$$

Proof. (1) is obvious.

To prove (2), for $g \in G$ and $\xi \in \mathrm{Lie}(G)$ compute

$$(R_g^* \mu)(\xi) = \mu((R_g)_* \xi) = (R_{g^{-1}})^* \xi = \mathrm{Ad}(g^{-1})\xi.$$

To prove (3), compute

$$\begin{aligned}\big(\mathrm{d}\mu + \tfrac{1}{2}[\mu \wedge \mu]\big)(\xi, \eta) &= \mathscr{L}_\xi(\mu(\eta)) - \mathscr{L}_\eta(\mu(\xi)) - \mu([\xi, \eta]) \\ &\quad + \frac{1}{2}([\mu(\xi), \mu(\eta)] - [\mu(\xi), \mu(\eta)]) \\ &= 0.\end{aligned}$$ ■

Exercise 3.24. Let $\rho\colon G \to H$ be a Lie group homomorphism. Prove that

$$\rho^* \mu_H = \mathrm{Lie}(\rho) \circ \mu_G.$$

3.5 Exponential map

Definition 3.25. Let G be a Lie group. The **exponential map** $\exp\colon \mathrm{Lie}(G) \to G$ is defined by

$$\exp(\xi) := \mathrm{flow}_\xi^1(\mathbb{1}).$$ •

Exercise 3.26. (1) Prove that exp is well-defined.

(2) Let $\rho\colon G \to H$ be a Lie group homomorphism. Prove that

$$\rho \circ \exp(\xi) = \exp \circ \mathrm{Lie}(\rho)(\xi).$$

(3) Prove that

$$C_g \circ \exp = \exp \circ \mathrm{Ad}_g.$$

Definition 3.27. Let X be smooth manifold. Let G be a Lie group. Let $L\colon\ G\times X\longrightarrow X$ be a smooth left action. The **infinitesimal action** of $\mathrm{Lie}(G)$ on X is the map $v=v^L\colon\ \mathrm{Lie}(G)\longrightarrow\mathrm{Vect}(X)$ defined by

$$v_\xi(x) := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} L_{\exp(t\xi)}(x).$$

•

Proposition 3.28. *Let X be smooth manifold. Let G be a Lie group. Let $L\colon\ G\times X\longrightarrow X$ be a smooth left action. Denote by $v\colon\ \mathrm{Lie}(G)\longrightarrow\mathrm{Vect}(X)$ the corresponding infinitesimal action.*

(1) *For every $\xi\in\mathrm{Lie}(G)$*
$$L_{\exp(t\xi)} = \mathrm{flow}^t_{v_\xi}.$$

(2) *For every $g\in G$ and $\xi\in\mathrm{Lie}(G)$*
$$v_{\mathrm{Ad}(g)\xi} = L^*_{g^{-1}}v_\xi$$

(3) *The infinitesimal action v is an Lie algebra anti-isomorphism; that is: for every $\xi,\eta\in\mathrm{Lie}(G)$*
$$v_{[\xi,\eta]} = -[v_\xi, v_\eta].$$

Remark 3.29. If R is a right action and L is the corresponding left-action, then $v^R=-v^L$. In particular, v^R is a Lie algebra homomorphism. ♣

Proof. (1) is obvious.

To prove (2), compute

$$\begin{aligned} v_{\mathrm{Ad}(g)\xi}(x) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} L_{g\exp(t\xi)g^{-1}}(x)\\ &= T_{L_g(x)}L_g\left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} L_{\exp(t\xi)}L_{g^{-1}}(x)\right)\\ &= T_{L_g(x)}L_g\Big(v_\xi(L_{g^{-1}}(x))\Big)\\ &= (L^*_{g^{-1}}v_\xi)(x). \end{aligned}$$

To prove (3), differentiate

$$v_{\mathrm{Ad}(\exp(t\xi))\eta} = L^*_{\exp(-t\xi)}v_\eta = \left(\mathrm{flow}^t_{v_\xi}\right)^* v_\eta.$$ ■

3.6 Haar volume form

Proposition 3.30. *Let G be a Lie group. Set $d:=\dim G$. There is a unique left-invariant volume form up to multiplication by a non-zero constant:*

$$\dim\Omega^d(G)^L := \{\nu\in\Omega^d(G) : L_g^*\nu=\nu\} = 1.$$

Definition 3.31. Let G be a Lie group. A **Haar volume form** on G is a left-invariant volume form ν on G. ν is **normalised** if $\int_G \nu = 1$. •

Proof of Proposition 3.30. If $\nu \in \Omega^d(G)$ is left-invariant, then

$$\nu_g = \nu_1 \circ \Lambda^d T_g L_{g^{-1}}.$$

Therefore, ν is uniquely determined by $\nu_1 \in \Lambda^d T_1^* G$. Conversely, every $\nu_1 \in \Lambda^d T_1^G$ determines a left-invariant $\nu \in \Omega^d(G)$. ■

Exercise 3.32. Let G be a Lie group. Let ν be a Haar volume form on G. For every $g \in G$, $R_g^* \nu$ is a Haar volume form. The **modular function** of G is the function $\Delta \in C^\infty(G, \mathbf{R}^\times)$ defined by

$$\Delta(g) := \frac{R_g^* \nu}{\nu}.$$

(1) Prove that $\Delta = 1$ if and only if G admits a right-invariant Haar measure. These groups are **unimodular**.

(2) Prove that $\Delta \colon G \to \mathbf{R}^\times$ is a Lie group homomorphism.

(3) Prove that $\Delta = 1$ if G is compact.

(4) Define $i \colon G \to G$ by $i(g) := g^{-1}$. Prove that $i^* \nu = \Delta \nu$.

(5) Consider the Lie group

$$G := \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} : x > 0, y \in \mathbf{R} \right\}.$$

Compute modular function of G.

3.7 The Killing form

Definition 3.33. Let $\mathfrak{g}$ be a Lie algebra. The **Killing form** $B \in S^2 \mathfrak{g}^*$ is defined by

$$B(\xi, \eta) := \operatorname{tr}(\operatorname{ad}(\xi) \circ \operatorname{ad}(\eta)).$$ •

Exercise 3.34. Prove that

$$B([\xi, \eta], \zeta) = B(\eta, [\xi, \zeta]).$$

Definition 3.35. A Lie algebra is called **semisimple** if B is negative definite. G **semisimple** if $\mathrm{Lie}(G)$ is semisimple. •

Exercise 3.36. Prove that if $\mathfrak{g} = \mathfrak{gl}(n)$, then

$$B(\xi, \eta) = 2n \operatorname{tr}(\xi \eta) - 2 \operatorname{tr}(\xi) \operatorname{tr}(\eta).$$

Exercise 3.37. Prove that if $\mathfrak{g} = \mathfrak{su}(n)$, then

$$B(\xi, \eta) = 2n \operatorname{tr}(\xi \eta).$$

3.8 de Rham cohomology of manifolds with G–actions

Lemma 3.38. *Let G be a Lie group. Let X be a smooth manifold together with an action $\rho\colon G \circlearrowright X$. Set*

$$\Omega^\bullet(X)^\rho \coloneqq \{\alpha \in \Omega^\bullet(X) : \rho_g^*\alpha = \alpha \text{ for every } g \in G\}.$$

If G is connected and compact, then the inclusion $\iota\colon \Omega^\bullet(X)^\rho \hookrightarrow \Omega^\bullet(X)$ induces an isomorphism

$$\mathrm{H}^\bullet(\iota)\colon \mathrm{H}^\bullet(\Omega^\bullet(X)^\rho) \cong \mathrm{H}^\bullet_{\mathrm{dR}}(X).$$

Proof. The proof relies on the construction of an inverse $\mathrm{H}^\bullet(\mathrm{av})$ of $\mathrm{H}^\bullet(\iota)$. Choose an orientation on G. Denote by

$$(\mathrm{pr}_X)_*\colon \Omega^\bullet(G\times X) \to \Omega^{\bullet-\dim G}(X)$$

the fibre integration map along $\mathrm{pr}_X\colon G\times X \to X$

$$(\mathrm{pr}_X)_* \circ \mathrm{pr}_G^*\alpha = \int_G \alpha \quad \text{and} \tag{3.39}$$

$$(\mathrm{pr}_X)_*(\beta \wedge \mathrm{pr}_X^*\gamma) = (\mathrm{pr}_X)_*\beta \wedge \gamma \tag{3.40}$$

for every $\alpha \in \Omega^\bullet(G)$, $\beta \in \Omega^\bullet(G\times X)$, $\gamma \in \Omega^\bullet(X)$. For every $\eta \in \Omega^{\dim G}(G)$ with $\int_G \eta = 1$ define

$$\mathrm{av}_\eta\colon \Omega^\bullet(X) \to \Omega^\bullet(X)$$

by

$$\mathrm{av}_\eta \coloneqq (\mathrm{pr}_X)_* \circ (\mathrm{pr}_G^*\eta \wedge \cdot) \circ \rho^*.$$

This is a cochain map because $(\mathrm{pr}_X)_*$, $\mathrm{pr}_G^*\eta \wedge \cdot$, and ρ^* are.

For every $g \in G$ and $\alpha \in \Omega^\bullet(X)$,

$$\rho_g^*\,\mathrm{av}_\nu(\alpha) = (\mathrm{pr}_X)_*(\mathrm{pr}_G^*\nu \wedge (\mathrm{id}_G \times \rho_g)^*\rho^*\alpha) = (\mathrm{pr}_X)_*(\mathrm{pr}_G^*\nu \wedge (R_g^* \times \mathrm{id}_X)^*\rho^*\alpha) = \mathrm{av}_\nu(\alpha).$$

Therefore, $\operatorname{im}\mathrm{av}_\nu \subset \Omega^\bullet(X)^\rho$. Denote by $\mathrm{av}\colon \Omega^\bullet(X) \to \Omega^\bullet(X)^\rho$ the cochain map by corestricting av_ν.

Since $\alpha \in \Omega^\bullet(X)^\rho$ if and only if $\rho^*\alpha = \mathrm{pr}_X^*\alpha$, and using (3.40),

$$\mathrm{av}\circ\iota(\alpha) = (\mathrm{pr}_X)_*(\mathrm{pr}_G^*\nu \wedge \mathrm{pr}_X^*\alpha) = (\mathrm{pr}_X)_*\mathrm{pr}_G^*\nu \wedge \alpha = \alpha.$$

Therefore, av is a left-inverse of ι. It remains to prove that $\mathrm{H}^\bullet(\iota\circ\mathrm{av}) = \mathrm{H}^\bullet(\mathrm{av}_\nu) = \mathrm{id}_{\mathrm{H}^\bullet_{\mathrm{dR}}(X)}$.

$\mathrm{H}^\bullet(\mathrm{pr}_G^*\eta \wedge \cdot)$ is independent of η. A moment's thought shows that there are a smooth map $\tilde\rho\colon G\times X \to X$ and a non-empty open subset $U \subset G$ such that $\tilde\rho$ is homotopic to ρ and $\tilde\rho|_{U\times X} = \mathrm{pr}_X|_{U\times X}$. Choose $\eta \in \Omega^{\dim G}(X)$ with $\int_G \eta = 1$ and $\operatorname{supp}\eta \subset U$. By direct computation

$$\begin{aligned}
\mathrm{H}^\bullet(\mathrm{av}_\nu) &= \mathrm{H}^\bullet((\mathrm{pr}_X)_*) \circ \mathrm{H}^\bullet(\mathrm{pr}_G^*\eta \wedge \cdot) \circ \mathrm{H}^\bullet(\rho^*) \\
&= \mathrm{H}^\bullet((\mathrm{pr}_X)_*) \circ \mathrm{H}^\bullet((\mathrm{pr}_G^*\eta \wedge \cdot) \circ \rho^*) \\
&= \mathrm{H}^\bullet((\mathrm{pr}_X)_*) \circ \mathrm{H}^\bullet((\mathrm{pr}_G^*\eta \wedge \cdot) \circ \tilde\rho^*) \\
&= \mathrm{H}^\bullet((\mathrm{pr}_X)_*) \circ \mathrm{H}^\bullet((\mathrm{pr}_G^*\eta \wedge \cdot) \circ pr_X^*) \\
&= \mathrm{H}^\bullet((\mathrm{pr}_X)_* \circ (\mathrm{pr}_G^*\nu \wedge \cdot) \circ \mathrm{pr}_X^*) \\
&= \mathrm{id}.
\end{aligned}$$

■

Remark 3.41. The advantage of not following the heuristic argument, is that one can (at least in principle) write down a chain homotopy h such that

$$i \circ \mathrm{av} - \mathrm{id} = \mathrm{d} \circ h + h \circ \mathrm{d}. \qquad ♣$$

Let us now use Lemma 3.38 to compute the de Rham cohomology in a few simple cases.

Example 3.42. $G = \mathrm{SO}(n+1)$ acts transitively on S^n. The stabilizer of any $x \in S^n$ is $\mathrm{SO}(T_x S^n) \cong \mathrm{SO}(n)$. A moments thought shows that

$$\begin{aligned} \Omega^\bullet(S^n)^G &= (\Lambda^* T_x S^n)^{\mathrm{SO}(T_x S^n)} \\ &= (\Lambda^*(\mathbf{R}^n)^*)^{\mathrm{SO}(n)} \\ &= \mathbf{R} \cdot 1 \oplus \mathbf{R} \cdot \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n \\ &= \mathbf{R}[0] \oplus \mathbf{R}[n]. \end{aligned}$$

The differential necessarily vanishes (for dimension reasons if $n > 1$); hence, this already is $\mathrm{H}^\bullet_{\mathrm{dR}}(S^n)$. The last step in the above computation is a fact from the representation theory of $\mathrm{SO}(n)$. ♠

Example 3.43. $G = \mathrm{U}(n+1)$ acts transitively on $\mathbf{C}P^n$ with stabiliser of any $\mathbf{C}\cdot z \in \mathbf{C}P^n$ isomorphic to $\mathrm{U}(z^\perp) = \mathrm{U}(T_z \mathbf{C}P^n) \cong \mathrm{U}(n)$. We compute

$$\Omega^\bullet(\mathbf{C}P^n)^{U(n+1)} \otimes \mathbf{C} = (\Lambda^*(\mathbf{C}^n)^*)^{\mathrm{U}(n)}.$$

The latter is generated as a $\mathbf{C}$*−algebra* by the standard symplectic form

$$\omega := \sum_{i=1}^n \frac{i\mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i}{2};$$

that is,

$$\begin{aligned} (\Lambda^*(\mathbf{C}^n)^*)^{\mathrm{U}(n)} &= \mathbf{C} \cdot 1 \oplus \mathbf{C} \cdot \omega \oplus \cdots \oplus \mathbf{C} \cdot \omega^n \\ &= \mathbf{C}[\omega]/(\omega^{n+1}). \end{aligned}$$

Since this complex is supported in even degrees, the differential vanishes and this already is $H^\bullet_{\mathrm{dR}}(\mathbf{C}P^n) \otimes \mathbf{C}$. ♠

Example 3.44. Let G be a Lie group. Let $\mathfrak{g} := \mathrm{Lie}(G)$. If we consider the L action of G on itself, then

$$\Omega^\bullet(G)^L = \Lambda^* \mathfrak{g}^* = \mathrm{Hom}(\Lambda^* \mathfrak{g}, \mathbf{R}).$$

The differential, which is usually denoted by δ, does not vanish. It can be computed to be

$$\begin{aligned} (\delta\alpha)(\xi_1, \ldots, x_{k+1}) = &\sum_{i=1}^{k+1} (-1)^{i+1} \xi_i \cdot \alpha(\xi_1, \ldots, \widehat{\xi_i}, \ldots, \xi_{k+1}) \\ &+ \sum_{i<j=1}^{k+1} (-1)^{i+j} \alpha([\xi_i, \xi_j], \xi_1, \ldots, \widehat{\xi_i}, \ldots, \widehat{\xi_j}, \ldots, \xi_{k+1}); \end{aligned}$$

in fact, since the Lie algebra acts trivially on $\mathbf{R}$ the first term vanishes. $(\mathrm{Hom}(\Lambda^*\mathfrak{g}, \mathbf{R}), \delta)$ is the **Chevalley–Eilenberg cochain complex** (although it was discovered decades before Chevalley–Eilenberg by Cartan). It is defined for every Lie algebra $\mathfrak{g}$. Its cohomology

$$\mathrm{H}^\bullet(\mathfrak{g}) \coloneqq \mathrm{H}^\bullet(C^\bullet(\mathfrak{g}), \delta).$$

is the **Lie algebra cohomology** of $\mathfrak{g}$.

If V is any representation of $\mathfrak{g}$, then δ as defined above makes $\mathrm{Hom}(\Lambda^*\mathfrak{g}, M)$ into a cochain complex. $\mathrm{H}^\bullet(\mathfrak{g}; V) \coloneqq \mathrm{H}^\bullet(\mathrm{Hom}(\Lambda^*\mathfrak{g}, V))$ is called the **Lie algebra cohomology** of $\mathfrak{g}$ with coefficients in V. Lemma 3.38 shows that $\mathrm{H}^\bullet_{\mathrm{dR}}(G) = \mathrm{H}^\bullet(\mathfrak{g}; \mathbf{R})$. The notion of Lie algebra cohomology goes back to Chevalley and Eilenberg [CE48]. ♠

Remark 3.45. Let $\rho\colon G \to \mathrm{GL}(V)$ be a Lie group representation. Consider the trivial vector bundle $\mathrm{pr}_G\colon \underline{V} = G \times V \to G$. G acts on $\underline{V}$ by $L \times \rho$. This turns $\underline{V}$ into a G*–invariant* vector bundle. The formula $\mathrm{d}_\nabla s \coloneqq \mathrm{d}s + (\mathrm{Lie}(\rho) \circ \mu)s$ defines a G*–invariant* flat connection on $\underline{V}$. A moment's thought shows that $\mathrm{H}^\bullet_{\mathrm{dR}}(G, \underline{V}) = \mathrm{H}^\bullet(\mathfrak{g}, V)$. ♣

Example 3.46. Let G be a connected compact Lie group. Let $H < G$ be a connected closed Lie subgroup. Set $\mathfrak{g} \coloneqq \mathrm{Lie}(G)$ and $\mathfrak{h} \coloneqq \mathrm{Lie}(H)$. Set $C^\bullet(\mathfrak{g}) \coloneqq \mathrm{Hom}(\Lambda^\bullet\mathfrak{g}, \mathbf{R})$ and define δ as above. Denote by $C^\bullet(\mathfrak{g}, \mathfrak{h})$ the subcomplex of those $\alpha \in C^\bullet(\mathfrak{g})$ with

$$i_\xi\alpha = 0 \quad \text{and} \quad i_\xi\delta\alpha = 0 \quad \text{for every} \quad \xi \in \mathfrak{h}.$$

The **relative Lie algebra cohomology** of $\mathfrak{g} \supset \mathfrak{h}$ is

$$\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{h}) \coloneqq \mathrm{H}^\bullet(C^\bullet(\mathfrak{g}, \mathfrak{h}), \delta).$$

The adjoint action of the Lie algebra $\mathfrak{h}$ on $\mathfrak{g}$ descends to $\mathfrak{g}/\mathfrak{h}$. Denote by $\mathrm{Hom}(\Lambda^\bullet\mathfrak{g}/\mathfrak{h}, \mathbf{R})^\mathfrak{h}$ the corresponding invariant subspace of $\mathrm{Hom}(\Lambda^\bullet\mathfrak{g}/\mathfrak{h}, \mathbf{R})$. $\mathrm{Hom}(\Lambda^\bullet\mathfrak{g}/\mathfrak{h}, \mathbf{R})^\mathfrak{h}$ can be regarded as a subspace $C^\bullet(\mathfrak{g})$. A moment's thought identifies it as $C^\bullet(\mathfrak{g}, \mathfrak{h})$. Moreover, $\mathrm{Hom}(\Lambda^\bullet\mathfrak{g}/\mathfrak{h}, \mathbf{R})^\mathfrak{h} \cong \Omega^\bullet(G/H)^H$ and the differentials δ and d agree. Therefore,

$$\mathrm{H}^\bullet_{\mathrm{dR}}(G/H) \cong \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{h}). \qquad ♠$$

Exercise 3.47. Show that $\mathrm{H}^1(\mathfrak{g}, \mathbf{R}) = (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*$.

Example 3.48. Set $\tilde{R}(g, h) \coloneqq hg^{-1}$. If we consider the action $L \times \tilde{R}$ of $G \times G$ on G, then

$$\Omega^\bullet(G)^{L \times \tilde{R}} = (\Lambda^\bullet\mathfrak{g}^*)^{\mathrm{Ad}}.$$

Here Ad denotes the coadjoint action. Suppose $\alpha \in \Omega^k(G)^{L \times \tilde{R}}$, that is, α is left invariant and right invariant. Since derivative of the map $i\colon G \to G, g \mapsto g^{-1}$ is

$$\mathrm{d}_g i = -\mathrm{d}L_{g^{-1}} \circ \mathrm{d}R_{g^{-1}},$$

we have

$$i^*\alpha = (-1)^k\alpha.$$

It follows that

$$\mathrm{d}\alpha = (-1)^k \mathrm{d}i^*\alpha = (-1)^k i^*\mathrm{d}\alpha = -\mathrm{d}\alpha;$$

hence, the differential vanishes on $\Omega^\bullet(G)^{L\times\tilde{R}}$ and

$$\mathrm{H}^\bullet_{\mathrm{dR}}(G) = (\Lambda^*\mathfrak{g}^*)^{\mathrm{Ad}}.$$

The formula

$$\gamma(\xi,\eta,\zeta) \coloneqq B([\xi,\eta],\zeta)$$

defines an element $\gamma \in (\Lambda^3\mathfrak{g}^*)^{\mathrm{Ad}}$. If G is semisimple, then $\gamma \neq 0$; hence $b_3(G) \geqslant 1$. ♠

Example 3.49. Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation. $\mathrm{Lie}(\rho)\colon \mathfrak{g} \to \mathfrak{gl}(V)$ can be regarded as an element of $\theta_\rho \in \mathfrak{g}^* \otimes \mathfrak{gl}(V)$. Evidently, θ_ρ is invariant under the action induces by Ad and ρ and so is $\theta_\rho^{\wedge k} \in \Lambda^k\mathfrak{g}^* \otimes \mathfrak{gl}(V)$. Therefore, $\mathrm{tr}(\theta_\rho^{\wedge k}) \in \Lambda^k\mathfrak{g}^*$. ♠

Remark 3.50. The multiplication map $m\colon G \times G \to G$ induces a map $\Delta\colon \mathrm{H}^\bullet(G) \to \mathrm{H}^\bullet(G) \otimes \mathrm{H}^\bullet(G)$. This turns $\mathrm{H}^\bullet(G)$ into a **Hopf algebra**. ♣

4 Principal bundles

4.1 Definition and examples

Definition 4.1. Let G be a Lie group. A G–**principal fibre bundle** is a smooth map $p\colon P \to B$ together with a right action $R\colon P \circlearrowleft G$ such that:

(1) p is G–invariant; that is: for every $x \in P$, $g \in G$, $p(xg) = p(x)$; and

(2) for every $b \in B$ there are an open subset $b \in U \subset B$, and a G–equivariant **local trivialisation** of $p|_{p^{-1}(U)}$; that is: a G–equivariant diffeomorphism $\tau\colon p^{-1}(U) \to U \times G$ such that

$$\mathrm{pr}_U \circ \tau = p|_{p^{-1}(U)}.$$

G is the **structure group** of (p,R). ●

Remark 4.2. In the situation of Definition 4.1, $p\colon P \to B$ is a quotient $P \to P/G$, R is free, fibre-preserving, and its restriction to any fibre $p^{-1}(b)$ is transitive. ♣

Remark 4.3. The action R is an important part of the data and cannot be recovered from p. ♣

Example 4.4. The **trivial** G–**principal bundle** over B is $\mathrm{pr}_B\colon B \times G \to B$ with $(b,g)\cdot h \coloneqq (b,gh)$. ♠

Example 4.5. Every smooth principal covering map $p\colon X \to B$ is an $\mathrm{Aut}(p)^{\mathrm{op}}$–principal fibre bundle. ♠

Example 4.6. Let B be a smooth manifold. Let $V \to B$ be a vector bundle of rank r. Denote by

$$\mathrm{Fr}(V) \coloneqq \{(b,\phi) : b \in B, \phi\colon \mathbf{R}^r \to V_b \text{ isomorphism}\}$$

the **frame bundle of** V. Denote by $p\colon\ \mathrm{Fr}(V) \to B$ the projection map. $\mathrm{GL}_r(\mathbf{R})$ acts on the right of $\mathrm{Fr}(V)$ via

$$(b,\phi)\cdot\tau \coloneqq (b,\phi\circ\tau).$$

There is a unique smooth structure on $\mathrm{Fr}(V)$ such that (p,R) is a $\mathrm{GL}_r(\mathbf{R})$–principal bundle. ♠

Example 4.7. Let $k, r \in \mathbf{N}$ with $k < r$. The **Stiefel manifold** $\mathrm{St}_k^*(\mathbf{R}^r)$ is the submanifold

$$\mathrm{St}_k^*(\mathbf{R}^r) := \{(v_1, \dots, v_k) \in (\mathbf{R}^r)^k : v_1, \dots, v_k \text{ linearly independent}\}$$

or, equivalently,

$$\mathrm{St}_k^*(\mathbf{R}^r) := \{A \in \mathrm{Hom}(\mathbf{R}^k, \mathbf{R}^r) : A \text{ is injective}\}.$$

$\mathrm{GL}_k(\mathbf{R})$ acts on the right of $\mathrm{St}_k^*(\mathbf{R}^r)$ via $R(A, \tau) := A \circ \tau$. The map $p\colon\ \mathrm{St}_k^*(\mathbf{R}^r) \to \mathrm{Gr}_k(\mathbf{R}^r)$ defined by

$$p(A) := \operatorname{im} A$$

together with R is a $\mathrm{GL}_k(\mathbf{R})$–principal bundle. Of course, $\mathrm{St}_k^*(\mathbf{R}^r)$ is the frame bundle of the tautological bundle over $\mathrm{Gr}_k(\mathbf{R}^r)$. ♠

Proposition 4.8. *Let G be a Lie group. Let P be smooth manifold. If $R\colon\ P \times G \to P$ is a proper free right action, then $p\colon\ P \to B := P/G$ together with R is a G–principal fibre bundle.*

Proof. This is a consequence of Proposition 3.12. ■

Exercise 4.9. The Hopf bundle $p\colon\ S^{2n+1} \to \mathbf{C}P^n$ together with the right action defined by $z \cdot e^{i\alpha} := ze^{i\alpha}$ is a $\mathrm{U}(1)$–principal bundle.

Exercise 4.10. Let $n \in \mathbf{N}$. $\mathrm{Sp}(1) := \{q \in \mathbf{H} : |q| = 1\}$ acts on $S^{4n+3} \subset \mathbf{H}^{n+1}$ by $R\colon\ S^{4n+3} \times \mathrm{Sp}(1) \to S^{4n+3}$ with

$$R_+(x, q) := q^{-1}x.$$

The quotient of R is the $\mathbf{H}P^n$, the space of $\mathbf{H}$–left modules $L \subset \mathbf{H}^{n+1}$ of dimension 1. The projection map $q\colon\ S^{4n+3} \to \mathbf{H}P^n$ together with R is an $\mathrm{Sp}(1)$–principal bundle—the **quaternionic Hopf bundle**.

4.2 The action of gauge transformations on connections

Definition 4.11. Let $(p\colon\ P \to B, R)$ and $(q\colon\ Q \to B, S)$ be G–principal fibre bundles. A **morphism** $(p, R) \to (q, S)$ is a G–equivariant smooth map $\phi\colon\ P \to Q$ satisfying $q \circ \phi = p$. A **gauge transformation** of (p, R) is an isomorphism $(p, R) \to (p, R)$. The **gauge group** of (p, R) is the group of all gauge transformations of (p, R) and denoted by

$$\mathscr{G}(p, R).$$ •

Exercise 4.12. Prove that every morphism of G–principal bundles is an isomorphism.

Exercise 4.13. Let (p, R) be a G–principal bundle. Prove that (p, R) is isomorphic to the trivial G–principal bundle if and only if p admits a section.

The gauge group plays a very important role. The following concrete description of the gauge group is quite useful.

Proposition 4.14. *Let $(p\colon\ P \to B, R)$ be G–principal fibre bundles. Denote by $C^\infty(P, G)^C$ the subspace of $u \in C^\infty(P, G)$ satisfying*

$$R_g^* u = C_{g^{-1}} u, \quad \text{i.e.,} \quad u(xg) = g^{-1}u(x)g$$

for every $g \in G$

(1) *For every $u \in C^\infty(P,G)^C$ the map $\check{u} \in C^\infty(P,P)$ defined by*

$$\check{u}(x) = x \cdot u(x)$$

is a gauge transformation.

(2) *The map $\check{\cdot}\colon C^\infty(P,G)^C \to \mathscr{G}(p,R)$ defined by the above is a bijection; in fact, a group isomorphism.*

Proof. To prove (1), it suffices to verify that $\tilde{u}$ is G–equivariant:

$$\check{u}(xg) = xg \cdot u(xg) = xg \cdot g^{-1}u(x)g = xu(x)g = \check{u}(x)g.$$

Evidently, the map $\check{\cdot}\colon C^\infty(P,G)^C \to \mathscr{G}(p,R)$ is injective. To prove that it is surjective, observe that if $\check{u} \in \mathscr{G}(p,R)$ then for every $x \in P$ there is a unique $u(x) \in G$ such that $\tilde{u}(x) = x \cdot u(x)$. The map $u \in \mathrm{Map}(P,G)$ thus defined is smooth. The G–equivariance of $\check{u}$ follows from the G–equivariance of u.

It remains to prove that $\check{\cdot}$ is a group homomorphism. To see this observe that

$$\check{v}(\check{u}(x)) = x \cdot u(x) \cdot v(x \cdot u(x)) = x \cdot v(x) \cdot u(x). \qquad \blacksquare$$

Definition 4.15. Denote by $\hat{\cdot}\colon \mathscr{G}(p,R) \to C^\infty(P,G)^C$ the inverse of $\check{\cdot}\colon C^\infty(P,G)^C \to \mathscr{G}(p,R)$. •

Example 4.16. If G is an abelian group, then $C^\infty(P,G)^C$ consists precisely of the G–invariant maps $C^\infty(P,G)^G \cong C^\infty(B,G)$. Therefore, $\mathscr{G}(p,R) \cong C^\infty(B,G)$.

Let G be an arbitrary Lie group. Denote by $Z(G) \coloneqq \{g \in G : gh = hg \text{ for every } h \in G\}$ the **center of** G. Evidently,

$$C^\infty(B,Z(G)) \cong C^\infty(P,Z(G))^C \hookrightarrow \mathscr{G}(p,R). \qquad \spadesuit$$

Example 4.17. Let $V \to B$ be a vector bundle of rank r. Consider the frame bundle $(p\colon \mathrm{Fr}(V) \to B, R)$. For every $\lambda \in \mathbf{R}^*$ the map $\varepsilon\colon \mathrm{Fr}(V) \to \mathrm{Fr}(V)$ defined by

$$\varepsilon(b,(v_1,\dots,v_r)) \coloneqq \varepsilon(b,(\lambda v_1,\dots,\lambda v_r))$$

is an example of a gauge transformation. ♠

Example 4.18. For the trivial G–principal bundle $(p\colon B \times G \to B, R)$ the map $C^\infty(B,G) \to \mathscr{G}(p,R)$ defined by $C^\infty(B,G) \ni \gamma \mapsto u_\gamma$ defined by $u_\gamma(b,g) \coloneqq (b,\gamma(b)g)$ is a group isomorphism. (Observe that for general G–principal bundle the left-multiplication is not available.) ♠

4.3 Pulling back

Proposition 4.19. *Let $p\colon P \to B$ with R be a G–principal bundle. Let $f\colon A \to B$ be a smooth map. Denote by $f^*p\colon f^*P \to A$ the pullback of $p\colon P \to B$. Define $f^*R\colon f^*P \times G \to f^*P$ by*

$$f^*R((a,p),g) = (a, R(p,g)).$$

(f^*p, f^*R) is a G–principal fibre bundle.

Definition 4.20. The G–principal bundle (f^*p, f^*R) constructed above is the **pullback** of (p, R) via f. •

❉

Example 4.21. Let G be a Lie group. Here is how to construct a fiber bundle $p\colon X \to B$ with fibres diffeomorphic to G but which cannot be turned into a G–principal fibre bundle. Let $\phi \in \mathrm{Diff}(G)$. Denote by $X_\phi := ([0,1] \times G)/{\sim}$ with $\sim$ generated by $(1, x) \sim (0, \phi(x))$ the mapping torus of ϕ. The projection $p\colon X_\phi \to S^1$ is a fibre bundle whose fibres are diffeomorphic to G.

The right action of G on $[0,1] \times G$ descends to X_ϕ if and only if $\phi(g) = hg$ for some $h \in G$. Therefore, usually, p cannot be turned into a G–principal fibre in the obvious way. In fact, often, p cannot be turned into a G–principal at all. To see this, observe that if there is a $g \in G$ such that g and $\phi(g)$ lie in the same connected component of G, then p admits a section. Therefore, p is isomorphic to $\mathrm{pr}_{S^1}\colon S^1 \times G \to S^1$. However, this implies that ϕ is isotopic to id_G.

To make this concrete consider the orientation reversing diffeomorphism $\phi \in \mathrm{Diff}(\mathrm{U}(1))$ defined by $\phi(e^{i\alpha}) := e^{-i\alpha}$. The mapping torus T_ϕ is the Klein bottle; hence, not diffeomorphic to $S^1 \times \mathrm{U}(1)$. However, the projection $p\colon T_\phi \to S^1$ admits a section $s(b) := [b, \mathbf{1}]$. ♠

4.4 G–principal connections

Proposition 4.22. *Let $(p\colon P \to B, R\colon P \times G \to P)$ be a G–principal fibre bundle. The map $\kappa\colon P \times \mathfrak{g} \to V_p$ defined by*

$$\kappa(p, \xi) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} R(p, \exp(t\xi)) = T_1 R_p \circ \mathrm{ev}_1(\xi).$$

is an isomorphism. ■

The isomorphism κ simplifies the theory of connections (or at least it makes it more concrete).

Definition 4.23. Let $(p\colon P \to B, R\colon P \circlearrowleft G)$ be a G–principal fibre bundle. For $\xi \in \mathfrak{g}$ define $v_\xi \in \Gamma(V_p)$ by

$$v_\xi(p) := \kappa(p, \xi).$$ •

Exercise 4.24. Prove that $\mathfrak{g} \to \mathrm{Vect}(P)$, $\xi \mapsto v_\xi$ is a Lie algebra homomorphism.

Exercise 4.25. Construct a fibre bundle $p\colon X \to B$ whose fibres are diffeomorphic to S^1 but which cannot be equipped with a S^1 action R making (p, R) into an S^1–principal bundle.

