Seminar: Riemannian Holonomy
Winter Semester 2024/25

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This seminar takes place every Wednesday 13–15 during the Winter Semester 2024/25 in Rudower Chaussee 25 Haus 1 Room 012.

What is this seminar about?

The choice of a Riemannian metric g on a smooth manifold X induces concepts of distance, geodesics, curvature, volume, etc. Riemannian Geometry is concerned with studying the ramifications of these concepts. Every smooth manifold X admits a Riemannian metric, but X may admit *additional structure*, e.g.: an almost complex structure I.

> What does it mean for such a structure to be *compatible* with the Riemannian metric?

A reasonable answer is to demand that the structure is *parallel* with respect to the Levi-Civita connection. The study of Riemannian manifolds together with a compatible additional structure is *refinement* of Riemannian geometry. Arguably, the most important such refinement Kähler geometry: the study of Riemannian manifolds (X, g) together with parallel (almost) complex structures I.

> What (other) *kind* of structures might be compatible with a Riemannian metric?

This question can be addressed systematically by studying $\mathrm{Hol}(g) \subset \mathrm{O}(n)$, the *holonomy group* "generated by parallel transport". In fact, the *holonomy principle* translates the above question into to the classification of (Riemannian) holonomy groups. This classification has been achieved by Berger [Ber55] in a tour de force. His answer is that, after various simplifications, the following possibilities remain:

$\dim X$	$\mathrm{Hol}(g)$	
n	$\mathrm{SO}(n)$	"generic"
$2n$	$\mathrm{U}(n)$	Kähler
$2n$	$\mathrm{SU}(n)$	Calabi–Yau
$4n$	$\mathrm{Sp}(n)$	hyperkähler
$4n$	$\mathrm{Sp}(1) \cdot \mathrm{Sp}(n)$	quaternion-Kähler
7	G_2	
8	$\mathrm{Spin}(7)$	

The purpose of the first two talks of this seminar is to explain what the above means.

The bulk of the seminar is devoted to understanding (or beginning to understand) the six geometries corresponding to the special holonomy groups. In particular, the talks will address the questions: What are these spaces? How can one work with them? How can one construct them? Moreover, each of these geometries leads to specific questions, conjectures, open problems, etc.

Prerequisites

The seminar assumes a firm grasp on the basic concepts of differential geometry and Riemannian geometry (at the level taught in Differential Geometry I, II at HU Berlin).

Talks

(1) Organisation (Date: 16.10)

(2) Introduction to Riemannian Holonomy; Riemannian Symmetric Spaces, de Rham Decomposition Theorem, Berger's Classification (Date: 30.10, Speaker: Thomas Walpuski)

   (a) Review of Riemannian Geometry; in particular: Levi-Civita connection, Riemann curvature tensor (and its symmetries). parallel transport (of tensor fields).
   (b) Review or introduce Lie groups and their Lie algebras.
   (c) Define the holonomy group $\mathrm{Hol}(g)$ of a Riemannian manifold; mention that it is Lie group.
   (d) Establish the holonomy principle.
   (e) Discuss the Ambrose–Singer Theorem, relating $\mathrm{Hol}(g)$ and the Riemann curvature tensor R_g.
   (f) Riemannian Symmetric Spaces: various characterization, determination of holonomy group; possibly: mention Cartan's classification
   (g) State the de Rham Decomposition Theorem; possibly: sketch proof.
   (h) State Berger's Classification of Riemannian Holonomy Groups: Provide a superficial sketch of Berger's proof.

[Joy00, §3.1, 3.2, 3.3, 3.4]

(3) Kähler manifolds I: (Date: 6.11, Speaker: Chen)

(a) Introduce (almost) complex manifolds.

(b) Provide examples: $\mathbf{C}P^n$, smooth projective varieties, Hopf manifolds, etc.

(c) Discuss the (p, q) decomposition of $\Omega^*(X; \mathbf{C})$ on almost complex manifolds, and the decomposition of the exterior derivative d.

(d) Motivate and introduce the Nijenhuis tensor. State the Newlander–Nirenberg Theorem.

(e) Introduce holomorphic vector bundles, holomorphic sections. Provide examples: $\mathcal{O}_{\mathbf{C}P^n(k)}$, canonical bundle K_X, etc. Possibly: mention Kodaira embedding.

[Joy00, §4.1, 4.2, 4.3; Huy05, §2]

(4) Kähler manifolds II: (Date: 13.11, Speaker: Robin Sluk)

(a) Define the concept of a Kähler manifold as a complex manifold equipped with a Kähler metric.

(b) Construct the Fubini–Study metric on $\mathbf{C}P^n$.

(c) Explain the relation between Kähler manifolds and holonomy $\mathrm{U}(n)$.

(d) Prove the local $\partial\bar{\partial}$–Lemma; state the global $\partial\bar{\partial}$–Lemma; introduce the concepts of Kähler potential, Kähler class, Kähler cone.

[Joy00, §4.4, 4.5; Huy05, §2]

(5) Kähler manifolds III: (Date: 20.11, Speaker: Apratim Choudhury)

(a) Review or summarise Hodge theory on compact Riemannian manifolds.

(b) Prove the Kähler identities.

(c) Explain the Hodge Theory of Kähler manifolds.

(d) Prove the global $\partial\bar{\partial}$–Lemma.

(e) Prove Serre duality.

(f) Prove that Hopf manifolds admits no Kähler metric.

[Joy00, §4.7; Huy05, Appendix A, §3]

(6) Calabi–Yau manifolds I (Date: 27.11, Speaker: Gerard Bargalló)

(a) Recall the canonical bundle K_X. Explain the connection between the curvature of the canonical bundle K_X and Ricci curvature. Introduce the concepts Calabi–Yau metric, Calabi–Yau manifold.