Definition 4.26. Let $(p\colon P \to B, R\colon P \circlearrowleft G)$ be a G–principal fibre bundle. A **G–principal connection** on (p, R) is a G–equivariant left splitting of the short exact sequence of vector bundles (equipped with right G actions)

$$P \times \mathfrak{g} \hookrightarrow TP \twoheadrightarrow p^*TB;$$

that is: an $A \in \Omega^1(P, \mathfrak{g})$ such that

$$A(v_\xi) = \xi$$

and

$$R_g^* A = \mathrm{Ad}(g^{-1}) \circ A.$$

The space of connections on (p, R) is denoted by $\mathscr{A}(p, R)$. •

Remark 4.27. The horizontal distribution $H_A \coloneqq \ker A$ of a G–principal connection is characterized by its G–invariance: that is $TR_g H_A = H_A$ for every $g \in G$. ♣

Proposition 4.28. *Every G–principal connection is complete.*

Proof. Exercise. ■

4.5 Equivariant differential forms

Definition 4.29. Let $(p\colon P \to B, R\colon P \times G \to P)$ be a G–principal fibre bundle. Let V be a finite-dimensional vector space. Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation of G. G acts on $\Omega^\bullet(P, V)$ via

$$g \cdot \alpha \coloneqq \rho(g) \circ R_g^* \alpha.$$

Set

$$\Omega^\bullet_{\mathrm{hor}}(P, V)^G = \Omega^\bullet_{\mathrm{hor}}(P, V)^\rho \coloneqq \{\alpha \in \Omega^\bullet_{\mathrm{hor}}(P, V) : g \cdot \alpha = \alpha \text{ for every } g \in G\}.$$ •

Remark 4.30. The above construction is particularly important for the adjoint representation $\rho\colon G \to \mathrm{GL}(\mathrm{Lie}(G))$. ♣

Proposition 4.31. *Let $(p\colon P \to B, R\colon P \times G \to P)$ be a G–principal fibre bundle. $\mathscr{A}(p, R)$ is an affine space modelled on $\Omega^1_{\mathrm{hor}}(P, \mathrm{Lie}(G))^{\mathrm{Ad}}$.*

Proof. Exercise; cf. ??. ■

Proposition 4.32. *Let $(p\colon P \to B, R\colon P \times G \to P)$ be a G–principal fibre bundle. Let $A \in \mathscr{A}(p, R)$. Denote by $\sigma_A\colon TP \to H_A$ the projection onto H_A. Let V be a finite-dimensional vector space. Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation of G. There is a unique linear map*

$$\mathrm{d}_A\colon \Omega^\bullet_{\mathrm{hor}}(P, V)^\rho \to \Omega^{\bullet+1}_{\mathrm{hor}}(P, V)^\rho$$

such that

$$\mathrm{d}_A \alpha = \mathrm{d}\alpha + (\mathrm{Lie}(\rho) A) \wedge \alpha.$$

Proof. Le $\alpha \in \Omega^k_{\mathrm{hor}}(P, V)^\rho$. Set $\mathrm{d}_A \alpha \coloneqq \mathrm{d}\alpha + (\mathrm{Lie}(\rho)\theta_A) \wedge \alpha$. By direct computation,

$$\begin{aligned}
R_g^* \mathrm{d}_A \alpha &= \mathrm{d} R_g^* \alpha + \mathrm{Lie}(\rho) R_g^* A \wedge R_g^* \alpha \\
&= \mathrm{d}\rho(g^{-1})\alpha + \mathrm{Lie}(\rho)\, \mathrm{Ad}(g^{-1}) A \wedge \rho(g^{-1})\alpha \\
&= \rho(g^{-1})(\mathrm{d}\alpha + \mathrm{Lie}(\rho) A \wedge \alpha) \\
&= \rho(g^{-1}) \mathrm{d}_A \alpha.
\end{aligned}$$

Therefore, $\mathrm{d}_A\alpha$ is G–equivariant. To verify that $\mathrm{d}_A\alpha$ is horizontal, the $\xi \in \mathfrak{g}$ and denote by $v_1, \dots, v_k$ G–invariant local vector field. By direct computation.

$$\begin{aligned}
\mathrm{d}\alpha(v_\xi, v_1 \dots, v_k) &= \mathscr{L}_{v_\xi}(\alpha(v_1, \dots, v_k)) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (R^*_{\exp(t\xi)}\alpha)(v_1, \dots, v_k)) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \rho(\exp(-t\xi)) \circ \alpha(v_1, \dots, v_k)) \\
&= -\operatorname{Lie}(\rho)(\xi) \circ \alpha(v_1, \dots, v_k) \\
&= -(\operatorname{Lie}(\rho)(\theta_A) \wedge \alpha)(v_\xi, v_1, \dots, v_k).
\end{aligned}$$ ■

Therefore, $\mathrm{d}_A\alpha$ is horizontal.

Remark 4.33. The maps $\mathrm{d}_A\colon \Omega^\bullet_{\mathrm{hor}}(P, V)^\rho \to \Omega^{\bullet+1}_{\mathrm{hor}}(P, V)^\rho$ are compatible with the usual operations on representations, in particular, $\otimes$ and $\oplus$. One consequence of this is that $\Omega^\bullet_{\mathrm{hor}}(P, V)$ is a (left-)module over $\Omega^\bullet_{\mathrm{hor}}(P)^G \cong \Omega^\bullet(B)$ and d_A satisfies the corresponding Leibniz rule. ♣

4.6 Curvature

Proposition 4.34. *Let $(p\colon P \to B, R\colon P \times G \to P)$ be a G–principal fibre bundle. Let $A \in \mathscr{A}(p, R)$.*

(1) *There is a unique G–invariant horizontal 2–form $F_A \in \Omega^2_{\mathrm{hor}}(P, \mathfrak{g})^{\mathrm{Ad}}$ such that for every $v, w \in \Gamma(H_A)$*

$$F_A(v, w) = -A([v, w]).$$

*F_A is the **curvature** of A.*

(2) *F_A can be computed as*

$$F_A = \mathrm{d}A + \frac{1}{2}[A \wedge A]$$

(3) *F_A satisfies the **Bianchi identity***

$$\mathrm{d}_A F_A = 0.$$

(4) *If $\rho\colon G \to \mathrm{GL}(V)$ is a finite-dimensional representation of G, then $\mathrm{d}_A\colon \Omega^\bullet_{\mathrm{hor}}(P, V)^\rho \to \Omega^{\bullet+1}_{\mathrm{hor}}(P, V)^\rho$ satisfies*

$$\mathrm{d}_A \circ \mathrm{d}_A = (\operatorname{Lie}(\rho) \circ F_A) \wedge \cdot.$$

Remark 4.35. One sometimes sees the formula $F_A = \mathrm{d}_A A$. This is correct, but it easily leads to confusion. The issue is that one is tempted to forget the original definition of d_A and use Proposition 4.32 ?? instead; however: A is (not at all) horizontal and this formula obviously does not hold for A. ♣

Proof of Proposition 4.34. Define $F_A \coloneqq \mathrm{d}A + \frac{1}{2}[A \wedge A]$ and compute

$$F_A(v, w) = \mathscr{L}_v A(w) - \mathscr{L}_w A(v) - A([v, w]) + [A(v), A(w)].$$

This matches $-A(v,w)$ for $v,w \in \Gamma(H_A)$.

If $\xi \in \mathfrak{g}$ and $w \in \Gamma(H_A)$ is G–invariant, then

$$F_A(v_\xi, w) = \mathscr{L}_w \xi - A([v_\xi, w]) = 0.$$

The first term vanishes because ξ is constant. The second term vanishes because w is G–invariant.

For $\xi, \eta \in \mathfrak{g}$

$$F_A(v,w) = \mathscr{L}_{v_\xi}\eta - \mathscr{L}_{v_\eta}\xi - A([v_\xi, v_\eta]) + [\xi,\eta].$$

The first two term vanish because ξ,η are constant. The last two terms cancel because $\xi \mapsto v_\xi$ is a Lie algebra homomorphism. (This is, of course, essentially the proof of the Maurer–Cartan equation.)

The G–invariance of F_A follows from the G–invariance of A. Thus (1) and (2) are proved.

$$\begin{aligned}
\mathrm{d}_A F_A &= (\mathrm{d} + [A \wedge \cdot])(\mathrm{d}A + \frac{1}{2}[A \wedge A]) \\
&= \frac{1}{2}([\mathrm{d}A \wedge A] - [A \wedge \mathrm{d}A]) + [A \wedge \mathrm{d}A] + \frac{1}{2}[A \wedge [A \wedge A]] \\
&= 0
\end{aligned}$$

because the first three term cancel and the last term vanishes by the Jacobi identity.

(4) follows by direct computation. ■

4.7 Parallel transport

Proposition 4.36. *Let $(p\colon P \to B, R\colon P \circlearrowleft G)$ be a G–principal fibre bundle. Let $A \in \mathscr{A}(p,R)$. Let $\gamma\colon [0,1] \to B$ be a piecewise smooth path. The parallel transport tra_γ^A is G–equivariant; that is: for every $x \in p^{-1}(\gamma(0))$ and $g \in G$*

$$\mathrm{tra}_\gamma^A(x) \cdot g = \mathrm{tra}_\gamma^A(x \cdot g).$$

Proof. This is a consequence of A being G–invariant. ■

Remark 4.37. Let $x_0 \in p^{-1}(b_0)$. There is a $g \in G$ such that $\mathrm{tra}_\gamma^A(x_0) = x_0 \cdot g$. Every element of $p^{-1}(b_0)$ is of the form $x_0 \cdot h$ for some $h \in G$. Since tra_A^γ is G–invariant,

$$\mathrm{tra}_\gamma^A(x_0 h) = \mathrm{tra}_\gamma^A(x_0)h = x_0 \cdot gh. \qquad ♣$$

Definition 4.38. Let $(p\colon P \to B, R\colon P \circlearrowleft G)$ be a G–principal fibre bundle. Let $A \in \mathscr{A}(p,R)$. Let $b_0 \in B$ and $x_0 \in p^{-1}(b_0)$. The **holonomy group** of A based at x_0 is the subgroup $\mathrm{Hol}_{x_0}(A) < G$ defined by

$$\mathrm{Hol}_{x_0}(A) \coloneqq \{g \in G : (\star)\}$$

with $(\star)$ meaning that there is a piecewise smooth loop $\gamma\colon [0,1] \to B$ based at b_0 with $\mathrm{tra}_\gamma^A(x_0) = x_0 \cdot g$. The **restricted holonomy group** of A based at x_0 is the subgroup $\mathrm{Hol}_{x_0}^0(A) < G$ defined by

$$\mathrm{Hol}_{x_0}^0(A) \coloneqq \{g \in G : (\dagger)\}$$

with (†) meaning that there is a null-homotopic piecewise smooth loop $\gamma\colon [0,1] \to B$ based at b_0 with $\mathrm{tra}_\gamma^A(x_0) = x_0 \cdot g$. •

Proposition 4.39. *The holonomy group and the restricted holonomy group are Lie subgroups of G.*

Proof sketch. $\mathrm{Hol}^0_{x_0}(A)$ is path-connected and therefore a Lie subgroup of G. Parallel transport defines a group homomorphism $\pi_1(B, b_0) \to \mathrm{Hol}_{x_0}(A)/\mathrm{Hol}^0_{x_0}(A)$. With Γ denoting its image

$$\mathrm{Hol}_{x_0}(A) = \coprod_{\gamma\in\Gamma} g \cdot \mathrm{Hol}^0_{x_0}(A).$$

Use this to construct a smooth structure on $\mathrm{Hol}_{x_0}(A)$. ■

Remark 4.40. The above underlines that G–principal connections are really much simpler than general Ehresmann connections. The holonomy group of an Ehresmann connection is a subgroup of $\mathrm{Diff}(p^{-1}(b_0))$, a possibly wild infinite-dimensional beast; while that of a G–principal connection sits inside a fixed finite dimensional Lie group. ♣

4.8 The action of gauge transformations on connections

Definition 4.41. Let $(p\colon P \to B, R\colon G \circlearrowright P)$ be a G–principal fibre bundle. $\mathscr{G}(p,R)$ acts on the right of $\mathscr{A}(p,R)$ by pullback:

$$A \cdot u = u^*A.$$ •

Recall the identification

$$\hat{\cdot}\colon \mathscr{G}(p,R) \to C^\infty(P,G)^C.$$

Proposition 4.42. *The above action has the following properties.*

(1) *With μ denoting the Maurer–Cartan form:*

$$u^*A = \mathrm{Ad}(\hat{u}^{-1}) \circ A + \hat{u}^*\mu.$$

(More informally: $u^*A = \hat{u}^{-1}A\hat{u} + \hat{u}^{-1}\mathrm{d}\hat{u}$.)

(2) *The horizontal subspaces of A and u^*A are related by*

$$H_{u^*A} = Tu^{-1}(H_A).$$

(3) *The pullback via the gauge transformation u preserves $\Omega^\bullet(P,V)^\rho$ and*

$$\mathrm{d}_{u^*A} = u^* \circ \mathrm{d}_A \circ (u^{-1})^*.$$

(4) *The curvatures of A and u^*A are related by*

$$F_{u^*A} = \mathrm{Ad}(\hat{u}^{-1})F_A.$$

(5) *The parallel transports of A and u^*A are related by*

$$\mathrm{tra}_\gamma^{u^*A} = u^{-1} \circ \mathrm{tra}_\gamma^A \circ u.$$

Proof. Evidently, $H_{u^*A} = Tu^{-1}(H_A)$.

Since

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} R_{g\exp(t\xi)}x = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} R_{\exp(t\xi)}R_g(x)$$
$$= v_\xi(R_g(x)),$$

the derivative of the map $x \mapsto R_{\hat{u}(x)}(y)$ is

$$v_{\hat{u}^*\mu}(R_{\hat{u}(x)}(y)).$$

Since $u(x) = R_{\hat{u}(x)}(x)$,

$$T_x u(\hat{x}) = T_x R_{\hat{u}(x)}(\hat{x}) + v_{(\hat{u}^*\mu)(\hat{x})}(R_{\hat{u}(x)}(x)).$$

Therefore,

$$\begin{aligned}\theta_{u^*A} &= u^*A \\ &= \mathrm{Ad}(\hat{u}^{-1}) \circ A + \hat{u}^*\mu.\end{aligned}$$

■

4.9 Associated fibre bundles

Proposition 4.43 (Construction of associated fibre bundles). *Let G be a Lie group. Let ($p\colon P \to B$, $R\colon P \circlearrowleft G$) be a G–principal fibre bundle. Let F be a smooth manifold together with a left action $\lambda\colon G \circlearrowright F$. Define the right action $S\colon P \times F \circlearrowleft G$ defined by*

$$S_g(x, f) \coloneqq (R_g(x), \lambda_g^{-1}(f)).$$

(1) *S admits a quotient*

$$q\colon P \times F \to (P \times F)/G =: P \times_\lambda F = P \times_G F.$$

Moreover, (q, S) is a G–principal fibre bundle.

(2) *The map $r\colon P \times_\lambda F \to B$ defined by*

$$r([x, f]) \coloneqq p(x)$$

is a fibre bundle—the ***associated fibre bundle of*** *(p, R)* ***and*** *λ.*

(3) *The diagram*

$$\begin{array}{ccc} P \times F & \xrightarrow{\mathrm{pr}_P} & P \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ P \times_\lambda F & \xrightarrow{r} & B \end{array}$$

is a pullback (along both p and r); that is: the map $\phi\colon P \times Ftop^*(P \times_\lambda F)$ defined by $\phi(x, f) \coloneqq (x, [x, f])$ is defines an isomorphism $\phi\colon \mathrm{pr}_P \to p^*r$ of fibre bundles over P; and the map $\psi\colon P \times F \to r^*P$ defined by $\psi(x, f) \coloneqq ([x, f], x)$ is defines an isomorphism $\psi\colon (q, S) \to r^*(p, R)$ of G–principal fibre bundles.

Proof. Let $\{U_i : i \in I\}$ be an open cover of B such that $p^{-1}(U_i)$ is G–equivariantly diffeomorphic to $U_i \times G$. Evidently, $(p^{-1}(U_i) \times F)/G \cong (U_i \times G \times F)/G \cong U_i \times F$ exists. This implies (1), (2), and (3). ∎

Example 4.44. Let $V \to B$ be a real (complex) vector bundle of rank r. Denote by $(p\colon \mathrm{Fr}(V) \to B, R)$ the frame bundle of $V \to B$. $\mathrm{GL}_r(\mathbf{R})$ ($\mathrm{GL}_r(\mathbf{C})$) acts on $\mathbf{R}P^n$ ($\mathbf{C}P^n$) The associated fibre bundle $\mathrm{Fr}(V) \times_{\mathrm{GL}_r(\mathbf{R})} \mathbf{R}P^n$ ($\mathrm{Fr}(V) \times_{\mathrm{GL}_r(\mathbf{R})} \mathbf{C}P^n$) is the **projectivisation** of V and denoted by $\mathbf{P}(V) \to B$. ♠

Example 4.45. Let $n \in \mathbf{Z}$. The **Hirzebruch surface** Σ_n is

$$\Sigma_n := \mathbf{P}(\mathscr{O}_{\mathbf{P}^1} \oplus \mathscr{O}_{\mathbf{P}^1}(n)). \quad ♠$$

Proposition 4.46 (sections and G–equivariant maps). *Assume the situation of Proposition 4.43. Consider the set of sections*

$$\Gamma(r) := \{s \in C^\infty(B, P \times_\lambda F) : r \circ s = \mathrm{id}_B\}$$

and the set of the G–equivariant maps

$$C^\infty(P,F)^\lambda := \{\hat{s} \in C^\infty(P,F) : \hat{s}(xg) = g^{-1}\hat{s}(x)\}.$$

There is a unique bijection

$$\hat{\cdot}\colon\ \Gamma(r) \to C^\infty(P,F)^\lambda$$

such that, for every $s \in \Gamma(r)$ and $x \in P$,

$$s(p(x)) = [x, \hat{s}(x)].$$

Proof. Clearly for every $s \in \Gamma(r)$ there is a unique $\hat{s} \in \mathrm{Map}(P,F)$ as above. Since $[xg, \hat{s}(xg)] = s(p(xg)) = s(p(x)) = [x, \hat{s}]$, $\hat{s}$ is G–equivariant. Considerations in local trivialisations show that $\hat{s}$ is smooth. This proves that $\hat{\cdot}$ is defined and injective. Evidently, it is also surjective. ∎

`The construction of an Ehresmann connection on` r `from an` G`-principal connection on` $(p\colon P \to B, R\colon P \circlearrowleft G)$ `is omitted.`

4.10 Associated vector bundles

Proposition 4.47. *Let G be a Lie group. Let $(p\colon P \to B, R\colon P \circlearrowleft G)$ be a G–principal fibre bundle. Let $\lambda\colon G \to \mathrm{GL}(V)$ be a finite-dimensional representation. Consider the associated fibre bundle $r\colon P \times_\lambda V \to B$.*

(1) *There are unique vector space structure on the fibres of r such that r becomes a vector bundle and the isomorphism of fibre bundles $p^*(P \times_\lambda V) \to P \times V$ is an isomorphism of vector bundles. This is the **vector bundle associated with** (p,R) **and** λ.*

(2) *If $\alpha \in \Omega^\bullet(B, P \times_\lambda V)$, then $\hat{\alpha} := \mathrm{pr}_V p^*\alpha \in \Omega^\bullet_{\mathrm{hor}}(P,V)^\lambda$. This defines a bijection*

$$\hat{\cdot}\colon\ \Omega^\bullet(B, P \times_\lambda V) \to \Omega^\bullet_{\mathrm{hor}}(P,V)^\lambda.$$

(3) *There is a unique covariant derivative* d_A *on* $P \times_\lambda V$ *such that*

$$\begin{tikzcd}
\Omega^\bullet(B, P \times_\lambda V) \arrow[r, "\mathrm{d}_A"] \arrow[d, "\cong"] & \Omega^{\bullet+1}(B, P \times_\lambda V) \arrow[d, "\cong"] \\
\Omega^\bullet_{\mathrm{hor}}(P, V)^\lambda \arrow[r, "\mathrm{d}_A"] & \Omega^{\bullet+1}_{\mathrm{hor}}(P, V)^\lambda.
\end{tikzcd}$$

Moreover, if $G = \mathrm{GL}(V)$ *and* $\lambda = \mathrm{id}$*, then every covariant derivative on* $P \times_\lambda V$ *arises from a unique* $\mathrm{GL}(V)$*–principal connection* A*.*

Proof. It suffices to prove (1) for the trivial bundle $\mathrm{pr}_B \colon B \times G \to B$. In this case the map $q\colon (B \times G) \times V \to B \times V$ defined by $((b, g), v) \mapsto (b, \lambda(g)v)$ is G–invariant and every G–invariant map from $(B \times G) \times V$ factors through q. Therefore, $(B \times G) \times_\lambda V$ is (isomorphic) to $B \times V$.

(2) is an analogous to the earlier statement about sections.

The first part of (3) follows from the fact that d_A is compatible with tensor products. It suffices to verify the second part for the trivial bundle. In this case $P = B \times \mathrm{GL}(V)$ and the quotient map $q\colon P \times V \to P \times_\lambda V = B \times V$ is $q(b, g, v) := (b, \lambda(g)v)$. The map $r\colon P \times_\lambda V = B \times V \to B$ is $r = \mathrm{pr}_B$. The product connection gives rise to the covariant derivative $\mathrm{d}\colon C^\infty(B, V) \to \Omega^1(B, V)$. Any other covariant derivative is of the form $\mathrm{d} + a$ with $a \in \Omega^1(B, \mathfrak{gl}(V))$. Denote by θ_0 the connection 1–form of the trivial connection. Then $\theta_0 + \mathrm{Ad}(\mathrm{pr}_G)^{-1} \cdot r^*a$ is a connection 1–form and induces $\mathrm{d} + a$. ■

Example 4.48. Let X be a smooth manifold of dimension n. Denote by $p\colon \mathrm{Fr}(TX) \to X$ the frame bundle of TX. The orientation double cover of X is the fibre bundle associated with (p, R) and the action of $\mathrm{GL}_n(\mathbf{R})$ on $\{\pm 1\}$ induced by the group homomorphism $\mathrm{GL}_n(\mathbf{R}) \to \{\pm 1\}$ given by

$$\phi \mapsto \frac{\det \phi}{|\det \phi|}.$$ ♠

Example 4.49. Let $s \geqslant 0$ and $n \in \mathbf{N}_0$. A s**–density** on $\mathbf{R}^n$ is a map $\mu\colon (\mathbf{R}^n)^{\times n} \to \mathbf{R}$ such that for every $A \in \mathrm{GL}_n(\mathbf{R})$ and $v_1, \dots, v_n \in \mathbf{R}^n$

$$\mu(\phi(v_1), \dots, \phi(v_n)) = |\det \phi|^s \mu(v_1, \dots, v_n).$$

The set of s–densities is a 1–dimensional vector space: $D^s(\mathbf{R}^n)$.

Let X be a smooth manifold of dimension n. Denote by $p\colon \mathrm{Fr}(TX) \to X$ the frame bundle of TX. The bundle of s–densities on X is

$$D^s(TX) := \mathrm{Fr}(TX) \times_{\mathrm{GL}_n(R)} D^s(\mathbf{R}^n).$$

A **density** is a 1–density and $D(TX) := D^1(TX)$. ♠

Exercise 4.50. Let X be a closed smooth manifold (possibly not oriented). Construct a linear map

$$\int_X \colon D(TX) \to \mathbf{R}$$

(worthy of its notation).

Example 4.51. Let $V \to B$ be a vector bundle rank r. Denote the corresponding frame bundle by $\mathrm{Fr}(V)$ of isomorphisms $\phi\colon \mathbf{R}^r \to V_x$. The map $\mathrm{ev}\colon \mathrm{Fr}(V) \times \mathbf{R}^r \to V$ defined by

$$((x,\phi),v) \coloneqq (x,\phi(v))$$

is G–invariant and exhibits V as (isomorphic to) $\mathrm{Fr}(V) \times_\lambda \mathbf{R}^r$ with $\lambda \coloneqq \mathrm{id}\colon\ \mathrm{GL}_r(\mathbf{R}) \to \mathrm{GL}_r(\mathbf{R})$. ♠

Remark 4.52. The frame bundle formalism is a convenient way to carry linear algebra constructions over to vector bundles:

(1) Denote by $\lambda^*\colon\ \mathrm{GL}_r(\mathbf{R}) \to \mathrm{GL}((\mathbf{R}^r)^*)$ the contragredient representation defined by

$$\lambda^*(g)\lambda \coloneqq \lambda \circ \lambda(g^{-1}).$$

$\mathrm{Fr}(V) \times_{\lambda^*} (\mathbf{R}^r)^* \cong V^*.$

(2) Denote by $\Lambda^k\lambda\colon\ \mathrm{GL}_r\, r(\mathbf{R}) \to \mathrm{GL}(\Lambda^k(\mathbf{R}^r))$ the representation defined by

$$(\Lambda^k\lambda)(g)\alpha \coloneqq \Lambda^k(\lambda(g))\alpha.$$

$\mathrm{Fr}(V) \times_{\Lambda^k\lambda} \Lambda^k\mathbf{R}^r \cong \Lambda^k V.$ ♣

4.11 Reduction of the structure group

Definition 4.53 (Extension of structure group). Let G, H be Lie groups, $\lambda\colon\ H \to G$ a Lie group homomorphism, and $(q\colon\ Q \to B, S\colon\ Q \circlearrowleft H)$ an H–principal bundle. The **extension of** (q,S) **along** λ is the G–principal bundle (p,R) with $p\colon\ P \coloneqq Q \times_\lambda G \to B$ denoting the fibre bundle associated with (p,R) and λ, and $R\colon\ P \circlearrowleft G$ defined by $R([x,g],h) \coloneqq [x,gh]$. •

Exercise 4.54. Prove that (p,R) is a G–principal fibre bundle.

This raises the question of when one can undo this construction.

Definition 4.55 (Reduction of structure group). Let G, H be Lie groups, $\lambda\colon\ H \to G$ a Lie group homomorphism, and $(p\colon\ P \to B, R\colon\ P \circlearrowleft G)$ be a G–principal fibre bundle. A **reduction of** (p,R) **along** λ is an H–principal fibre bundle $(q\colon\ Q \to B, S\colon\ Q \circlearrowleft H)$ together with a smooth H–equivariant map $\phi\colon\ Q \to P$ satisfying $p \circ \phi = q$. Two reductions $(q_1, S_1; \phi_1)$ and $(q_2, S_2; \phi_2)$ are **isomorphic** if there is an isomorphism $\psi\colon\ (q_1,S_1) \to (q_2,S_2)$ such that $\phi_2 \circ \psi = \phi_1$. Denote by $\mathrm{Red}_\lambda(p,R)$ the set of isomorphism classes of reductions of (p,R) along λ. •

Remark 4.56. In the situation of Definition 4.55, the map

$$Q \times_H G \to P, [q,g] \mapsto i(q)g$$

is an isomorphism of G–principal bundles. ♣

Example 4.57. A trivialization is a reduction to the trivial group. ♠

Example 4.58. Let (V,g) be a Euclidean vector bundle over B. Denote by $p\colon\ \mathrm{Fr}(V) \to B$ and $q\colon\ \mathrm{Fr}_{\mathrm{O}}(V) \to B$ the frame bundle and the orthogonal frame bundle. The inclusion $\mathrm{Fr}_{\mathrm{O}}(V) \to \mathrm{Fr}(V)$ is a reduction. ♠

Example 4.59. Let V be an oriented Euclidean vector bundle over B of rank r. A **spin structure** on V is a reduction of $\mathrm{Fr}_{\mathrm{SO}}(V)$, the positive orthonormal frame bundle, along $\mathrm{Spin}(r) \to \mathrm{SO}(r)$. ♠

Reductions along a general Lie group homomorphism $\lambda\colon H \to G$ can be quite deliciate. However, if λ is the inclusion of a closed subgroup, then the problem can be understood very concretely.

Proposition 4.60. *Let G be a Lie group, $H \looparrowright G$ an immersed Lie subgroup, and $(p\colon P \to B, R\colon P \circlearrowleft G)$ a G–principal bundle. Denote by $c^\infty(P, G/H)^G$ the set of those $\hat{s} \in \mathrm{Map}(P, G/H)^G$ $b \in B$, there is an open subset $U \subset B$ with $b \in U$ such that $\hat{s}|_{p^{-1}(U)}$ lifts to $C^\infty(p^{-1}(U), G)$.*

1. *Let $(q, S; \phi)$ be a λ–reduction of (p, R). There is a unique $\hat{s} \in c^\infty(P, G/H)^G$ such that for every $x \in P$*
$$x\hat{s}(x) \in \operatorname{im}\phi.$$
2. *Let $\hat{s} \in c^\infty(P, G/H)^G$. $Q := \hat{s}^{-1}(\mathbf{1}H)$ is an immersed submanifold and H preserves Q. Moreover, $(q\colon Q :=\to B, S := R|_Q; \phi : Q \hookrightarrow P)$ is a λ–reduction of (p, R).*

The above establishes a bijection
$$\mathrm{Red}_\lambda(p, R) \cong c^\infty(P, G/H)^G.$$

Remark 4.61. If $H < G$ is a closed Lie subgroup, then the quotient G/H exists as a smooth manifold and $c^\infty(P, G/H)^G = C^\infty(P, G/H)^G \cong \Gamma(P \times_G G/H)$. (It's tempting to pretend that G/H is smooth manifold.) ♣

Proof of Proposition 4.60. Evidently, $\hat{s} \in \mathrm{Map}(P, G/H)^G$ is uniquely determined. To verify that $\hat{s}$ has local smooth lifts it suffices to consider the case where q, p are trivial. In this case $\phi\colon B \times H \to B \times G$. Define $\gamma\colon B \to G$ by $\phi(b, \mathbf{1}) = (b, \gamma(b))$ Observe that $(b, g) \cdot g^{-1}\gamma(b) \in \operatorname{im}\phi$ and $(b, g) \mapsto g^{-1}\gamma(b)$ is smooth.

Since $\hat{s}$ has local smooth lifts which are transverse to H, by equivariance, $Q \subset P$ is an immersed submanifold. Moreover, the action of H preserves Q. To prove that $(q := p|_Q\colon Q \to B, S)$ is an H–principal fibre bundle it remains to exhibit local trivialisations. For this it suffices to consider the trivial G–principal fibre bundle, in which case it is trivial. ■

Example 4.62. Let V be a real vector bundle over B of rank n. Denote by $\mathrm{Fr}(V) \to B$ its frame bundle.

1. A reduction of $\mathrm{Fr}(V)$ to $\mathrm{GL}(\mathbf{C}^{n/2})$ corresponds to a section of $\mathrm{Fr}(V) \times_{\mathrm{GL}(\mathbf{R}^n)} \mathrm{GL}(\mathbf{R}^n)/\mathrm{GL}(\mathbf{C}^{n/2})$. The latter correspond precisely to the almost complex structures on V.
2. A reduction of $\mathrm{Fr}(V)$ to $\mathrm{GL}^+(n)$ corresponds to a section of $\mathrm{Fr}(V) \times_{\mathrm{GL}(\mathbf{R}^n)} \mathrm{GL}(\mathbf{R}^n)/\mathrm{GL}^+(\mathbf{R}^n)$. The latter correspond precisely to the orientations on V.
3. A reduction of $\mathrm{Fr}(V)$ to $\mathrm{O}(n)$ corresponds to a section of $\mathrm{Fr}(V) \times_{\mathrm{GL}(\mathbf{R}^n)} \mathrm{GL}(\mathbf{R}^n)/\mathrm{O}(n)$. The latter correspond precisely to Euclidean inner products on V. ♠

4.12 The holonomy bundle

It is a natural question to ask: given a G–principal bundle, what is the smallest possible reduction? This question is quite difficult. But if the reduction is required to be compatible with a connection, it becomes easy.

Theorem 4.63 (Reduction to the holonomy group). *Let $(p\colon P \to B, R\colon P \circlearrowleft G)$ be a G–principal bundle with B connected, $A \in \mathscr{A}(p,R)$, and $x \in P$. Define*

$$P_{A,x} \coloneqq \{y \in P : \text{there is an } A\text{–horizontal path from } x \text{ to } y\}.$$

The following hold:

(1) *$P_{A,x} \subset P$ is an immersed submanifold and $(q \coloneqq p|_{P_{A,x}}\colon P_{A,x} \to B, S \coloneqq R|_{P_{A,x}\times \mathrm{Hol}_x(A)})$ is a $\mathrm{Hol}_p(A)$–principal bundle—the* ***holonomy bundle*** *of A based at x.*

(2) *There is a unique connection $\tilde{A} \in \mathscr{A}(q,S)$ with $H_{\tilde{A},y} = H_{A,y}$ for every $y \in P_{A,x}$.*

(3) *$P_{A,x} \xrightarrow{\cdot g} P_{A,xg}$ is a isomorphism of principal bundles.*

Proof. Set $H \coloneqq \mathrm{Hol}_p(A)$. Define a map $\hat{s}\colon P \to G/H$ as follows. For every $y \in P$, pick a path $\gamma\colon [0,1] \to B$ from $p(x)$ to $p(y)$. Let $\tilde{\gamma}\colon [0,1] \to P$ be a horizontal lift starting at y. Then there is a unique $g \in G$ such that $y = \tilde{\gamma}(1)\cdot g$. Define $\hat{s}(y) = gH$. This defines an $\hat{s} \in c^\infty(P, G/H)$. By Proposition 4.60 there is an associated structure reduction to H; indeed: the reduction is exactly $P_{A,x}$. This proves (1).

Suppose $y \in P_{A,x}$ and $v \in H_{A,y}$. Let γ be a horizontal curve starting in q with $\dot{\gamma}(0) = v$. We can extend γ to a horizontal curve passing though x for some $t < 0$. This shows that $H_{A,y} \subset T_yP_{A,x}$. Thus A induces a connection on $P_{A,x}$.

The last assertion is clear. ■

Remark 4.64. $P_{A,x}$ is the minimal reduction of P compatible with A. ♣

Definition 4.65. A is **irreducible** if $P_{A,p} = P$; otherwise it is **reducible**. •

4.13 Ambrose–Singer Theorem

Definition 4.66. Let (p,R) be a G–principal bundle. Let $A \in \mathscr{A}(p,R)$. The **holonomy Lie algebra** $\mathfrak{hol}_x(A)$ based at $x \in P$ is the Lie algebra of the Lie group $\mathrm{Hol}_x(A) \hookrightarrow G$. •

Theorem 4.67 (Ambrose and Singer [AS53]). *The holonomy Lie algebra satisfies*

$$\mathfrak{hol}_x(A) = \langle\{F_A(u,v) : y \in P_{A,x}, u,v \in T_yP\}\rangle \subset \mathfrak{g}.$$

Proof. By Theorem 4.63, without loss of generality $\mathrm{Hol}_p(A) = G$ and $P_{A,x} = P$. Set

$$\mathfrak{f} \coloneqq \langle\{F_A(u,v) : x \in P, u,v \in T_yP\}\rangle \subset \mathfrak{g}.$$

Denote by $\underline{\mathfrak{f}} \subset TP$ the corresponding subbundle of V_p.

By definition of curvature, $[H_A, H_A] \subset \underline{\mathfrak{k}}$. Because the distribution H_A is G–invariant, $[H_A, \underline{\mathfrak{k}}] \subset H_A$. Moreover for every $\xi \in \mathfrak{k}$,

$$\begin{aligned}
[F_A(u,v), \xi] &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{Ad}(\exp(-t\xi))F_p(u,v) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} F_A(\mathrm{d}R_{\exp(t\xi)}u, \mathrm{d}R_{\exp(t\xi)}v) \in \mathfrak{k}.
\end{aligned}$$

Therefore, $H_A \oplus \underline{\mathfrak{k}}$ is involutive.

Denote by Q the maximal connected integral submanifold through x. Since $y \in Q$ if and only if there is a path γ from x to y with $\dot{\gamma} \in E$, clearly $P \subset Q$.

This proves that $H_A \oplus \underline{\mathfrak{k}} = TP$ and consequently $\mathfrak{k} = \mathfrak{hol}_x(A)$. ■

Exercise 4.68. Give an example of a connection with $\mathfrak{hol}_x(A)$ is not spanned by the curvature at x itself.

4.14 Chern–Weil theory

The following notion is very useful to distinguish G–principal bundles.

Definition 4.69. Let $\mathrm{H}^\bullet$ be a cohomology theory, e.g., $\mathrm{H}^\bullet = \mathrm{H}^\bullet_{\mathrm{dR}}$. A **characteristic class** is an assignment of any G–principal bundle $(p\colon P \to B, R)$ to a cohomology class $c(p,R) \in \mathrm{H}^\bullet(B)$ such that if $f\colon A \to B$, then $c(f^*(p,R)) = f^*c(p,R)$. •

Remark 4.70. An excellent reference for the theory of characteristic classes is Milnor and Stasheff [MS74]. A systematic approach uses the **classifying space** BG. This is a topological space together with a G–principal bundle $(p\colon EG \to BG, R)$ such that up to isomorphism every G–principal bundle arises as $f^*(p,R)$ for some $f\colon B \to BG$. This exhibits a bijection between the set of isomorphism classes of G–princpal bundles over B and

$$[B, BG],$$

the set of homotopy classes of continuous maps $B \to BG$. Knowing this, a characteristic class is simply a cohomology class $c \in \mathrm{H}^\bullet(BG)$. For $G = \mathrm{U}(1)$, $B\mathrm{U}(1) = \mathbf{C}P^\infty$, $E\mathrm{U}(1) = S^\infty$, and $p\colon E\mathrm{U}(1) \to B\mathrm{U}(1)$ is a (version of the) Hopf fibration. Unfortunately, BG is not a (finite dimensional) smooth manifold. BG can be approximated by smooth manifolds (or be regarded as a smooth stack $BG = [*/G]$). ♣

The above approach still doesn't give us concrete characteristic classes.

❈

Let V be a vector space. A **polynomial** of degree k is a linear map $p \in \mathrm{Hom}(S^kV, \mathbf{R})$. Here S^kV denotes the k–th symmetric product. If $V = \mathbf{R}^n$, then

$$e_1^{k_1} \odot \cdots \odot e_n^{k_n}$$

with $\sum k_i = k$ form a basis of $S^k V$. This gives us a polynomial $P \in \mathbf{R}[x_1, \ldots, x_k]$ of degree k by

$$P := \sum_{k_1+\ldots+k_n=k} p(e_1^{k_1} \odot \cdots \odot e_n^{k_n}) \cdot x_1^{k_1} \cdots x_n^{k_n}.$$

A **formal power series** on V is an element of

$$\mathrm{Hom}(S^\bullet V, \mathbf{R}) = \mathrm{Hom}(\bigoplus_{k=0}^{\infty} S^k V, \mathbf{R}) = \prod_{k=0}^{\infty} \mathrm{Hom}(S^k V, \mathbf{R}).$$

If $V = \mathfrak{g} := \mathrm{Lie}(G)$, then G acts on $\mathrm{Hom}(S^\bullet \mathfrak{g}, \mathbf{R})$ via $g \mapsto S^\bullet \mathrm{Ad}_g$. Denote the Ad–**invariant** formal power series by

$$\mathrm{Hom}(S^\bullet \mathfrak{g}, \mathbf{R})^{\mathrm{Ad}}.$$

(This space is very computable using Lie theory, as we will see concretely later.)

Example 4.71. If $\mathfrak{g} = \mathfrak{u}(n)$, then $p(A) = \mathrm{tr}(A)$, $q(A, B) = \mathrm{tr}(AB)$, $r(A, B, C) = \mathrm{tr}(ABC + BAC + ACB)$ are Ad–invariant polynomials of degree 1, 2, 3. ♠

Let $s_k \in \mathrm{Hom}(S^k \mathfrak{g}, \mathbf{R})^{\mathrm{Ad}}$. Let $(p\colon P \to B, R)$ be a G–principal bundle. Let $A \in \mathscr{A}(p, R)$. Since $F_A \in \Omega^2_{\mathrm{hor}}(P, \mathfrak{g})^{\mathrm{Ad}}$,

$$F_A^{\wedge k} := \underbrace{F_A \wedge \ldots \wedge F_A}_{k \text{ times}} \in \Omega^{2k}_{\mathrm{hor}}(P, S^k \mathfrak{g})^{\mathrm{Ad}}$$

and

$$s_k(F_A^{\wedge k}) \in \Omega^{2k}_{\mathrm{hor}}(P)^G.$$

Since the latter is horizontal and G–invariant, there is a unique

$$\widecheck{s_k(F_A^{\wedge k})} \in \Omega^{2k}(B) \quad \text{with} \quad p^*[\widecheck{s_k(F_A^{\wedge k})}] = s_k(F_A^{\wedge k}).$$

$\check{-}$ is the inverse of the map $\hat{-}$ from Proposition 4.47 corresponding to the trivial representation of G on $\mathbf{R}$. Observe that, on $\Omega^\bullet_{\mathrm{hor}}(P)^G$, $\mathrm{d}_A = \mathrm{d}$.

Theorem 4.72 (The Chern–Weil construction). *Let $(p\colon P \to B, R\colon P \circlearrowleft G)$ be a G–principal bundle. Let $s \in \mathrm{Hom}(S^\bullet \mathfrak{g}, \mathbf{R})$.*

(1) *Let $A \in \mathscr{A}(p, R)$. The differential form*

$$\mathrm{CW}_A(s) := \widecheck{s(F_A^{\wedge \bullet})} \in \Omega^\bullet(B)$$

is closed.

(2) *The de Rham cohomology class*

$$\mathrm{CW}_{p,R}(s) := [\mathrm{CW}_A(s)] \in \mathrm{H}^\bullet_{\mathrm{dR}}(B)$$

is independent of $A \in \mathscr{A}(p, R)$.

(3) *The map* $\mathrm{CW}_{p,R}\colon \mathrm{Hom}(S^\bullet\mathfrak{g},\mathbf{R}) \to \mathrm{H}^\bullet_{\mathrm{dR}}(B)$ *is an* $\mathbf{R}$*–algebra homomorphism, the* ***Chern–Weil homomorphism****.*

(4) *If* $f\colon C \to B$ *is smooth, then*

$$\mathrm{CW}_{f^*(p,R)}(s) = f^*\mathrm{CW}_{p,R}(s).$$

Proof of Theorem 4.72. Suppose that $s \in \mathrm{Hom}(S^k\mathfrak{g},\mathbf{R})$. To prove (1), use the Bianchi identity and to compute

$$\mathrm{d}_A s(F_A^{\wedge k}) = s(\mathrm{d}_A(F_A^k)) = 0.$$

To prove (2), set

$$a := A_1 - A_0 \in \Omega^1_{\mathrm{hor}}(P,\mathfrak{g}) \quad \text{and} \quad A_t := A_0 + ta.$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t}F_{A_t} = \mathrm{d}_{A_t}a$$

and using the

$$\begin{aligned} s(F_{A_1}^{\wedge k}) - s(F_{A_0}^{\wedge k}) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} s(F_{A_t}^{\wedge k})\,\mathrm{d}t \\ &= k\int_0^1 s(\mathrm{d}_{A_t}a \wedge F_{A_t}^{\wedge k-1})\,\mathrm{d}t \\ &= \mathrm{d}\tau(A_0,A_1) \end{aligned}$$

with

$$\tau(A_0,A_1) := k\int_0^1 s(a\wedge F_{A_t}^{\wedge k-1})\,\mathrm{d}t.$$

Assertions (4) and (3) are obvious. ■

The following theorem asserts that the Chern–Weil homomorphism constructs all characteristic classes (up to torsion).

Theorem 4.73. $\mathrm{Hom}(S^k\mathfrak{g},\mathbf{R})^{\mathrm{Ad}} = \mathrm{H}^k(BG,\mathbf{R})$.

Sadly, the proof is outside of the scope of this course.

❊

Let us consider the case $G = \mathrm{GL}_n(\mathbf{C})$. For convenience of notation we use the obvious adaptation of the Chern–Weil homomorphism to $\mathbf{C}$ (instead of $\mathbf{R}$).

Proposition 4.74. *The restriction*

$$\mathrm{res}_\Delta\colon \mathrm{Hom}_{\mathbf{C}}(S^\bullet\mathfrak{gl}_n(\mathbf{C}),\mathbf{C})^{\mathrm{Ad}} \to \mathbf{C}[[x_1,\dots,x_n]]^{S_n}$$

to diagonal matrices in $\mathfrak{u}(n)$*, is an isomorphism.*

Proof. Diagonalizable matrices are dense in $\mathfrak{gl}(\mathbf{C}^n)$, Therefore, any $s \in \mathrm{Hom}(S^\bullet\mathfrak{gl}_n(\mathbf{C}), \mathbf{C})^{\mathrm{GL}_n(\mathbf{R})}$ is determined by its values on diagonal matrices. The space of diagonal matrices is $\mathbf{C}^n$ and the stabiliser of this subspace is $S_n \subset \mathrm{GL}(\mathbf{C}^n)$. ■

Remark 4.75. This is a special case of Chevalley's restriction theorem: If G is a complex connected semi-simple Lie group, $\mathfrak{t} \subset \mathfrak{g}$ is a Cartan subalgebra and W is the Weyl group, then

$$\mathrm{res}\colon\ \mathrm{Hom}(S^\bullet\mathfrak{g}, \mathbf{C})^{\mathrm{Ad}} \to \mathrm{Hom}(S^\bullet\mathfrak{t}, \mathbf{C})^W$$

is an isomorphism.

This is also the algebraic incarnation of what is sometimes called the splitting lemma. ♣

For $X \in \mathfrak{gl}_n(\mathbf{C})$ consider the characteristic polynomial

$$\det\left(\mathbf{1} + \lambda\frac{iX}{2\pi}\right) = \sum_{k=0}^{n} p_k(X)\lambda^k.$$

Clearly, $s_k \in \mathrm{Hom}(S^k\mathfrak{gl}_n(\mathbf{C}), \mathbf{R})^{\mathrm{Ad}}$. In terms of Proposition 4.74 we have

$$\mathrm{res}_\Delta(s_k) = \left(\frac{i}{2\pi}\right)^k \sum_{1\leqslant i_1<\ldots<i_k\leqslant n} x_{i_1}\cdots x_{i_k}.$$

Up to the prefactor these are the **elementary symmetric polynomials** of degree k in n–variables. It is not too difficult to see that these generate $\mathbf{C}[[x_1, \dots, x_n]]^{S_n}$.

Definition 4.76. The k–th **Chern class** of P is the characteristic class defined by

$$c_k(p, R) := \mathrm{CW}_{p,R}(s_k) \in H^{2k}_{\mathrm{dR}}(B; \mathbf{C})$$

The **total Chern class** is

$$c(p, R) := \sum_{k=0}^{\infty} c_k(p, R).$$

If E is a complex vector bundle of rank n, we also call the Chern classes of $\mathrm{Fr}_{\mathrm{GL}_n(\mathbf{C})}(E)$ the Chern classes of E. •

Exercise 4.77. Write a explicit formulae for $c_0(p, R)$, $c_1(p, R)$ and $c_2(p, R)$ in terms of a connection on (p, R).

Exercise 4.78. Compute $c(TCP^1)$.

Remark 4.79. The normalization $\frac{i}{2\pi}$ might seem strange at this point. It ensures that $c_k(P)$ is **integral**, i.e.,

$$c_k(p, R) \in \mathrm{im}(\mathrm{H}^{2k}(B; \mathbf{Z}) \to \mathrm{H}^{2k}(B; \mathbf{C}))$$

and also is needed to make c_k agree with other definitions of the Chern class. ♣

Exercise 4.80. Prove that if E is a complex rank r vector bundles, then $c_k(E) = 0$ for $k > r$.

Exercise 4.81. Prove that E_1 and E_2 are complex vector bundles, then

$$c(E_1 \oplus E_2) = c(E_1) \cup c(E_2).$$

For $X \in \mathfrak{gl}_n(\mathbf{C})$ consider

$$\operatorname{tr}\exp\left(\lambda \frac{iX}{2\pi}\right) = \sum_{\lambda=1}^{\infty} t_k(X)\lambda^k.$$

In terms of Proposition 4.74 we have

$$\operatorname{res}_\Delta(t_k) = \left(\frac{i}{2\pi}\right)^k \sum_{i=1}^{n} \frac{x_i^k}{k!}.$$

These too, like the s_k, generate all of $\operatorname{Hom}(S^\bullet \mathfrak{gl}_n(\mathbf{C}), \mathbf{C})^{\mathrm{Ad}}$. Expressions of the form $\sum_{i=1}^n x_i^k$ are called **power sums**.

Definition 4.82. The k–th **Chern character** of P is the characteristic class defined by

$$\operatorname{ch}_k(p, R) := \mathrm{CW}_{p,R}(t_k) \in \mathrm{H}^{2k}_{\mathrm{dR}}(M; \mathbf{C})$$

The **total Chern character** is

$$\operatorname{ch}(P) := \sum_{k=0}^{\infty} \operatorname{ch}_k(p, R).$$

If E is a complex vector bundle, then we also call the Chern characters of $\mathrm{Fr}_{\mathrm{GL}_n(\mathbf{C})}(E)$ the Chern characters of E. •

Exercise 4.83. Show that

$$\operatorname{ch}(E_1 \oplus E_2) = \operatorname{ch}(E_1) + \operatorname{ch}(E_2)$$

and

$$\operatorname{ch}(E_1 \otimes E_2) = \operatorname{ch}(E_1) \cup \operatorname{ch}(E_2).$$

Since both the p_k and the q_k generate $\mathbf{C}[\mathfrak{g}]^G$, it follows from Proposition 4.74 that the c_k can be expressed as a function of the ch_k. The following formulae are used often.

Proposition 4.84. *If E is a complex vector bundle, then*

$$\begin{aligned}
\operatorname{ch}_1(E) &= c_1(E), \\
\operatorname{ch}_2(E) &= \frac{1}{2}(c_1(E)^2 - 2c_2(E)) \quad \textit{and} \\
\operatorname{ch}_3(E) &= \frac{1}{6}(c_1(E)^3 - 3c_1(E)c_2(E) + 3c_3(E)).
\end{aligned}$$

Proof. The first is obvious since $p_1 = q_1$. For the second note that

$$\sum_{i=1}^{n} x_i^2 = \left(\sum_{i=1}^{n} x_i\right)^2 - 2 \sum_{1 \leqslant i < j \leqslant n} x_i x_j.$$

I leave the last identity as an exercise. ■

Remark 4.85. Underlying the above proposition are certain combinatorial identities know as the **Newton identities**. ♣

❊

Complexification induces an inclusion

$$\iota\colon\ \mathrm{GL}_n(\mathbf{R}) \hookrightarrow \mathrm{GL}_n(\mathbf{C})$$

Therefore, the characteristic classes for $\mathrm{GL}_n(\mathbf{C})$ induce characteristic classes for $\mathrm{GL}_n(\mathbf{R})$:

$$\mathrm{Hom}(S^\bullet\mathfrak{gl}_n(\mathbf{C}),\mathbf{C})^{\mathrm{Ad}} \xrightarrow{\iota^*} \mathrm{Hom}(S^\bullet\mathfrak{gl}_n(\mathbf{R}),\mathbf{C})^{\mathrm{Ad}}.$$

Definition 4.86. The k–th **Pontryagin class** of a real vector bundle $E \to B$ is

$$p_k(E) = (-1)^k c_{2k}(E \otimes_{\mathbf{R}} \mathbf{C}) \in \mathrm{H}^{4k}_{\mathrm{dR}}(B;\mathbf{C}). \qquad \bullet$$

Exercise 4.87. Show that $c_{2k+1}(E \otimes_{\mathbf{R}} \mathbf{C}) = 0$.

Exercise 4.88. Suppose B is a Riemannian closed 4–manifold. Let G be a semi-simple Lie group and P a principal G–bundle. Then minus the Killing form is a metric on $\mathfrak{g}_P := P \times_{\mathrm{Ad}} \mathfrak{g}$. Show that there are constants $c_1 > 0$ and $c_2 \in \mathbf{R}$ such that for any $A \in \mathscr{A}(P)$

$$\mathrm{YM}(A) = c_1 \int_X |F_A + *F_A|^2 + c_2 \int_X p_1(\mathfrak{g}_P).$$

Since the second term on the right-hand side depends only on P, this shows, in particular, that anti-self-dual instantons are absolute minima of YM (and not just critical points).

Finally, let me introduce the Euler class. This requires some linear algebra.

Proposition 4.89. *If $X \in \mathfrak{o}(2n)$, then there exists a $g \in \mathrm{SO}(2n)$ such that gXg^{-1} is block diagonal with blocks of the form*

$$\begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix}$$

The stabilizer of the space of block diagonal matrices is S_n.

Remark 4.90. If $X \in \mathfrak{o}(2m+1)$, an analogous result holds but one needs to allow for one block of the form (0). ♣

Definition 4.91. The **Pfaffian** is a $\mathrm{SO}(2n)$–invariant degree n polynomial on $\mathfrak{o}(2n)$ defined by

$$\mathrm{Pf}(A) = \prod_{i=1}^{n} \lambda_i. \qquad \bullet$$

Remark 4.92. It is clear from the definition that $\mathrm{Pf}(A)^2 = \det(A)$; ♣

Exercise 4.93. If $X \in \mathfrak{o}(2m)$ and $\omega := \sum_{i<j} X_{ij} e_i \wedge e_j$, then

$$\mathrm{Pf}(X) \cdot e_1 \wedge \cdots \wedge e_{2n} = \frac{\omega^n}{n!}.$$

Definition 4.94. If (p, R) is a principal $\mathrm{SO}(2m)$–bundle, then its **Euler class** is

$$e(p, R) \coloneqq \mathrm{CW}_{p,R}\left(\frac{\mathrm{Pf}}{(2\pi)^n}\right).$$

Define the **Euler class** of an oriented (Euclidean) vector bundle of rank $2m$ by as the Euler class of its $\mathrm{SO}(2m)$ frame bundle. ●

Remark 4.95. Here is a **warning**: the Pfaffian really is attached to $\mathrm{SO}(2m)$ and *not* $\mathrm{GL}^+(2m)$. Therefore, a $\mathrm{GL}^+(2m)$–principal bundle might very well admit a flat connection, but a reduction of structure group to $\mathrm{SO}(2m)$ does not have a trivial Euler class! ♣

Exercise 4.96. If $(p\colon P \to B, R)$ is a $\mathrm{U}(n)$–principal bundle and $(q\colon Q \coloneqq P \times_{\mathrm{U}(n)} \mathrm{SO}(2n), S)$ is its associated $\mathrm{SO}(2n)$–principal bundle, then

$$e(q, S) = c_n(p, R).$$

Example 4.97. Let (Σ, g) be an Riemann surface. If (e_1, e_2) is a local orthonormal frame and (e^1, e^2) is the dual coframe, then R_g is of the form

$$R_g = K_g \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes e^1 \wedge e^2$$

for some function K_g. A moments thought shows that K_g does not depend on the choice of local frame and, hence, defines a function $K_g \in C^\infty(M)$. K_g is called the **Gauss curvature** of g. In an *arbitrary* basis (e_1, e_2) we have

$$K_g \coloneqq -\frac{\langle R_g(e_1, e_2)e_1, e_2\rangle}{|e_1|^2|e_2|^2 - \langle e_1, e_2\rangle^2}.$$

If Σ is oriented, then we can define $e(T\Sigma)$ and the above shows that

$$e(T\Sigma) = \frac{1}{2\pi}[K_g \cdot \mathrm{vol}_g]. \qquad ♠$$

See my Riemannian geometry notes from 2021 for a discussion of the Chern–Gauß–Bonnet theorem.

4.15 G–structures on smooth manifolds

Definition 4.98. Let $\rho\colon G \to \mathrm{GL}_n(\mathbf{R})$ be a Lie group homomorphism Let X be a smooth manifold of dimension n. A G**–structure** on X is a ρ reduction of $\mathrm{Fr}(TX)$; that is: a G–principal bundle $(p\colon P \to X, R)$ together with an isomorphism

$$P \times_\rho \mathrm{GL}_n(\mathbf{R}) \cong \mathrm{Fr}(TX).$$

●

Example 4.99. A $\mathrm{GL}_n^+(\mathbf{R})$–structure on X is equivalent to an orientation of X. ♠

Example 4.100. An $\mathrm{O}(n)$–structure on X is equivalent to a Riemannian metric on X. ♠

Example 4.101. An $\mathrm{GL}_{n/2}(\mathbf{C})$–structure on X is equivalent to an almost complex structure. ♠

Example 4.102. An $\mathrm{U}(n/2)$–structure on X is equivalent to an almost Hermitian structure. ♠

Example 4.103. For $n \geqslant 3$, $\pi_1(\mathrm{SO}(n), \mathbf{1}) \cong \mathbf{Z}/2\mathbf{Z}$. The universal cover of $\mathrm{SO}(n)$ is a Lie group, $\mathrm{Spin}(n)$, and comes with the covering map is a Lie group homomorphism: $\rho\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$. A **spin structure** on a *Riemannian manifold* (X, g) is a ρ–reduction of $\mathrm{Fr}_{\mathrm{SO}}(TX, g)$. The above description of $\mathrm{Spin}(n)$, unfortunately, is not that useful. $\mathrm{Spin}(n)$ has more representations than $\mathrm{SO}(n)$ and to understand those one needs (either) a more concrete description of $\mathrm{Spin}(n)$ (or some knowledge of Lie theory). We will learn more about spin geometry in the context of Seiberg–Witten theory. ♠

For an affine connection, that is: a covariant derivative ∇ on TX, there is a notion of torsion $T_\nabla \in \Omega^2(X, TX)$ defined by

$$T_\nabla(v, w) \coloneqq \nabla_v w - \nabla_w v - [v, w]$$

for $v, w \in \mathrm{Vect}(X)$. In the context of G–structures this can be formulated using the solder form.

Definition 4.104. Let $(p\colon P \to X, R)$ be a G–principal bundle. Let $\rho\colon G \to \mathrm{GL}(V)$ be a finite-dimensional representation. A **(ρ–)solder form** σ on (p, R) is a horizontal G–equivariant 1–form

$$\sigma \in \Omega^1_{\mathrm{hor}}(P, V)^\rho$$

such that the map $TX \to P \times_\rho V$, defined by

$$(p(x), v) \mapsto [x, \sigma(\tilde{v})]$$

with $\tilde{v}$ denoting any lift of v, is an isomorphism. •

Example 4.105. Let $(p\colon \mathrm{Fr}(TX) \to X, R)$ be the frame bundle of TX. The canonical solder form σ on (p, R) is defined as follows. For $x \in X$ and $\phi\colon \mathbf{R}^n \to T_x X$ a frame define

$$\sigma_{(x,\phi)}(v) \coloneqq \phi^{-1} \circ T_{(x,\phi)} p(v).$$

By restriction this induces a solder form for every G–structure on X. ♠

Remark 4.106. If (p, R) admits a solder form σ, then that induces an isomorphism $P \times_G \mathrm{GL}(V) \cong \mathrm{Fr}(TX)$ such that σ is the restriction of the canonical solder form. Therefore, it is convenient to regard a G–structure as a G–principal bundle (p, R) together with a solder form. ♣

Proposition 4.107. *Let A be a $\mathrm{GL}_n(\mathbf{R})$–principal connection on $(p\colon \mathrm{Fr}(TX) \to X, R)$. Denote by ∇ the corresponding affine connection. Denote by σ the canonical solder form. The isomorphism $\Omega^2(X, TX) \cong \Omega^2_{\mathrm{hor}}(\mathrm{Fr}(TX), \mathbf{R}^n)^{\mathrm{GL}_n(\mathbf{R})}$ maps the torsion T_∇ of ∇ to*

$$\mathrm{d}_A \sigma = \mathrm{d}\sigma + [A \wedge \sigma].$$

Proof. Let $v, w \in \mathrm{Vect}(X)$. Let

$$\hat{v}, \hat{w} \in C^\infty(\mathrm{Fr}(TX), \mathbf{R}^n)^{\mathrm{GL}_n(\mathbf{R})}$$

be the lifts to maps. Denote by $\tilde{v}, \tilde{w} \in \mathrm{Vect}(P)$ (arbitrary) lifts to vector fields. The solder form relates these by

$$\sigma(\tilde{v}) = \hat{v} \quad \text{and} \quad \sigma(\tilde{w}) = \hat{w}.$$

According to Proposition 4.47, $\nabla v, \nabla w \in \Omega^1(X, TX)$ lift to

$$\mathrm{d}_A\hat{v}, \mathrm{d}_A\hat{w} \in \Omega^1_{\mathrm{hor}}(\mathrm{Fr}(TX), \mathbf{R}^n)^{\mathrm{GL}_n(\mathbf{R})}.$$

Therefore, $T_\nabla(v, w)$ lifts to

$$\mathrm{d}_A\hat{v}(\tilde{w}) - \mathrm{d}_A\hat{w}(\tilde{v}) - \sigma([\tilde{v}, \tilde{w}]) \in C^\infty(\mathrm{Fr}(TX), \mathbf{R}^n)^{\mathrm{GL}_n(\mathbf{R})}.$$

This is precisely $(\mathrm{d}_A\sigma)(\tilde{v}, \tilde{w})$. ■

Remark 4.108. The equation $\mathrm{d}_A\sigma = \mathrm{d}\sigma + [A \wedge \sigma]$ is sometimes called **Cartan's first structure equation**. **Cartan's second structure equation** is $F_A = \mathrm{d}A + \frac{1}{2}[A \wedge A]$.

If you read these phrases anywhere, then typically the solder form is called a co-frame and written as in components: $\sigma = (\omega^i)$. Similarly, the connection A is written in components $A = (\theta^i_j)$. The first structure equation then takes the form

$$\mathrm{d}\omega^i = -\theta^i_j \wedge \omega^j + \frac{1}{2}T^i_{jk}\omega^j \wedge \omega^k$$

with T^i_{jk} denoting the torsion. The second structure equation becomes

$$\mathrm{d}\theta^i_j = -\theta^i_k \wedge \theta^k_j + \frac{1}{2}R^i_{jk\ell}\omega^k \wedge \omega^\ell.$$

(Throughout, Einstein's summation convention is assumed.) If you squint your eyes, this looks like the "usual coordinate" expressions. ♣

We know that X always admits a torsion-free affine connection, even one compatible with a choice of Riemannian metric, i.e. a $\mathrm{O}(n)$–structure. What about other G–structures?

Definition 4.109. Let (p, R) be G–principal bundle on X together with a solder form σ. Let $A \in \mathscr{A}(p, R)$. The **torsion** of A is $\mathrm{d}_A\sigma$. If A' is another G–principal connection, then $A' = A + a$ with $a \in \Omega^1_{\mathrm{hor}}(P, \mathfrak{g})^{\mathrm{Ad}}$ and

$$\mathrm{d}_{A'}\sigma = \mathrm{d}_A\sigma + \rho(a) \wedge \sigma.$$

The **intrinsic torsion** of $(p, R; \sigma)$ is defined by

$$T(p, R; \sigma) \coloneqq [\mathrm{d}_A\sigma] \in \mathrm{coker}\big[\rho(-) \wedge \sigma\colon\ \Omega^1_{\mathrm{hor}}(P, \mathfrak{g})^{\mathrm{Ad}} \to \Omega^2_{\mathrm{hor}}(P, V)^\rho\big].$$

If $T(p, R; \sigma) = 0$, then $(p, R; \sigma)$ is **torsion-free**. •

Obviously:

Proposition 4.110. *(p, R) admits a torsion-free connection if and only if the intrinsic torsion vanishes. Moreover, in this case, the space of torsion-free connection is an affine space modelled on*

$$\ker\big[\rho(-) \wedge \sigma\colon\ \Omega^1_{\mathrm{hor}}(P, \mathfrak{g})^{\mathrm{Ad}} \to \Omega^2_{\mathrm{hor}}(P, V)^\rho\big].$$

■

Exercise 4.111. Prove that the map

$$\rho(-) \wedge \sigma\colon \ \Omega^1_{\mathrm{hor}}(P, \mathfrak{o}(n))^{\mathrm{Ad}} \to \Omega^2_{\mathrm{hor}}(P, \mathbf{R}^n)^{\mathrm{O}(n)}$$

is an isomorphism. (This algebraic fact implies the fundamental theorem of Riemannian geometry: the existence and uniqueness of the Levi–Civita connection.)

Exercise 4.112. Let X be a smooth manifold of dimension $2n$. Let I be an almost complex structure on TX. Prove that the corresponding $\mathrm{GL}_n(\mathbf{C})$–structure has vanishing intrinsic torsion if and only if the Nijenhuis tensor N_I vanishes.

Remark 4.113. It is an interesting and complicated problem to determine for which subgroup $G < \mathrm{O}(n)$ irreducible torsion-free G–structures exists. The irreducibility condition is important, since flat manifolds exist. By the Ambrose–Singer theorem, it amounts to ensuring that $\mathrm{Hol}(A) = G$. This question has been solved by Berger [Ber55]. ♣

5 Aspects of Yang–Mills theory

5.1 The Yang–Mills functional

Let (X, g) be an oriented pseudo-Riemannian manifold. Let G be a Lie group. Set $\mathfrak{g} := \mathrm{Lie}(G)$

Definition 5.1. Let $(p\colon\ P \to X, R)$ be a G–principal bundle. The **adjoint bundle** associated with (p, R) is the vector bundle

$$\mathrm{Ad}(P) := P \times_{\mathrm{Ad}} \mathfrak{g} \to X. \qquad \bullet$$

Ad(P) plays an important role because

$$\Omega^k_{\mathrm{hor}}(P, \mathfrak{g})^{\mathrm{Ad}} \cong \Omega^k(X, \mathrm{Ad}(P)).$$

Therefore, $\mathscr{A}(p, R)$ can be regarded as an affine spaces modelled on $\Omega^1(X, \mathrm{Ad}(P))$, and the curvature F_A of a connection A can and (mostly) will be regarded as an $\mathrm{Ad}(P)$–valued 2–form.

Every Ad–invariant bilinear form $B \in \mathrm{Hom}(S^2\mathfrak{g}, \mathbf{R})^{\mathrm{Ad}}$ induces a bilinear form on $\mathrm{Ad}(P)$. If G is a matrix group; that is: $G < \mathrm{GL}_n(\mathbf{R})$, then a natural choice is

$$B(\xi, \eta) := \mathrm{tr}(\xi\eta).$$

In fact, there always is a canonical choice.

Definition 5.2. The **Killing form** is the Ad–invariant bilinear form $B \in \mathrm{Hom}(S^2\mathfrak{g}, \mathbf{R})^{\mathrm{Ad}}$ defined by

$$B(\xi, \eta) := \mathrm{tr}(\mathrm{ad}(\xi) \circ \mathrm{ad}(\eta)). \qquad \bullet$$

Choose a $B \in \mathrm{Hom}(S^2\mathfrak{g}, \mathbf{R})^{\mathrm{Ad}}$. Define

$$|F_A|^2 \in C^\infty(X, \mathbf{R})$$

using B and the pseudo-Riemannian metric g.

Definition 5.3. The **Yang–Mills functional** $\mathrm{YM}\colon \mathscr{A}(p,R) \to \mathbf{R}$ is defined by

$$\mathrm{YM}(A) \coloneqq \frac{1}{2}\int_X |F_A|^2 \operatorname{vol}_g. \qquad \bullet$$

If $u \in \mathscr{G}(p,R)$, then

$$\mathrm{YM}(u^*A) = \mathrm{YM}(A).$$

(Prove this!) Therefore, YM descends to a map

$$\mathrm{YM}\colon \mathscr{A}(p,R)/\mathscr{G}(p,R) \to \mathbf{R}.$$

The Yang–Mills functional should be regarded as an energy functional. (At this stage, despite the notation, $|F_A|^2$ need not be non-negative. This should not deter us.)

Suppose that B is non-degenerate. The covariant derivative

$$\mathrm{d}_A\colon \Omega^\bullet(X,\mathrm{Ad}(P)) \to \Omega^{\bullet+1}(X,\mathrm{Ad}(P))$$

has a formal adjoint

$$\mathrm{d}_A^*\colon \Omega^\bullet(X,\mathrm{Ad}(P)) \to \Omega^{\bullet-1}(X,\mathrm{Ad}(P))$$

with respect to B and g.

Proposition 5.4. *For $A \in \mathscr{A}(p,R)$ and $a \in \Omega^1(X,\mathrm{Ad}(P))$*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{YM}(A+ta) = \int_X \langle \mathrm{d}_A^* F_A, a\rangle \operatorname{vol}_g$$

Proof. This is a consequence of

$$F_{A+ta} = F_A + t\mathrm{d}_A a + \frac{1}{2}t^2[a\wedge a].$$

Indeed,

$$\begin{aligned}\mathrm{YM}(A+ta) &= \frac{1}{2}\int_X |F_A + t\mathrm{d}_A a|^2 \operatorname{vol}_g + O(t^2)\\ &= \mathrm{YM}(A) + t\int_X \langle F_A, \mathrm{d}_A a\rangle \operatorname{vol}_g + O(t^2)\\ &= \mathrm{YM}(A) + t\int_X \langle \mathrm{d}_A^* F_A, a\rangle \operatorname{vol}_g + O(t^2). \qquad \blacksquare\end{aligned}$$

Definition 5.5. The **Yang–Mills equation** is

$$\mathrm{d}_A^* F_A = 0. \qquad \bullet$$

Remark 5.6. The Yang–Mills equation should be understood as a second order equation on the connection A. ♣

The Yang–Mills equation stands at the interface of physics and geometry, similar to (the vacuum) Einstein equations. Mysteriously, its study has had remarkable impact on mathematics: ranging across representation theory, algebraic geometry, partial differential equations, and topology.

5.2 Maxwell's equations

Maxwell's equations (without charges and currents) governing electro-magnetism are an instance of the Yang–Mills equation. In Maxwell's theory, there are two fields: the electric field $E = (E_1, E_2, E_3)$ and the magnetic field $B = (B_1, B_2, B_3)$ satisfying

$$\nabla \cdot B = 0, \quad \nabla \times E + \partial_t B = 0, \quad \nabla \cdot E = 0, \quad \text{and} \quad \nabla \times B - \partial_t E = 0.$$

In the presence of charges and currents, the last two equations are modified. The first two equations imply that there are ϕ, the electric potential, and $\mathbf{A} = (A_1, A_2, A_3)$, the vector potential, such that

$$E = \nabla\phi - \partial_t \mathbf{A} \quad \text{and} \quad B = \nabla \times \mathbf{A}.$$

Of course, $\mathbf{A}$ and ϕ are not uniquely determined by E and B.

It is convenient to package $\mathbf{A}$ and ϕ together as

$$A \coloneqq i \sum_{a=1}^{3} A_a \mathrm{d}x_a + \phi \mathrm{d}t \in \Omega^1(\mathbf{R}^4, \mathfrak{u}(1)).$$

This can be regarded as a U(1)–principal connection on the trivial U(1)–principal bundle over $\mathbf{R}^4$. Its curvature is

$$F_A = \mathrm{d}A = \frac{i}{2} \sum_{a,b=1}^{3} \underbrace{(\partial_a A_b - \partial_b A_a)}_{B_c} \mathrm{d}x_a \wedge \mathrm{d}x_b + i \sum_{a=1}^{3} \underbrace{(\partial_a \phi - \partial_t A_a)}_{E_a} \mathrm{d}x_a \wedge \mathrm{d}t.$$

The Bianchi equation $\mathrm{d}_A F_A = 0$ encodes precisely the first two of Maxwell's equations. The ambiguity in choosing $\mathbf{A}$ and ϕ corresponds the possibly gauge transformations u^*A of A which (because U(1) is abelian) have the same curvature $F_{u^*A} = F_A$.

To obtain the last two of Maxwell's equations, equip $\mathbf{R}^4$ with the Minkowski metric

$$g = \mathrm{d}x_1 \odot \mathrm{d}x_1 + \mathrm{d}x_2 \odot \mathrm{d}x_2 + \mathrm{d}x_3 \odot \mathrm{d}x_3 - \mathrm{d}t \odot \mathrm{d}t.$$

Denote by ε_{abc} the Levi–Civita symbol. A brief computation reveals that

$$\begin{aligned} \mathrm{d}_A^* F_A &= i(\partial_2 B_3 - \partial_3 B_2 - \partial_t E_1)\mathrm{d}x_1 + i(\partial_3 B_1 - \partial_1 B_3 - \partial_t E_2)\mathrm{d}x_2 \\ &\quad + i(\partial_1 B_2 - \partial_2 B_1 - \partial_t E_3)\mathrm{d}x_3 - i \sum_{a=1}^{3} \partial_a E_a \, \mathrm{d}t. \end{aligned}$$

Therefore, $\mathrm{d}_A^* F_A = 0$ is equivalent to the last two of Maxwell's equations.