(b) Review or introduce the first Chern class; in particular: $c_1(K_X)$.

(c) Discuss the Calabi conjecture (in the case $c_1(K_X) = 0$) and Yau's Theorem.

(d) Introduce the tools to compute $c_1(K_X)$ for $X = \mathbf{C}P^n$ and complete intersections in $\mathbf{C}P^n$: Euler sequence, adjunction formula etc. Prove that a smooth quartic $Q \subset \mathbf{C}P^3$ admits a Calabi–Yau metric.

[Joy00, §5, §6]

(7) Calabi–Yau manifolds II (Date: 4.12, Speaker: Jacek Rzemieniecki)

(a) Discuss the Calabi–Ansatz. Construct Eguchi–Hanson metric on $\mathcal{O}_{\mathbf{C}P^1}(-2) \cong T^*S^2$.

(b) Discuss the construction of the Stenzel–Metric on T^*S^n.

alternatively: Discuss the Tian–Todorov Theorem on the unobstructedness of deformation theory of Calabi–Yau manifolds.

[Lye23; Huy05, §6]

(8) Hyperkähler 4–manifolds (Date: 11.12, Speaker: Zhengxiong Cao)

(a) Explain the coincidence $\mathrm{Sp}(1) = \mathrm{SU}(2)$

(b) Introduce the concept of hyperkähler 4–manifold.

(c) Discuss the Gibbons–Hawking construction

(d) Construct the (multi-center) Eguchi–Hanson and (multi-center) Taub–NUT metrics using the Gibbons–Hawking construction. Determine their asymptotic geometry at infinity.

(e) Introduce the concept of ALE hyperkähler 4–manifolds ("gravitational instantons"); maybe: mention the work of Kronheimer;

(f) Possibly: summarise the recent developments on "gravitational instantons"

[GH78; GRG97]

(9) Hyperkähler manifolds (Date: 18.12, Speaker: Viktor Majewski)

(a) Introduce the concepts hyperkähler manifold, complex symplectic manifold. Explain their relation via Yau's theorem.

(b) Sketch (possibly very superficially) construction methods of compact hyperkähler manifolds;

(c) Explain the hyperkähler quotient construction.

(d) Sketch the construction of the Eguchi–Hanson space as a hyperkähler quotient. Maybe: recall/mention the work of Kronheimer and give a glimpse of his construction.

[Joy00, §7; HKLR87; Hit92]

(10) Quaternionic Kähler manifolds (Date: 8.1, Speaker: Robin Sluk?)

(a) Introduce the concept of quaternionic Kähler manifolds.
(b) Discuss the known compact examples of quaternionic Kähler manifolds; mention the LeBrun–Salamon conjecture.
(c) Discuss the Quaternionic Kähler/hyperkähler correspondence.

[Joy00, §7.5.2; Bes87, Chapter 14; Swa91; Hay08; Hit13]

(11) G_2–MANIFOLDS (Date: 15.1, Speaker: Jacek Rzemieniecki)

(a) Explain what the exceptional Lie group G_2 is. Discuss how it connects with $\phi \in \Lambda^3(\mathbf{R}^7)^*$ and $\psi \in \Lambda^4(\mathbf{R}^7)^*$. Explain how ϕ (or ψ) determines a Euclidean inner product.
(b) Introduce the concepts (almost) G_2–manifold, G_2–structure, torsion of a G_2–structure; explain their relation.
(c) Prove Fernandez's Theorem on the torsion of a G_2–structure; along the way discuss salient points of the representation theory of G_2.

[Joy96a; Joy96b; Joy00, §10; Bry06]

(12) G_2–MANIFOLDS (Date: 22.1, Speaker: Sven Jüttermann)

(a) Explain Bryant–Salamon's construction complete non-compact G_2–manifolds
(b) Discuss the G_2–Teichmüller space and Joyce's theorem on the unobstructedness of the deformation theory.

[BS89]

(13) G_2–MANIFOLDS (Date: 29.1, Speaker: Fabian Lehmann)

(a) State Joyce's G_2 perturbation theorem: "almost torsion-free" $\rightsquigarrow$ torsion-free.
(b) Sketch Joyce's generalised Kummer construction.

[Joy96a; Joy96b; Joy00, §11, §12]

(14) Spin(7)–MANIFOLDS (Date: 5.2, Speaker: Sven Jüttermann)

(a) Explain what Spin(7) is and its embedding $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$.
(b) Introduce the concept Spin(7)–manifold.
(c) Explain the relation of Spin(7) to G_2, Sp(2), SU(4).
(d) Sketch the statement of Joyce's Spin(7) perturbation theorem.
(e) Sketch Joyce's "new" construction of Spin(7)–manifolds.

[Joy99; Joy00, §13, §15]

(15) CALIBRATIONS (Date: 12.2, Speaker: Jacek Rzemieniecki)

(a) Introduce the concepts calibration, calibrated submanifold. Explain the connection between calibrated submanifolds and minimal submanifolds (homologically volume minimising submanifolds).

(b) Discuss that Riemannian manifolds with special holonomy carry calibrations, and explain some construction methods for calibrated submanifolds.

(c) Options:

  (i) Discuss the deformation theory of (certain) calibrated submanifolds. Sketch Joyce's and Donaldson–Thomas's ideas regarding enumerative invariants from calibrated submanifolds.

  (ii) Discuss the Thomas–Yau conjecture regarding special Lagrangians.

  (iii) Discuss the Angle Theorem and the Nance calibration.

[HL82]

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