5.3 Anti-self-duality

Let (X, g) be an oriented Riemannian manifold. Let B be an Euclidean inner product on $\mathfrak{g}$.

Flat connections trivially satisfy the Yang–Mills equation. Indeed, they are absolute minima of the Yang–Mills functional. Moreover, the condition to be flat is a first order equation on the metric, while the Yang–Mills equation is a second order equation. Chern–Weil theory gives rise to numerous obstructions for G–principal bundles to admit flat connections.

Suppose that
$$\dim X = 4.$$

The miracle of anti-self-duality appears:
$$*\colon\ \Omega^2(X) \to \Omega^2(X) \quad \text{and} \quad ** = \mathrm{id}.$$

Therefore, $*$ has two eigenvalues $+1$ and -1. By the Bianchi identity,
$$*F_A = \pm F_A \implies \mathrm{d}_A^* F_A = 0.$$

Definition 5.7. A connection $A \in \mathscr{A}(p, R)$ is **anti-self-dual (ASD)** if
$$*F_A = -F_A.$$
•

Remark 5.8. Initially, whether one studies the anti-self-duality equation $*F_A = -F_A$ or the (possibly more natural seeming) self-duality equation $*F_A = F_A$ seems to not matter. After all, one is free to flip the orientation on X and that exchanges these notions. It turns out, however, that for Kähler 4–manifolds, the complex structure selects a preferred orientation and for that orientation the anti-self-duality equation interacts well with the theory of holomorphic vector bundles. ♣

Remark 5.9. There are versions of anti-self-duality in higher dimension, but these all require X to have special geometry; e.g., it must be Kähler manifold, a G_2–manifold, or a $\mathrm{Spin}(7)$–manifold, etc. ♣

Example 5.10. Flat connections are ASD. ♠

Example 5.11. If $G = \mathrm{U}(1)$, then $F_A \in \Omega^2(X, i\mathbf{R})$. By the Bianchi identity, $\mathrm{d}F_A = \mathrm{d}_A F_A = 0$. Therefore $[F_A] \in \mathrm{H}^2_{\mathrm{dR}}(X, i\mathbf{R})$. If $[\Omega] \in \mathrm{im}(\mathrm{H}^2(X, \mathbf{Z}) \to \mathrm{H}^2_{\mathrm{dR}}(X, \mathbf{R}))$, then there always is a $\mathrm{U}(1)$–principal bundle with a connection A such that $F_A = -2\pi i\Omega$. Therefore, $\mathrm{U}(1)$ ASD connections (up to gauge transformations) are essentially classified by the anti-self-dual harmonic 2–forms in $\mathrm{im}(\mathrm{H}^2(X, \mathbf{Z}) \to \mathrm{H}^2_{\mathrm{dR}}(X, \mathbf{R}) \cong \mathscr{H}^2(X, g))$ ♠

Proposition 5.12. *Suppose that G is semi-simple and $-B$ is the Killing form. For every $A \in \mathscr{A}(p, R)$*
$$\mathrm{YM}(A) = \mp 4\pi^2 \int_X p_1(\mathrm{Ad}(P)) + \frac{1}{4}\int_X |F_A \pm *F_A|^2 \,\mathrm{vol}_g$$

In particular, (anti-)self-dual connections are absolute minima of the Yang–Mills functional.

Proof. By definition,
$$p_1(\mathrm{Ad}(P)) = -c_2(\mathrm{Ad}(P) \otimes \mathbf{C}).$$

Since
$$c_2 = -\mathrm{ch}_2 + \frac{1}{2}\mathrm{ch}_1^2, \quad \mathrm{ch}_k(\mathrm{Ad}(P) \otimes \mathbf{C}) = \left[\mathrm{tr}\,\frac{1}{k!}\left(\tfrac{i}{2\pi}\,\mathrm{ad} \circ F_A\right)^{\wedge k}\right],$$

and tr vanishes on $\mathfrak{o}(n)$,

$$
\begin{aligned}
p_1(\mathrm{Ad}(P)) &= -\frac{1}{8\pi^2}[\mathrm{tr}(\mathrm{ad}\circ F_A \wedge \mathrm{ad}\circ F_A)] \\
&= \frac{1}{8\pi^2}\langle F_A \wedge F_A\rangle.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{1}{4}\int_X |F_A \pm *F_A|^2 \operatorname{vol}_g &= \frac{1}{4}\int_X \langle (F_A \pm *F_A) \wedge (*F_A \pm F_A)\rangle \\
&= \frac{1}{2}\int_X |F_A|^2 \pm \frac{1}{2}\langle F_A \wedge F_A\rangle \\
&= \mathrm{YM}(A) \pm 4\pi^2 p_1(\mathrm{Ad}(P)).
\end{aligned}
$$

■

Remark 5.13. In dimension 4, YM depends on the conformal class of g only; that is: YM computed with respect to g is identical to YM computed with respect to $e^{2f}g$. Moreover, $*\colon \Lambda^2 T^*X \to \Lambda^2 T^*X$ also depends only on the conformal class of g. Even more is true: $*$ determines the conformal structure. More precisely: the wedge product defines an symmetric bilinear form of signature $(3,3)$ on $\Lambda^2(\mathbf{R}^4)^*$. If $\Lambda^+ \subset \Lambda^2(\mathbf{R}^4)^*$ is a maximal positive definite subspace, then there is a unique conformal class $[g]$ such that Λ^+ is the $+1$–eigenspace of $*\colon \Lambda^2(\mathbf{R}^4)^* \to \Lambda^2(\mathbf{R}^4)^*$. ♣

5.4 The BPST instanton

The BPST instanton is an important example of an anti-self-dual connection on the trivial $\mathrm{SU}(2)$–principal bundle over $\mathbf{R}^4$ discovered by Belavin, Polyakov, Schwartz, and Tyupkin [BPST75]. Much of the following discussion stems from Atiyah's wonderful book [Ati79].

To understand Belavin, Polyakov, Schwartz, and Tyupkin [BPST75]'s construction it is useful to use the quaternions $\mathbf{H} = \mathbf{R}\langle 1, i, j, k\rangle$. Denote by $q \in C^\infty(\mathbf{H}, \mathbf{H})$ the identity map and define $q_0, q_1, q_2, q_3 \in C^\infty(\mathbf{H})$ by

$$q =: q_0 + q_1 i + q_2 j + q_3 k.$$

Denote by $\bar{q}$ the conjugate; that is:

$$\bar{q} = q_0 - q_1 i - q_2 j - q_3 k.$$

Define

$$- \wedge - \colon\ \Omega^\bullet(X, \mathbf{H}) \otimes \Omega^\bullet(X, \mathbf{H}) \to \Omega^\bullet(X, \mathbf{H})$$

as the composition $-\wedge-\colon\ \Omega^\bullet(X,\mathbf{H}) \otimes \Omega^\bullet(X,\mathbf{H}) \to \Omega^\bullet(X, \mathbf{H}\otimes\mathbf{H})$ and multiplication $\mathbf{H}\otimes\mathbf{H} \to \mathbf{H}$. Since $\mathbf{H}$ is not commutative, $- \wedge -$ is not graded commutative. Indeed,

$$
\begin{aligned}
\mathrm{d}q \wedge \mathrm{d}\bar{q} = &-2(\mathrm{d}q_0 \wedge \mathrm{d}q_1 + \mathrm{d}q_2 \wedge \mathrm{d}q_3) \otimes i \\
&- 2(\mathrm{d}q_0 \wedge \mathrm{d}q_2 + \mathrm{d}q_3 \wedge \mathrm{d}q_1) \otimes j \\
&- 2(\mathrm{d}q_0 \wedge \mathrm{d}q_3 + \mathrm{d}q_1 \wedge \mathrm{d}q_2) \otimes k
\end{aligned}
$$

but

$$
\begin{aligned}
\mathrm{d}\bar{q} \wedge \mathrm{d}q = 2(&\mathrm{d}q_0 \wedge \mathrm{d}q_1 - \mathrm{d}q_2 \wedge \mathrm{d}q_3) \otimes i \\
&+ 2(\mathrm{d}q_0 \wedge \mathrm{d}q_2 - \mathrm{d}q_3 \wedge \mathrm{d}q_1) \otimes j \\
&+ 2(\mathrm{d}q_0 \wedge \mathrm{d}q_3 - \mathrm{d}q_1 \wedge \mathrm{d}q_2) \otimes k.
\end{aligned}
$$

Observe that the coefficients of $\mathrm{d}q \wedge \mathrm{d}\bar{q}$ are a basis of $\Lambda^+\mathbf{H}^*$ and the coefficients of $\mathrm{d}\bar{q} \wedge \mathrm{d}q$ span $\Lambda^-\mathbf{H}^*$.

The Lie group $\mathrm{Sp}(1) \coloneqq \{q \in \mathbf{H} : q\bar{q} = 1\}$ is isomorphic to $\mathrm{SU}(2)$ ($\mathbf{H} = \mathbf{C} \oplus \mathbf{C}j$). Observe that

$$
\mathfrak{sp}(1) \coloneqq \mathrm{Lie}(\mathrm{Sp}(1)) = \mathrm{Im}\,\mathbf{H}.
$$

Therefore, a $\mathrm{Sp}(1)$–connection on the trivial $\mathrm{Sp}(1)$–bundle over $\mathbf{H}$ can be regarded as a 1–form $A \in \Omega^1(\mathbf{H}, \mathrm{Im}\,\mathbf{H})$. $\mathrm{Sp}(1)$ acts on $\mathbf{H}$ by

$$
R(g)x \coloneqq xg^*
$$

and on $\mathrm{Im}\,\mathbf{H}$ by

$$
\mathrm{Ad}(g)\xi = g\xi g^*.
$$

If A is required to satisfy the invariance condition

$$
[R(q)]^* A = \mathrm{Ad}(q)A,
$$

then it must be of the form

$$
A = 2f(|q|^2)\,\mathrm{Im}(\bar{q}\mathrm{d}q) = f(|q|^2)(\bar{q}\mathrm{d}q - \mathrm{d}\bar{q}q)
$$

To facilitate the computation of F_A, observe that

$$
2\,\mathrm{Im}(\bar{q}\mathrm{d}q) = 2\bar{q}\mathrm{d}q - \mathrm{d}|q|^2 = -2\mathrm{d}\bar{q}q + \mathrm{d}|q|^2.
$$

Hence,

$$
\begin{aligned}
4\,\mathrm{Im}(\bar{q}\mathrm{d}q) \wedge \mathrm{Im}(\bar{q}\mathrm{d}q) &= (-2\mathrm{d}\bar{q}q + \mathrm{d}|q|^2) \wedge (2\bar{q}\mathrm{d}q - \mathrm{d}|q|^2) \\
&= -4|q|^2\mathrm{d}\bar{q} \wedge \mathrm{d}q + 2\mathrm{d}|q|^2 \wedge (\bar{q}\mathrm{d}q - \mathrm{d}\bar{q}q) \\
&= -4|q|^2\mathrm{d}\bar{q} \wedge \mathrm{d}q + \mathrm{d}|q|^2 \wedge 4\,\mathrm{Im}(\bar{q}\mathrm{d}q).
\end{aligned}
$$

Therefore, the curvature of A can be computed to be

$$
F_A = \left[2f(|q|^2) - 4|q|^2 f(|q|^2)^2\right]\mathrm{d}\bar{q} \wedge \mathrm{d}q + \left[f'(|q|^2) + 2f(|q|^2)^2\right]\mathrm{d}|q|^2 \wedge 2\,\mathrm{Im}(\bar{q}\mathrm{d}q).
$$

The first term is anti-self-dual. To make the second term vanish, one needs to solve the ODE $f' + 2f^2 = 0$:

$$
f(|q|^2) = \frac{1}{2}\frac{\mu^2}{\mu^2|q|^2 + 1}
$$

with $\mu > 0$. Therefore, we arrive at

$$A_\mu = \frac{\operatorname{Im}(\mu^2 \bar{q} \mathrm{d}q)}{\mu^2 |q|^2 + 1}.$$

The above computation singles out the origin in $\mathbf{H}$. The **BPST instanton** of scale μ and center b is

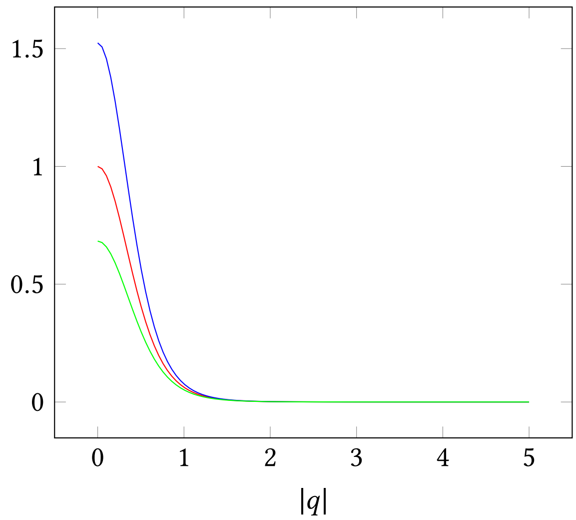
$$A_{\mu,b} \coloneqq \frac{\operatorname{Im}(\mu^2 (\bar{q} - \bar{b}) \mathrm{d}q)}{\mu^2 |q - b|^2 + 1}.$$

A without any indices shall always refer to $A_{1,0}$.

Observe that

$$F_{A_{\mu,b}} = \frac{\mu^2 \mathrm{d}\bar{q} \wedge \mathrm{d}q}{(\mu^2 |q-b|^2 + 1)^2}.$$

Here are plots of $1/(|q|^2 + 1/\mu^2)^2$ for $\mu^2 \in \{0.9, 1, 1.1\}$.



Let us compute $\mathrm{YM}(A)$.

$$F_A = \frac{\mathrm{d}\bar{q} \wedge \mathrm{d}q}{(|q|^2 + 1)^2}.$$

To compute the norm of i, j, k with respect to the the negative of the Killing form observe that $\mathrm{ad}(\xi)$ vanishes on ξ and acts as 2ξ on $\xi^\perp$. Therefore,

$$-B(i,i) = -B(j,j) = -B(k,k) = 8.$$

Consequently,

$$|\mathrm{d}\bar{q} \wedge \mathrm{d}q|^2 = 3 \cdot 8 \cdot 8 = 192.$$

Therefore,

$$\mathrm{YM}(A) = 96 \cdot \mathrm{vol}(S^3) \int_0^\infty \frac{r^3}{(r^2+1)^4} \mathrm{d}r = 16\pi^2.$$

This uses $\mathrm{vol}(S^3) = 2\pi^2$ and evaluates the integral to $1/12$.

Exercise 5.14. Compute $\mathrm{YM}(A)$!

Exercise 5.15. Prove that the parameters μ, b are determined by $|F_{A_{\mu,b}}|$ and, hence, the gauge equivalence class of $A_{\mu,b}$.

Remark 5.16. $\mathrm{Sp}(1)$ also acts on $\mathbf{H}$ via $L(g)x := gx$. This leads to a similar expression with q and $\bar{q}$ exchanged. The corresponding connection has self-dual curvature. ♣

5.4.1 The BPST instanton on S^4

Here is another perspective on the BPST instanton. $\mathrm{Sp}(1)$ acts on the right of

$$S^7 := \{(q_1, q_2) \in \mathbf{H}^2 : |q_1|^2 + |q_2|^2 = 1\}$$

via

$$R((q_1, q_2), q) := (q_1 q, q_2 q).$$

The quotient

$$\mathbf{H}P^1 := S^7/\mathrm{Sp}(1)$$

parametrizes rank 1 right $\mathbf{H}$–submodules $\ell \subset \mathbf{H}^2$. Denote by $p\colon S^7 \to \mathbf{H}P^1$ the canonical projection. Define $\theta_A \in \Omega^1(S^7, \mathfrak{sp}(1))$ by

$$\theta_A := \mathrm{Im}(\bar{q}_1 \mathrm{d}q_1 + \bar{q}_2 \mathrm{d}q_2).$$

A moment's thought shows that θ_A is a $\mathrm{Sp}(1)$–principal connection 1–form. Denote the corresponding $\mathrm{Sp}(1)$–principal connection by A.

Define $\iota_\pm\colon \mathbf{H} \to \mathbf{H}P^1$ by $\iota_+(q) := (q, 1)$ and $\iota_-(q) := (1, q)$. The map $s_\pm\colon \mathbf{H} \to S^7$ defined by

$$s_+(q) := \frac{(q,1)}{\sqrt{|q|^2+1}} \quad \text{and} \quad s_-(q) := \frac{(1,q)}{\sqrt{|q|^2+1}}$$

is trivialisation of $\iota_\pm^*(p, R)$. Moreover, a short computation reveals that

$$s_\pm^* \theta_A = \frac{\mathrm{Im}(\bar{q}\mathrm{d}q)}{|q|^2+1}.$$

Remark 5.17. The above discussion shows the BPST instanton $A := A_{1,0}$ on $\mathbf{H}$ can be extended to the conformal compactifications $\mathbf{H}P^1$. Uhlenbeck's removable singularities theorem [Uhl82b] says that this can always be done provided $\mathrm{YM}(A) < \infty$. ♣

Remark 5.18. Since $\mathrm{YM}(A) = 16\pi^2$, $p_1(\mathrm{Ad}(P)) = -4$. This is consistent with the fact that the underlying rank 2 complex bundle E has $c_1(E) = 0$ and $c_2(E) = 1$. Let E be a Hermitian vector bundle of rank n. Denote by P the corresponding $\mathrm{PU}(n)$–principal bundle. The complexification of $\mathrm{Ad}(P)$ is $\mathrm{End}_0(E)$ with the subscript meaning trace-free. A simple computation using Exercise 4.83 and Proposition 4.84 shows that

$$p_1(\mathrm{Ad}(P)) = r(c_1(E)^2 - 2c_2(E)) = -2r\left(c_2(E) - \frac{1}{2}c_1(E)^2\right). \qquad ♣$$

Remark 5.19. The orientation preserving conformal inversion $\jmath$ of $\mathbf{H}^\times$ defined by $f(q) \coloneqq q^{-1}$ lifts to S^7 ($\mathbf{H}P^1$) as $(q_1, q_2) \mapsto (q_2, q_1)$ ($[q_1 : q_2] \mapsto [q_2 : q_1]$). Obviously, θ_A is invariant under this inversion. This shows that $\jmath^* A$ is gauge equivalent to A. Indeed, the gauge transformation is $q \mapsto u(q) = q/|q|$. This can also be verified by direct computation. ♣

Set

$$S^4 \coloneqq \{(q,t) \in \mathbf{H} \oplus \mathbf{R} : |q|^2 + t^2 = 1\}.$$

Define the stereographic projections $\sigma_\pm \colon U_\pm \coloneqq S^4 \backslash \{(0, \mp 1)\} \to \mathbf{H}$ by

$$\sigma_\pm(q,t) \coloneqq \frac{q}{1 \mp t}.$$

The map $\phi \colon S^4 \to \mathbf{H}P^1$ defined by

$$\phi(q,t) \coloneqq \begin{cases} \iota_+(\sigma_+(q,t)) & \text{if } (q,t) \in U^+, \\ \iota_-(\overline{\sigma_-(q,t)}) & \text{if } (q,t) \in U^- \end{cases}$$

is a diffeomorphism. Since $\sigma_\pm$ are conformal, $\phi^* A$ is an anti-self-dual connection on $\phi^*(p, R)$ defined over S^4.

Remark 5.20. $\mathbf{H}P^1$ carries a natural metric g_{FS}, the Fubini–Study metric. The standard metric on $S^7 \subset \mathbf{H}^2$ descends along p because it is $\mathrm{Sp}(1)$–invariant. To obtain a formula proceed as follows. Denote by $\hat{p} \colon \mathbf{H}^2 \backslash \{0\} \to \mathbf{H}P^1$ the canonical projection. The Riemannian metric g_{FS} is characterised by the condition that

$$(\hat{p}^* g_{\mathrm{FS}})_x(v,w) = |x|^{-2}\langle v, w\rangle = |x|^{-2}\operatorname{Re}(w^* v)$$

whenever $v, w \perp x \cdot \mathbf{H}$, that is: $x^* v = x^* w = 0$. Since $v \mapsto v - x x^* v/|x|^2$ is the projection to $(x \cdot \mathbf{H})^\perp$,

$$\begin{aligned}
(\hat{p}^* g_{\mathrm{FS}})_x(v,w) &= |x|^{-2}\operatorname{Re}[(w - x x^* w/|x|^2)^*(v - x x^* v/|x|^2)] \\
&= |x|^{-2}\operatorname{Re}[(w^* - w^* x x^*/|x|^2)^*(v - x x^* v/|x|^2)] \\
&= \frac{\operatorname{Re}(w^* v)}{|x|^2} - \frac{\operatorname{Re}((w^* x)(x^* v))}{|x|^4}.
\end{aligned}$$

Therefore,

$$(\iota_+^* g_{\mathrm{FS}})_q(v,w) = \frac{\operatorname{Re}(w^* v)}{1 + |q|^2} - \frac{|q|^2 \operatorname{Re}(w^* v)}{(1+|q|^2)^2} = \frac{\operatorname{Re}(w^* v)}{(1+|q|^2)^2}.$$

This reveals that: $4 g_{\mathrm{FS}} = g_{S^4}$. Moreover, it shows that with respect to this metric $|F_{A_{1,0}}|$ is constant! ♣

Set

$$\begin{aligned}
\mathrm{SL}_2(\mathbf{H}) &\coloneqq \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbf{H}) : ad - bc = 1 \right\}, \\
\mathrm{Sp}(2) &\coloneqq \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbf{H}) : |a|^2 + |c|^2 = |b|^2 + |d|^2 = 1, \bar{a}b + \bar{c}d = 0. \right\}, \\
\mathrm{PSL}_2(\mathbf{H}) &\coloneqq \mathrm{SL}_2(\mathbf{H})\{\pm 1\}, \quad \text{and} \\
\mathrm{PSp}(2) &\coloneqq \mathrm{Sp}(2)/\{\pm 1\}.
\end{aligned}$$

$\mathbf{P}\mathrm{SL}_2(\mathbf{H})$ acts on $\mathbf{H}P^1$ via

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} [q_1 : q_2] = [aq_1 + bq_2, cq_1 + dq_2].$$

In fact, $\mathbf{P}\mathrm{SL}_2(\mathbf{H})$ is the orientation-preserving conformal group of $\mathbf{H}P^1$. This action does not lift to S^7, but the underlying action of $\mathrm{SL}_2(\mathbf{H})$ does. The best way to see this is to observe that $\mathrm{SL}_2(\mathbf{H})$ acts on $\mathbf{H}^2\backslash\{0\}$ and to identify $S^7 = (\mathbf{H}^2\backslash\{0\})/\mathbf{R}^+$.

Remark 5.21. $\mathbf{P}\mathrm{Sp}(2)$ acts on

$$V := \left\{ \begin{pmatrix} t & \bar{q} \\ q & -t \end{pmatrix} : t \in \mathbf{R}, q \in \mathbf{H} \right\} \cong \mathbf{H} \oplus \mathbf{R},$$

the quaternionic self-adjoint matrices, via conjugation. The latter have an natural inner product and, obviously, this action orthogonal (and orientation preserving). This exhibits an isomorphism $\mathbf{P}\mathrm{Sp}(2) \cong \mathrm{SO}(V) = \mathrm{SO}(5)$ and $\mathrm{Sp}(2) = \mathrm{Spin}(5)$. However, the diffeomorphism $\phi\colon S^4 \to \mathbf{H}P^1$ is *not* $\mathrm{SO}(5)$–equivariant. There is another inclusion $\mathrm{SO}(5) = \mathrm{Isom}(S^4) = \mathrm{Isom}(\mathbf{H}P^1) \hookrightarrow \mathbf{P}\mathrm{SL}_2(\mathbf{H})$. ♣

An simple computation shows that $\mathrm{Sp}(2) < \mathrm{SL}_2(\mathbf{H})$ preserves θ_A. Via $\iota_+\colon \mathbf{H} \to \mathbf{H}P^1$ this gives the partially defined action

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} q = (aq + b)(cq + d)^{-1}$$

by Möbius transformations. The map $q \mapsto \mu(q + b)$ lifts to

$$\begin{bmatrix} \sqrt{\mu} & \sqrt{\mu} b \\ 0 & 1/\sqrt{\mu} \end{bmatrix} = \begin{bmatrix} \sqrt{\mu} & 0 \\ 0 & 1/\sqrt{\mu} \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}.$$

Every element of $g \in \mathbf{P}\mathrm{SL}_2(\mathbf{H})$ can be written uniquely as $g = nak$ with

$$n = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}, \quad a = \begin{bmatrix} \sqrt{\mu} & 0 \\ 0 & 1/\sqrt{\mu} \end{bmatrix}$$

and $g \in \mathbf{P}\mathrm{Sp}(2)$. (This is an Iwasawa decomposition of $\mathbf{P}\mathrm{SL}_2(\mathbf{H})$.) Let us compute

$$\begin{pmatrix} \sqrt{\mu} & \sqrt{\mu} b \\ 0 & 1/\sqrt{\mu} \end{pmatrix}^* \theta_A = \frac{\mathrm{Im}[\mu^2(\bar{q}_1 - \bar{q}_2\bar{b})\mathrm{d}q_1 + (\mu^2\bar{q}_1 + \bar{q}_2)\mathrm{d}q_2]}{\mu^2|q_1 + bq_2|^2 + |q_2|^2}.$$

Obviously this yields $A_{\mu,b}$ (by construction). This shows that the BPST instantons all arise from the actions of $\mathrm{SL}_2(\mathbf{H})$ on S^7 and $\mathbf{H}P^1$. Therefore, the space of BPST instantons is

$$\mathrm{SL}_2(\mathbf{H})/\mathrm{Sp}(2) = \mathbf{P}\mathrm{SL}_2(\mathbf{H})/\mathbf{P}\mathrm{Sp}(2) \cong \mathbf{H} \times \mathbf{R}^+.$$

5.4.2 The BPST instanton on $\mathbf{R} \times S^3$

Consider the conformal diffeomorphism $\phi\colon \mathbf{R} \times \mathrm{Sp}(1) \to \mathbf{H}^\times$ defined by $\phi(t, g) := e^t g$. A brief computation shows that

$$A^{\mathrm{cyl}} := \phi^* A = \frac{g^{-1}\mathrm{d}g}{1 + e^{-2t}} = \frac{\mu_{\mathrm{Sp}(1)}}{1 + e^{-2t}}$$

with μ denoting the Maurer–Cartan form on $\mathrm{Sp}(1)$. As $t \to -\infty$, $\phi^* A$ tends to the trivial connection θ_0. As $t \to +\infty$, $\phi^* A$ tends to $u^*\theta_a$ with $u = \mathrm{id}_{\mathrm{Sp}(1)}\colon \mathrm{Sp}(1) \to \mathrm{Sp}(1)$.

Observe that

$$\partial_t A^{\mathrm{cyl}}(t, -) = \frac{2e^{-2t}\mu_{\mathrm{Sp}(1)}}{(1 + e^{-2t})^2}$$

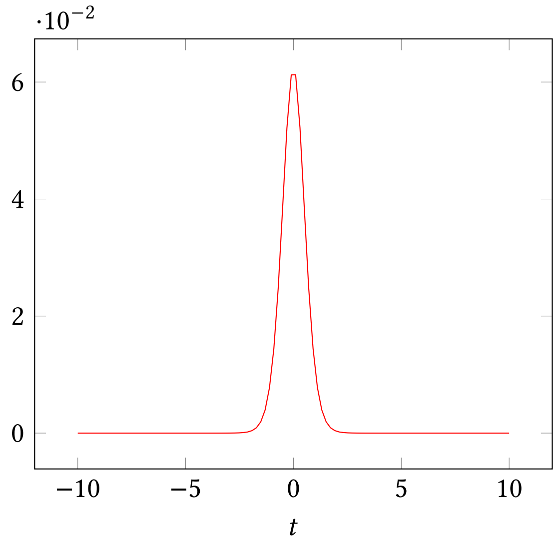
and

$$\begin{aligned} F_{A^{\mathrm{cyl}}(t,-)} &= \frac{\mathrm{d}\mu}{1 + e^{-2t}} + \frac{[\mu \wedge \mu]}{2(1 + e^{-2t})^2} \\ &= \left(-\frac{1}{2(1 + e^{-2t})} + \frac{1}{2(1 + e^{-2t})^2}\right)[\mu \wedge \mu] \\ &= -\frac{e^{-2t}}{2(1 + e^{-2t})^2}[\mu \wedge \mu]. \end{aligned}$$

Since $[\mu \wedge \mu] = 4 * \mu$, this shows that

$$\partial_t A^{\mathrm{cyl}}(t, -) = -F_{A^{\mathrm{cyl}}(t,-)}.$$

Here is a plot of $\frac{e^{-4t}}{2(1+e^{-2t})^4}$.



Exercise 5.22. What is the effect of the scale parameter μ in this perspective?

5.5 Hyperkähler manifolds and hyperkähler reduction

Atiyah, Drinfeld, Hitchin, and Manin [ADHM78] managed to construct every ASD instanton over $\mathbf{H}$ with finite Yang–Mills energy and $G = \mathrm{SU}(r)$ using "linear algebra". This was one of the early major achievements in mathematical gauge theory. The proper context for their construction are hyperkähler manifolds and hyperkähler reduction.

Definition 5.23. Let X be a smooth manifold of dimension $4n$. A **hyperkähler structure** on X consists of a Riemannian metric g and a triple of almost complex structures I_a ($a = 1, 2, 3$) such that

$$g(I_a -, I_a -) = g, \quad I_1 I_2 = I_3, \quad \text{and} \quad \nabla I_a = 0.$$

(This is equivalent to a torsion-free $\mathrm{Sp}(n)$–structure.) A **hyperkähler manifold** is a manifold with a hyperkähler structure.

Observe that $\omega_a := g(I_a -, -) \in \Omega^2(X)$ defines a symplectic (or Kähler) form on X. It is often convenient to encode I_a and ω_a ($a = 1, 2, 3$) as a **hypercomplex structure** and a **hyperkähler form**

$$\begin{aligned} \mathbf{I} &:= i^* \otimes I_1 + j^* \otimes I_2 + k^* \otimes I_3 \in (\operatorname{Im}\mathbf{H})^* \otimes \Gamma(\operatorname{End}(X)) \quad \text{and} \\ \boldsymbol{\omega} &:= i^* \otimes \omega_1 + j^* \otimes \omega_2 + k^* \otimes \omega_3 \in (\operatorname{Im}\mathbf{H})^* \otimes \Omega^2(X). \end{aligned}$$ •

It is a mildly non-trivial fact, that $\boldsymbol{\omega}$ determines $\mathbf{I}$ and g.

Remark 5.24. The hypercomplex structure $\mathbf{I}$ equips every tangent space T_xX with the structure of an $\mathbf{H}$ left-module: for $q = t + \xi$ with $t \in \mathbf{R}$ and $\xi \in \operatorname{Im}\mathbf{H}$

$$q \cdot v := tv + I_\xi v \quad \text{with} \quad I_\xi := \langle \mathbf{I}, \xi \rangle.$$ ♣

Example 5.25. For every $n \in \mathbf{N}$, $\mathbf{H}^n$ is a hyperkähler manifold with g denoting the standard inner product, $I_1 = i$, $I_2 = j$, $I_3 = k$. In this case,

$$\boldsymbol{\omega} = \sum_{a=1}^{n} \mathrm{d}q_a \wedge \mathrm{d}\bar{q}_a.$$

If $\Lambda \subset \mathbf{H}^n$ is a lattice, then $\mathbf{H}^n/\Lambda$ is a hyperkähler manifold. ♠

Non-flat compact hyperkähler manifolds are notoriously difficult to construct. The first example is the $K3$ surface and the construction of the hyperkähler structure requires the use of Yau's solution of the Calabi conjecture. For mysterious(?) reasons, non-compact hyperkähler manifolds habitually emerge as moduli spaces in gauge theory. (More about that later.)

If S is a hyperkähler manifold and G acts on S, then S/G (if the quotient exists) is typically not hyperkähler.

Definition 5.26. Let S by a hyperkähler manifold with hyperkähler form $\boldsymbol{\omega}$. Let G be a compact, connected Lie group. Set $\mathfrak{g} := \mathrm{Lie}(G)$ A **hypersymplectic action** of G on X is a smooth action $\lambda\colon G \to \mathrm{Diff}(S)$ such that for every $g \in G$

$$\lambda(g)^* \boldsymbol{\omega} = \boldsymbol{\omega}.$$

For $\xi \in \mathfrak{g}$ set

$$v_\xi := \mathrm{Lie}(\lambda)(\xi) \in \mathrm{Vect}(S).$$

A **hyperkähler moment map** for λ is a G–equivariant smooth map $\mu\colon S \to (\mathrm{Im}\,\mathbf{H})^* \otimes \mathfrak{g}^*$ such that for every $x \in S$ and $\xi \in \mathfrak{g}$

$$\langle T_x\mu, \xi\rangle = i_{v_\xi(x)}\boldsymbol{\omega}. \tag{5.27}$$

•

Definition 5.28. (1) A **quaternionic Hermitian vector space** is a left $\mathbf{H}$–module S together with an inner product $\langle\cdot,\cdot\rangle$ such that i, j, k acts by isometries.

(2) The **unitary symplectic group** $\mathrm{Sp}(S)$ is the subgroup of $\mathrm{GL}_{\mathbf{H}}(S)$ preserving $\langle\cdot,\cdot\rangle$. A **quaternionic representation** of a Lie group G is a Lie group homomorphism $\lambda\colon G \to \mathrm{Sp}(S)$.

(3) The **distinguished hyperkähler moment map** of λ is the map $\mu\colon S \to (\mathrm{Im}\,\mathbf{H})^* \otimes \mathfrak{g}^*$ defined by

$$\langle \mu(x), q \otimes \xi\rangle := \frac{1}{2}\langle q \cdot \mathrm{Lie}(\rho)(\xi)x, x\rangle.$$

•

Exercise 5.29. $\mathbf{H}^2$ is a quaternionic Hermitian vector space. Consider the quaternionic representation $\lambda\colon \mathrm{U}(1) \to \mathrm{Sp}(\mathbf{H}^2)$ defined by

$$\lambda(e^{i\alpha})(q_1, q_2) := (q_1e^{i\alpha}, q_2e^{i\alpha}).$$

Compute μ.

Example 5.30. Let G be a compact, connected Lie group. $\mathbf{H} \otimes \mathfrak{g}$ is a quaternionic Hermitian vector space. Consider the quaternionic representation $\lambda\colon G \to \mathrm{Sp}(\mathbf{H} \otimes \mathfrak{g})$ defined by

$$\lambda(g) := \mathbf{1} \otimes \mathrm{Ad}_g\,.$$

Identifying $(\mathrm{Im}\,\mathbf{H})^* = \mathrm{Im}\,\mathbf{H}$ and $\mathfrak{g}^* = \mathfrak{g}$,

$$\begin{aligned}\mu(\xi) &= \frac{1}{2}[\xi, \xi] \\ &= ([\xi_2, \xi_3] + [\xi_0, \xi_1]) \otimes i + ([\xi_3, \xi_1] + [\xi_0, \xi_2]) \otimes j + ([\xi_1, \xi_2] + [\xi_0, \xi_3]) \otimes k\end{aligned}$$

for $\xi = \xi_0 \otimes 1 + \xi_1 \otimes i + \xi_2 \otimes j + \xi_3 \otimes k \in \mathbf{H} \otimes \mathfrak{g}$. A computation shows that

$$|\mu|^2 = \frac{1}{2}\sum_{a,b=0}^{4} |[\xi_a, \xi_b]|^2.$$

Therefore, $\mu(\xi) = 0$ if and only if the components of ξ are in an abelian subalgebra of $\mathfrak{g}$. ♠

The following is a direct consequence of (5.27):

Proposition 5.31. *In the above situation, $x \in S$ is a regular point of μ if and only if the stabilizer G_x is discrete; indeed:*

$$T_1G_x = \{\xi \in \mathfrak{g} : v_\xi(x) = 0\} = \{\xi \in \mathfrak{g} : \langle T_x\mu, \xi\rangle = 0\}.$$

■

Proposition 5.32. *Let $x \in S$. Set*

$$V_x := \{v_\xi(x) : \xi \in \mathfrak{g}\} \quad \textit{and} \quad H_x := \ker T_x\mu \cap V_x^\perp.$$

H_x and $V_x \oplus (\ker T_x\mu)^\perp$ are perpendicular $\mathbf{H}$–submodules of T_xS. Moreover,

$$V_x \oplus (\ker T_x\mu)^\perp = \mathbf{H} \cdot V_x,$$

Proof. By (5.27), $\ker T_x\mu = (\operatorname{Im}\mathbf{H} \cdot V_x)^\perp$. Therefore, $H_x = (\mathbf{H} \cdot V_x)^\perp$; hence: it is an $\mathbf{H}$–submodule. Since $\mathbf{H} \cdot V_x = H_x^\perp = V_x \oplus (\ker T_x\mu)^\perp$, the latter is an $\mathbf{H}$–submodule. ∎

Since G is compact,

$$\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{z} \quad \text{with} \quad \mathfrak{z} := \ker(\mathrm{ad}\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})).$$

Define

$$\mathfrak{z}^* := [\mathfrak{g}, \mathfrak{g}]^0 \subset \mathfrak{g}^*,$$

the annihilator of $[\mathfrak{g}, \mathfrak{g}]$. One can identify $\mathfrak{z}^*$ with the dual of $\mathfrak{z}$. (This justifies the notation.) The importance of $\mathfrak{z}^*$ is that its elements are G–invariant. Since μ is G–invariant, for $\zeta \in (\operatorname{Im}\mathbf{H})^* \otimes \mathfrak{z}^*$, G acts on $\mu^{-1}(\zeta)$. Denote by

$$\Delta$$

the interior of the set of regular value of μ in $(\operatorname{Im}\mathbf{H})^* \otimes \mathfrak{z}^*$.

Proposition 5.33. *For $\zeta \in \Delta$ set*

$$P_\zeta := \mu^{-1}(\zeta) \quad \textit{and} \quad X_\zeta := \mu^{-1}(\zeta)/G.$$

(1) *For $x \in P_\zeta$, G_x is finite. Therefore, X_ζ is an orbifold and*

$$p_\zeta\colon P_\zeta \to X_\zeta$$

is an orbifold principal G–bundle. (If you don't know what an orbifold is, then just assume that G_x is trivial.)

(2) *For $x \in P_\zeta$, $V_x = \ker T_xp_\zeta$.*

(3) *$H_\zeta := \coprod_{x \in P_\zeta} H_x \subset TP_\zeta$ defines a principal G–connection A_ζ on P_ζ.*

(4) *The 2–form $\boldsymbol{\omega}|_{P_\zeta}$ is G–invariant and H_ζ–horizontal. It descends to a hyperkähler structure $\check{\boldsymbol{\omega}}_\zeta$ on X_ζ.*

(5) *The curvature $F_{A_\zeta} \in \Omega^2(P_\zeta, \mathfrak{g})$ satisfies*

$$\langle F_{A_\zeta}, \boldsymbol{\omega}|_{X_\zeta} \rangle = 0.$$

Proof. (2) is obvious.

(3) and (4) follow from Proposition 5.32.

(5) is [GN, Theorem 1.3]. It can be proved directly as follows. Let $v, w \in \Gamma(H_\zeta)^G$ and $\xi \in \mathfrak{g}$. Since $\langle v_\xi, v\rangle = \langle v_\xi, w\rangle = 0$, $\mathscr{L}_{v_\xi}\langle v, w\rangle = 0$, and $[v_\xi, v] = [v_\xi, w] = 0$,

$$\begin{aligned}\langle F_{H_\zeta}(v,w), \xi\rangle &= \langle [v,w], v_\xi\rangle \\ &= \langle \nabla_v w - \nabla_w v, v_\xi\rangle \\ &= \langle v, \nabla_w v_\xi\rangle - \langle w, \nabla_v v_\xi\rangle \\ &= \langle v, \nabla_{v_\xi} w\rangle - \langle w, \nabla_{v_\xi} v\rangle = 2\langle v, \nabla_{v_\xi} w\rangle.\end{aligned}$$

Therefore and since $\mathbf{I}$ is parallel,

$$F_{H_\zeta}(I_\xi -, I_\xi -) = F_{H_\zeta}. \qquad \blacksquare$$

Definition 5.34. The hyperkähler manifold $(X_\zeta, \check{\boldsymbol{\omega}}_\zeta)$ is the **hyperkähler quotient of** $(S, \boldsymbol{\omega})$ **by** G **at level** ζ. This is often denoted as

$$X_\zeta := S /\!\!/\!\!/_\zeta G.$$

If ζ is omitted, then $\zeta = 0$ is assumed. •

Remark 5.35. *Here is an important observation.* If $\dim X_\zeta = 4$, then A_ζ is an ASD instanton on $p_\zeta \colon P_\zeta \to X_\zeta$. ♣

Example 5.36. Consider the adjoint representation $G \to \mathrm{Sp}(\mathbf{H} \otimes \mathfrak{g})$. Let $T < G$ be a maximal torus. Set $\mathfrak{t} = \mathrm{Lie}(T) \subset \mathfrak{g}$. If $\mu(\xi) = 0$ then there is a $g \in G$ such that $\mathrm{Ad}_g(\xi) \in \mathbf{H} \otimes \mathfrak{t}$. If $\xi \in \mathbf{H} \otimes \mathfrak{t}$ and $\mathrm{Ad}_g(\xi) \in \mathbf{H} \otimes \mathfrak{t}$, then g is in the normaliser $N(T)$ and T acts trivially. Therefore,

$$(\mathbf{H} \otimes \mathfrak{g}) /\!\!/\!\!/ G = (\mathbf{H} \otimes \mathfrak{t})/W$$

with $W := N(T)/T$ denoting the Weyl group. For $G = \mathrm{U}(n)$, $\mathfrak{t} = i\mathbf{R}^n$ and $W = S_n$. Therefore,

$$(\mathbf{H} \otimes \mathfrak{u}(n)) /\!\!/\!\!/ \mathrm{U}(n) = \mathbf{H}^n / S_n =: \mathrm{Sym}^n \mathbf{H},$$

the n**–fold symmetric product** of $\mathbf{H}$. This is, of course, an orbifold, but varying ζ generically gives smooth hyperkähler manifolds. ♠

As ζ varies in a connected component of Δ, the diffeomorphism type X_ζ persists but $\check{\boldsymbol{\omega}}_\zeta$ varies. The question of how to determine this variation (in cohomology) has been considered by Duistermaat and Heckman [DH82, §2]. [XXX: skip this in class.]

Proposition 5.37. *Set*

$$P := \{(x, \zeta) \in S \times \Delta : \mu(x) = \zeta\} \quad \textit{and} \quad X := P/G.$$

The tangent bundle TP decomposes as

$$T_{(x,\zeta)}P = H \oplus K \oplus V$$

with

$$V := \ker T\pi, \quad H := \ker T\mu \cap V^\perp, \quad \textit{and} \quad K := (\ker T\mu)^\perp = (H \oplus V)^\perp.$$

(1) *The projection* $\pi\colon P \to X$ *is a principal* G*–bundle. The projection* $\rho\colon X \to \Delta$ *is a fiber bundle.*

(2) $H \oplus K$ *is the horizontal distribution of a* G*–principal connection on* π*.*

(3) K *is a* G*–invariant Ehresmann connection on* $\rho \circ \pi$*. Hence, it descends to an Ehresmann connection on* $\check{K}$ *on* ρ*. The vertical tangent bundle of* ρ *lifts to* H*.*

(4) *The form* $(\mathrm{pr}_S^*\boldsymbol{\omega})^{2,0}$ *is* G*–equivariant and* $(H \oplus K)$*–horizontal. Therefore, it descends to* $\check{\boldsymbol{\omega}}$ *on* X*. It is of bi-degree* $(0,2)$ *with respect to* $\check{K}$ *and satisfies*

$$\check{\boldsymbol{\omega}}|_{X_\zeta} = \check{\boldsymbol{\omega}}_\zeta.$$

(5) *Denote by* $\tau \in (\operatorname{Im}\mathbf{H})^* \otimes \mathfrak{z}^* \otimes \Omega^1(\Delta)$ *the tautological* 1*–form on* Δ*. Denote by* $\pi_\mathfrak{z} F_H \in \Omega^{0,2}(X,\mathfrak{z})$ *the* $(0,2)$*–form defined by*

$$\pi_\mathfrak{z} F_H|_{X_\zeta} \coloneqq \pi_\mathfrak{z} F_{H_\zeta}.$$

With respect to $\check{K}$*,*

$$\mathrm{d}\check{\boldsymbol{\omega}} = \mathrm{d}^{1,0}\check{\boldsymbol{\omega}} = -\langle \rho^*\tau \wedge \pi_\mathfrak{z} F_H\rangle.$$

(6) *Suppose that* $\check{K}$ *is a complete Ehresmann connection. Let* $\boldsymbol{\zeta} \in C^\infty([a,b],\Delta)$ *smooth path. With* $\mathrm{tra}_{\boldsymbol{\zeta}}\colon X_{\boldsymbol{\zeta}(a)} \to X_{\boldsymbol{\zeta}(b)}$ *denoting parallel transport along* $\boldsymbol{\zeta}$*,*

$$\check{\boldsymbol{\omega}}_{\boldsymbol{\zeta}(a)} = (\mathrm{tra}_{\boldsymbol{\zeta}})^*\check{\boldsymbol{\omega}}_{\boldsymbol{\zeta}(b)} - \int_a^b \langle \dot{\boldsymbol{\zeta}}(t), (\mathrm{tra}_{\boldsymbol{\zeta}|_{[a,t]}})^*\pi_\mathfrak{z} F_{H_{\boldsymbol{\zeta}(t)}}\rangle \,\mathrm{d}t.$$

Proof. (1), (2), and (3) hold by construction.

The triple of 2–forms $\mathrm{pr}_S^*\boldsymbol{\omega} \in (\operatorname{Im}\mathbf{H})^* \otimes \Omega^2(P)$ is closed and G–invariant. However, it fails to be $(H \oplus K)$–horizontal: it has components of bi-digree $(2,0)$ and a $(1,1)$. In fact, by Proposition 5.32 $\mathrm{pr}_S^*\boldsymbol{\omega} \in (\operatorname{Im}\mathbf{H})^* \otimes \Gamma(\Lambda^2 H^* \oplus K^* \otimes V^*)$. This implies (4).

Since $\mathrm{d}\mathrm{pr}_S^*\boldsymbol{\omega} = 0$,

$$\pi^*\mathrm{d}\check{\boldsymbol{\omega}} = \mathrm{d}^{1,0}(\mathrm{pr}_S^*\boldsymbol{\omega})^{2,0} = -\mathrm{d}^{2,-1}(\mathrm{pr}_S^*\boldsymbol{\omega})^{1,1} = -(\mathrm{pr}_S^*\boldsymbol{\omega}_S)^{1,1}(F_{H\oplus K}(\cdot,\cdot),\cdot).$$

Since $\boldsymbol{\omega}|_V = 0$, the above is a section of $(\operatorname{Im}\mathbf{H})^* \otimes \Lambda^2(H \oplus K)^* \otimes K^*$ with the $\Lambda^2(H\oplus K)^*$ factor arising from $F_{H\oplus K}$. By (5.27), $\mathrm{pr}_S^*\boldsymbol{\omega}(v_\xi,\cdot) = \langle T_x\mu \circ \mathrm{pr}_S, \xi\rangle$. Moreover, $T_x\mu \circ \mathrm{pr}_S = (\rho\circ\pi)^*\tau$. Therefore,

$$\mathrm{d}^{1,0}(\mathrm{pr}_S^*\boldsymbol{\omega})^{2,0} = -\langle \pi^*\rho^*\tau, F_{H\oplus K}\rangle = -\langle \pi^*\rho^*\tau, \pi_\mathfrak{z} F_{H\oplus K}\rangle.$$

It remains to show that $\pi_\mathfrak{z} F_{H\oplus K} = \pi_\mathfrak{z} F_H$. Let $v, w \in \Gamma(H\oplus K)^G$ and $\xi \in \mathfrak{z}$. As in the proof of Proposition 5.33 (5),

$$\begin{aligned}
\langle F_{H_\zeta}(v,w),\xi\rangle &= \langle [v,w], v_\xi\rangle \\
&= \langle \nabla_v w - \nabla_w v, v_\xi\rangle \\
&= \langle v, \nabla_w v_\xi\rangle - \langle w, \nabla_v v_\xi\rangle \\
&= \langle v, \nabla_{v_\xi} w\rangle - \langle w, \nabla_{v_\xi} v\rangle = 2\langle v, \nabla_{v_\xi} w\rangle.
\end{aligned}$$

Therefore, for $w = \gamma(p)v_\eta \in \Gamma(K)$

$$\langle F_{H_\zeta}(v,w),\xi\rangle = 2\langle v,\gamma(p)\nabla_{v_\xi}v_\eta\rangle = 2\langle v,\gamma(p)v_{[\xi,\eta]}\rangle = 0.$$

Consequently, $\pi_3 F_{H\oplus K} = \pi_3 F_H$. Therefore, $\mathrm{d}\check{\omega} = -\langle \rho^*\tau \wedge \pi_\zeta F_H\rangle$. Since the latter is of bi-degree $(1,2)$, this finishes the proof of (5).

By (5),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{tra}_{\zeta|_{[a,t]}})^*\check{\omega}_{\zeta(t)} &= \frac{\mathrm{d}}{\mathrm{d}\tau}\Big|_{\tau=0}(\mathrm{tra}_{\zeta|_{[a,t]}})^*(\mathrm{tra}_{\zeta|_{[t,t+\tau]}})^*\check{\omega}_{\zeta(t+\tau)}\\ &= -\langle\dot{\zeta}(t),(\mathrm{tra}_{\zeta|_{[a,t]}})^*\pi_3 F_{H_{\zeta(t)}}\rangle.\end{aligned}$$

Integration proves (6). ■

5.6 Aside: The Gibbon–Hawking ansatz

[XXX: The following is a little of an aside and will be discussed in the problem session. It at least shows that there are a lot of hyperkähler 4–manifolds.]

Let U be an open subset of $\mathbf{R}^3$. Denote by $g_{\mathbf{R}^3}$ the restriction of the standard metric on $\mathbf{R}^3$ to U. Let $\pi\colon X \to U$ be a principal $\mathrm{U}(1)$–bundle. Denote by $\partial_\alpha \in \mathrm{Vect}(X)$ the generator of the $\mathrm{U}(1)$–action. Let $i\theta \in \Omega^1(X, i\mathbf{R})$ be a $\mathrm{U}(1)$–connection 1–form and let $f \in C^\infty(U,(0,\infty))$ be a positive smooth function such that

$$\mathrm{d}\theta = - *_3\,\mathrm{d}f. \tag{5.38}$$

Set

$$g := f\pi^* g_{\mathbf{R}^3} + \frac{1}{f}\theta\otimes\theta$$

and define complex structures I_1, I_2, I_3 by

$$I_i\partial_\alpha = f^{-1}\partial_{x_i} \quad\text{and}\quad I_i\partial_{x_j} = \sum_{k=1}^3 \varepsilon_{ijk}\partial_{x_k}.$$

The corresponding Hermitian forms are

$$\omega_i := \theta\wedge \mathrm{d}x_i + \frac{1}{2}\sum_{j,k=1}^3 \varepsilon_{ijk} f\,\mathrm{d}x_j\wedge\mathrm{d}x_k.$$

Writing (5.54) as

$$\mathrm{d}\theta = -\frac{1}{2}\sum_{\ell,j,k=1}^3 \varepsilon_{\ell jk}\partial_{x_\ell} f\,\mathrm{d}x_j\wedge\mathrm{d}x_k,$$

we see that

$$\mathrm{d}\omega_i = \mathrm{d}\theta\wedge\mathrm{d}x_i + \frac{1}{2}\sum_{j,k=1}^3\varepsilon_{ijk}\mathrm{d}f\wedge\mathrm{d}x_j\wedge\mathrm{d}x_k = 0.$$

Therefore, we have proved the following.

Proposition 5.39. *(X, g, I_1, I_2, I_3) is hyperkähler manifold.*

This construction is called the **Gibbons–Hawking ansatz**.

Remark 5.40. By construction, the length of the $\mathrm{U}(1)$–orbit over $x \in U$ is $f(x)^{-1/2}$. ♣

Remark 5.41. The fact that

$$i(\partial_\alpha)\omega_i = -\mathrm{d}x_i$$

means that the map $\pi\colon X \to U \subset \mathbf{R}^3$ is a hyperkähler moment map for the action of $\mathrm{U}(1)$ on X (with $\mathbf{R}^3$ and $(\mathfrak{u}(1) \otimes \operatorname{Im}\mathbf{H})^*$ identified suitably. ♣

Remark 5.42. By (5.54),

$$\Delta f = 0.$$

Conversely, suppose that $f\colon U \to \mathbf{R}$ is harmonic and the cohomology class of $*_3\mathrm{d}f$ lies in $\operatorname{im}(H^2(U, 2\pi\mathbf{Z}) \to H^2(U, \mathbf{R}))$, then there is a $\mathrm{U}(1)$–bundle X over U together a connection $i\theta$ satisfying

$$\mathrm{d}\theta = - *_3 \mathrm{d}f.$$

♣

Example 5.43 ($\mathbf{R}^4$)**.** Let $U = \mathbf{R}^3\backslash\{0\}$ and define $f\colon U \to \mathbf{R}$ by

$$f(x) = \frac{1}{2|x|}.$$

This function is harmonic and satisfies

$$- *_3 \mathrm{d}f = \frac{1}{2}\mathrm{vol}_{S^2}.$$

Since $\mathrm{vol}(S^2) = 4\pi$, there is a $\mathrm{U}(1)$–bundle X over U together with a connection $i\theta$ such that (5.54). Therefore, the Gibbins–Hawking ansatz yields a hyperkähler metric on X.

By Chern–Weil theory the first Chern number of the restriction of X to S^2 is

$$\int_{S^2} i\frac{i}{4\pi}\mathrm{vol}_{S^2} = -1.$$

Up to is isomorphism, there is only one principal $\mathrm{U}(1)$–bundle over S^2: the Hopf bundle $\pi\colon S^3 \to S^2$ and the $\mathrm{U}(1)$–action given by $e^{i\alpha}\cdot(z_0, z_1) = (e^{-i\alpha}z_0, e^{-i\alpha}z_1)$. If g_{S^3} denotes the standard metric on S^3, then

$$\theta = g_{S^3}(-\partial_\alpha, \cdot)$$

satisfies

$$\mathrm{d}\theta = \pi^*\mathrm{vol}_{S^2}.$$

It follows that

$$X = S^3 \times (0, \infty) = \mathbf{R}^4\backslash\{0\}$$

and the Gibbons–Hawking ansatz gives the metric

$$g = 2r\,\theta \otimes \theta + \frac{1}{2r}(\mathrm{d}r \otimes \mathrm{d}r + r^2 g_{S^2}).$$

The change of coordinates $\rho = \sqrt{2r}$ rewrites this metric as

$$g = \mathrm{d}\rho \otimes \mathrm{d}\rho + \rho^2(\theta \otimes \theta + \frac{1}{4}g_{S^2}) = \mathrm{d}\rho \otimes \mathrm{d}\rho + \rho^2 g_{S^3}.$$

This means that the Gibbons–Hawking ansatz yield the standard metric on $\mathbf{R}^4$. ♠

Example 5.44 (Taub–NUT). Let $U = \mathbf{R}^3\backslash\{0\}$, let $c > 0$, and define $f_c\colon U \to \mathbf{R}$ by

$$f_c(x) = \frac{1}{2|x|} + c.$$

This function is harmonic and we have

$$\mathrm{d}f_c = \mathrm{d}f.$$

By the preceding discussion, $X = S^3 \times (0, \infty)$ and the Gibbons–Hawking ansatz gives the metric

$$g = \left(\frac{1}{2r} + c\right)^{-1} \theta \otimes \theta + \left(\frac{1}{2r} + c\right)(\mathrm{d}r \otimes \mathrm{d}r + r^2 g_{S^2}).$$

As r tends to zero this metric is asymptotic to

$$c^{-1}\theta \otimes \theta + g_{\mathbf{R}^3}.$$

Although, the metric appears singular at $r = 0$, the coordinate change $\rho = \sqrt{2r}$ rewrites it as

$$(1 + c\rho^2)\mathrm{d}\rho \otimes \mathrm{d}\rho + \rho^2 \left((1 + c\rho^2)^{-1}\theta \otimes \theta + (1 + c\rho^2)\frac{1}{4}g_{S^2}\right)$$

which is smooth.

This metric is called the **Taub–NUT metric**. It is non-flat hyperkähler metric on $\mathbf{R}^4$. It was first discovered by Taub [Tau51] and Newman, Tamburino, and Unti [NTU63]. The Taub–NUT space is the archetype of an **ALF space**. ♠

Remark 5.45. It was observed by LeBrun [LeB91] that the Taub–NUT metric is in fact Kähler for the standard complex structure on $\mathbf{C}^2$. Thus it yields a non-flat Ricci-flat Kähler metric on $\mathbf{C}^2$. ♣

Example 5.46 ($(\mathbf{R}^4\backslash\{0\})/\mathbf{Z}_k$). Let $k \in \{1, 2, 3, \ldots\}$ Let $U = \mathbf{R}^3\backslash\{0\}$ and define $f\colon U \to \mathbf{R}$ by

$$f(x) := \frac{k}{2|x|}.$$

This function is harmonic and it satisfies

$$- *_3 \mathrm{d}f = k\mathrm{vol}_{S^2}.$$

Thus, the Gibbons–Hawking ansatz applies. Denote by (X_k, g_k) the Riemannian manifold obtained in this way. If $k = 1$, then this $\mathbf{R}^4$ with its standard metric. Let us understand the cases $k \geqslant 2$.

The restriction of X_k to S^2 has Chern number $-k$. This U(1)–bundle is $S^3/\mathbf{Z}_k \to S^2$. Consequently,

$$X_k = S^3/\mathbf{Z}_k \times (0, \infty) = \mathbf{R}^4/\mathbf{Z}_k.$$

We can choose the connection 1-form $i\theta_k$ on X_k such that its pullback to X_1 is $ik\theta_1$. It follows that the pullback of g_k to X_1 can be written as

$$2kr\,\theta \otimes \theta + \frac{k}{2r}(\mathrm{d}r \otimes \mathrm{d}r + r^2 g_{S^2}).$$

Up to a coordinate change $r \mapsto kr$ this is the standard metric on $\mathbf{R}^4$. It follows that g_k is the metric induced by the standard metric on $\mathbf{R}^4$. ♠

Example 5.47 (Eguchi–Hanson and multi-center Gibbons–Hawking). Let $x_1, \ldots, x_k$ be k distinct points in $\mathbf{R}^3$. Set $U := \mathbf{R}^3 \backslash \{x_1, \ldots, x_k\}$ and define $f\colon U \to \mathbf{R}$ by

$$f(x) = \sum_{i=1}^{k} \frac{1}{2|x - x_i|}.$$

From the in discussion Example 5.43 it is clear that the Gibbons–Hawking ansatz for f produces a Riemannian manifold whose apparent singularities over $x_1, \ldots, x_k$ can be removed. Denote the resulting manifold by (X, g).

Since

$$f(x) = \frac{k}{2|x|} + O(|x|^{-2}) \quad \text{as} \quad |x| \to \infty,$$

(X, g) is asymptotic at infinity to $\mathbf{R}^4/\mathbf{Z}_k$. These spaces are called **ALE spaces of type** A_{k-1}. For $k = 2$, this metric was discovered by Eguchi and Hanson [EH79]. The metrics for $k \geqslant 3$ were discovered by Gibbons and Hawking [GH78].

Let us understand the geometry and topology of these spaces somewhat more. Suppose γ is an arc in $\mathbf{R}^3$ from x_i to x_j avoiding all the other points x_k. The pre-image in X of any interior point of γ is an S^1 while the pre-images of the end points are points. Therefore,

$$\pi^{-1}(\gamma) \subset X$$

is diffeomorphic to S^2. Suppose γ is straight line segment in $\mathbf{R}^3$ with unit tangent vector

$$v = \sum_{i=1}^{3} a_i \partial_{x_i}$$

with $a_1^2 + a_2^2 + a_3^2 = 1$. The tangent spaces to $\pi^{-1}(\gamma)$ are spanned by ∂_α and v. In particular, they are invariant with respect to the complex structure

$$I_v := a_1 I_1 + a_2 I_2 + a_3 I_3.$$

Its volume is given by

$$\int_{\pi^{-1}(\gamma)} a_1\omega_1 + a_2\omega_2 + a_3\omega_3.$$

Therefore,

$$[\pi^{-1}(\gamma)] \neq 0 \in H_2(X, \mathbf{Z}).$$

If necessary we can reorder the points x_i so that for $i = 1, \dots, k-1$, there is a straight-line segment γ_i joining x_i and x_{i+1}. Set

$$\Sigma_i := \pi^{-1}(\gamma_i).$$

It is not difficult to see that $[\Sigma_1], \dots, [\Sigma_{k-1}]$ generate $H_2(M; \mathbf{Z})$. It is an exercise to show that

$$[\Sigma_i] \cdot [\Sigma_j] = \begin{cases} -2 & \text{if } i = j, \\ 1 & \text{if } i \neq j. \end{cases}$$

♠

Remark 5.48. Kronheimer [Kro89b] gave an alternative construction of the ALE spaces of type A_{k-1} (in fact, all ALE spaces) as hyperkähler quotients. He also classified these spaces completely [Kro89a]. ♣

Example 5.49. Let $x_1, \dots, x_k$ be k distinct points in $\mathbf{R}^3$ and let $c > 0$. Set $U := \mathbf{R}^3 \backslash \{x_1, \dots, x_k\}$ and define $f\colon U \to \mathbf{R}$ by

$$f(x) = \sum_{i=1}^{k} \frac{1}{2|x - x_i|} + c.$$

The Gibbons–Hawking ansatz for f gives rise to the so-called **multi-center Taub–NUT** metric. ♠

Example 5.50. The following is due to Anderson, Kronheimer, and LeBrun [AKL89]. Let $x_1, x_2, \dots$ be an infinite sequence of distinct points in $\mathbf{R}^3$ and denote by U the complement of these points. If

$$\sum_{j=2}^{\infty} \frac{1}{x_1 - x_j} < \infty,$$

then

$$f(x) := \sum_{j=1}^{\infty} \frac{1}{2|x - x_j|}$$

defines a harmonic function on U. The Gibbons–Hawking ansatz gives rise to a hyperkähler manifold X whose second homology $H_2(X, \mathbf{Z})$ is infinitely generated. Anderson, Kronheimer, and LeBrun prove that the metric g is complete. ♠

This is not a complete list of interesting examples of hyperkähler manifold which can be produced using the Gibbons–Hawking ansatz. The most egregious omission is that of the Ooguri–Vafa metric.

5.7 Anti-self-duality as a moment map

Let G be a compact connecte semi-simple Lie group. Fix $k \in \{2, 3, \dots\}$. (The choice of k ultimately turns out to be quite insubstantial. For concreteness one could take $k = 2$.) Set

$$\mathscr{A} := W^{k,2}\Omega^1(\mathbf{H}, \mathfrak{g})$$

Here the prefix $W^{k,2}$ denotes taking the completion with respect to the norm $\|-\|_{W^{k,2}}$ defined by

$$\|A\|^2_{W^{k,2}} := \sum_{\ell=0}^{k} \int_{\mathbf{H}} |\nabla^k A|^2.$$

This is a Hilbert manifold (indeed: a Hilbert space). The L^2–inner product defines a Riemannian metric on $\mathscr{A}$ and, obviously, $\operatorname{Im}\mathbf{H}$ acts on $T\mathscr{A}$: for $a \in T_A\mathscr{A}$ and $\xi \in \operatorname{Im}\mathbf{H}$

$$I_\xi a := -a(\xi \cdot -).$$

(The minus sign is neccessary to preserve $I_i I_j = I_k$.)

Therefore, $\mathscr{A}$ is an infinite dimensional hyperkähler manifold.

Set

$$\mathscr{G}_0 := \exp(W^{k+1,2}(\mathbf{H}, \mathfrak{g})) := \{u = \exp(\xi) : \xi \in W^{k+1,2}(\mathbf{H}, \operatorname{End}(\mathfrak{g}))\}.$$

The subscript zero indicates that the gauge transformations $u \in \mathscr{G}_0$. $\mathscr{G}_0$ acts on the right of $\mathscr{A}$ via

$$u^*A := \operatorname{Ad}(u) \circ A + \mu(u) = \text{"}u^{-1}Au + u^{-1}\mathrm{d}u\text{"}.$$

This action preserves the hyperkähler structure on $\mathscr{A}$. The infinitesimal action of $\xi \in \operatorname{Lie}(\mathscr{G}_0) = W^{k+1,2}(\mathbf{H}, \mathfrak{g})$ is

$$v_\xi(A) = \mathrm{d}_A \xi.$$

This is a (Hilbert space) quaternionic representation. Let us compute the distinguished hyperkähler moment map: for $A \in \mathscr{A}$, $q \in \operatorname{Im}\mathbf{H}$, and $\hat{u} \in \operatorname{Lie}(\mathscr{G}_0) = W^{k+1,2}(\mathbf{H}, \mathfrak{g})$

$$\langle \mu(A), q \otimes \xi \rangle = \frac{1}{2}\langle I_q \mathrm{d}_A \xi, A\rangle.$$

To digest this expression observe that

$$\begin{aligned} \mathrm{d}^*(f\omega_q) &= \sum_{a=1}^{4} -(\partial_a f) i_{\partial_{x_a}} \omega_q = \sum_{a=1}^{4} -(\partial_a f)\langle I_q \partial_{x_a}, -\rangle = \sum_{a=1}^{4} (\partial_a f)\langle \partial_{x_a}, I_q -\rangle \\ &= \sum_{a=1}^{4} (\partial_a f)\mathrm{d}x_a \circ I_q = -I_q \mathrm{d}f. \end{aligned}$$

Therefore,

$$\langle \mu(A), q \otimes \xi\rangle = -\frac{1}{2}\langle \mathrm{d}_A^*(\xi \cdot \omega_q), A\rangle = -\frac{1}{2}\langle \xi \cdot \omega_q), \mathrm{d}_A A\rangle = -\frac{1}{2}\langle \xi \cdot \omega_q, F_A\rangle = -\frac{1}{2}\langle \xi \cdot \omega_q, F_A^+\rangle.$$

Identifying, $\operatorname{Lie}(\mathscr{G}_0)$ and its dual and $\operatorname{Im}\mathbf{H}^* = \operatorname{Im}\mathbf{H} = \Lambda^+\mathbf{H}$ via $q^* \mapsto -\frac{1}{2}\omega_q$ exhibits

$$\mu(A) = F_A^+.$$

Therefore, the anti-self-dual part of the curvature is the hyperkähler moment map! (The phenomenon that an interesting non-linear partial differential equation appears as a moment maps is suprisingly common.)

Now the hyperkähler reduction of the $\mathscr{A}$ by $\mathscr{G}_0$ is

$$\mathscr{M}_G^{\mathrm{fr}} := \{A \in \mathscr{A} : F_A^+ = 0\}/\mathscr{G}_0.$$

This is the **moduli space of framed G ASD instantons on H**. The adjective framed has to do with the fact that the quotient is by gauge transformations which decay to the identity at infinity. $\mathscr{M}_G^{\mathrm{fr}}$ has an action by G. The quotient is the actual moduli space (but it does not have the feature of being hyperkähler).

For every $[A] \in \mathscr{M}_G^{\mathrm{fr}}$

$$k := \frac{1}{4\pi^2}\mathrm{YM}(A) \in \mathbf{N}_0.$$

This number k is the **instanton number** or **charge** of A. It is customary to decompose

$$\mathscr{M}_G^{\mathrm{fr}} = \coprod_{k\in\mathbf{N}_0} \mathscr{M}_{G,k}^{\mathrm{fr}}.$$

5.8 Preparation: projections and connections

Let X be a smooth manifold. Denote by $\underline{\mathbf{R}}^n := \mathbf{R}^n \times X$ the trivial vector bundle of rank N over X. Let $P \in \Gamma(\mathrm{End}(\underline{\mathbf{R}}^n))$ be a projection of constant rank; that is:

$$P^2 = P \quad \text{and} \quad \mathrm{rk}\, P = r.$$

Define the complementary projection by

$$Q := \mathbf{1} - P.$$

These define a decomposition

$$\underline{\mathbf{R}}^n = E \oplus F \quad \text{with} \quad E := \mathrm{im}\, P \quad \text{and} \quad F := \mathrm{im}\, Q.$$

With respect to this decomposition

$$\mathrm{d} = \begin{pmatrix} P\mathrm{d} & P\mathrm{d}Q \\ Q\mathrm{d}Q & Q\mathrm{d} \end{pmatrix}.$$

The diagonal components define covariant derivatives $\nabla := P\mathrm{d}$ on E and $\nabla := Q\mathrm{d}$ on F. Since the roles of E and F are interchangable, let us focus on E. If P is an orthogonal projection; that is: $P^* = P$, then ∇ is an orthogonal covariant derivative on E. (Similarly, if P is complex linear, etc.)

To compute the curvature of ∇ it is convenient to define the covariant derivative $\bar{\nabla}$ on $\underline{\mathbf{R}}^n$ by

$$\bar{\nabla} s := (\mathrm{d} + A)s \quad \text{with} \quad A := -Q\mathrm{d}P = Q\mathrm{d}Q.$$

The covariant derivative $\bar{\nabla}$ preserves the subbundle $E \subset \underline{\mathbf{R}}^N$ and induces ∇. The curvature $F_{\bar{\nabla}}$ of $\bar{\nabla}$ is

$$F_{\bar{\nabla}} = \mathrm{d}Q \wedge \mathrm{d}Q = \mathrm{d}P \wedge \mathrm{d}P$$

To see this observe that

$$Q\mathrm{d}Q \wedge Q\mathrm{d}Q = Q(\mathrm{d}Q - Q\mathrm{d}Q) \wedge \mathrm{d}Q = 0$$

Therefore,

$$F_\nabla = P(\mathrm{d}P) \wedge (\mathrm{d}P)P.$$

Henceforth, let us assume that P is orthogonal. Suppose that $R\colon \underline{\mathbf{R}}^n \to \underline{\mathbf{R}}^{n-r}$ is a surjective vector bundle morphism and that

$$E = \ker R \quad \text{and} \quad F = \operatorname{im} R^*.$$

A moment's thought shows that

$$P = \mathbf{1} - Q \quad \text{and} \quad Q = R^*(RR^*)^{-1}R$$

(This becomes particularly simple if $RR^* = \mathbf{1}$.) Since $RP = 0$ and $PR^* = 0$, the above considerations yields

$$F_\nabla = P(\mathrm{d}R^*) \wedge (RR^*)^{-1}(\mathrm{d}R)P.$$

Remark 5.51. Let $k, r \in \mathbf{N}_0$.

(1) Denote by $\mathrm{Gr}_r(\mathbf{R}^{k+r})$ the Grassmannian of r–planes in $\mathbf{R}^{k+r}$. There is a tautological vector bundle $p\colon V \to \mathrm{Gr}_r(\mathbf{R}^{k+r})$ (indeed, a subbundle of $\underline{\mathbf{R}}^{k+r} \to \mathrm{Gr}_r(\mathbf{R}^{k+r})$ defined by

$$V := \left\{(\Pi, v) \in \mathrm{Gr}_r(\mathbf{R}^{k+r}) \times \mathbf{R}^{k+r} : v \in \Pi\right\}.$$

V inherits an Euclidean inner product from $\underline{\mathbf{R}}^{k+r}$.

(2) The procedure discussed above defines a covariant derivative ∇ on V.

(3) Let X is a smooth manifold. If $f\colon X \to \mathrm{Gr}_r(\mathbf{R}^{k+r})$ is a smooth map, then $f^*p\colon f^*V \to X$ is a Euclidean rank r vector bundle. Upto isomorphism every Euclidean rank r vector bundle over X comes from such a map for some value of k. (This is a basic result in the theory of vector bundles: a baby version of Whitney's embedding theorem.) Of course, ∇ defines a connection on f^*V. It turns out that all ASD instantons on $\mathbf{H}P^1 = S^4$ can be obtained in this way.

(4) Denote by

$$\begin{aligned}
\mathrm{St}_k^*(\mathbf{R}^{k+r}) &:= \left\{R \in \mathrm{Hom}(\mathbf{R}^{k+r}, \mathbf{R}^k) : R \text{ is surjective}\right\} \quad \text{and} \\
\mathrm{St}_k(\mathbf{R}^{k+r}) &:= \left\{R \in \mathrm{St}_k^*(\mathbf{R}^{k+r}) : RR^* = \mathbf{1}\right\}
\end{aligned}$$

the Stiefel manifold and the orthogonal Stiefel manifold. (To match this with our earlier definition, observe that R and R^* are equivalent data.) The maps

$$\ker\colon \mathrm{St}_k^*(\mathbf{R}^{k+r}) \to \mathrm{Gr}_r(\mathbf{R}^{k+r}) \quad \text{defined by} \quad \ker(R) := \ker R$$

makes $\mathrm{St}_k^*(\mathbf{R}^{k+r})$ into a $\mathrm{GL}_k(\mathbf{R})$–principal bundle and $\mathrm{St}_k(\mathbf{R}^{k+r})$ into a $\mathrm{O}(k)$–principal bundles. Indeed, they are the frame bundle and the orthogonal frame bundle of $V^\perp(= \operatorname{im} R^*)$ respectively.

(5) The diffeomorphism $\mathrm{Gr}_k(\mathbf{R}^{k+r}) \cong \mathrm{Gr}_r(\mathbf{R}^{k+r})$ turns $\mathrm{St}_r(\mathbf{R}^{k+r})$ into an $\mathrm{O}(r)$–principal bundle over $\mathrm{Gr}_r(\mathbf{R}^{k+r})$. Indeed, it is the orthogonal frame bundle of V. Therefore, it inherits a $\mathrm{O}(r)$–principal connection A_∇ from the covariant derivative ∇ on V.

(6) $\mathrm{St}_r(\mathbf{R}^{k+r})$ has a canonical $\mathrm{O}(r)$–invariant Riemannian metric g; indeed:

$$g(\hat{R}_1, \hat{R}_2) = \mathrm{tr}(\hat{R}_1^* \hat{R}_2).$$

This equips it with an $\mathrm{O}(r)$–principal connection A_g. Indeed:

$$A_g = A_\nabla!$$

(7) The Gram–Schmidt process defines a map

$$\mathrm{GS}\colon\ \mathrm{St}_k^*(\mathbf{R}^{k+r}) \to \mathrm{St}_k(\mathbf{R}^{k+r}).$$

Indeed,

$$\mathrm{GS}(R) = (RR^*)^{-1/2} R.$$

(8) Finally, the above precedure can be thought of (universally) as computing the the pullback to $\mathrm{St}_k(\mathbf{R}^{k+r})$ of the canonical $\mathrm{O}(r)$–principal connection A on $\mathrm{St}_r(\mathbf{R}^{k+r}) \to \mathrm{Gr}_r(\mathbf{R}^{k+r})$ and determining its curvature F_A. The total space of this the pullback bundle is

$$\left\{(R, S) \in \mathrm{St}_k^*(\mathbf{R}^{k+r}) \times \mathrm{Hom}(\mathbf{R}^{k+r}, \mathbf{R}^r) : S^* R = 0, SS^* = \mathbf{1}\right\}.$$

The formulae for the connection and the curvature from above apply immediately. ♣

5.9 The ADHM construction

The ADHM construction due to Atiyah, Drinfeld, Hitchin, and Manin [ADHM78] is one of the early groundbreaking discoveries in mathematical gauge theory: it gives a concrete description of $\mathscr{M}^{\mathrm{fr}}_{\mathrm{SU}(r),k}$ as a finite-dimensional hyperkähler reduction. This makes use of ideas from many areas of geometry and has ultimately impacted much of mathematics itself. In the following I will only discuss the construction and not give a complete treatment. If you want to learn more about this, read [DK90, §3.3], [Ati79], and/or [ADHM78].

The following perspective on the ADHM construction is taken from [Ati79]. Define a $\mathrm{Sp}(1)$–connection $A \in \Omega^1(\mathbf{H}^k, \mathfrak{sp}(1))$ by

$$A \coloneqq \sum_{a=1}^{k} \frac{\mathrm{Im}(\bar{q}_a \mathrm{d}q_a)}{1 + |q|^2}.$$

This connection satisfies

$$F_A = \frac{\sum_{a=1}^{k} r\mathrm{d}\bar{q}_a \mathrm{d}q_a}{(1 + |q|^2)^2};$$

in particular,

$$\langle F_A, \boldsymbol{\omega} \rangle = 0 \in C^\infty(\mathbf{H}^k, \mathfrak{sp}(1) \otimes \mathrm{Im}\,\mathbf{H}^*).$$

The latter is an higher dimensional analogue of the ASD condition. The idea is to obtain ASD instantons on $\mathbf{H}$ by pulling back A with a *suitable* map $f\colon \mathbf{H} \to \mathbf{H}^k$. Of course,

$$u^*A = \sum_{a=1}^{k} \frac{\operatorname{Im}(\bar{u}_a \mathrm{d}u_a)}{1+|u|^2}.$$

Atiyah [Ati79] makes the **ansatz**

$$u(q) = \lambda(B-q)^{-1}$$

with $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbf{H}^k$ (a row vector) and B a *symmetric* $k \times k$–matrix of quaternions.

It turns that this gives a ASD instanton if and only if

$$B^*B + \lambda^*\lambda \quad \text{is a a real } k \times k \text{ matrix}$$

and for every $q \in \mathbf{H}$

$$\ker\left(\lambda B - q\right) = 0.$$

The first condition can be seen to correpond to $\mu(B,\lambda) = 0$ for the distinguished hyperkähler moment map of $\mathrm{O}(k)$ acting on

$$S_{1,k} \coloneqq \mathrm{Sym}(\mathbf{H}^k) \oplus \mathbf{H}^k.$$

For $k = 1$ this gives the BPST instantons (after applying a conformal inversion $q \mapsto q^{-1}$). If $B = \mathrm{diag}(b_1, \dots, b_n)$ with the entries distinct and $\lambda_1, \dots, \lambda_k > 0$, then the above give an type of ASD instanton discovered by 't Hooft.

Theorem 5.52 (Atiyah, Drinfeld, Hitchin, and Manin [ADHM78]). *Every* $\mathrm{Sp}(1)$ *ASD instanton with instanton number* k *arises from some choice of* (λ, B) *as above. Two such ASD instantons are gauge equivalent if and only if there are* $g \in \mathrm{Sp}(1)$ *and* $T \in \mathrm{O}(k)$ *with*

$$\lambda' = q\lambda T \quad \text{and} \quad B' = T^{-1}BT.$$

This can also be phrased as

$$\mathscr{M}^{\mathrm{fr}}_{\mathrm{Sp}(1),k} = S^{\circ}_{1,k} /\!\!/\!\!/ \mathrm{O}(k).$$

Here the superscript $\circ$ indicates imposing the non-degeneracy condition.

A similar story works for $\mathrm{Sp}(r)$, $\mathrm{SU}(r)$, $\mathrm{O}(r)$. For $\mathrm{Sp}(r)$, λ is replaced by $\Lambda \in \mathbf{H}^{r\times k}$ and u is replaced by

$$U(q) = \Lambda(B-x)^{-1}.$$

The corresponding connection is of the form

$$(\mathbf{1} + U^*U)^{-1/2}(U^*\mathrm{d}U)(\mathbf{1} + U^*U)^{-1/2} + (\mathbf{1} + U^*U)^{1/2}\mathrm{d}(\mathbf{1} + U^*U)^{-1/2}.$$

Ultimately,

$$\mathscr{M}^{\mathrm{fr}}_{\mathrm{Sp}(r),k} = S^{\circ}_{r,k} /\!\!/\!\!/ \mathrm{O}(k) \quad \text{with} \quad S_{r,k} \coloneqq \mathrm{Sym}(\mathbf{H}^k) \oplus \mathbf{H}^{r\times k}.$$

I still owe you an explanation for why the above connections indeed are ASD instantons. It is not impossible to do this by a brute-force computation, but here is a nicer explanation. Let

$$C, D \in \mathrm{Hom}_{\mathbf{H}}(\mathbf{H}^{k+r}, \mathbf{H}^k)$$

by $k \times (k+r)$ matrices of quaternions. The subscript $\mathbf{H}$ denotes linearity with respect to the right $\mathbf{H}$–module structure. For $q = (q_1, q_2) \in \mathbf{H}^2 \backslash \{0\}$ set

$$R(q) := q_1 C + q_2 D.$$

Assume that $R(q)$ is surjective for every $q \in \mathbf{H}^2\backslash\{0\}$. The $\mathbf{H}$ right $\mathbf{H}$–module $\ker R(q)$ depends only on $[q] \in \mathbf{H}P^1$ (which we take to be the left quotient). This defines a quaternionic vector bundle

$$E := \ker R \subset \underline{\mathbf{H}}^{k+r}.$$

The curvature of the induced covariant derivative ∇ on E can be computed using the technology from the previous subsection. It suffices to do this over $\mathbf{H} \hookrightarrow \mathbf{H}P^1$. Over $\mathbf{H}$,

$$R(q) = qC + D.$$

Therefore,

$$F_\nabla = PC^* \mathrm{d}\bar{q} \wedge (RR^*)^{-1} \mathrm{d}qCP.$$

If the matrix RR^* is always real for every $q \in \mathbf{H}$, then the above is anti-self-dual. (In fact, this is an if and only if.)

Now R can be brought into the normal form

$$R = \begin{pmatrix} \Lambda^* & (B-q)^* \end{pmatrix}$$

Therefore,

$$RR^* = \Lambda^*\Lambda + B^*B - (B^*q + \bar{q}B) + |q|^2$$

is real if and only if

$$\Lambda^*\Lambda + B^*B \quad \text{and} \quad B^*q + \bar{q}B$$

are real. The latter condition is equivalent to B being symmetric. The final ingredient is to observe that

$$\begin{pmatrix} -\mathbf{1} \\ U \end{pmatrix} (1 + U^*U)^{-1/2}$$

parametrises $\ker R$ and the corresponding connection is also

$$(\mathbf{1} + U^*U)^{-1/2}(U^*\mathrm{d}U)(\mathbf{1} + U^*U)^{-1/2} + (\mathbf{1} + U^*U)^{1/2}\mathrm{d}(\mathbf{1} + U^*U)^{-1/2}.$$

The following perspective is useful. The Grassmannian $\mathrm{Gr}_k^{\mathbf{H}}(\mathbf{H}^{k+r})$ carries a natural $\mathrm{Sp}(r)$–bundle with a connection. The data above specifies a map $\mathbf{H}(\to \mathbf{H}P^1) \to \mathrm{Gr}_k(\mathbf{H}^{k+r})$ which pulls back the standard connection to an ASD instanton.

Here is the ADHM description for $\mathrm{SU}(r)$. Let $r \in \mathbf{N}$, $k \in \mathbf{N}_0$. Set

$$S_{r,k} := \mathrm{Hom}_{\mathbf{C}}(\mathbf{C}^r, \mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^k) \oplus \mathbf{H} \otimes_{\mathbf{R}} \mathfrak{u}(k)$$

The tensor product $\otimes_{\mathbf{C}}$ uses the $\mathbf{C}$ right-module structure that arises from right-multiplication by $i \in \mathbf{H}$. $T \in \mathbf{H} \otimes_{\mathbf{R}} \mathfrak{u}(k)$ defines an endomorphism of $\mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^k$ given by the composition of multiplication in $\mathbf{H}$ and the action of $\mathfrak{u}(k)$ on $\mathbf{C}^k$. For $(\Psi, T) \in S_{r,k}$ and $x \in \mathbf{H}$ define

$$R_x \colon (\mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^k) \oplus \mathbf{C}^r \to \mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^k$$

by

$$R_x(\phi, v) := (T - x^*)(\phi) + \Psi(v).$$

(Ψ, T) is **non-degenerate** if R_x is surjective for every $x \in \mathbf{H}$. In this case,

$$V := \coprod_{x \in \mathbf{H}} \ker R_x \subset \mathbf{H} \times [(\mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^k) \oplus \mathbf{C}^r]$$

is a Hermitian subbundle of rank

$$\mathrm{rk}\, V = r$$

and inherits a covariant dervivative $\nabla = \nabla_{\Psi,T} \colon \Gamma(V) \to \Omega^1(\mathbf{H}, V)$

$$\nabla s := (\mathrm{d}s)^\perp$$

with $(-)^\perp$ denoting the orthogonal projection onto V. Set

$$S^\circ_{r,k} := \big\{(\Psi, T) \in S_{r,k} : (\Psi, T) \text{ is non-degenerate}\big\}.$$

$G := \mathrm{U}(k)$ acts on $S_{r,k}$ and $S^\circ_{r,k}$ by the defining representation on $\mathbf{C}^k$ and the adjoint representation on $\mathfrak{u}(k)$. Evidently, if (Ψ, T) and $g(\Psi, T)$ give rise to gauge equivalent covariant derivatives. The action of G on $S_{r,k}$ is a quaternionic representation. It turns out that the vanishing of the distinguished hyperkähler moment map is precisely the condition for $\nabla_{\Psi,T}$ to have anti-self-dual curvature (in complete analogy with the ∞–dimensional case considered earlier.)

Again, the data (Ψ, T) defines a map to

$$\mathrm{Gr}^{\mathbf{C}}_r(\mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^k \oplus \mathbf{C}^r) = \mathrm{Gr}^{\mathbf{C}}_r(\mathbf{C}^{2k+r}).$$

The factor $\mathbf{C}^r$ has a particular meaning. To see this, observe that R can be extended from $\mathbf{H}$ to $\mathbf{H}^2\backslash\{0\}$ as

$$R(q_0, q_1) = \begin{pmatrix} q_0^* T - q_1^* & q_0^* \cdot \end{pmatrix}$$

The kernel of R depends only on $[q_0, q_1] \in \mathbf{H}P^1$. The fiber over $\infty = [0, 1]$ is precisely $\mathbf{C}^r$.

5.10 Dimensional reduction

Let G be a Lie group. Let X be an oriented Riemannian manifold. Let $d \in \mathbf{N}$. Let $p\colon P \to X$ be a G–principal bundle. Denote by $\mathrm{pr}_X\colon X \times \mathbf{R}^d \to X$ the canonical projection map. Consider $\mathrm{pr}_Y^* p\colon \mathrm{pr}_Y^* P = P \times \mathbf{R}^d \to \mathbf{R} \times Y$. Let $\mathbf{A} \in \mathscr{A}(\mathrm{pr}_Y^* p) \subset \Omega^1(P \times \mathbf{R}^d, \mathfrak{g})^{\mathrm{Ad}}$. $\mathbf{A}$ can be decomposed as

$$\mathbf{A} = A + \sum_{a=1}^{d} \xi_a \mathrm{d}t_a.$$

with $A \in \Omega^1(P \times \mathbf{R}^d, \mathfrak{g})^{\mathrm{Ad}}$ satisfying $i_{\partial_{t_a}} A = 0$ and $\xi_a \in C^\infty(P \times \mathbf{R}^d, \mathfrak{g})^{\mathrm{Ad}}$ ($a \in \{1, \ldots, d\}$). It is useful to think:

$$A \in C^\infty(\mathbf{R}^d, \mathscr{A}(p)) \quad \text{and} \quad \xi_a \in C^\infty(\mathbf{R}^d, \Gamma(\mathrm{Ad}(P)))$$

The curvature of $\mathbf{A}$ is

$$F_{\mathbf{A}} = F_A - \sum_{a=1}^{d} \big(\partial_{t_a} A - \mathrm{d}_A \xi_a\big) \wedge \mathrm{d}t_a + \frac{1}{2} \sum_{a,b=1}^{d} \big(\partial_{t_a} \xi_b - \partial_{t_a} \xi_b + [\xi_a, \xi_b]\big) \mathrm{d}t_a \wedge \mathrm{d}t_b.$$

Here F_A denotes the curvature of A restricted to the slices $\{t\} \times Y$.

Dimensional reduction is to *impose*

$$\partial_{t_a} A = 0 \quad \text{and} \quad \partial_{t_a} \xi_b = 0 \quad (a, b \in \{1, \ldots, d\}).$$

In this case, the above expression for $F_{\mathbf{A}}$ simplifies to

$$F_{\mathbf{A}} = F_A + \sum_{a=1}^{d} \mathrm{d}_A \xi_a \wedge \mathrm{d}t_a + \frac{1}{2} \sum_{a,b=1}^{d} [\xi_a, \xi_b] \mathrm{d}t_a \wedge \mathrm{d}t_b.$$

The dimensional reduction of the Yang–Mills functional yields the following **Yang–Mills–Higgs functional**

$$\mathrm{YMH}(A, \xi) \coloneqq \frac{1}{2} \int_X |F_A|^2 + \sum_{a=1}^{d} |\mathrm{d}_A \xi_a|^2 + \frac{1}{4} \sum_{a,b=1}^{d} |[\xi_a, \xi_b]|^2.$$

More generally, for any representation $\rho\colon G \to \mathrm{O}(V)$ and G–invariant function $Q\colon V \to \mathbf{R}$ one can consider the **Yang–Mills–Higgs functional**

$$\mathrm{YMH}(A, \phi) \coloneqq \frac{1}{2} \int_X |F_A|^2 + |\mathrm{d}_A \phi|^2 + Q(\phi)$$

for $A \in \mathscr{A}(p)$ and $\phi \in \Gamma(P \times_\rho V)$.

The dimensional reductions of the anti-self-duality equation to dimensions 3, 2, 1 give rise to the Bogomolny equation (monopoles), the Hitchin equation (Higgs bundles), and Nahm's equation. Let us derive these equations. I'll give you a little survey of these equations afterwards.

Proposition 5.53. *Let (Y, g) be an oriented Riemannian 3–manifold. Let $(p\colon P \to Y, R)$ be a G–principal bundle. Let $A \in \mathscr{A}(p, R)$ and $\xi \in \Gamma(\mathrm{Ad}(P))$. The G–principal connection*

$$\mathbf{A} \coloneqq A + \xi \mathrm{d}t \in \mathscr{A}(\mathrm{pr}_Y^*(p, R))$$

*is anti-self-dual on $Y \times \mathbf{R}$ if and only if the **Bogomolny equation***

$$F_A = - * \mathrm{d}_A \xi \tag{5.54}$$

holds.

Proof. To prove this, one needs to understand how the Hodge-$*$-operator on $X \coloneqq Y \times \mathbf{R}$ and Y are related. The orientation on X is so that $\mathrm{vol}_X = \mathrm{vol}_Y \wedge \mathrm{d}t$. As a consequence for $\alpha, \beta \in \Omega^2(Y)$,

$$\alpha \wedge *_X \beta = \langle \alpha, \beta \rangle \mathrm{vol}_X = \langle \alpha, \beta \rangle \mathrm{vol}_Y \wedge \mathrm{d}t = \alpha \wedge *_Y \beta \wedge \mathrm{d}t.$$

Therefore,

$$*_X \beta = *_Y \beta \wedge \mathrm{d}t$$

Similarly, for $\alpha \in \Omega^1(Y)$

$$*_X(\alpha \wedge \mathrm{d}t) = *_Y \alpha.$$

Therefore,

$$*_X F_{\mathbf{A}} = (*_Y F_A) \wedge \mathrm{d}t + *_Y(\mathrm{d}_A \xi).$$

$*_X F_{\mathbf{A}} = -F_{\mathbf{A}}$ thus amounts to the above equation. ■

Proposition 5.55. *Let (Σ, g) be an oriented Riemann surface Let $(p\colon P \to \Sigma, R)$ be a G–principal bundle. Let $A \in \mathscr{A}(p, R)$ and $\xi_1, \xi_2 \in \Gamma(\mathrm{Ad}(P))$. The G–principal connection*

$$\mathbf{A} \coloneqq A + \xi_1 \mathrm{d}t_1 + \xi_2 \mathrm{d}t_2 \in \mathscr{A}(\mathrm{pr}_Y^*(p, R))$$

*is anti-self-dual if and only if **Hitchin's equation***

$$\begin{aligned} F_A + [\xi_1, \xi_2] &= 0, \\ \mathrm{d}_A \xi_1 + * \mathrm{d}_A \xi_2 &= 0. \end{aligned} \tag{5.56}$$

holds.

Proof. By the above,

$$F_{\mathbf{A}} = F_A + \mathrm{d}_A \xi_1 \wedge \mathrm{d}t_1 + \mathrm{d}_A \xi_2 \wedge \mathrm{d}t_2 + [\xi_1, \xi_2] \mathrm{d}t_1 \wedge \mathrm{d}t_2.$$

The orientation on $X = \mathbf{R}^2 \times Y$ is $\mathrm{vol}_X = \mathrm{d}t_1 \wedge \mathrm{d}t_2 \wedge \mathrm{vol}_\Sigma$. Therefore,

$$*_X(\mathrm{d}t_1 \wedge \mathrm{d}t_2) = \mathrm{vol}_\Sigma$$

and for $\alpha \in \Omega^1(\Sigma)$

$$*_X(\alpha \wedge \mathrm{d}t_1) = -(*_\Sigma \alpha) \wedge \mathrm{d}t_2 \quad \text{and} \quad *_X(\alpha \wedge \mathrm{d}t_2) = (*_\Sigma \alpha) \wedge \mathrm{d}t_1.$$ ■

Proposition 5.57. *Let I be an interval. Let $\xi_0, \xi_1, \xi_2, \xi_3 \in C^\infty(I, \mathfrak{g})$ The G–principal connection*

$$\mathbf{A} := \xi_0 \mathrm{d}t_0 + \xi_1 \mathrm{d}t_1 + \xi_2 \mathrm{d}t_2 + \xi_3 \mathrm{d}t_3$$

is anti-self-dual if and only if ***Nahm's equation***

$$\begin{aligned} \dot{\xi}_1 + [\xi_0, \xi_1] + [\xi_2, \xi_3] &= 0, \\ \dot{\xi}_2 + [\xi_0, \xi_2] + [\xi_3, \xi_1] &= 0, \\ \dot{\xi}_3 + [\xi_0, \xi_3] + [\xi_1, \xi_2] &= 0 \end{aligned} \tag{5.58}$$

holds.

Proof. By the above,

$$F_{\mathrm{A}} = \sum_{a=1}^{3} (\dot{\xi}_a + [\xi_0, \xi_a]) \mathrm{d}t_0 \wedge \mathrm{d}t_a + \frac{1}{2} \sum_{a,b=1}^{3} [\xi_a, \xi_b] \mathrm{d}t_a \wedge \mathrm{d}t_b.$$

Of course,

$$*(\mathrm{d}t_0 \wedge \mathrm{d}t_a) = \frac{1}{2} \sum_{b,c=1}^{3} \varepsilon_{abc} \mathrm{d}t_b \wedge \mathrm{d}t_c$$

This implies the assertion directly. ■

5.11 The Bogomolny equation

Bogomolny [Bog76] Prasad and Sommerfield [PS75]

Proposition 5.59. *Let (Y, g) be an oriented Riemannian 3–manifold. Let $(p\colon P \to Y, R)$ be a G–principal bundle. If $A \in \mathscr{A}(p, R)$ and $\xi \in \Gamma(\mathrm{Ad}(P))$ is a solution of the Bogomolny equation (5.54), then*

$$\mathrm{d}_A * \mathrm{d}_A \xi = 0.$$

If Y is closed, then

$$\mathrm{d}_A \xi = 0 \quad \textit{and} \quad F_A = 0.$$

Proof. The Bianchi identity $\mathrm{d}_A F_A = 0$ immediately implies that $\mathrm{d}_A^* \mathrm{d}_A \xi = 0$. Therefore,

$$\int_Y |\mathrm{d}_A \xi|^2 = \int_Y \langle \xi, \mathrm{d}_A^* \mathrm{d}_A \xi \rangle.$$ ■

A consequence of the above on typically studies the Bogomolny equation on a non-compact Y or admits A and ξ to have singularities. In fact, the study of the Bogomolny equation has largely focused on $\mathbf{R}^3$.

Examples. Here are two important examples.

Example 5.60. Denote by $(p\colon S^3 \to S^2, R)$ the Hopf bundle. The adjoint bundle $\mathrm{Ad}(S^3)$ is trivial bundle $i\underline{\mathbf{R}}$. The $\mathrm{U}(1)$–principal connection B induced by the standard Riemannian metric on S^3 satisfies

$$F_B = -\frac{i}{2}\mathrm{vol}_{S^2}.$$

For $k \in \mathbf{Z}$ define $\lambda_k\colon \mathrm{U}(1) \to \mathrm{U}(1)$ by $\lambda_k(z) := z^k$ and denote by $p_k\colon P_k := S^3 \times_{\lambda_k} \mathrm{U}(1) \to S^2$ the corresponding $\mathrm{U}(1)$–principal bundle. B induced a $\mathrm{U}(1)$–principal connection B_k on p_k satisfying

$$F_{B_k} = -\frac{ik}{2}\mathrm{vol}_{S^2}.$$

Denote by $\pi\colon \mathbf{R}^3\backslash\{0\} \to S^2$ the projection map. Let $m \in \mathbf{R}$. The **Dirac monopole** of mass m and charge k defined by

$$A_k^{\mathrm{Dirac}} := \pi^* B_k \quad \text{and} \quad \xi_{m,k}^{\mathrm{Dirac}} := \left(m - \frac{k}{r}\right)\frac{i}{2}$$

satisfies (5.54). ♠

Remark 5.61 (Scaling monopoles). Let $\lambda > 0$. Define $s_\lambda\colon \mathbf{R}^3 \to \mathbf{R}^3$ by $s_\lambda(x) := \lambda x$. If (A, ξ) is a solution to (5.54), the so is

$$(s_\lambda^* A, \lambda s_\lambda^* \xi).$$

The mass parameter can be varied by scaling. ♣

Example 5.62. Let $m \in \mathbf{R}$. Let $k \in \mathbf{N}$. Let $x_1, \dots, x_k \in \mathbf{R}^3$. Set

$$\xi := \left(m - \sum_{a=1}^{k} \frac{1}{|x - x_a|}\right)\frac{i}{2}.$$

There is a $\mathrm{U}(1)$–principal bundle $(p\colon P \to \mathbf{R}^3\backslash\{x_1, \dots, x_k\}, R)$ and a connection $A \in \mathscr{A}(p, R)$ which together with ξ satisfies (5.54). ♠

Example 5.63. Bogomolny [Bog76] and Prasad and Sommerfield [PS75] discovered the **BPS monopole**, a particular solution of (5.54). Identify $\mathbf{R}^3 = \mathfrak{sp}(1)$. In particular, $S^2 \subset \mathfrak{sp}(1)$. Denote by $\tau \in C^\infty(S^2, \mathfrak{sp}(1))$ the inclusion map. A brief computation shows that

$$\begin{aligned}
\mathrm{d}\tau &= \frac{1}{2}[\tau, *_{S^2}\mathrm{d}\tau], \\
*_{S^2}\mathrm{d}\tau &= -\frac{1}{2}[\tau, \mathrm{d}\tau], \\
[\mathrm{d}\tau \wedge \mathrm{d}\tau] &= 4\tau\mathrm{vol}_{S^2}, \\
[(*_{S^2}\mathrm{d}\tau) \wedge (*_{S^2}\mathrm{d}\tau)] &= 4\tau\mathrm{vol}_{S^2}.
\end{aligned}$$

Therefore, the *ansatz*

$$A = f(r) *_{S^2} \mathrm{d}\tau \quad \text{and} \quad \xi = g(r)\tau$$

leads to

$$F_A = f'\mathrm{d}r \wedge *_{S^2}\mathrm{d}\tau + 2(-f + f^2)\tau\mathrm{vol}_{S^2} \quad \text{and}$$
$$\mathrm{d}_A\xi = g'\tau\mathrm{d}r + (g - 2fg)\mathrm{d}\tau.$$

Since $*\mathrm{d}r = r^2\mathrm{vol}_S^2$, the Bogomolny equation therefore amounts to the ODE

$$f' = g - 2fg \quad \text{and}$$
$$r^2g' = 2(f - f^2).$$

Here is a family of solutions ($m > 0$) to this equation

$$f_m(r) = -\frac{1}{2}\Big(\frac{mr}{\sinh mr} - 1\Big) \quad \text{and} \quad g_m(r) = \frac{m}{2}\bigg(\frac{1}{\tanh mr} - \frac{1}{mr}\bigg).$$

Observe that

$$f_m = \frac{1}{2} + O(mre^{-2mr}). \quad \text{and} \quad g_m = \bigg(\frac{m}{2} - \frac{1}{2r}\bigg) + O(e^{-2mr}).$$

This gives the 1–parameter family of solutions

$$(A_m^{\mathrm{BPS}}, \xi_m^{\mathrm{BPS}}) \quad m > 0$$

of (5.54). ♠

Remark 5.64.

(1) Let X be a manifold of dimension at most 3. Let $L \longrightarrow X$ be a Hermitian line bundle. Set $E \coloneqq L \oplus L^* \longrightarrow X$. Eventhough, L might not be trivial, E always is. Here is why. Since $3 < 4$, a generic section s of E is nowhere-vanishing. Since E inherits a complex structure i from L, E has at two linearly independent sections: s, is. The Hermitian inner product, defines an i–anti-linear map $j\colon\ L \longrightarrow L^*$ that can be extended to a futher almost complex structure j on E satifying $ij = -ji$. Consequently, E is a quaternionic line bundle and s, is, js, ijs are linearly independent.

(2) As a concrete instantance of the above but in setting of $\mathrm{U}(1)$– and $\mathrm{Sp}(1)$–principal bundles. The Hopf bundle $p\colon\ P \coloneqq S^3 \longrightarrow S^2$ is non-trivial. Define $\rho\colon\ \mathrm{U}(1) \longrightarrow \mathrm{Sp}(1)$ by

$$\rho(e^{i\alpha}) = e^{i\alpha}.$$

The associated $\mathrm{Sp}(1)$–principal bundle $q\colon\ Q \coloneqq S^3 \times_\rho \mathrm{Sp}(1) \longrightarrow S^2$ is trivial. Indeed, the section $s\colon\ S^2 = S^3/\mathrm{U}(1) \longrightarrow Q$ defined by

$$s([x]) \coloneqq [x, x^*]$$

trivialises Q. Here $S^3 = \mathrm{Sp}(1)$.

(3) The Dirac monopole is built from the standard $\mathrm{U}(1)$–principal connection 1–form $\theta \in \Omega^1(S^3, i\mathbf{R})$ and the constant $\mathrm{U}(1)$–equivariant map $i \in C^\infty(P, i\mathbf{R})^{\mathrm{Ad}}$. The latter induces the $\mathrm{Sp}(1)$–equivariant map $\xi \in C^\infty(Q, \mathfrak{sp}(1))^{\mathrm{Ad}}$ by

$$\xi([x, g]) := \mathrm{Ad}(g)^{-1}(\mathrm{Lie}\,\rho)(i) = g^* i g.$$

The pullback via s satisfies

$$s^*\eta([x]) = xix^*.$$

This is the map $\tau\colon S^2 = S^3/\mathrm{U}(1) \to \mathfrak{sp}(1)$. This shows that the Higgs field of the BPS monopole is (exponentially) asymptotic to the Higgs field of the Dirac monopole.

The same is true for the connection. The connection 1–form θ on P can be written as

$$\theta = i\,\mathrm{Re}(\mathrm{d}x^* x i) = \frac{1}{2}(i\mathrm{d}x^* x i + x^*\mathrm{d}x).$$

The induced connection on Q descends from

$$\frac{1}{2}\,\mathrm{Ad}(g)^{-1}(i\mathrm{d}x^* x i + x^*\mathrm{d}x) + g^{-1}\mathrm{d}g.$$

with $x \in P$ and $g \in \mathrm{Sp}(1)$. The pullback via s is

$$\frac{1}{2}x(i\mathrm{d}x^* x i + x^*\mathrm{d}x)x^* + x\mathrm{d}x^* = \frac{1}{2}(xi\mathrm{d}x^* x i x^* + x\mathrm{d}x^*).$$

Therefore, for $\tilde{\tau}(x) = xix^*$,

$$\begin{aligned} -\frac{1}{4}[\tilde{\tau}, \mathrm{d}\tilde{\tau}] &= -\frac{1}{4}[xix^*, \mathrm{d}xix^* + xi\mathrm{d}x^*] \\ &= -\frac{1}{4}(xix^*\mathrm{d}xix^* - x\mathrm{d}x^* + \mathrm{d}xx^* - xi\mathrm{d}x^* xix^*) \\ &= \theta. \end{aligned}$$

This proves the statement about the connection. ♣

Energy and charge. Henceforth, we restrict to $G = \mathrm{SU}(2) = \mathrm{Sp}(1)$ and, in fact, shortly to $Y = \mathbf{R}^3$. Consider the Yang–Mills–Higgs functional

$$\mathrm{YMH}(A, \xi) = \frac{1}{2}\int_Y |F_A|^2 + |\mathrm{d}_A\xi|^2.$$

Obviously,

$$\mathrm{YMH}(A, \xi) = \frac{1}{2}\int_Y |F_A \pm *\mathrm{d}_A\xi|^2 \mp 4\pi N \quad \text{with} \quad N := \frac{1}{4\pi}\int_Y \langle F_A \wedge \mathrm{d}_A\xi\rangle.$$

For $Y = \mathbf{R}^3$, by Stokes' theorem

$$N = \frac{1}{4\pi}\lim_{r\to\infty}\int_{\partial B_r(0)} \langle F_A, \xi\rangle$$

Under suitable boundary conditions this limit exits and indeed agrees with the degree of $\xi\colon S^2_\infty \to S^2 \subset \mathfrak{sp}(1)$. In particular,

$$N \in \mathbf{Z}.$$

This is the anlogue of the energy identity for the (anti-)self-duality equation.

Moduli spaces. It is customary study (5.54) on the trivial $\mathrm{Sp}(2)$–bundle over $\mathbf{R}^3$ with the following boundary condition that, with respect to the isomorphism $\mathrm{Sp}(1) \times (\mathbf{R}^3 \backslash \{0\}) \cong Q$ discussed above,

$$(A, \xi) = (A_k^{\mathrm{Dirac}}, \xi_{1,k}^{\mathrm{Dirac}}) + \text{lower order terms at infinity.}$$

(The mass parameter can be adjusted by scaling.) Denote the corresponding space of configurations of (A, ξ) by $\mathscr{C}_k$. The relevant group of gauge transformations $\mathscr{G}_0$ is required to decay to the identity at infinity. The **framed moduli space of charge k monopoles** is then

$$\mathscr{N}_k := \{(A, \xi) \in \mathscr{C}_k : (5.54)\}/\mathscr{G}_0.$$

Theorem 5.65. *$\mathscr{N}_k$ is a hyperkähler manifold of dimension $4k$. $S^1 \times \mathbf{R}^3$ acts freely on $\mathscr{N}_k$.*

The **strongly centred framed monopole moduli space of charge k** is

$$\tilde{\mathscr{N}}_k^0 := \widetilde{\frac{\mathscr{N}_k}{S^1 \times \mathbf{R}^3}}.$$

$\tilde{\mathscr{N}}_2^0$ is **Atiyah–Hitchin manifold**, already a tantalising geometric object.

The fact that $\mathscr{N}_k$ is always non-empty is one of the first major mathematical achievments of Taubes. His idea is to start with a Dirac monopole with k (well-separated) singularities and to repair the singularities by gluing in BPS monopoles. Using cut-off functions this can be done approximately. To make these approximate solutions actual solutions requires a delicate analysis; cf. Jaffe and Taubes [JT80].

Theorem 5.66 (Donaldson [Don84]). *A choice of isometry $\mathbf{R}^3 = \mathbf{R} \times \mathbf{C}$ defines a bijection between $\mathscr{N}_k$ and the space of rational maps $f\colon \mathbf{C}P^1 \to \mathbf{C}P^1$ of degree k satisfying $f([1:0]) = [0:1]$.*

This theorem was the first to give a global uniform understanding of $\mathscr{N}_k$ for all k. This was conjectured in Murray [Mur83, Appendix B]. Donaldson's proof is rather roundabout. Jarvis [Jar00] gave a direct and geometric proof (of an extension of Theorem 5.66.

the cyclic group C_k acts on $\tilde{\mathscr{N}}_k^0$. Denote by μ_k the k–th roots of unity. There are canonical maps $\lambda_\ell\colon C_k \to \mu_k$ which sends the generator of C_k to $e^{2\pi i \ell/k}$. Therefore, it also acts on the cohomology $\mathrm{H}^i(\tilde{\mathscr{N}}_k^0, \mathbf{C})$. Denote by

$$\mathrm{H}_\ell^i(\tilde{\mathscr{N}}_k^0, \mathbf{C})$$

the subspace in which the action of C_k agrees with λ_ℓ. There is an analoge of the above spaces cohomology replaced by L^2 harmonic forms $\mathscr{H}$. **Sen's conjecture** [Sen94] asserts that

1. If k, ℓ are coprime, then $\mathscr{H}_\ell^{2k-2}(\tilde{\mathscr{N}}_k^0, \mathbf{C}) \cong \mathbf{C}$ and vanishes otherwise.
2. If k, ℓ are not coprime, then $\mathscr{H}_\ell^i(\tilde{\mathscr{N}}_k^0, \mathbf{C})$ vanishes.

This conjecture was a main driving force in the study of the geometry of moduli spacee $\tilde{\mathscr{N}}_k^0$. The cohomological version of theses statements are have been proved by Segal and Selby [SS96] The first part of these conjectures has now been proved by Fritzsch, Kottke, and Singer [FKS18] (modulo the appearance of part 2 of that paper?).

Monopoles and scattering. Let I be an interval. Let Σ be a Riemann surface. Let $E \to \Sigma$ be a Hermitian vector bundle with $\det E = \mathbf{C}$. Let d_A be a compatible covariant derivative on E and let $\xi \in \Gamma(\mathfrak{su}(E))$. The covariant derivative d_A splits as

$$\mathrm{d}_A = \partial_A + \bar\partial_A + \mathrm{d}t \wedge \nabla_{A,\partial_t}.$$

Therefore,

$$F_A = \mathrm{d}_A^2 = \bar\partial_A\partial_A + \partial_A\bar\partial_A + \mathrm{d}t \wedge ([\nabla_{A,\partial_t}, \bar\partial_A] + [\nabla_{A,\partial_t}, \partial_A]).$$

and

$$*\mathrm{d}_A\xi = \nabla_{A,\partial_t}\xi \cdot \mathrm{vol}_\Sigma + i\mathrm{d}t \wedge \partial_A\xi - i\mathrm{d}t \wedge \bar\partial_A\xi$$

By direct inspection, (5.54) is equivalent to

$$\begin{aligned} [\nabla_{A,\partial_t} + i\xi, \bar\partial_A] = 0 \quad &\text{and} \\ \bar\partial_A\partial_A + \partial_A\bar\partial_A = \nabla_{A,\partial_t}&\xi\mathrm{vol}_\Sigma. \end{aligned}$$

Suppose now that $I = [0,1]$. Restriction to t defines a holomorphic structure $\bar\partial_{A,t}$ on E for every $t \in [0,1]$. Consider parallel transport $T_t \colon E \to E$ from $t = 0$ to t associated with the covariant derivative $\nabla_A + i\xi$. The first of the above equations shows that T_t defines an isomorphism of holomorphic vector bundles $\mathscr{E}_0 := (E, \bar\partial_{A,0}) \to \mathscr{E}_t := (E, \bar\partial_{A,t})$. This map is (sometimes) called the **scattering map**. Under suitable boundary conditions/stability conditions, upto suitable equivalences, etc., given $\bar\partial_A$ and $\nabla_{A,\partial_t} + i\xi$ satisfying the first of the above equations, (A, ξ) can be recovered so that the second equation also holds. This is usually studied when (A, ξ) has singularities. In this case the scattering map is not an isomorphism, but a Hecke modification (an isomorphism in the complement of a bunch of points). See Norbury [Nor11] and Charbonneau and Hurtubise [CH11] to learn more about this. Hurtubise [Hur85] gave an explanation of Theorem 5.66 via scattering maps.

5.12 Hitchin's equation

[This will be discussed in the problem session]

[Nit87]

5.13 The moduli space of ASD instantons

Let (X, g) be an closed oriented Riemannian 4–manifold. Let G be a compact semi-simple Lie group. Let $(p\colon P \to X, R)$ be a G–principal bundle. The **moduli space of ASD instantons** on (p, R) is

$$\mathscr{M} := \{A \in \mathscr{A} : F_A^+ = 0\}/\mathscr{G}.$$

At this stage, $\mathscr{M}$ is a topological space. The purpose of this section is to equip $\mathscr{M}$ with more structure and understand its geometry better. (Here and throughout, to ease notation, (p, R) is dropped from the notation.)

It turns out to be benificial to construct the quotient

$$\mathscr{B} := \mathscr{A}/\mathscr{G}$$

and then construct $\mathscr{M}$. A useful framework to proceed in is that of Banach manifolds. This requires us to enlarge $\mathscr{A}$ and $\mathscr{G}$.

5.13.1 Sobolev spaces

Let $U \subset \mathbf{R}^n$ be a bounded open subset (with smooth boundary).

[Review definition of $W^{k,p}(U)$; check what is known: L^p? distributions? weak derivatives?; mention scaling weight $k - n/p$]

Theorem 5.67 (Sobolev inequality/Sobolev embedding theorem). *Let $k, \ell \in \mathbf{N}_0$, $p, q \in [1, \infty)$. If*

$$k > \ell, \quad \text{and} \quad k - \frac{n}{p} \geqslant \ell - \frac{n}{q},$$

then $W^{k,p}(U) \subset W^{\ell,q}(U)$ and the inclusion map is continuous.

Theorem 5.68 (Rellich–Kondrachov). *Let $k, \ell \in \mathbf{N}_0$, $p, q \in [1, \infty)$. If*

$$k > \ell, \quad \text{and} \quad k - \frac{n}{p} > \ell - \frac{n}{q},$$

then the inclusion map $W^{k,p}(U) \subset W^{\ell,q}(U)$ is compact.

Theorem 5.69 (Morrey inequality). *Let $k, r \in \mathbf{N}_0$, $p \in [1, \infty)$, and $\alpha \in (0, 1)$. If*

$$r + \alpha = k - \frac{n}{p},$$

then every $W^{k,p}$ function on U is (representable) by a $C^{r,\alpha}$ function on $\bar{U}$ and the map

$$W^{k,p}(U) \to C^{r,\alpha}(\bar{U})$$

is continuous. ∎

Theorem 5.70 (Sobolev multiplication). *Let $k \in \mathbf{N}_0$, $p \in [1, \infty)$. If*

$$k - \frac{n}{p} > 0,$$

then the multiplication map

$$W^{k,p}(U) \times W^{k,p}(U) \to W^{k,p}(U)$$

is continuous. ∎

If (X, g) is a Riemannian manifold, V is an Euclidean vector space, $E \to X$ is an Euclidean vector bundle equipped with a covariant derivative ∇, then we define Sobolev spaces $W^{k,p}(X, V)$, $W^{k,p}\Omega^\bullet(X, V)$, $W^{k,p}\Gamma(E)$, $W^{k,p}\Omega^\bullet(X, E)$, etc. with norms

$$\|s\|_{W^{k,p}} \coloneqq \sum_{\ell=0}^{k} \left(\int_X |\nabla^k s|^p \mathrm{vol}_g \right)^{1/p}.$$

These are Banach spaces. Of course, there are analogous definitions of with $C^{r,\alpha}$ instead of $W^{k,p}$. These are Banach spaces too. The results mentioned above carry over mutatis mutandis.

5.13.2 Sobolev connections and gauge transformations

Let (X, g) be a closed connected oriented Riemannian manifold of dimension n. Let G be a compact semi-simple Lie group. Let $(p\colon P \to X, R)$ be a G–principal bundle. The theory of connections developed in the smooth case largely carries over to the Sobolev setting. Throughout, let $k \in \mathbf{N}_0$, $p \in (1, \infty)$ with

$$k + 2 - \frac{n}{p} > 0.$$

(The significance of this restriction shall be explained shortly.)

Definition 5.71.

(1) A $W^{k+1,p}$ **connection on** (p, R) is a $W^{k+1,p}$ 1–form

$$\theta_A \in W^{k+1,p}\Omega^1(P, \mathfrak{g})$$

such that for almost every $x \in P$ and $\xi \in \mathfrak{g}$

$$\theta_A(v_\xi(x)) = \xi$$

and for every $g \in G$

$$R_g^* \theta_A = \mathrm{Ad}(g)^{-1}\theta_A.$$

Denote the set of $W^{k+1,p}$ connections by $W^{k+1,p}\mathscr{A}(p, R)$.

(2) A $W^{k+2,p}$ **gauge transformation of** (p, R) is a $W^{k+2,p}$ map

$$u\colon\ W^{k+2,p}(P, G)^C$$

with the super-script C indicating that for every $g \in G$

$$u \circ R_g = C_g^{-1} u.$$

Denote the set of $W^{k+2,p}$ gauge transformations by $W^{k+2,p}\mathscr{G}(p, R)$. •

The theory developed in the smooth case carries over to the Sobolev setting (provided the regularity suffices to write the formulae.) Here are some facts (and consequences of the theory of Sobolev spaces):

(1) $W^{k+1,p}\mathscr{A}(p, R)$ is an affine space modelled on

$$W^{k+1,p}\Omega^1(X, \mathrm{Ad}(P)).$$

(2) Every $A \in W^{k+1,p}\mathscr{A}(p, R) \cap L^{2p}\mathscr{A}(p, R)$ has a curvature

$$F_A \in W^{k,p}\Omega^2(X, \mathrm{Ad}(P)).$$

Indeed, the curvature map

$$F\colon\ W^{k+1,p}\mathscr{A}(p, R) \cap L^{2p}\mathscr{A}(p, R) \to W^{k,p}\Omega^2(X, \mathrm{Ad}(P))$$

is analytic.

(3) $W^{k+2,p}\mathscr{G}(p,R)$ is a Banach Lie group with Lie algebra

$$W^{k+2,p}\Gamma(\mathrm{Ad}(P)),$$

and it acts smoothly on (the right of) $W^{k+1,p}\mathscr{A}(p,R)$:

$$\theta_{u^*A} = \mathrm{Ad}(u)^{-1}\theta_A + u^*\mu_G.$$

with $\mu_G \in \Omega^1(G,\mathfrak{g})$ denoting the Maurer–Cartan form on G.

Only the last point requires any justification. Ultimately, this fact follows from the Sobolev multiplication theorem. Naively, applything that theorem suggest that we should have imposed the much stronger condition

$$k+2-\frac{n+\dim G}{p} > 0.$$

The crucial point is that gauge transformations are G–equivariant and, therefore, the Sobolev multiplication theorem in dimension n can be used.

5.13.3 A slice theorem

Continue with the situation of the previous subsection, but we drop the pre-scripts $W^{k+1,p}$ and $W^{k+2,p}$ of $\mathscr{A}$ and $\mathscr{G}$ (in an attempt to not go insane). Our goal is to understand the quotient

$$\mathscr{B} := \mathscr{A}/\mathscr{G}.$$

This is a task for the slice theorem.

Proposition 5.72. *The action of $\mathscr{G}$ on $\mathscr{A}$ is proper.*

Proof sketch. Let (u_n) be a sequence $\mathscr{G}$ and (A_n) a sequence in $\mathscr{A}$. We have to prove that if (A_n) converges to A in $W^{k+1,p}$ and $u_n^*A_n$ converges to B in $W^{k+1,p}$, then (u_n) has a convergent subsequence in $W^{k+2,p}$. To do this one has to meditate over the identity

$$\theta_{u_n^*A_n} = \mathrm{Ad}(u_n)^{-1}\theta_{A_n} + u_n^*\mu_G = u_n^{-1}\theta_{A_n}u_n + u_n^{-1}\mathrm{d}u_n$$

and use the idea of bootstrapping.

By hypothesis, $\theta_{u_n^*A_n}$ and θ_{A_n} converge in $W^{k+1,p}$. Moreover, *since G is compact*, $\|u_n\|_{L^\infty}$ is bounded.

By the hypothesis $\|\mathrm{Ad}(u_n)^{-1}\theta_{A_n}\|_{L^p}$ is bounded. But then $\|u_n^*\mu_G\|_{L^p}$ is bounded. This implies that $\|\mathrm{d}u_n\|_{L^p}$ is bounded. That is $\|u_n\|_{W^{1,p}}$ is bounded. Using Sobolev embedding, Hölder's inequality etc. this argument can be iterated to obtain that $\|u_n\|_{W^{k+2,p}}$ is bounded. (The details of this are not hard, but a little fiddly.)

Using Rellich–Kondrachov one sees that a subsequence of u_n converges in $W^{k+1,q}$. This is not quite enough. We wanted convergence in $W^{k+2,p}$. The last missing point is to use the above identity to see that in this case $\mathrm{d}u_n$ must also converge in $W^{k+2,p}$. ■

The discussion from Section 3.3 extends to proper actions and Banach manifolds. The action of $\mathscr{G}$ on $\mathscr{A}$ however is *not free.*

Definition 5.73. Let $A \in \mathscr{A}$. The **isotropy group** of A is defined by

$$\Gamma_A \coloneqq \{u \in \mathscr{G} : u^*A = A\}. \qquad \bullet$$

Henceforth, fix $x_0 \in P$. Recall the holonomy group $\mathrm{Hol}_{x_0}(A)$ from Definition 4.38.

Proposition 5.74.

(1) *The evaluation map* $\mathrm{ev} = \mathrm{ev}_{x_0} \colon \mathscr{G} \to G$ *defined by* $\mathrm{ev}(u) = u(x_0)$ *defines an injection* $\mathrm{ev} \colon \Gamma_A \hookrightarrow G$.

(2) *The image of* Γ_A *in* G *is precisely the* $C_G(\mathrm{Hol}_{x_0}(A))$ *the centraliser of the holonomy group.*

Proof. By equivariance, $u(x_0)$ determines u on $p^{-1}(p(x_0))$. If u preserves A, then u commutes with the A–parallel transport. Since X is connected, this completely determines u. This proves (1).

The fact that u commutes with A–parallel transport also implies that $\mathrm{ev}(\Gamma_A) \subset C_G(\mathrm{Hol}_{x_0}(A))$. To prove the reverse conclusion observe that if $g \in C_G(\mathrm{Hol}_{x_0}(A))$ then it can be extended a G–equivariant map $p^{-1}(p(x_0)) \to G$ and to a G–equivariant map $u \colon P \to G$ by A–parallel transport. Since u is was constructed to commute with A–parallel transport, it preserves A. ∎

As a consequence of the above, Γ_A *always* contains $Z(G)$, the center of G.

Definition 5.75. A connection A is **irreducible** if $\mathrm{ev}(\Gamma_A) = Z(G)$. Denote the subset of irreducible connections in $\mathscr{A}$ by

$$\mathscr{A}^*.$$

A is **reducible** if it is not irreducible. •

Remark 5.76. The terminology "irreducible" is common but not ideal. $\mathrm{Hol}_{x_A}(A) < G$ might very well be a proper subgroup with centralizer $C(G)$. ♣

The slice theorem now constructs the quotient

$$\mathscr{B}^* \coloneqq \mathscr{A}^*/(\mathscr{G}/Z(G)) = \mathscr{A}^*/\mathscr{G}$$

as a Banach manifold. If $A_0 \in \mathscr{A}^*$ and $\varepsilon > 0$ is sufficiently small, then a chart of $\mathscr{A}^*/\mathscr{G}$ around $[A_0]$ can be constructed as follows. Consider the *local slice*

$$U_{A_0,\varepsilon} \coloneqq \{A + a : \mathrm{d}_{A_0}^* a = 0, \|a\|_{W^{k+1,p}} < \varepsilon\}.$$

The map $U_{A_0,\varepsilon} \to \mathscr{A}^*/\mathscr{G}$ is (the inverse of) a chart. Because every $u \in \Gamma_A$ takes values in the center $C(G)$ it acts trivially on $U_{A_0,\varepsilon}$. The slice condition

$$\mathrm{d}_{A_0}^* a = 0$$

is precisely the condition to be L^2 orthogonal to the action of infinitesimal gauge transformations of A_0:

$$\mathrm{d}_{A_0} \colon W^{k+2,p}\Gamma(\mathrm{Ad}(P)) \to W^{k+1,p}\Omega^1(X, \mathrm{Ad}(P)).$$

The above gives us a very useful description within which to understand the moduli space of irreducible ASD instantons. The application we have in mind requires a description of reducible ASD instantons.

Let $\Gamma < \mathscr{G}$ be any subgroup such that $\mathrm{ev}\colon \Gamma \to G$ is injective. Set

$$\begin{aligned}\mathscr{A}_{(\Gamma)} &:= \{A \in \mathscr{A} : \Gamma_A \text{ is conjugate to } \Gamma\},\\ \mathscr{A}_{\Gamma} &:= \{A \in \mathscr{A} : \Gamma_A = \Gamma\}, \quad \text{and}\\ W_{\mathscr{G}}(\Gamma) &:= N_{\mathscr{G}}(\Gamma)/\Gamma.\end{aligned}$$

A moment's thought shows that the inclusion map induces a homeomorphism

$$\mathscr{A}_{\Gamma}/W_{\mathscr{G}}(\Gamma) \cong \mathscr{A}_{(\Gamma)}/\mathscr{G} =: \mathscr{B}_{(\Gamma)}.$$

This former quotient can again be constructed as a Banach manifold using the slice theorem. Here is how to understand the charts. Let $A_0 \in \mathscr{A}_{\Gamma}$. The tangent space

$$T_{A_0}\mathscr{A} = \Omega^1(X, \mathrm{Ad}(P))$$

decomposes into a Γ–invariant part and its L^2 orthogonal complement. A local slice of the quotient $\mathscr{A}_{\Gamma}/W_{\mathscr{G}}(\Gamma)$ is

$$U^{\Gamma}_{A_0,\varepsilon} := \{A_0 + a : a \text{ is } \Gamma\text{–invariant}, \mathrm{d}^*_{A_0}a = 0, \|a\|_{W^{k+1,p}} < \varepsilon\}.$$

Γ acts trivially on $U^{\Gamma}_{A_0,\varepsilon}$ and the map $U^{\Gamma}_{A_0,\varepsilon} \to \mathscr{A}_{\Gamma}/W_{\mathscr{G}}(\Gamma)$ is (the inverse of) a chart.

Remark 5.77. A gauge tranformation $u \in \mathscr{G} = C^\infty(P, G)^C$ acts on $a \in T_A\mathscr{A} = \Omega^1_{\mathrm{hor}}(P, \mathfrak{g})^{\mathrm{Ad}}$ by

$$u^*a = \mathrm{Ad}(u)^{-1}a.$$

Decompose

$$\mathfrak{g} = \mathfrak{K} \oplus \mathfrak{m} \quad \text{with} \quad \mathfrak{k} = \mathfrak{g}^{\Gamma_A} = \{\xi \in \mathfrak{g} : \mathrm{Ad}(g)\xi = \xi \text{ for every } g \in \Gamma_A\} \quad \text{and} \quad \mathfrak{m} = \mathfrak{k}^{\perp}.$$

$\mathfrak{k} \subset \mathfrak{g}$ is a Lie subalgebra. It contains the holonomy Lie algebra hol but might be larger. Denote by $K < G$ the corresponding Lie subgroup containing Hol. The bundle P admits a reduction Q of structure group to K. $T_A\mathscr{A}_{\Gamma}$ is $\Omega^1(X, \mathrm{Ad}(Q))$. In fact, $\mathscr{A}_{\Gamma}(P) = \mathscr{A}^*(Q)$. ♣

The upshot of the discussion so far is that

$$\mathscr{B} = \mathscr{B}^* \amalg \coprod_{\Gamma \neq Z(G)} \mathscr{B}_{(\Gamma)}$$

with all of the pieces being Banach manifolds.

Here is description of a neigborhood of A_0 in $\mathscr{B}$.

Proposition 5.78. *Let $\varepsilon > 0$ be sufficiently small. Set*

$$U_{A_0,\varepsilon} := \{A_0 + a : \mathrm{d}^*_{A_0}a = 0, \|a\|_{W^{k+1,p}} < \varepsilon\}.$$

The map $\phi\colon U_{A_0,\varepsilon}/\Gamma \to \mathscr{B}$ is an open embedding. Moreover, the stabliser of a in Γ is precisely Γ_{A_0+a}.

More globally, there is a vector bundle $\mathscr{V} \to \mathscr{B}_{(\Gamma)}$ obtained as the decend of the vector bundle over $\mathscr{A}_\Gamma$ whose fiber over A_0 is

$$\ker \mathrm{d}_{A_0} \cap \left[W^{k+1,p}\Omega^1(X, \mathrm{Ad}(P))^\Gamma\right]^\perp.$$

Γ acts on $\mathscr{V}$ and the structure of $\mathscr{B}$ normal to $\mathscr{B}_{(\Gamma)}$ is modelled on $\mathscr{V}/\Gamma$.

∼

In our application, we specialise to $G = \mathrm{Sp}(1)$. In this case the only options for Γ are

$$\{\pm \mathbf{1}\} = Z(\mathrm{Sp}(1)), \quad S^1, \quad \text{and} \quad \mathrm{Sp}(1).$$

(This is *not* a completely trivialy fact.) $\Gamma = \mathrm{Sp}(1)$ corresponds to connections A with holonomy in $\{\pm\mathbf{1}\}$. In particular, A must be flat. These, will cannot appear in the spaces we are interested in. If $\Gamma = S^1 = \{\exp t\xi : t \in \mathbf{R}\}$ $\xi \in S^2 \subset \mathrm{Im}\,\mathbf{H}$, then the holonomy is among the following

$$\text{a finite subgroup of } S^1 \quad \text{and} \quad S^1$$

Again, the former correspond to flat connections and cannot appear. In both cases, the rank 2 Hermitian vector bundle E corresponding to P splits as $E = L \oplus L^*$. In this case,

$$\mathrm{Ad}(P) = i\mathbf{R} \oplus L^2$$

and

$$\Omega^1(X, \mathrm{Ad}(P)) = \Omega^1(X) \oplus \Omega^1(X, L^2).$$

The former summand is Γ invariant and Γ acts on the later by multiplication with unit complex numbers (squared). Therefore,

$$\mathscr{B}_{(S^1)} \cong \{A_0 + a \in \Omega^1(X, i\mathbf{R}) : \mathrm{d}^* a = 0\}.$$

(This means: there is a global slice for $\mathscr{B}_{(S^1)}$. This is true in all abelian gauge theories.) The normal structure of $\mathscr{B}_{(S^1)}$ at A_0 is modelled on

$$\{a \in \Omega^1(X, L^2) : \mathrm{d}_{A_0}^* a\}/S^1.$$

Somewhat informally, the latter is

$$\mathbf{C}^\infty/S^1 = \text{the cone on } \mathbf{C}P^\infty.$$

Exercise 5.79. What is the local model around the trivial connection?

5.13.4 Kuranishi models for the moduli space

We are now in an excellent position to understand

$$\mathscr{M} := \{A \in \mathscr{A} : F_A^+ = 0\}/\mathscr{G}.$$

We continue with dropping the Sobolev prescripts. I will explain at the end why this is ultimately justified if one only cares about $\mathscr{M}$.

Over $\mathscr{A}$ there is a trivial Banach space bundle

$$\tilde{\mathscr{E}} := \mathscr{A} \times W^{k,p}\Omega^+(X, \mathrm{Ad}(P)).$$

The anti-self dual part of the curvature defines a section of $\tilde{\mathscr{E}}$:

$$A \mapsto F_A^+.$$

$\tilde{\mathscr{E}}$ descends to a Banach space bundle $\mathscr{E} \to \mathscr{B}$ and s defines as section of $\mathscr{E}$. By definition:

$$\mathscr{M} = s^{-1}(0) \subset \mathscr{B}.$$

Proposition 5.80. *Let A be an ASD instanton. Set*

$$V_{A,\varepsilon} := \left\{A + a : \|a\|_{W^{k+1,p}} < \varepsilon, \mathrm{d}_A^* a = 0, F_{A+a}^+ = \mathrm{d}_A^+ a + \tfrac{1}{2}[a \wedge a]^+ = 0\right\}.$$

The map $V_{A,\varepsilon}/\Gamma_A \to \mathscr{M}$ is an open embedding. Moreover, the stabiliser of a in Γ_A is Γ_{A+a}.

The term $\frac{1}{2}[a \wedge a]^+$ can be treated a small perturbation (as we will see shortly). It is therefore crucial to understand the linearised operator

$$\delta_A := (\mathrm{d}_A^*, \mathrm{d}_A^+) \colon W^{k+1,p}\Omega^1(X, \mathrm{Ad}(P)) \to W^{k,p}\Omega^0(X, \mathrm{Ad}(P)) \oplus W^{k,p}\Omega^+(X, \mathrm{Ad}(P)).$$

Proposition 5.81. *δ_A is an elliptic operator and, therefore,* ***Fredholm****; that is:*

$$\dim \ker \delta_A < \infty \quad \textit{and} \quad \dim \operatorname{coker} \delta_A < \infty.$$

Moreover,

$$\operatorname{index} \delta_A = \dim \ker \delta_A - \dim \operatorname{coker} \delta_A = -2p_1(\mathrm{Ad}(P)) + \dim G \cdot (b^1(X) - 1 - b^+(X)).$$

It is an easy exercise to compute the symbol of δ_A and verify ellipticity. The index can be computed using the Atyiah–Singer index theorem [see Atiyah–Hitchin–Singer].

Proposition 5.82. *Let X, Y be Banach spaces. Let $U \subset X$ be an open neighborhood of $0 \in X$. Let $f \colon U \to Y$ be a smooth map with $f(0) = 0$. Suppose that $T_0 f \colon X \to Y$ is Fredholm. Choose decompositions*

$$X = \ker T_0 f \oplus \operatorname{coim} T_0 f \quad \textit{and} \quad Y = \operatorname{coker} T_0 f \oplus \operatorname{im} T_0 f.$$

There is an open neighborhood V of 0 in X and a diffeomorphism $\phi \colon V \to U$ and a linear isomorphism $I \colon \operatorname{coim} T_0 f \to \operatorname{im} T_0 f$ such that

$$(f \circ \phi)(x, y) = (g(x, y), I(y)).$$

Apply this with

$$U \subset \ker(\mathrm{d}_A^*\colon\ W^{k+1,p}\Omega^1(X,\mathrm{Ad}) \to W^{k,p}\Omega^0(X,\mathrm{Ad})), Y := W^{k+1,p}\Omega^2(X,\mathrm{Ad}(P))$$

and $a := F_{A+a}^+$. It follows that a neighborhood of $[A] \in \mathscr{M}$ is modelled on

$$f^{-1}(0)/\Gamma_A$$

for some Γ_A–equivariant smooth map

$$f\colon\ \ker\delta_A \to \operatorname{coker}\mathrm{d}_A^+.$$

5.13.5 Digression: the deformation complex

[possibly skip this]

If A is an ASD instanton, then

$$0 \to \Omega^0(X,\mathrm{Ad}(P)) \xrightarrow{\mathrm{d}_A} \Omega^1(X,\mathrm{Ad}(P)) \xrightarrow{\mathrm{d}_A^+} \Omega^+(X,\mathrm{Ad}(P)) \to 0$$

is an elliptic complex. Its cohomology groups H_A^0, H_A^1, H_A^2 correpond to infinitesimal gauge transformations, infinitesimal deformations, and infinitesimal obstructions. In fact,

$$H_A^1 \cong \ker\delta_A \quad \text{and} \quad H_A^2 \cong \operatorname{coker}\mathrm{d}_A^+.$$

This encodes the infinitesimal deformation somewhat more naturally. The above map f can be understood as map $f\colon\ H_A^1 \to H_A^2$.

5.13.6 Freed–Uhlenbeck transversality theorem

Theorem 5.83 (Freed–Uhlenbeck). *If $G = \mathrm{SU}(2)$ or $\mathrm{SO}(3)$, then for a generic Riemannian metric g and every irreducible ASD instanton A,* $\operatorname{coker}\mathrm{d}_A^+ = 0$.

As a consequence, in the situation of the Freed–Uhlenbeck theorem $\mathscr{M}^*$, the moduli space of *irreducible* ASD instantons, is a smooth manifold of dimension

$$-2p_1(\mathrm{Ad}(P)) + \dim G \cdot (b^1(X) - 1 - b^+(X)).$$

5.14 Uhlenbeck compactness

The moduli space $\mathscr{M}$ is very rarely compact. Most applications of ASD instantons require either some understanding of how compactness fails or even a suitable compactification $\overline{\mathscr{M}}$. If you want to understand this issue in detail, then [Weh04] is an excellent reference. Let me begin by discussing what can possibly go wrong:

(1) If A is a connection over $\mathbf{R}^n$, $s_\lambda(x) = \lambda^{-1}x$, then $A_\lambda := s_\lambda^* A$ satisfies

$$\begin{aligned}\mathrm{YM}(A_\lambda) &= \frac{1}{2}\int_{\mathbf{R}^n} |F_{A_\lambda}|^2(x)\mathrm{d}x\\ &= \frac{1}{2}\int_{\mathbf{R}^n} \lambda^{-4}|F_A|^2(\lambda^{-1}x)\lambda^n\mathrm{d}(\lambda^{-1}x)\\ &= \lambda^{n-4}\mathrm{YM}(A).\end{aligned}$$

That is the scaling-weight is $n-4$. As a conseqeunce: if $n \leqslant 3$, then for a connection to minimise its Yang–Mills energy it is not beneficial to "scale down"; if $n \geqslant 5$, then for a connection to minimise its Yang–Mills energy it is beneficial to "scale down"; and $n = 4$ is the boarder line. Therefore, one should *expect* that: the compactness problem for Yang–Mills connections in dimension $n \leqslant 3$ should be quite easy ("sub-critical"); for $n \geqslant 5$ it should be very hard ("super-critical"), and for $n = 4$ it is something in between ("critical").

(2) In dimension 4, the Yang–Mills functional is not just scaling invariant. It is in fact invariant under conformal changes: $g \mapsto \lambda^2 g$ for $\lambda \in C^\infty(X, (0,\infty))$. This means that the conformal group acts on Yang–Mills solutions. Since the conformal group is non-compact, this might be a source of non-compactness.

(3) We already have examples of the failure of compactness. The curvature of the BPST instanton $A_{\mu,b}$ on $\mathbf{H}$ is given satisfies

$$|F_{A_{\mu,b}}| = \frac{192^{1/2}\mu^2}{(\mu^2|q-b|^2+1)^2}.$$

As μ tends to ∞, this fails to converge at $q = b$. Away from $q = b$, however, this converges to zero and so does the $A_{\mu,b}$. That is: $A_n = A_{\mu_n,b}$ converges on almost all of $\mathbf{R}^4$, but there is one point at which something goes wrong. This point is identifiable by the fact that the Yang–Mills energy in a small (n–independent) ball around x stays quite large:

$$\liminf_{r\to 0} \liminf_{n\to\infty} \mathrm{YM}(A_n|_{B_r(x)}) > 0.$$

The Yang–Mills energy *concentrates* at x.

Another further issue that complicates the compactness analysis in Yang–Mills theory is that because the gauge group $\mathscr{G}$ is severely non-compact one cannot expect any sequence (A_n) of connections to converge without pasing to a gauge transformed seqeunce $(u_n^* A_n)$. (Of course, since $\mathscr{M} \subset \mathscr{A}/\mathscr{G}$ this is not an actuall issue, but it means that one has to be carefull about what to expect.)

The following discussion focuses on dimension $n = 4$ and ASD instantons. In a sense this is ideal, because it gives us a topological energy bound

$$\mathrm{YM}(A) = -\frac{1}{4\pi^2} p_1(\mathrm{Ad}(p))$$

to get started. The theory in dimension $n \leqslant 3$ is much simpler (and can easily be derived from that in dimension $n = 4$). The theory in dimension $n \geqslant 5$ is quite a bit more complicated (and indeed not fully understood). Some extra ideas and observations are needed for $n \geqslant 5$, in particular: the monotonicity formula due to Price [Pri83] and a more delicate ε–regularity theorem.

5.14.1 Uhlenbeck gauge fixing

The upcoming discussion is local. Let $G < \mathrm{O}(N)$ be a Lie group. On $B_1(0) \subset \mathbf{R}^n$ consider the trivial G–principal bundle $(p\colon G \times B_1(0) \to B_1(0), R)$. A **gauge** is a section of p. The trivial gauge is $x \mapsto (\mathbf{1}, x)$. Of course, any other gauge be identified with a map $u\colon B_1(0) \to G$. By comparison with the trivial gauge it can be identified with a gauge transformation. We regard connection on (p, R) as 1–forms on $B_1(0)$ with value in $\mathfrak{g}$.

Theorem 5.84 (Uhlenbeck [Uhl82a, Theorem 2.1]). *Let $n/2 < p$. There are constants $\varepsilon = \varepsilon(p, G)$ and $c = c(p, G)$ such that the following holds. If $A \in \mathscr{A}(p, R)$ satisfies*

$$\|F_A\|_{L^p} \leqslant \varepsilon,$$

then there is a $W^{2,p}$ gauge u such that u^*A satisfies the gauge fixing conditions

$$\begin{aligned} \mathrm{d}^*(u^*A) &= 0, \\ i(\partial_r)(u^*A) &= 0 \quad \text{on } \partial B_1(0) \end{aligned} \tag{5.85}$$

and

$$\|u^*A\|_{W^{1,p}} \leqslant c\|F_A\|_{L^p}. \tag{5.86}$$

Remark 5.87. Uhlenbeck [Uhl82a, Theorem 2.1] is somewhat stronger than Theorem 5.84 (the smallness condition is on $\|F_A\|_{L^{n/2}}$). Uhlenbeck [Uhl82a, Theorem 1.3] proved an even more delicate result at the Sobolev border line $p = n/2$. (In this case u might not be even be continuous.) It turns out that one can get away with the above result, by working a little harder later. ♣

Remark 5.88. The restriction $p < n$ might appear somewhat strange. It has to do with wanting to avoid the borderline Sobolev embedding at $W^{1,n}$. ♣

Sketch of proof of Theorem 5.84. The proof is based on the *continuity method.* The set

$$\mathscr{A}_\varepsilon := \{A \in W^{1,p}\mathscr{A} : \|F_A\|_{L^p} \leqslant \varepsilon\}.$$

is connected. Indeed, $[0,1] \ni \lambda \mapsto s_\lambda^*A$ with $s_\lambda(x) = \lambda x$ joins any connection in $\mathscr{A}_\varepsilon$ to the connection 0. (The scaling weight of the L^p norm of a 2–form is $\lambda^{(n-2p)/p}$.)xs The strategy is to prove that

$$\mathscr{A}_\varepsilon^\star := \{A \in \mathscr{A}_\varepsilon : \exists u \in W^{2,p}\mathscr{G} : \mathrm{d}(u^*A) = 0, i(\partial_r)(u^*A) = 0 \text{ on } \partial B_1(0), \|u^*A\|_{W^{1,p}} \leqslant c\|F_A\|_{L^p}.\}$$

is open and closed.

Evidently, $\mathscr{A}_\varepsilon^*$ is $W^{2,p}\mathscr{G}$–invariant. The conditions

$$\begin{aligned} \mathrm{d}^*(u^*A) &= 0, \\ i(\partial_r)(u^*A) &= 0, \\ \|u^*A\|_{W^{1,p}} &\leqslant c\|F_A\|_{L^p} \end{aligned}$$

are closed.

To see that $\mathscr{A}_\varepsilon^\star$ is closed, let (A_n) be a sequence in $\mathscr{A}_\varepsilon^\star$ which converges to $A \in \mathscr{A}$. Denote by u_n the gauge transformations such that $u_n^* A_n$ satisfies the gauge fixing condition. The sequence $u_n^* A_n$ has a weak limit in $W^{1,p}$. The argument from the proof that the action of $\mathscr{G}$ on $\mathscr{A}$ is proper shows that u_n converges weakly in $W^{2,p}$ to a limit u. This suffices to obtain the gauge fixing conditions on $u^* A$. Therefore, $A \in \mathscr{A}_\varepsilon^\star$. This explains why $\mathscr{A}_\varepsilon^\star$ is closed (regardless of the choice of c)

To prove openness of $\mathscr{A}_\varepsilon^\star$ it suffices (by $\mathscr{G}$–invariance) to show that if A satisfies the gauge fixing condition and $\|A\|_{W^{1,p}} \leqslant c\|F_A\|_{L^p} \leqslant c\varepsilon$, then for a with $\|a\|_{W^{1,p}} < \delta \ll 1$ there is a gauge transformation u such that $u^*(A+a)$ satisfies the gauge fixing condition and $\|u^*(A+a)\|_{W^{1,p}} \leqslant c\|F_A\|_{L^p}$. Here is an assertion proving that the last condition will be automatic.

Proposition 5.89. *There are constant $c = c(n, G)$ and $\varepsilon_0 = \varepsilon_0(n, G)$ such that the following holds. If $A \in \mathscr{A}_\varepsilon$ satisfies (5.85) and $\|A\|_{W^{1,p}} < c_s \varepsilon_0$, then*

$$\|A\|_{W^{1,p}} \leqslant c\|F_A\|_{L^p}.$$

Proof. If (5.85) holds, then

$$\begin{aligned}(\mathrm{d} \oplus \mathrm{d}^*)A &= F_A - \frac{1}{2}[A \wedge A] \\ i(\partial_r)A &= 0.\end{aligned}$$

[An integration by parts argument proves that

$$\int_{B_1(0)} |\nabla A|^2 + \int_{\partial B_1(0)} |A|^2 = \int_{B_1(0)} |\mathrm{d}A|^2$$

This shows that the operator on the LHS is well-behaved on $W^{1,2}$. This is also the case for $W^{1,p}$.] The linear operator on the left-hand side of the above equations is an elliptic operator with trivial kernel. (See [Weh04] for a discussion of such operators.) Therefore, there is a constant $c = c(n, G) > 0$ such that

$$\|A\|_{W^{1,p}} \leqslant c\|F_A - \frac{1}{2}[A \wedge A]\|_{L^p} \leqslant c\|F_A\|_{L^p} + c\|A\|_{L^{2p}}^2.$$

Hölder's inequality and Sobolev embedding gives

$$\|A\|_{L^{2p}}^2 \leqslant c_S \|A\|_{W^{1,p}}^2.$$

(This is exactly the condition $p > n/2$). For $\varepsilon \ll 1$, the term $c_S\|A\|_{W^{1,p}}$ is at most $\frac{1}{2}$ and can be absorbed into the left-hand side. ■

To prove that $\mathscr{A}_\varepsilon^\star$ with $c = c(n, G)$ from above is an now application of the implicit function theorem (to find the gauge transformation, because the condition $\|A\|_{L^n} < cc_s\varepsilon_0$ is open). We omit details of the proof, but here is the crucial point. If A is in Uhlenbeck gauge and a is small, then $u = \exp(\xi)$ puts $A + a$ in Uhlenbeck gauge if and only if

$$\begin{aligned}\mathrm{d}^*\mathrm{d}\xi &= \mathrm{d}^*(e^{-\xi}(A+a)e^{\xi}) \\ \partial_r \xi &= 0 \quad \text{on the boundary.}\end{aligned}$$

The linearisation of the equation at $a = 0$, $\xi = 0$ is the LHS. By elliptic theory this is surjective, so one can apply the IFT. ■

Let ε as above.

Proposition 5.90. *There are constants $c_k > 0$ such that the following holds. If $A \in \mathscr{A}$ on $B_1(0)$ satisfies*

$$F_A^+ = 0,$$
$$\|F_A\|_{L^p} \leqslant \varepsilon,$$

then there is a (smooth) gauge transformation u such that

$$\|u^*A\|_{W^{k,p}} \leqslant c_k$$

for all k.

Proof sketch. Theorem 5.84, we can assume that A is already in Uhlenbeck gauge (i.e. $u = \mathbf{1}$). The result then follows from elliptic theory applied to $A \mapsto (\mathrm{d}^+A, \mathrm{d}^*A, i(\partial_r)A)$ and a little elliptic bootstrapping. ■

By Rellich–Kondrachov, these $W^{k,p}$–bounds will give smooth convergence (on compact subsets). The question is now: where do we get the bound $\|F_A\|_{L^p} \leqslant \varepsilon$ from? At this stage we can only assume a global L^2 bound: $\|F_A\|_{L^2} = -\frac{1}{2}\sqrt{p_1(\mathrm{Ad}(P))}$.

5.14.2 ε–regularity

We continue with the situation on $B_1(0)$. The following result is essentially due to Uhlenbeck [Uhl82b, Theorem 3.5]

Theorem 5.91. *There are constants $c_\mathrm{R}, \varepsilon > 0$ such that the following holds. Let A be an anti-self-dual instanton on $B_1(0)$. If*

$$\|F_A\|_{L^2(B_1(0))} \leqslant \varepsilon,$$

then

$$\|F_A\|_{L^\infty(B_1(0))} \leqslant c_\mathrm{R}\|F_A\|_{L^2(B_1(0))}.$$

Proof. By the Weitzenböck formula implies that

$$\nabla_A^*\nabla_A F_A = (\mathrm{d}_A^* + \mathrm{d}_A)^2 F_A + \{F_A, F_A\} = \{F_A, F_A\}$$

with $\{-,-\}$ denoting a universal bilinear form. Therefore,

$$\Delta|F_A|^2 = 2\langle\nabla_A^*\nabla_A F_A, F_A\rangle - 2|\nabla_A F_A|^2 \leqslant c|F_A|^3.$$

This implies

$$\Delta|F_A| \leqslant c|F_A|^2.$$

It turns out that such an inequality automatically implies the above assertion. [This is discussed next.] ■

Theorem 5.92 (ε–regularity). *Consider $B_1(0) \subset \mathbf{R}^4$. There are constants $c, \varepsilon > 0$ such that the following holds. If $f \in W^{1,2}(B_1(0)) \cap L^\infty(B_1(0))$, $f \geqslant 0$,*

$$\Delta f \leqslant f^2$$

holds weakly, and

$$\|f\|_{L^2(B_1(0))} \leqslant \varepsilon,$$

then

$$\|f\|_{L^\infty(B_{1/2}(0))} \leqslant c\|f\|_{L^2(B_1(0))}.$$

Proof using the mean value inequality. The mean value inequality implies that if $f \in C^\infty(B_r(0), [0,\infty))$ satisfies

$$\Delta f \leqslant \Lambda,$$

then

$$f(0) \leqslant c\big(r^{-2}\|f\|_{L^2(B_r(0))} + \Lambda r^2\big).$$

(The mean value inequality is very easy for $\mathbf{R}^n$. For non-flat backgrounds a proof is contained in [GT01, Theorem 9.20]; but see below.)

Define the auxiliary function $\phi\colon B_1(0) \to [0,\infty)$ by

$$\phi(x) \coloneqq (1-|x|)^2 f(x).$$

It suffices to prove that

$$\|\phi\|_{L^\infty} \leqslant c\varepsilon \quad \text{with} \quad \varepsilon = \|f\|_{L^2},$$

provided $\varepsilon \ll 1$.

Since ϕ vanishes on $\partial B_1(0)$, it achieves a maximum at some point $x_0 \in B_1(0)$. Set

$$r_0 \coloneqq \frac{1}{2}(1-|x_0|) \quad \text{and} \quad a_0 \coloneqq f(x_0).$$

The task is then to prove that

$$r_0^2 a_0 \leqslant c\varepsilon.$$

For every $x \in B_{r_0}(x_0)$,

$$1-|x| \geqslant 1-|x_0|-r_0 = r_0.$$

[draw a picture of this] Therefore, since $\phi(x) \leqslant \phi(x_0)$,

$$f(x) = (1-|x|)^{-2}\phi(x) \leqslant (1-|x|)^{-2}\phi(x_0) = \left(\frac{1-|x_0|}{1-|x|}\right)^2 f(x_0) \leqslant 4a_0.$$

Therefore, by the mean value inequality for every $0 \leqslant r \leqslant r_0$

$$a_0 \leqslant c_0\big(r^{-2}\varepsilon + r^2 a_0^2\big),$$

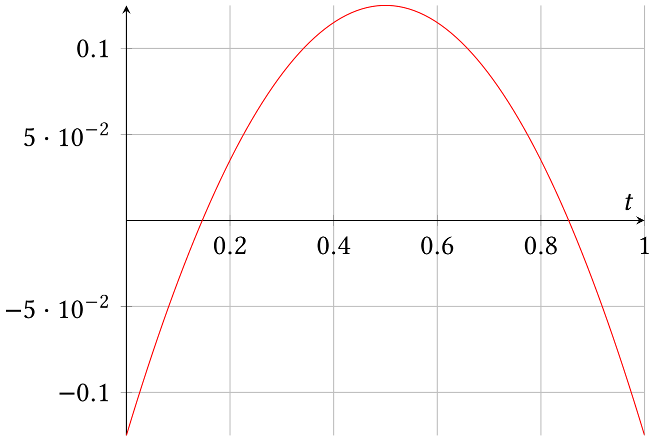
or, equivalently,

$$r^2 a_0 \leqslant c_0\big(\varepsilon + r^4 a_0^2\big).$$

That is for $t(r) := r^2 a_0$:

$$t(1 - c_0 t) - c_0 \varepsilon \leqslant 0.$$

This inequality holds for every $r \in [0, r_0]$ and t is non-negative. An inspection of the graph of the polynomial on the left-hand side shows that, provided $\varepsilon \ll 1$, $t \leqslant 2c_0\varepsilon$. For $r = r_0$ this proves the assertion. ■



Remark 5.93. The polynomial

$$p(t) = t(1 - ct) - \varepsilon$$

has the roots

$$t_0 = \frac{1}{2c}\left(1 - \sqrt{1 - 4c\varepsilon}\right) \quad \text{and} \quad t_1 = \frac{1}{2c}\left(\sqrt{1 - 4c\varepsilon} + 1\right).$$

As long as $\varepsilon \leqslant 1/4c$, the roots are both real and positive. As long as $\varepsilon \ll_c 1$,

$$t_0 \leqslant 2\varepsilon$$

♣

5.14.3 Convergence away from finitely many points

Theorem 5.94. *Let (X, g) be a closed oriented Riemannian 4−manifold. Let $G < \mathrm{O}(N)$ be a Lie group (such that the embedding is compatible with minus the Killing form). Let $(p\colon P\colon X, R)$ be a principal G−bundle. Let (A_n) be a sequence in of ASD instantons on (p, R). After passing to a subsequence the following holds. There are*

(1) *finitely many points $x_1, \ldots, x_k \in X$ and numbers $m_1, \ldots, m_k \in \mathbf{N}$,*

(2) *an ASD instanton A on $(p, R)|_{X\setminus\{x_1,\ldots,x_k\}}$, and*

(3) *a sequence of gauge transformations $u_k \in \mathscr{G}((p, R)|_{X\setminus\{x_1,\ldots,x_k\}})$*

such that

(4) *The sequence of measures* $|F_{A_n}|^2\mathrm{vol}$ *weakly converges to*

$$|F_A|^2\mathrm{vol} + \sum_{a=1}^{k} 8\pi^2 m_a \delta_{x_k}.$$

(5) *For every compact subset* $K \subset X\backslash\{x_1, \dots, x_k\}$, $u_n^* A_n|_K$ *converges to* $A|_K$ *(in the* C^∞ *topology).*

Proof sketch. Conisder the sequence of measures $\mu_n := |F_{A_n}|^2\mathrm{vol}$. The total mass of thesse measures is

$$c_{\mathrm{YM}} := -2\pi^2 p_1(\mathrm{Ad}(P));$$

in particular: it is uniformly bounded. Therefore, after passing to a subsequence μ_n weakly converges to a measure μ.

Let $x \in X$. Let ε be as in the ε–regularity theorem. If there exisits an $r > 0$ such that

$$\liminf_{n\to\infty} \int_{B_r(x)} |F_{A_n}|^2 < \varepsilon^2,$$

then on $B_r(x)$ a subsequence of A_n converges after gauge transformation. Therefore, it is crucial to understand the points for which this fails; that is: points with

$$\lim_{r\to 0} \liminf_{n\to\infty} \int_{B_r(x)} |F_{A_n}|^2 \geqslant \varepsilon^2,$$

If there are at least k points with this property, then

$$k\varepsilon^2 \leqslant \lim_{r\to 0} \liminf_{n\to\infty} \sum_{a=1}^{k} \int_{B_r(x_a)} |F_{A_n}|^2 \leqslant c_{\mathrm{YM}}.$$

This yields an apriori bound $k \leqslant c_{\mathrm{YM}}/\varepsilon^2$.

This identifies the points $\{x_1, \dots, x_k\}$. After passing to a subsequence, away from these points, (A_n) converges locally upto gauge transformations. These local gauge transformations can be patched together. (This is not trivial; see Donaldson and Kronheimer [DK90, §4.4.2] and Waldron [Wal19, §2.5].) This proves the convergence statement for A_n.

By Fatou's lemma

$$\delta = \mu - |F_A|^2\mathrm{vol}$$

is a non-negative measure. It must be supported on $\{x_1, \dots, x_k\}$. This proves the convergence statement for the measures with m_a non-necessarily integers.

To prove that m_a is an integer one first has to prove that A extends to all of X but a bundle which might be different from (p, R). [This *might* be proved in the next section.] Then, by construction for $r \ll 1$

$$\begin{aligned} m_a &= \frac{1}{8\pi^2} \lim_{n\to\infty} \int_{B_r(x_a)} |F_{A_n}|^2 - |F_A|^2 \\ &= \frac{1}{8\pi^2} \lim_{n\to\infty} \int_{B_r(x_a)} \langle F_{A_n} \wedge F_{A_n}\rangle - \langle F_A \wedge F_A\rangle. \end{aligned}$$

The last term can be rewritten as a Chern–Simons term and is known to be an integer. [The details are omitted.] ■

5.14.4 Uhlenbeck's removable singularities theorem

[discussed in problem session] sketch

(1) decay [Otway's unpublished argument]

(2) go to cylinder

(3) prove asymptotically flat

(4) gauge transform

(5) go back to ball

5.15 Digression: $\mathscr{H}^{\pm}(X, g)$ and product connections

Let (X, g) be a closed oriented Riemannian n–manifold. Consider the differential operator

$$\mathrm{d} + \mathrm{d}^* \colon \ \Omega^\bullet(X) \to \Omega^\bullet(X).$$

Denote by

$$\mathscr{H}^k(X, g) := \ker(\mathrm{d} + \mathrm{d}^*) \cap \Omega^k(X)$$

the space of harmonic k–forms on X. By Hodge theory,

$$\mathscr{H}^k(X, g) \cong \mathrm{H}^k_{\mathrm{dR}}(X).$$

The Hodge–$*$–operator induces an isomorphism

$$* \colon \ \mathscr{H}^k(X, g) \to \mathscr{H}^{n-k}(X, g).$$

If $n = 4k$, then $*$ has eigenvalues ± 1 on $\mathscr{H}^{2k}(X, g)$. The eigenspaces are denoted by

$$\mathscr{H}^{\pm}(X, g) \subset \mathscr{H}^{2k}(X, g).$$

Under the Hodge isomorphism, these correspond to the positive and negative definite subspace of the intersection form

$$Q \colon \ S^2\mathrm{H}^{2k}_{\mathrm{dR}}(X) \to \mathbf{R}$$

defined by

$$Q([\alpha], [\beta]) := \int_X \alpha \wedge \beta.$$

The **refined Betti number**

$$b^{\pm}(X) := \dim \mathscr{H}^{\pm}(X, g)$$

is independent of g.

Let us now specialise to $n = 4$. Consider the trivial bundle $(p \colon \ P := X \times G \to X, R)$ with the product (or trivial) connection A_0. According to Section 5.13, we can understand a neighborhood of $[A_0] \in \mathscr{M}$ as follows:

(1) If $u \in \mathscr{G} = C^\infty(X, G)$ fixes A_0, then u must be constant. Therefore, the isotropy group of A_0 is
$$\Gamma_{A_0} = G.$$

(2) The operator
$$\delta_{A_0} = (\mathrm{d}^*_{A_0}, \mathrm{d}^+_{A_0}) \colon \Omega^1(X, \mathfrak{g}) \to \Omega^0(X, \mathfrak{g}) \oplus \Omega^+(X, \mathfrak{g})$$
is simply $\delta \otimes \mathrm{id}_{\mathfrak{g}}$ with
$$\delta \coloneqq (\mathrm{d}^*, \mathrm{d}^+) \colon \Omega^1(X) \to \Omega^0(X) \oplus \Omega^+(X).$$
Evidently, $\mathscr{H}^1(X, g) \subset \ker \delta$. In fact, if $\delta\alpha = 0$, then $\mathrm{d}^*\alpha = 0$ and
$$0 = \int_X 2\langle \mathrm{d}^+\alpha, \mathrm{d}\alpha\rangle = \int_X 2\langle \mathrm{d}^*\mathrm{d}^+\alpha, \alpha\rangle = \int_X \langle \mathrm{d}^*\mathrm{d}\alpha, \alpha\rangle = \int_X |\mathrm{d}\alpha|^2.$$
Therefore, $\mathrm{d}\alpha = 0$. This shows that
$$\ker \delta_{A_0} = \mathscr{H}^1(X, g) \otimes \mathfrak{g}.$$
Moreover,
$$\operatorname{coker} \mathrm{d}^+_{A_0} = \mathscr{H}^+(X, g) \otimes \mathfrak{g}.$$

(3) Therefore, a neighborhood of $[A_0]$ is modelled on
$$f^{-1}(0)/G$$
for smooth map
$$f \colon \mathscr{H}^1(X, g) \otimes \mathfrak{g} \supset U \to \mathscr{H}^+(X, g) \otimes \mathfrak{g}.$$
In fact, $f = 0$. Thus the model is
$$\big(\mathscr{H}^1(X, g) \otimes \mathfrak{g}\big)/G = \big(\mathscr{H}^1(X, g) \otimes \mathfrak{t}\big)/W$$
with $\mathfrak{t}$ denoting a maximal abelian subalgebra and W denoting the Weyl group.

5.16 A sketch of Taubes' gluing theorem

Uhlenbeck's compactness theorem suggests that ASD instantons can degenerate by concentrating at points. At these points one might expect BPST instantons to "bubble off". Taubes' gluing theorem is concerned with the question of whether one can construct such degenerating ASD instantons. The idea is to glue BPST instantons into a product (trivial) connection.

Throughout, suppose that (X, g) is an closed oriented Riemannian 4–manifold with
$$b^+(X) = 0.$$
Denote by A_0 the product connection on the trivial $\mathrm{Sp}(1)$–principal bundle over X. Recall that the BPST instanton is given by
$$A^{BPST} \coloneqq \frac{\operatorname{Im}(\bar{q}\mathrm{d}q)}{|q|^2 + 1}$$

on the on the trivial $\mathrm{Sp}(1)$−principal bundle over $\mathbf{H} = \mathbf{R}^4$. The first task is to scale A^{BPST} down by $0 < \lambda \ll 1$ and glue it into A_0 to obtain an almost ASD instanton $\tilde{A}_\lambda$.

Identify a neighborhood of x_0 with $B_{2\varepsilon}(0)$. To simply our live, we will also assume that the metric g is the Euclidean metric on $B_\varepsilon(0)$. The local connection 1−form of A_0 simply vanishes. We would like to "glue" the scaled down version of A^{BPST} with A_0 over the annulus

$$B_{2\varepsilon}(0) \backslash \bar{B}_\varepsilon(0).$$

If $s_\lambda(q) \coloneqq q/\lambda$, then

$$s_\lambda^* A^{BPST} \coloneqq \frac{\mathrm{Im}(\bar{q}\mathrm{d}q)}{|q|^2 + \lambda^2}$$

Unfortunately, the restriction of this to $B_{2\varepsilon}(0) \backslash \bar{B}_\varepsilon(0)$ is not at all small. (This not unexpected: because otherwise we might be on our way to construct a non-flat ASD instanton on the trivial bundle—which is impossible.) Consider the gauge transformation $u(q) = q/|q|$ defined on $\mathbf{H} \backslash \{0\}$. A computation reveals that

$$u^* A^{BPST} = -\frac{\mathrm{Im}(\mathrm{d}q\bar{q})}{|q|^2(1 + |q|^2)}$$

and

$$s_\lambda^*(u^* A^{BPST}) = -\lambda^2 \frac{\mathrm{Im}(\mathrm{d}q\bar{q})}{|q|^2(\lambda^2 + |q|^2)}$$

The restriction of this to $B_{2\varepsilon}(0) \backslash \bar{B}_\varepsilon(0)$ is small if $\lambda \ll 1$.

Define a $\mathrm{Sp}(1)$−principal bundle over X by gluing the trivial $\mathrm{Sp}(1)$−principal bundles over $X \backslash \bar{B}_\varepsilon(0)$ and $B_{2\varepsilon}(0)$ over $B_{2\varepsilon}(0) \backslash \bar{B}_\varepsilon(0)$ via the gauge transformation $u(q) = q/|q|$. Choose a cut-off function $\chi \colon [0, 2) \to [0, 1]$ which is equal to one on $[0, 1]$ and has compact support. Define a connection $\tilde{A}_\lambda$ to agree with

$$\chi(|q|/\varepsilon) s_\lambda^* A^{BPST}$$

on $B_{2\varepsilon}(0)$ and with A_0 on $X \backslash \bar{B}_{2\varepsilon}(0)$. Since

$$u^*(\chi(|q|/\varepsilon) s_\lambda^* A^{BPST}) = -\chi(|q|/\varepsilon) \lambda^2 \frac{\mathrm{Im}(\mathrm{d}q\bar{q})}{|q|^2(\lambda^2 + |q|^2)},$$

this gives us the desired interpolation.

How small is $F^+_{\tilde{A}_\lambda}$? Certainly, $F^+_{\tilde{A}_\lambda}$ vanishes outside of the annulus $B_{2\varepsilon}(0) \backslash \bar{B}_\varepsilon(0)$. To simplify notation, set

$$a_\lambda \coloneqq -\lambda^2 \frac{\mathrm{Im}(\mathrm{d}q\bar{q})}{|q|^2(\lambda^2 + |q|^2)}.$$

Observe that for $\varepsilon \leqslant |q| \leqslant 2\varepsilon$,

$$|a_\lambda| \leqslant c(\varepsilon) \cdot \lambda^2$$

[Note: while ε should be thought of as small, it is also fixed and λ is much smaller than ε.]

We compute

$$F_{\tilde{A}_\lambda} = \chi(|q|/\varepsilon) F_{u^* s_\lambda^* A^{BPST}} + \varepsilon^{-1}\chi'(|q|/\varepsilon)\mathrm{d}|q| \wedge a_\lambda - \frac{1}{2}(\chi(|q|/\varepsilon)^2 - \chi(|q|/\varepsilon))[a_\lambda \wedge a_\lambda]$$

The first term is anti-self-dual. Therefore, it suffices to estimate the last two terms. Therefore,

$$\left\| F^+_{\tilde{A}_\lambda} \right\|_{L^\infty} \leqslant c(\varepsilon, \chi) \cdot \lambda^2.$$

The task at hand is now to find $a = a(\lambda)$ such that

$$F^+_{\tilde{A}_\lambda + a} = F^+_{\tilde{A}_\lambda} + \mathrm{d}^+_{\tilde{A}_\lambda} a + \frac{1}{2}[a \wedge a]^+ = 0.$$

To break the gauge symmetry it is customary to supplement this equation with

$$\mathrm{d}^*_{\tilde{A}_\lambda} a = 0.$$

The full system of equations is then

$$\delta_{\tilde{A}_\Lambda} a + \frac{1}{2}[\alpha \wedge a]^+ + F^+_{\tilde{A}_\lambda} = 0.$$

Remark 5.95. Schematically, this is of the form

$$Lx + \mathscr{N}(x) + E = 0.$$

with L linear, $\mathscr{N}$ non-linear, and E denoting the initial (pre-gluing) error. There is a standard approach towards solving such equations. Let us pretend that $\mathscr{N} = 0$ and that the problem is finite-dimensional. In this case, we can certainly always solve the equation provided that L is surjective. Indeed, if R is a right-inverse of L (that is: $LR = \mathbf{1}$), then

$$x = -RE$$

is the desired solution.

If $\mathscr{N}$ does not vanish, then the equation can still be rewritten as follows by setting $x = Ry$

$$y = -(\mathscr{N}(Ry) + E).$$

This is a fixed-point equation. It can be solved in $B_\rho(0)$ (uniquely) using Banach's fixed-point theorem provided $y \mapsto -(\mathscr{N}(Ry) + E)$ is a contraction for $|y| \leqslant \rho \ll 1$. ♣

The above scheme can be carried out in the situation at hand with

$$L_\lambda := \delta_{\tilde{A}_\lambda} \colon W^{1,p}\Omega^1(X, \mathrm{Ad}(P)) \to L^p\Omega^0(X, \mathrm{Ad}(P)) \oplus L^p\Omega^+(X, \mathrm{Ad}(P))$$

and $p > 2$ (so that $W^{1,p} \hookrightarrow L^{2p}$ with a constant independent of λ by Kato). The hard part is to construct a right-inverse $R_\lambda \colon L^p \to W^{1,p}$ of L_λ with

$$\|R_\lambda\|_{\mathscr{L}} \leqslant c_1 \lambda^{-\alpha}$$

with $0 \leqslant \alpha < 1$. (Here $\|-\|_{\mathscr{L}}$ denotes the operator norm.) (IMHO the best way to do this is to patch right-inverses for the models. For the trivial connection the right-inverse is easy to obtain. For the BPST instanton one has to think a bit.) The easy part is to observe that

$$\left\|F^+_{\tilde{A}_\lambda}\right\|_{L^p} \leqslant c_2\lambda^2$$

and that $\mathscr{N}(a) := \frac{1}{2}[a \wedge a]^+$ satisfies

$$\|\mathscr{N}(a_1) - \mathscr{N}(a_2)\|_{L^p} \leqslant c_3(\|a_1\|_{W^{1,p}} + \|a_2\|_{W^{1,p}})\|a_1 - a_2\|_{W^{1,p}}.$$

Therefore,

$$a \mapsto -\left(\frac{1}{2}[R_\lambda a \wedge R_\lambda a]^+ + F^+_{\tilde{A}_\lambda}\right)$$

is a contraction on $\bar{B}_\rho(0) \subset L^p$ provided

$$c_3c_2^2\lambda^{-2\alpha}\rho < 1 \quad \text{and} \quad c_3c_2^2\lambda^{-2\alpha}\rho^2 + c_2\lambda^2 \leqslant \rho.$$

Since $\alpha < 1$, a suitable ρ can be found.

The upshot of all of this is that if $b^+(X) = 0$, then for every $x \in X$ and every $0 < \lambda \ll 1$ we can construct an ASD instanton $A_{\lambda,x}$ which is modelled (very closely) on a λ–scaled down BPST instanton in a neighborhood of x.

5.17 Donaldson's diagonalisation theorem

Theorem 5.96 (Donaldson [Don83, Theorem 1]). *Let X be a closed oriented* smooth 4*–manifold with $\pi_1(X) = \mathbf{1}$. If the intersection form $Q\colon \mathrm{H}^2(X,\mathbf{Z}) \otimes \mathrm{H}^2(X,\mathbf{Z}) \to \mathbf{Z}$ is positive or negative definite, then it is diagonalisable over $\mathbf{Z}$.*

This is a remarkable theorem. Over $\mathbf{Z}$ it is far from true that every symmetric bilinear form is diagonalisable. The quadratic form given by the matrix

$$E_8 = \begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & -1 & \\ & & & & -1 & 2 & & -1 \\ & & & & -1 & & 2 & \\ & & & & & -1 & & 2 \end{pmatrix}$$

is positive definite. However, E_8 is not diagonalisable over $\mathbf{Z}$. It is possible to construct a closed oriented *topological* 4–manifold X with $\pi_1(X) = \mathbf{1}$ with $Q = E_8$ (the "E_8 manifold"). Q is even and its signature $\sigma(X) = 8$. If X were smooth, then it would admit a spin structure Rohklin's theorem would imply that $\sigma(X)$ is divisible by 16. Therefore, X cannot be equipped with a smooth structure. Donaldson's theorem yields the same conclusion, of course.

The above theorem should be contrasted with the following.

Theorem 5.97 (Freedman [Fre82, Theorem 1.5]). *Let Q be an integral unimodular quadratic form. There is a closed topological 4–manifold X with $\pi_1(X)$ realising Q as its intersection form.*

Donaldson's theorem shows that many of Freedman's manifolds cannot be equipped with smooth structures.

Sketch of proof of Theorem 5.96. There is no loss in assuming that Q is negative definite; i.e.: $b^+(X) = 0$. Denote by $(p\colon P \to X, R)$ the SU(2)–principal bundle with $c_2(P) = 1$. (By a Theorem of Dold and Whitney c_2 specifies (p, R) up to isomorphism.) Choose a generic metric on g in the sense of Freed–Uhlenbeck. Denote by $\bar{\mathscr{M}}$ the Uhlenbeck compactification of the moduli space of ASD instantons on (p, R) with respect to g. The proof is based on a detailed understanding of $\mathscr{M}$.

By the Freed–Uhlenbeck theorem, the subset $\mathscr{M}^*$ of irreducible ASD instantons carries the structure of a smooth manifold of dimension

$$\dim \mathscr{M}^* = 8c_2 + 3(b^1 - 1 - b^+) = 5.$$

We need to understand:

(1) The locus of reducible ASD instantons $\mathscr{M}^{\mathrm{red}} := \mathscr{M}\backslash\mathscr{M}^*$ and how it fits with $\mathscr{M}^*$.

(2) The locus of ideal ASD instantons $\partial\mathscr{M} := \overline{\mathscr{M}}\backslash\mathscr{M}$ and how it fits with $\mathscr{M}^*$.

The first part is by far the easier given the discussion in Section 5.13. The isotropy group Γ_A of a reducible ASD instanton A on (p, R) is either $\mathbf{1}$ or S^1. If $\Gamma_A = \mathbf{1}$, then A is flat; but (p, R) carries not flat connections (by Chern–Weil theory). If $\Gamma_A = S^1$, then as we already discussed, the Hermitian rank 2 vector bundle E associated with (p, R) splits as

$$E = L \oplus L^*$$

and A arises from an ASD instanton on L. Since $b_1(X) = 0$ and $b^+(X) = 0$, L admits unique ASD instanton up to gauge transformations. The bundle L must satisfy

$$1 = c_2(E) = -c_1(L)^2.$$

Indeed, the elements of $\mathscr{M}^{\mathrm{red}}$ precisely corresponds to the pairs $\pm x \in \mathrm{H}^2(X, \mathbf{Z})$ of solutions of

$$Q(x, x) = -1.$$

Denote the number of those solutions by

$$n(Q).$$

Describes $\mathscr{M}^{\mathrm{red}}$ as a finite set. How does it fit the rest of $\mathscr{M}^*$? The discussion in Section 5.13 says that a neigborhood of $[A] \in \mathscr{M}^{\mathrm{red}}$ is modelled on

$$f^{-1}(0)/\Gamma_A$$

with $f\colon \ker\delta_A \supset U \to \operatorname{coker} \mathrm{d}_A^+$ a smooth map with $f(0) = 0$ and $T_0 f = 0$. Here we restrict $\delta_A = (\mathrm{d}_A^*, \mathrm{d}_A^+)$ to $\Omega^1(X, L^2) \to \Omega^0(X, L^2) \oplus \Omega^+(X, L^2)$ Therefore, δ_A and d_A^+ are complex vector spaces and S^1 acts by multiplication with the square unit complex numbers. An application of the index theorem proves that

$$\dim_{\mathbf{C}} \ker \delta_A - \dim_{\mathbf{C}} \operatorname{coker} \mathrm{d}_A^+ = 3.$$

It follows ultimately from the Freed–Uhlenbeck theorem, that $\operatorname{coker} \mathrm{d}_A^+ = 0$. Therefore, a neighborhood of A is modelled on

$$\mathbf{C}^3/S^1 = \operatorname{cone}(\mathbf{C}P^2).$$

[DRAW PICTURE OF WHAT IS KNOWN SO FAR.]

The next task is to identify $\partial\mathscr{M}$. By the energy identity, every $[A] \in \mathscr{M}$ has $\mathrm{YM}(A) = 8\pi^2$. Therefore, if $[A_n]$ in $\mathscr{M}$ converges to $[A_0, \sum_{a=1}^k m_a x_a] \in \partial\mathscr{M}$, then A_0 must be flat, $k = 1$, and $m_1 = 1$ by the convergence statement about the measures. Since $\pi_1(X) = \mathbf{1}$, A_0 must be trivial. Therefore, $\partial\mathscr{M} \subset X$. Taubes' theorem proves that $\partial\mathscr{M} = X$. This suggests that $\overline{\mathscr{M}}\backslash\mathscr{M}^{\mathrm{red}}$ is a smooth manifold with boundary. While this is true, but requires some actual work to construct the charts on the boundary. (This is Donaldson's collar theorem.)

[UPDATE PICTURE.]

The upshot of the above analysis is that $\overline{\mathscr{M}}$ furnishes us with a compact cobordism between X and $n(Q)$ copies $\mathbf{C}P^2$. The proof can now be completed as follows.

Lemma 5.98. *If Q is a negative definite quadratic form over $\mathbf{Z}$, then $n(Q) \leqslant \operatorname{rk} Q$. Equality holds if and only if Q is diagonal.*

Proof. This proved by induction on $r := \operatorname{rk} Q$. If $Q(x) = -1$, then $\mathbf{Z}^r = \mathbf{Z}\alpha \perp (\mathbf{Z}\alpha)^\perp$ via

$$y \mapsto (\langle y, x\rangle \cdot x, y - \langle y, x\rangle \cdot x).$$

Of course, the new intersection form Q' on $(\mathbf{Z}\alpha)^\perp$ has $n(Q') = n(Q) - 1$ and $\operatorname{rk}(Q') = \operatorname{rk}(Q) - 1$. ∎

Since Q is negative definite, $\operatorname{rk}(Q) = \sigma(Q)$. The signature is invariant of oriented cobordisms. Therefore,

$$\operatorname{rk}(Q) = \sigma(Q) = \sum_{a=1}^{n(Q)} \varepsilon_a \sigma(\mathbf{C}P^2) = \sum_{a=1}^{n(Q)} \varepsilon_a \leqslant n(Q)$$

with $\varepsilon_a \in \{\pm 1\}$ according to the orientation the corresponding of $\mathbf{C}P^2$. It follows that

$$n(Q) = \operatorname{rk} Q$$

and, therefore, Q is diagonalisable. ∎

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